CORRELATED BIVARIATE SEQUENCES FOR QUEUEING AND RELIABILITY APPLICATIONS

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ABSTRACT

We derive bivariate exponential, gamma, Coxian or hyperexponential distributions. To obtain a positive correlation, we define a linear relation between the variates X and Y of the form $Y = aX + Z$ where a is a positive constant and Z is independent of X. By fixing the marginal distributions of X and Y, we characterize the distribution of Z. To obtain negative correlations, we define $X = aP + V$ and $Y = bQ + W$ where P and Q are exponential antithetic random variables.

Our bivariate models are useful in introducing dependence between the interarrivals and service times in a queueing model and in the failure process in multicomponent systems. The primary advantage of our model in the context of queueing analysis is that it remains mathematically tractable because the Laplace Transform of the joint distribution is a rational function, that is a ratio of polynomials. Further, the variates can be very easily generated for computer simulation. These models can also be used for the study of transmission controlled

queueing networks.

1. INTRODUCTION

The single variable exponential distribution has been extensively used in queueing, life testing, reliability and many other models. This is primarily because its memorylessness property lends itself to easy analysis. To make the models represent the real life models, many distributions, like gamma, hyperexponential and Coxian, or more generally, phase type distributions have been derived from the exponential distribution. In fact it has been shown that the set of phase type distributions is dense and is suited especially for use in situations where the moments and/or the tail of the observed distribution does not match that of the exponential. Whenever there are two or more variables in a system model, in many analyses they are assumed to be independent. Consider, for example, a queuing system. The input process to this system is a bivariate sequence $\{Y_n, X_n\}$ where Y_n is the interarrival time between the arrivals of the n^{th} and $(n+1)^{\text{th}}$ customer and X_n is the service time of the n^{th} customer. Analytical queueing models have mostly concentrated on the GI/GI/m kind of models where it is assumed that the Y_n and the X_n are each i.i.d. random variables and there is neither autocorrelation in the Y_n or X_n nor a crosscorrelation between them. In most real life environments these independence assumptions do not quite hold. We illustrate the need to model autocorrelations and cross correlations in bivariate sequences using two specific examples – one from queueing analysis of packet communication networks and a second one for reliability and life testing models.

First consider packet traffic. Analysing network traffic traces (available from, for example, `http://ita.ee.lbl.gov` and `http://pma.nlanr.net/Traces/`), it has been observed that there can be significant crosscorrelation between the Y_n and the X_n in addition to the autocorrelations. Further, while the autocorrelations are typically positive, the crosscorrelation takes both positive and negative values and are significant at low lags and depends on the traffic mix. From our knowledge of the physical phenomena, we attempt a partial explanation. Data is exchanged in the network over TCP connections and a connection may be opened for each item like an image or a file. Typically, these items are quite large, and so a typical transfer will involve a large number of maximum sized packets sent with a small "interarrival" time between them. Thus transfer of bulk items leads to long packets and short interarrival times. Further, in a connection, long packets are followed by long packets. Now consider transactions on the network, like for example in applications like `telnet`, or the opening of `http` connections. In these applications typically only short packets are exchanged. Since each of these need to be processed at the other end before the reply is sent, the interarrival times will be relatively long. This and other similar behavior was first discussed in the packet train model of Jain and Routhier (1).

There have been models in the queueing literature that introduce correlations between Y_n and X_n. Cidon et. al (2) consider the model

$$Y_n = \Omega_n X_n + Z_n \tag{1.1}$$

where X_n are exponential, Ω_n are discrete i.i.d. with a finite support and independent of X_n, and Z_n is another nonnegative random variable independent of Ω_n and X_n. Note that for $Z_n = 0$, the marginal of Y_n becomes a mixture of exponentials. In a previous paper, Cidon et al. (3) had considered modeling the relationship between X_n and Y_{n-1} in a similar manner. The queueing analysis obtained explicit expressions for the mean waiting time in a single server queue when Z_n is exponential. It is shown that there is a significant decrease in the mean waiting time due to the correlation which further motivates the requirement of models that can incorporate specified marginals for the two variables.

In this paper, we do not aim to do queueing analysis. Rather, we wish to introduce a class of models for bivariate series that we expect to be of independent interest in other modeling situations, like for example reliability. In fact, later in this section, we will provide one example in reliability. The bivariate sequences (X_n, Y_n) that we define in this paper are of the form

$$Y_n = aX_n + Z_n, \tag{1.2}$$

where the distributions of Y_n and X_n will be specified to be exponential, Erlang, hyperexponential or Coxian. Although this appears to be a more restricted version of (1.1) we highlight the following important differences.

1. In (1.1) the distribution of Z_n is specified, while for our model of (1.2) the distribution of Z_n will be derived based on the marginals specified for X_n and Y_n. For example, in the case when both Y_n and X_n are exponential with parameters λ and μ respectively, we will show that Z_n will be the product of two independent random variables – a Bernoulli(ρ) random variable with $\rho = a\,\mu/\lambda$, and an exponential random variable with parameter λ. Our interest is in deriving correlated bivariate series (X_n, Y_n) with specified marginals. We show existence by characterising the Z_n corresponding to the specified marginal distributions of X_n and Y_n.

