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BIVARIATE EXPONENTIAL DISTRIBUTIONS USING LINEAR STRUCTURES

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SUMMARY. We derive bivariate exponential distributions using independent auxiliary random variables. We develop separate models for positive and negative correlations between the exponentially distributed variates. To obtain a positive correlation, we define a linear relation between the variates X and Y of the form $Y = aX + Z$ where a is a positive constant and Z is independent of X. To obtain exponential marginals for X and Y we show that Z is a product of a Bernoulli and an Exponential random variables. To obtain negative correlations, we define $X = aP + V$ and $Y = bQ + W$ where either P and Q or V and W or both are antithetic random variables. For the case of positive correlations, we also characterize a bivariate Poisson process generated by using the bivariate exponential as the interarrival distribution.

Our bivariate exponetial model is useful in introducing dependence between the interarrivals and service times in a queueing model and in the failure process in multicomponent systems. The primary advantage of our model is that the resulting queueing and reliability analysis remains mathematically tractable because the Laplace Transform of the joint distribution is a ratio of polynomials of s. Further, the variates can be very easily generated for computer simulation.

1. **Introduction**

The single variable exponential distribution has long been the favorite of analysts working with queueing systems, life testing and reliability models.

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This is because its memorylessness property lends itself to easy analysis. To provide a flavor of real life models, many distributions, like Gamma, hyper exponential, and Coxian, have been derived from the exponential distribution. Whenever there are two or more variables in a system model, in many analyses they are assumed to be independent. Consider, for example, the interarrival and the service times in a queuing system which are assumed independent in queuing theory while it can be shown that this is not true in most realistic situations particularly in packet communication networks. As another example consider the failure analysis of a multicomponent system in which the lifetime of one component influences the lifetime of the other component in which case it becomes necessary to introduce a dependence between the two variables. To continue to use analytically amenable distributions we will need to assume them to be marginally exponential and introduce multivariate exponential distributions that can model the dependence between the variables. However, unlike the normal distribution, there is no unique natural extension of the univariate exponential distribution and many models have been proposed. In this paper we consider a family of bivariate distributions that are marginally exponential. Mathematical analysis remains fairly straightforward with this family of distributions in spite of the dependence that is introduced among the variates. This is not true of many bivariate exponential distributions in literature. Further, these variables are very easy to simulate on a computer.

The most well known of the bivariate exponential distributions was derived by Marshall and Olkin(1967a,b) by considering shock models. If X and Y are the lifetimes of two components of a system, then the distributions of X and Y are exponential with parameters $\lambda_1+\lambda_{12}$ and $\lambda_2+\lambda_{12}$ and the joint distribution of X and Y is $F(x,y) = 1-\exp(-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y))$. This distribution is not absolutely continuous but satisfies the lack of memory property (LMP),

$$P[X > t_1 + s_1, Y > t_2 + s_2 / X > s_1, Y > s_2] = P[X > t_1, Y > t_2]. \qquad (1)$$

In fact it has been shown that the above is the only bivariate distribution that has exponential marginals and which satisfies LMP. Properties of this Marshall Olkin model are described in detail in Barlow and Proschan(1975). Block and Basu (1974) consider a bivariate distribution whose marginals are mixtures of exponentials and having an absolutely continuous joint distribution. Arnold and Strauss (1991) characterize bivariate distributions where the conditional distributions are in prescribed exponential families. A survey of various bivariate and multivariate exponential models can be found in Basu (1988).

In this paper we present a bivariate exponential distribution that derives its primary motivation from packet communication networks but can be used in other models. Consider a high speed packet communication link that can be modeled as a single server First Come-First Served (FCFS) queue. Let X be the length of a packet that has just joined the output queue of a network node at its tail and Y be the time till the arrival of the next packet. In most networks, a correlation can be established between X and Y described above. The nature of the correlation would depend on the kind of application generating the packets. For example in bulk transfer applications like `ftp`, the correlation will be negative. In some interactive applications like telnet, the correlations could be positive. In the next section we define three models - one for positive correlation and two for negative correlation in which the marginals are exponential, and the Laplace-Steiltjes transform (LST) of the joint distribution is a ratio of polynomials. In section 3 we consider a joint counting process (N, M) in which time between the increments of the M process is X and that between N is Y and obtain the mass function for the joint counting process.