2. It can be seen that $\mathtt{cov}(Y_n, X_n) = E(\Omega_n)\sigma_X^2$ where σ_X^2 is the variance of X_n. Since Ω_n are necessarily nonnegative, only positive crosscorrelations are addressed in (1.1). As we show later in the paper, negative crosscorrelation is modelled using two separate equations of the form given in (1.2) for X_n and Y_n and connecting them using antithetic variables.

3. We can specify the marginal distribution of Y_n and X_n from a very large class of distributions.

4. Cidon et al. (2) discuss in detail application of the model in (1.1) to study the effect of control mechanisms in packet communication networks. The model is used to describe a spacer controller to limit the peak rate at which a source can submit to a network. The model can also capture the effect of interactions between packets from a given source and other traffic streams inside the network. Using the spacer to randomise the gaps between successive packets helps in avoiding correlation between traffic streams of distinct sources. This in turn can help prevent malicious sources from adversely affecting network performance by cooperating to generate a large burst of data into the network. Finally, the random proportionality constant Ω_n can be used to enforce priorities for different types of packets. These applications also apply to the model (1.2). Note that the random proportional dependency Ω_n can be incorporated in (1.2) by letting X_n possess a mixture distribution. In (1.2), we seek to enforce the control

while keeping the marginal traffic characteristics, that is the distributions of X_n and Y_n, unaltered. Thus the control is non-intrusive in the sense of not affecting the marginals. Further, it is easier to choose the control parameter a in (1.2), after estimating the marginal distributions of X_n and Y_n. Most importantly, queueing analysis remains tractable for a large choice of marginal distributions for X_n and Y_n as we will show in a separate paper. In contrast, in (1.1) one has to choose the distribution of Z_n to achieve the desired control and keep the corresponding queueing analysis tractable.

To see an application of this model to a multicomponent system, assume for the moment that $a = 1$, that is, $Y = X + Z$. Consider a backup system protected by two safety devices D_Y and D_X that are subject to shocks from the same source(s). The marginal lifetime distributions of D_Y and D_X are exponential with parameters λ_Y and λ_X respectively. The shocks are first absorbed by device D_X. As soon as device D_X fails, device D_Y takes over. So, the lifetime Y of D_Y is the lifetime X of D_X plus a random amount Z. We will show later that Z will be a product of a Bernoulli random variable with $P[Z = 0] = \rho = \lambda_Y/\lambda_X$, and an independent exponential with parameter λ_Y. Thus, $Z = 0$ or $X = Y$ with probability ρ. Thus ρ is the probability that the shock is strong enough to destroy both the devices simultaneously. If the shock is not strong enough and destroys only the component D_X (this happens with probability $1 - \rho$), then device D_Y takes over and it lasts a lifetime that is exponentially distributed with parameter λ_Y. If $a \neq 1$, say $a > 1$, then we have a situation where D_Y suffers wear and tear from stresses, which also effect D_X proportionately. In addition D_Y suffers deterioration from an independent source. Failure of component D_X, at time X will require typically (that is with some probability $a\,\rho$), replacement of component D_Y at time $Y = aX$.

The exponential is the most widely used distribution in queueing and reliability models. Thus it is not surprising that much of the literature on nonnegative bivariate distributions motivated by these applications is for exponential distributions. The most well known of the bivariate exponential distributions was derived by Marshall and Olkin (4,5) by considering shock models. If X and Y are the lifetimes of two components of a system, then the distributions of X and Y are exponential with parameters $\lambda_1+\lambda_{12}$ and $\lambda_2+\lambda_{12}$ and the joint distribution of X and Y is $F(x,y) = 1-\exp(-\lambda_1 x-\lambda_2 y-\lambda_{12}\max(x,y))$. This distribution is not absolutely continuous but satisfies the lack of memory property (LMP). In fact it has been shown that the above is the only bivariate distribution that has exponential marginals and which satisfies LMP. Properties of this Marshall-Olkin model are described in detail in Barlow and Proschan (6). Block and Basu (7) consider a bivariate distribution whose marginals are mixtures of exponentials and having an absolutely continuous joint distribution. A survey of various bivariate and multivariate exponential models can be found in Basu (8). Multivariate Gamma distributions have also been considered in the reliability literature. See, for example, Krishnaiah (9). It is important to note that, like in the model of (1.1), all the bivariate models described above provide for only positive correlation between the Y_n and X_n. Arnold and Strauss (10) characterize bivariate distributions with exponential conditional distributions (and negative correlations). Arnold and Strauss (11) characterize bivariate distributions where the conditional distributions are in prescribed exponential families. Arnold, Castillo and Sarabia (12) is devoted to the study of models with conditional specification.

The authors have, in an earlier work Iyer and Manjunath (13), explored the bivariate exponential using linear structures. In this paper we extend the results from that work to include other marginal distributions and also to consider bivariate sequences. We consider a family of bivariate distributions that are marginally exponential, Gamma, hyperexponential or Coxian. Mathematical analysis remains fairly straightforward with this family of distributions in spite of the dependence that is introduced among the variates. This is not true of many bivariate exponential distributions in literature. Further, these variables are very easy to simulate on a computer. This work should be considered as an attempt at modeling short range dependencies in bivariate sequences. We also model negative crosscorrelation by the use of antithetic variables.

2. CROSSCORRELATIONS USING A LINEAR STRUCTURE

The bivariate distributions to introduce crosscorrelation between the variates that we develop in the following are based on the exponential autoregressive sequences of Gaver and

Lewis (14). While borrowing from the work of Gaver and Lewis (14), we would like to point out the differences with that as well. The major difference is that while Gaver and Lewis construct an autoregressive AR(1) univariate sequence X_n such that X_n and X_{n+1} are correlated and the X_n have identically distributed exponential marginals, our interest is to construct bivariate distributions with specified marginals and crosscorrelations.

2.1. POSITIVE CROSSCORRELATION

Let us first consider positive correlations. Also, for now we will assume that the autocorrelation in Y_n and X_n are zero. In this case, taking inspiration from Gaver and Lewis (14), we use an auxiliary random variable Z_n and let

$$Y_n = aX_n + Z_n \tag{2.1}$$

where a is non negative constant and Z_n is independent of X_n. Let $\mathcal{X}(s)$ be the Laplace Stieltjes transform (LST) of the distribution of X_n and $\mathcal{Y}(s)$ that of Y_n. We see from (2.1) that $\mathcal{Z}(s)$ is

$$\mathcal{Z}(s) = \frac{\mathcal{Y}(s)}{\mathcal{X}(as)} \tag{2.2}$$

Let $\mathtt{cov}(Y_n, X_n)$ denote the covariance of Y_n and X_n. It is easy to see that $\mathtt{cov}(Y_n, X_n) = a\sigma_X^2$ where σ_X^2 is the variance of X_n. Thus (2.2) defines a bivariate sequence with nonnegative correlations. Note that (2.1) imposes the feasibility condition that $\sigma_Y > a\sigma_X$. Further, when Y_n and X_n are related as in (2.1), $\mathcal{XY}(s_1, s_2)$, the two dimensional LST of the joint distribution of Y_n and X_n, will be

$$\begin{aligned}
\mathcal{XY}(s_1, s_2) &= E\left(e^{-s_1X_n - s_2Y_n}\right) = E\left(e^{-s_1X_n - s_2(aX_n + Z_n)}\right) \\
&= E\left(e^{-(s_1+as_2)X_n}\right) E\left(e^{-s_2Z_n}\right) = \mathcal{X}(s_1 + as_2)\mathcal{Z}(s_2) \\
&= \frac{\mathcal{X}(s_1 + as_2)\mathcal{Y}(s_2)}{\mathcal{X}(as_2)}
\end{aligned} \tag{2.3}$$

Let us now consider some example distributions for X_n and Y_n and see the form of the random variable Z and the form of the two dimensional LST of the joint distribution of X_n and and Y_n. For the specific case of exponential distribution for both Y_n and X_n, we also derive the joint failure rate.

2.1.1 EXPONENTIAL X_n AND EXPONENTIAL Y_n

First, let Y_n and X_n be exponential, with rates λ and μ respectively and let $\rho = \lambda/\mu$ with $a\rho \leq 1$. The following is easily obtained.

$$\mathcal{Z}(s) = \rho\frac{as+\mu}{s+\lambda} = (1-a\rho)\frac{\lambda}{s+\lambda} + a\rho \tag{2.4}$$

$$\mathcal{XY}(s_1, s_2) = \left[\frac{\mu}{\lambda + s_1 + as_2}\right]\left[(1-a\rho)\frac{\lambda}{s_2+\lambda} + (a\rho)\right]$$

$$r_{YX} = a\rho$$

where r_{YX} is the crosscorrelation between Y_n and X_n. Inverting $\mathcal{Z}(s)$, we find that the distribution of Z to be $a\rho + (1-a\rho)(1-e^{-\lambda z})$, $z \geq 0$, i.e., Z is a product of two independent random variables - a Bernoulli random variable with mean $(1-a\rho)$ and an exponential random variable with parameter λ.

The following is an interpretation of the model for a packet communication network where it is reasonable to model packet transmissions as bursty processes in which during the burst, interarrival times are strongly correlated and bursts arrive independently. Applying (1.2) to this case would imply that the burst length, number of packets in the burst, is geometrically distributed with mean $1/(a\rho)$ and the time from the end of one burst till the beginning of the next one will be exponentially distributed with parameter λ. During the burst, the time at which the "next packet" arrives will be a constant multiple of the current packet length. The marginal interarrival times and packet lengths will of course be exponential with parameter μ and λ respectively.

In reliability applications the failure rate is an important parameter and in the following, we derive it for the case exponential marginals of Y_n and X_n. In the process we also obtain the joint distribution and density functions. Rewrite (2.1) without the subscript n as

$$Y = aX + Z, \tag{2.5}$$

where Y, X and Z have the same interpretation as before. From (2.5) and the form of Z, it is clear that the support of the joint distribution $F(y, x)$ of (Y, X) is the region $\{y \geq ax\}$

with a jump on the line $y = ax$.

$$\begin{aligned} F(y,x) &= P(Y \le y, X \le x) \qquad\qquad \text{for } y \ge ax \ge 0. \\ &= P(aX + Z \le y, X \le x) \end{aligned}$$

$$= p\ P\left(X \le \frac{y}{a}, X \le x\right) + (1-p)\ P\left(X \le \frac{y-W}{a}, X \le x\right), \tag{2.6}$$

where W is an exponential random variable with parameter λ independent of X and $p = a\rho$. Since $y \ge ax$, the first term in (2.6) is

$$pP(X \le \frac{y}{a}, X \le x) = p(1 - e^{-\mu x}) \tag{2.7}$$

Now consider the second term of (2.6).