2. The Bivariate Exponential

The bivariate models that we develop are based on Exponential Autoregressive sequences of Gaver and Lewis (1980). While borrowing freely from the ideas in the Gaver and Lewis (1980) paper, we would like to point out the differences as well. The major difference is that while Gaver and Lewis construct an autoregressive AR(1) sequence X_n such that X_n and X_{n+1} are correlated and the X_n have identically distributed exponential marginals, our interest is to construct a bivariate exponential distribution. The primary purpose of the models introduced by Gaver and Lewis is to introduce a correlation between the successive interarrivals or service times in a queue. Our main interest is to introduce a cross correlation between the service time and the next interarrival. Further, in the model proposed for introducing a negative cross correlation, we do not take recourse to (a fictitious) antithetic queue.

2.1 *Positive cross correlation.* Consider two random variables X and Y whose marginals are exponential with parameters λ_x and λ_y respectively. Let

$$Y = aX + Z \tag{2}$$

where $a > 0$ is a constant and Z is a random variable independent of X. The constant a can be thought of as a scaling factor. Let us first characterize Z. Let $\mathcal{X}(s)$, $\mathcal{Y}(s)$ and $\mathcal{Z}(s)$ be the LST of the distributions of X, Y and Z

respectively. From equation (2), we see that

$$\mathcal{Y}(s) = \mathcal{X}(as)\mathcal{Z}(s)$$

From this we obtain

$$\mathcal{Z}(s) = \frac{\mathcal{Y}(s)}{\mathcal{X}(as)} = \frac{\lambda_y}{s+\lambda_y}\frac{\lambda_x + as}{\lambda_x}$$

Define $\lambda_y/\lambda_x = \rho$. After some algebra, for $0 \leq a\rho \leq 1.0$, we get

$$\mathcal{Z}(s) = \rho\frac{\lambda_x + a(s+\lambda_y - \lambda_y)}{\lambda_y + s} = (1-\rho a)\frac{\lambda_y}{s+\lambda_y} + (\rho a) \tag{3}$$

It is easy to see that (3) corresponds to the LST of the distribution of a random variable that is a product of two independent random variables - a Bernoulli random variable with mean $(1 - a\rho)$ and an exponential random variable with parameter λ_y. Thus the distribution of Z will be $a\rho + (1 - a\rho)(1 - e^{-\lambda_y z})$, $z \geq 0$. The LST of the joint distribution is easily obtained as follows.

$$\begin{aligned}
\mathcal{XY}(s_1, s_2) &= E\left(e^{-s_1 X - s_2 Y}\right) = E\left(e^{-(s_1+s_2 a)X}\right)E(e^{-s_2 Z}) \\
&= \left[\frac{\lambda_x}{\lambda_x + s_1 + s_2 a}\right]\left[(1-a\rho)\frac{\lambda_y}{s_2+\lambda_y} + (a\rho)\right] \qquad (4)
\end{aligned}$$

Let r_{xy} denote the correlation between X and Y. $E(XY)$, and hence r_{xy} is obtained as follows.

$$\begin{aligned}
E(XY) &= E(X(aX+Z)) = aE(X^2) + E(X)E(Z) \\
&= a\frac{2}{\lambda_x^2} + \frac{1}{\lambda_x}\frac{1-a\rho}{\lambda_y} \qquad (5) \\
r_{xy} &= \frac{a\frac{2}{\lambda_x^2} + \frac{1}{\lambda_x}\frac{1-a\rho}{\lambda_y} - \frac{1}{\lambda_y\lambda_x}}{\frac{1}{\lambda_y\lambda_x}} = a\rho \qquad (6)
\end{aligned}$$