$$P(X \le \frac{y-W}{a}, X \le x) = \int_0^{y-ax} (1 - e^{-\mu x})\, \lambda e^{-\lambda u} du + \int_{y-ax}^{y} \left(1 - e^{-\mu \frac{(y-u)}{a}}\right) \lambda e^{-\lambda u} du$$

$$= (1 - e^{-\mu x}) + e^{-\lambda y}\left(e^{-(\mu - a\lambda)x} - 1\right) - \frac{a\lambda}{a\lambda - \mu} e^{-\lambda y}\left(e^{-(\mu - a\lambda)x} - 1\right) \tag{2.8}$$

Observing that $-a\lambda/(a\lambda - \mu) = p/(1-p)$, and substituting from (2.7) and (2.8) into (2.6), we get

$$F(y,x) = (1 - e^{-\mu x}) + e^{-\lambda y}(e^{-(\mu - a\lambda)x} - 1) \qquad y \ge ax \ge 0. \tag{2.9}$$

Note that

$$F(\infty, x) = 1 - e^{-\mu x}, \qquad x \ge 0, \tag{2.10}$$

and

$$F(y, \infty) = F(y, \frac{y}{a}) = 1 - e^{-\lambda y}, \qquad y \ge 0. \tag{2.11}$$

The joint density of (Y, X) is easily seen to be

$$f(y,x) = p\lambda e^{-\lambda y}\delta_0(y - ax) + (1-p)\lambda\mu e^{-(\mu - a\lambda)x} e^{-\lambda y} I_{\{y > ax\}}, \qquad y \ge ax \ge 0, \tag{2.12}$$

where δ_0 is the Dirac delta function at 0.

The joint failure rate of (Y, X) is defined as

$$\begin{aligned} r(y,x) &= \frac{f(y,x)}{P[Y > y, X > x]}, \qquad y \ge ax \ge 0, \\ &= \frac{f(y,x)}{1 + F(x,y) - F(\infty, x) - F(y, \infty)}. \end{aligned} \tag{2.13}$$

Substituting from (2.9)-(2.12), we get

$$r(y,x) = \frac{p\lambda e^{-\lambda y}\delta_0(y-ax) + (1-p)\lambda\mu e^{-(\mu-a\lambda)x}e^{-\lambda y}I_{\{y>ax\}}}{e^{-\lambda y}e^{-(\mu-a\lambda)x}}, \qquad y \geq ax \geq 0. \tag{2.14}$$

The interesting point to be noted from (2.14) is that although the failure rates of the marginals are constant, the joint failure rate is not. On the line $y = ax$, the failure rate is $p\lambda\exp((\mu - a\lambda)x)$, which implies an increasing failure rate since $p = a\lambda/\mu < 1$. If $y > ax$, then the failure rate is a constant $(1-p)\lambda\mu$.

2.1.2. ERLANG X_n AND GAMMA Y_n

Here we assume that X_n and Y_n are a sequences of i.i.d. random variables having Erlang(ℓ, μ) and Gamma(α, λ) distributions respectively where ℓ is a positive integer and $\alpha \geq \ell$. The LST of Gamma(α, λ) distribution is

$$\mathcal{Y}(s) = \left(\frac{\lambda}{s+\lambda}\right)^{\alpha} \tag{2.15}$$

In this case the LST of the distribution of the variable Z is given by,

$$\mathcal{Z}(s) = \left(\frac{\lambda}{s+\lambda}\right)^{\alpha}\left(\frac{\mu+as}{\mu}\right)^{\ell} = \left(\frac{\lambda}{s+\lambda}\right)^{\alpha-\ell}\left(\rho\frac{\mu+as}{\mu}\right)^{\ell} \tag{2.16}$$

This is the sum of two random variables – a Gamma$(\alpha - \ell, \lambda)$ distributed random variable and another that is the sum of ℓ i.i.d. random variables that are the product of a Bernoulli and an exponential like the Z_n in Section 2.1.1. For a queueing like application this model allows the interarrival times to be described by a mixture of a random number (up to ℓ) of exponentials of rate λ.

2.1.3. EXPONENTIAL X_n AND HYPEREXPONENTIAL Y_n

Next we consider the case when X_n is exponential and Y_n is hyperexponential. In its most general form, the hyperexponential (also called a mixture of exponentials) density function has the form

$$\sum_{i=1}^{K} p_i\lambda_i e^{-\lambda_i x} \tag{2.17}$$

with $0 \leq p_i \leq 1$ and $\sum_{i=1}^{K} p_i = 1.0$. Using (2.2) the LST of Z_n is given by

$$\mathcal{Z}(s) = \sum_{i=1}^{K} p_i\left[(1-a\rho_i)\frac{\lambda_1}{s+\lambda_i} + a\rho_i\right], \tag{2.18}$$

where $\rho_i = \lambda_i/\mu$. Evidently, Z_n is a mixture of random variables with LST of the form given by (2.4). The joint LST of X_n and Y_n will satisfy

$$\mathcal{XY}(s_1, s_2) = \left[\frac{\mu}{\mu + s_1 + as_2}\right]\left[\sum_{i=1}^{K} p_i \left\{(1 - a\rho_i)\frac{\lambda_i}{s_1 + \lambda_i} + (a\rho_i)\right\}\right] \tag{2.19}$$

2.1.4. EXPONENTIAL X_n AND COXIAN Y_n

Now let Y_n be a k-stage Coxian with p_i being the probability that stage i is entered and λ_i the rate of the exponential of stage i. The LST of this distribution is given by

$$\mathcal{Y}(s) = \sum_{i=1}^{k}(1 - p_{i+1})\prod_{j=1}^{i} p_j \frac{\lambda_j}{s + \lambda_j}$$

with $0 \leq p_i \leq 1.0$ for $i = 1, 2, \cdots k$. We will assume continuous Coxian with $p_1 = 1$ and $p_{k+1} = 0$. For simplicity we take $k = 2$ and rewrite the LST of Y_n as

$$\mathcal{Y}(s) = p_1 \frac{\lambda_1}{s + \lambda_1} + (1 - p_1)\frac{\lambda_1}{s + \lambda_1}\frac{\lambda_2}{s + \lambda_2}$$

Let $\rho_1 = \frac{\lambda_1}{\mu}$. With this, we can write $\mathcal{Z}(s)$ as

$$p_1\rho_1\frac{as + \mu}{s + \lambda_1} + (1 - p_1)\left[a\rho_1\frac{\lambda_2}{s + \lambda_2} + [1 - a\rho_1]\frac{\lambda_1\lambda_2}{(s + \lambda_1)(s + \lambda_2)}\right]$$

Thus with probability p_1, Z_n is a product of a Bernoulli and an exponential random variable (see (2.4)), and with probability $1 - p_1$ it is a two stage Coxian. Once again, for a queueing model this would imply that the burst length is geometric with parameter $ap_1\frac{\lambda_1}{\mu}$ and the inter burst times is a mixture of an exponential and a Coxian. This can now be extended to the case $k > 2$. Using (2.3) we get

$$\mathcal{XY}(s_1, s_2) = \left(\frac{\mu + as_2}{\mu + s_1 + as_2}\right)\left[\sum_{i=1}^{k}(1 - p_{i+1})\prod_{j=1}^{i} p_j \frac{\lambda_j}{s_2 + \lambda_j}\right]$$

2.2. NEGATIVE CROSSCORRELATION

To obtain a negative crosscorrelation between Y_n and X_n, we use two separate linear equations for Y_n and X_n and introduce a negative correlation between them by using antithetic variables. More precisely consider the following equation

$$Y = aP + V \qquad X = bQ + W, \tag{2.20}$$

where (Y_n, X_n) is an i.i.d. sequence of random vectors distributed as (Y, X). Here a and b are non negative constants, P and V are independent of each other and so are Q and W. The random variables P and Q are assumed to be exponential. The distributions of Y and X could be Gamma, Hyperexponential or Coxian. We do not assume that Y and X have the same type of distribution. The random variables V and W are independent and are also independent of P and Q. The distributions of V and W have been characterised in Section 2.1.

Following Gaver and Lewis (14), to induce a negative correlation between Y and X, the variables P and Q would be considered to be antithetic. that is, we define P and Q through U which is a uniformly distributed random variable in $(0, 1)$, as follows

$$P = -\frac{1}{\lambda_p}\ln(U) \qquad Q = -\frac{1}{\lambda_q}\ln(1-U) \tag{2.21}$$

Under this definition P and Q are exponential random variables with parameter λ_p and λ_q respectively. Note that

$$\mathtt{cov}(Y, X) = ab\,\mathtt{cov}(P, Q) = \frac{ab}{\lambda_p \lambda_q}\left(1 - \frac{\pi^2}{6}\right) < 0 \tag{2.22}$$

If Y and X are exponential with parameters λ and μ respectively, then the correlation coefficient between Y and X, r_{YX}, is given by

$$r_{YX} = \frac{ab\lambda\mu}{\lambda_p \lambda_q}\left(1 - \frac{\pi^2}{6}\right) \qquad \text{for } 0 \le \left(\frac{a\lambda}{\lambda_p}\right), \left(\frac{b\mu}{\lambda_q}\right) \le 1 \text{ and } a, b \ge 0 \tag{2.23}$$

From (2.23) it follows that $\left(1 - \frac{\pi^2}{6}\right) \le r_{YX} \le 0$.

The LST of the joint distribution of Y and X, $\mathcal{XY}(s_1.s_2)$, is obtained as follows.

$$\begin{aligned}
\mathcal{XY}(s_1, s_2) &= E(e^{-s_1 X - s_2 Y}) = E(e^{-s_2 V - s_1 W})E(e^{-as_2 P - bs_1 Q}) \\
&= \left[\left(1 - \frac{a\lambda}{\lambda_p}\right)\frac{\lambda}{s_2 + \lambda} + \left(\frac{a\lambda}{\lambda_p}\right)\right]\left[\left(1 - \frac{b\mu}{\lambda_q}\right)\frac{\mu}{s_1 + \mu} + \left(\frac{b\mu}{\lambda_q}\right)\right] \times \\
&\quad \times \int_0^1 u^{as_2/\lambda_p}(1-u)^{bs_1/\lambda_q}\,du
\end{aligned} \tag{2.24}$$

Note that the integral on the RHS of (2.24) corresponds to the beta function.

3. MODELING AUTOCORRELATION

Among the many models available to model autocorrelation in a time series, the autoregressive (AR) model is the simplest one. The standard AR models assume that the innovation process is a zero mean Gaussian. However, this can lead to the time series taking on negative values. In the context of teletraffic models, the interarrival times and the service times are always nonnegative. Thus, the standard AR model is not quite useful in modeling these variables individually. To overcome this difficulty, we can use the AR process with nonnegative innovations as described in Gaver and Lewis (14). Let Y_n be the non-negative series corresponding to either the interarrival time or the service time and consider the usual linear, additive autoregressive equation of the form,

$$Y_n = aY_{n-1} + Z_n \tag{3.1}$$

where Z_n is the innovation process and a is a constant such that $0 \leq a < 1$. This sequence has various attractive features. Assuming that the marginal distribution of Y_n is given, the distribution for Z_n is obtained as follows. Since Y_{n-1} and Z_n are independent, the LST of the distribution of Z_n is obtained by taking LST of both sides of (3.1).

$$\begin{aligned} \mathcal{Y}_n(s) &= E[\exp(-sY_n)] = E[\exp(-s(aY_{n-1} + Z_n))] \\ &= \mathcal{Y}_{n-1}(as)\mathcal{Z}_n(s) \end{aligned}$$

Therefore, assuming that the series is stationary, we get,

$$\mathcal{Z}(s) = \frac{\mathcal{Y}(s)}{\mathcal{Y}(as)} \tag{3.2}$$

If a is negative then the LST will not exist for $s > 0$ and there will be no solution for (3.2). The covariance between Y_n and Y_{n+j}, $\mathtt{cov}(Y_n, Y_{n+j})$, is obtained as follows.

$$\begin{aligned} \mathtt{cov}(Y_n, Y_{n+j}) &= E(Y_n Y_{n+j}) - E(Y_n)E(Y_{n+j}) \\ &= E((a^j Y_n + a^{j-1}Z_{n+1} + a^{j-2}Z_{n+2} + \cdots + Z_{n+j})Y_n) - \\ &\qquad -E(a^j Y_n + a^{j-1}Z_{n+1} + a^{j-2}Z_{n+2} + \cdots Z_{n+j})E(Y_n) \\ &= E(a^j Y_n^2) - a^j (E(Y_n))^2 = a^j \sigma_Y^2 \end{aligned}$$

where σ_Y is the standard deviation of Y_n (and Y_{n+j}). The autocorrelation between Y_n and Y_{n+j} at lag j, $r_{YY}(j)$ is given by

$$r_{YY}(j) = \frac{\mathtt{cov}(Y_n, Y_{n+j})}{\sigma_Y^2} = a^j$$

Note that, in this method, we can only characterise positive autocorrelations in the Y_n process. We will now solve (3.2) for various marginal distributions of the random variable Y_n, viz., exponential distribution, hyperexponential distribution, Gamma distribution and the Coxian distribution. The cases of exponential, Gamma and Hyperexponential marginal distributions have been considered in Gaver and Lewis (14) and we only reproduce these results. The case of the Coxian marginal is derived here for the first time.

3.1. EXPONENTIAL DISTRIBUTION

Let the Y have an exponential marginal distribution with parameter λ. The probability density function of Y, $f_Y(y)$ is given by

$$f_Y(y) = \lambda e^{-\lambda y}$$

Thus for this case, (3.2) reduces to

$$\mathcal{Z}(s) = \frac{\lambda}{s+\lambda}\frac{as+\lambda}{\lambda} = a + (1-a)\frac{\lambda}{s+\lambda}$$

It is easy to see that this LST corresponds to a non-negative random variable which takes a value 0 with probability a and a positive value with exponential distribution of parameter λ with probability $(1-a)$. This is very easy to simulate.

3.2. GAMMA DISTRIBUTION

Let the marginal distribution of Y_n be a Gamma(α, λ). The probability density function of Y_n, $f_Y(y)$ is given by

$$f_Y(y) = \frac{\lambda^\alpha y^{\alpha-1} e^{-\lambda y}}{\Gamma(\alpha)}$$

$\Gamma(\alpha)$ is the gamma integral for the real number α. The LST of this distribution is

$$\mathcal{Y}(s) = \left(\frac{\lambda}{s+\lambda}\right)^\alpha$$

Solving (3.2) gives

$$\begin{aligned}\mathcal{Z}(s) &= \left(\frac{\lambda+as}{\lambda+s}\right)^\alpha = \left(a + (1-a)\frac{\lambda}{\lambda+s}\right)^\alpha \\ &= \sum_{i=0}^{\alpha} \binom{\alpha}{i} a^i (1-a)^{\alpha-i} \left(\frac{\lambda}{\lambda+s}\right)^{\alpha-i}\end{aligned}$$

Thus in general for Y_n a Gamma(α), Z_n is a random sum of Gamma(α) distributed variables.

3.3 HYPEREXPONENTIAL DISTRIBUTION

Now consider Y_n to be sequence having a marginal hyperexponential distribution like in (2.17). We will solve (3.2) for the simple case of $K = 2$.

$$f_Y(y) = p_1\lambda_1 e^{-\lambda_1 y} + p_2\lambda_2 e^{-\lambda_2 y}, p_1 + p_2 = 1.$$

$$\begin{aligned}\mathcal{Y}(s) &= p_1\frac{\lambda_1}{\lambda_1+s} + p_2\frac{\lambda_2}{\lambda_2+s} = \frac{\lambda_2}{\lambda_2+s}\left[p_2 + p_1\frac{\lambda_1}{\lambda_2}\left(\frac{\lambda_2+s}{\lambda_1+s}\right)\right] \\ &= \frac{\lambda_2}{\lambda_2+s}\left[1 - p_1\left(1-\frac{\lambda_1}{\lambda_2}\right) + p_1\left(1-\frac{\lambda_1}{\lambda_2}\right)\frac{\lambda_1}{\lambda_1+s}\right]\end{aligned}$$

The necessary and sufficient condition for $f_Y(y)$ to be a distribution function is

$$\gamma = p_1\left(1 - \lambda_1/\lambda_2\right) < 1$$

When this condition is satisfied we can solve (3.2) to get $\mathcal{Z}(s)$ as

$$\begin{aligned}\mathcal{Z}(s) &= \left(\frac{p_1\lambda_1}{\lambda_1+s} + \frac{p_2\lambda_2}{\lambda_2+s}\right)\left(\frac{p_1\lambda_1}{\lambda_1+as} + \frac{p_2\lambda_2}{\lambda_2+as}\right)^{-1} \\ &= \frac{(\lambda_1+as)(\lambda_2+as)(\lambda_1\lambda_2+p_1\lambda_1 s+p_2\lambda_2 s)}{(\lambda_1+s)(\lambda_2+s)(\lambda_1\lambda_2+p_1\lambda_1 as+p_2\lambda_2 as)} \\ &= a + (1-a)\left(\gamma_1\frac{\lambda_1}{\lambda_1+s} + \gamma_2\frac{\lambda_2}{\lambda_2+s} + \gamma_3\frac{C/a}{C/a+s}\right)\end{aligned}$$

where

$$\begin{aligned} C &= \tfrac{\lambda_1\lambda_2}{p_1\lambda_1+(1-p_1)\lambda_2} & \gamma_1 &= \tfrac{(\lambda_2-a\lambda_1)(C-\lambda_1)}{(\lambda_2-\lambda_1)(C-a\lambda_1)} \\ \gamma_2 &= \tfrac{(\lambda_2-C)(a\lambda_2-\lambda_1)}{(\lambda_2-\lambda_1)(a\lambda_2-C)} & \gamma_3 &= a\tfrac{(C-\lambda_1)(\lambda_2-C)}{(C-a\lambda_1)(C-a\lambda_2)}\end{aligned} \tag{3.3}$$

Without loss of generality, let $\lambda_1 < \lambda_2$. From (3.3),

1. $\lambda_1 < C < \lambda_2$; γ_1 is always positive.
2. γ_2 and γ_3 are positive if $0 < a \leq \lambda_1/\lambda_2$;
3. $\gamma_2 < 0$ and $\gamma_3 > 0$ if $\lambda_1/\lambda_2 < a < C/\lambda_2$;
4. $\gamma_2 > 0$ and $\gamma_3 < 0$ if $C/\lambda_2 < a < 1$;

Note that $\gamma_1 + \gamma_2 + \gamma_3 = 1.0$. Thus, under case 2 above, i.e., $0 < a \leq \lambda_1/\lambda_2$, $\mathcal{Z}(s)$ will be the LST of a valid density and Z_n can be interpreted as a product of a Bernoulli and a hyperexponential random variable.

3.4. COXIAN DISTRIBUTION

Now we consider the case when Y_n has a Coxian distribution. A Coxian distribution is best described in terms of a service mechanism in a queue. Consider a service of k stages. Each stage of service has an exponential distribution and after every stage of service, the next service stage may be invoked (with some probability) or the customer leaves the service facility. Thus the Coxian density function of k stages is

$$f_Y(y) = \sum_{i=1}^{k}(1-p_i)\left(\prod_{j=1}^{i-1} p_i\right)\lambda_j e^{-\lambda_j y}$$

and the LST of the Coxian distribution is

$$\mathcal{Y}(s) = \sum_{i=1}^{k}(1-p_i)\left(\prod_{j=1}^{i-1} p_i\right)\frac{\lambda_j}{s+\lambda_j}$$

Consider the simple case of a two stage Coxian with probability 1 that an incoming packet takes service at the initial stage, the LST of the distribution is

$$\mathcal{Y}(s) = p_1\frac{\lambda_1}{\lambda_1+s} + (1-p_1)\frac{\lambda_1}{\lambda_1+s}\frac{\lambda_2}{\lambda_2+s}$$

Substituting in (3.2) and solving for $\mathcal{Z}(s)$ we get

$$\begin{aligned}
\mathcal{Z}(s) &= \frac{(\lambda_1\lambda_2+p_1\lambda_1 s)(as+\lambda_1)(as+\lambda_2)}{(s+\lambda_1)(s+\lambda_2)(\lambda_1\lambda_2+p_1\lambda_1 as)} \\
&= a + \frac{A}{\lambda_1+s} + \frac{B}{\lambda_2+s} + \frac{C}{\lambda_2+p_1 as}
\end{aligned}$$

where

$$\begin{aligned}
A &= \frac{(\lambda_2-p_1\lambda_1)(\lambda_1-a\lambda_1)(\lambda_2-a\lambda_1)}{(\lambda_2-\lambda_1)(\lambda_2-p_1\lambda_1 a)} \\
B &= \frac{\lambda_2(1-p_1)(\lambda_1-a\lambda_2)(1-a)}{(\lambda_1-\lambda_2)(1-p_1 a)} \\
C &= \frac{a\lambda_1\lambda_2^2(a-1)(p_1\lambda_1-\lambda_2)(p_1-q1)}{\lambda_2(-\lambda_2+\lambda_1 p_1 a)(-1+p_1 a)}
\end{aligned}$$

Thus we see that the distribution of Z_n has a positive mass at 0 with probability a and the density is a mixture of three exponential densities. Note that the values of the constants A, B and C above may not always satisfy the conditions for $\mathcal{Z}(s)$ to be the LST of a distribution function. In fact, it can be shown that at least one of A, B, C is negative for any values of λ_1, λ_2, μ, a or p_1. Thus, in choosing the values for the parameters of the Coxian, we need to ensure that the density function of Z_n is always be positive.

3.5. REMARKS

In this section we have not considered negative autocorrelations. See Gaver and Lewis (14) for a discussion on modeling negative autocorrelations which is applicable here as well. We remark here that from the correlation analysis of packet traffic data, our original motivation for these models, negative autocorrelations do not seem to be significant in practice and hence the reduction in the modeling power of the kind of models that we consider is not very significant.

4. MODELING BOTH AUTOCORRELATION AND CROSSCORRELATION

In the previous sections we have discussed separately the modelling of crosscorrelations among the elements of a bivariate sequence (Y_n, X_n) and the autocorrelations in Y_n or X_n using linear structures. In this section, we consider jointly modeling the crosscorrelation and autocorrelations.

4.1. POSITIVE AUTOCORRELATIONS AND POSITIVE CROSSCORRELATIONS

Let the variates Y_n and X_n be related as follows.

$$X_n = cY_n + W_n \qquad\qquad Y_{n+1} = bY_n + Z_n \tag{4.1}$$

For this model, the autocorrelation between the variables $\{X_n\}$ at lag k, $r_{XX}(k)$, is given by:

$$r_{XX}(k) = \frac{c^2 b^k \sigma_Y^2}{\sigma_X^2} \tag{4.2}$$

4.2. POSITIVE AUTOCORRELATIONS AND NEGATIVE CROSSCORRELATION

As in Section 2.2, to generate random variables having negative crosscorrelation we use antithetic variables, except that now we also need to have autocorrelations. Consider the equations

$$Y_n = aP_n + V_n \qquad\qquad X_n = bQ_n + W_n \tag{4.3}$$

We can make either P_n and Q_n antithetic, or we can make V_n and W_n antithetic. It is more convenient to let P_n and Q_n be antithetic. This also gives greater flexibility in terms of the choice of marginal distributions. We take P_n and Q_n to be exponential. X_n and Y_n can be gamma, hyperexponential or Coxian. V_n and W_n will then be characterised as in Section 2.1. We can introduce a autocorrelation in the Y_n sequence by introducing an autocorrelation in either the P_n or the V_n sequence. First, consider the case where the autocorrelation is in V_n while the P_n are independent. Assume that the V_n are related as $V_{n+1} = cV_n + Z_n$. The Z_n would then be characterised as in Section 3. Similarly, we can introduce an autocorrelation in W_n, which is independent of the autocorrelation in V_n. The two relations of (4.3) are related only through the antithetic variables P_n and Q_n, which are independent among themselves, i.e., there is no autocorrelation among the P_n and the Q_n. In this case the autocorrelation in Y_n is the same as that in V_n and the autocorrelation in X_n is the same as that in W_n.

An alternative to the above would be to introduce an autocorrelations in the P_n. In this case assume that the P_n are related as $P_{n+1} = cP_n + Z_n$. In this model, since P_n and Q_n are antithetic, clearly the Q_n will have some correlation between them. It can be shown that the Q_n have a positive autocorrelation. However, in this case we cannot obtain a closed form expression for the autocorrelations in the X_n.

5. CONCLUDING REMARKS

We have described the method of generating bivariate sequences using linear structures to relate the components of the sequence. The primary motivation for this has come from teletraffic applications for packet networks where the bivariate sequence is naturally applied to the sequence of packet lengths and packet interarrival times. Our investigation of some well known packet network traffic traces indicates the existence of a significant crosscorrelation between the interarrival times and the packet lengths. Further, the linear dependence model that we describe in (2.1) is a natural definition for a packet communication network and its interpretation to explain the bulk transfer (open loop) process has been explained before. Similarly, our models may have limited structural appeal in reliability problems but they provide simple analytical mechanisms to model correlations in the system. A further advantage of our bivariate distribution models is the ease of simulation and the small number of parameters that we need to use. This later feature simplifies the parameter estimation process. The models proposed in this paper have a limitation. The models with positive and negative correlations cannot be expressed in a simple unified framework. Thus, one must have some knowledge of the structure of the problem to be able to choose either the model with positive or negative correlation as can be infered from the examples cited in the paper.

The bivariate distributions available in the literature do not lend themselves to easy queueing analysis. The queues driven by bivariate sequences constructed from the linear structure models of this paper are tractable, especially when the cross correlations are non negative. This is developed in a companion paper. In fact we note that much of the theory of GI/GI/1 queues will be directly applicable when (Y_n, X_n) is treated as the interarrival time and service time. See Cidon et al. (2,3) for some related work. Studying the effect of crosscorrelations is made easy because the crosscorrelation parameter can be chosen independent of the marginals and the effect of just the zero-lag crosscorrelation on the waiting time statistics can be easily modelled.

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