To see the application of this model to a multicomponent reliability system, assume for the moment that $a = 1$. That is, $Y = X + Z$. Consider a system protected by two safety devices D_1 and D_2 that are subject to shock from the same source(s). The marginal lifetime distributions of D_1 and D_2 are exponential with parameters λ_x and λ_y respectively. The shocks are first absorbed by device D_1. As soon as the device D_1 fails, the device D_2 takes over. So, the lifetime Y of D_2 is the lifetime X of D_1 plus a random amount

Z. From (2) we see that $Z = 0$ or $Y = X$ with probability ρ. Thus ρ is the probability that the shock is strong enough to destroy both the devices simultaneoulsy. If the device D_2 survives the shock that destroys D_1, then it lasts a lifetime that is exponentially distributed with parameter λ_y.

As a second application, consider a packet communication network with packet transmission times denoted by X and having an exponential distribution. Let Y be the packet interarrival time which also have an exponential distribution. In most networks, packet transmissions are best modelled as burst processes in which during the burst, consecutive packets have strongly correlated arrival times, typically a constant or a constant multiple of the packet length, and bursts arrive independently. If we use the above model to model such packet processes, the burst length, number of packets in the burst, will be geometrically distributed with mean $1/(a\rho)$ and the time from the end of one burst till the beginning of the next one will be exponentially distributed with parameter λ_y. During the burst, the time at which the "next packet" arrives will be a constant multiple of the current packet length.

Apart from modeling correlations in network traffic the models considered in this paper can be used to study transmission control schemes over a network. A tansmitter sends packets after a delay which is distributed as a constant multiple of the length of the previous packet (transmission time) plus a random amount. Packet sizes are synonymous with the service times. Using the linear structure and the fact that the marginals have exponential distributions, queueing analysis can be carried out. This type of control will help in increasing bandwith utilisation while reducing congestion.

2.2 *Negative cross correlation.* To obtain a negative cross correlation between X and Y, we take a different but similar approach than the one in the previous section. Consider the following equation

$$X = aP + V \qquad\qquad Y = bQ + W \tag{7}$$

Here a and b are non negative constants, P and V are independent of each other and so are Q and W. Further, from the previous section, if P and Q are exponential with parameters λ_p and λ_q respectively, and V and W are like Z of equation (2), the marginal distributions of X and Y will be exponential. To induce a negative correlation between X and Y, we could make either P and Q or V and W antithetic random variables and choose the other two to make the marginals of X and Y exponential. We will consider both these options separately.

Model 1. First, consider P and Q to be antithetic exponential random variables. Further let V and W be independent. Following Gaver and Lewis (1980), we define P and Q through U which is a uniformly distributed random variable in $(0,1)$, as follows

$$P = -\frac{1}{\lambda_p}\ln(U) \qquad Q = -\tfrac{1}{\lambda_q}\ln(1-U) \tag{8}$$

Under this definition of P and Q, V is the product of a Bernoulli variable with mean $(1 - a\frac{\lambda_x}{\lambda_p})$ and an exponential random variable with parameter λ_x. Similarly, W is the product of a Bernoulli variable with mean $(1 - b\frac{\lambda_y}{\lambda_q})$ and an exponential random variable with parameter λ_y. Also, a and b are to be chosen such that $0 \le a\frac{\lambda_x}{\lambda_p} \le 1$ and $0 \le b\frac{\lambda_y}{\lambda_q} \le 1$. It is easy to see that $\mathtt{cov}(X,Y) = E(XY) - E(X)E(Y) = ab\mathtt{cov}(P,Q)$. Therefore, the correlation coefficient between X and Y, r_{xy}, is given by

$$r_{xy} = \frac{ab\lambda_x\lambda_y}{\lambda_p\lambda_q}\left(1 - \frac{\pi^2}{6}\right) \qquad \text{for } 0 \le \left(\tfrac{a\lambda_x}{\lambda_p}\right), \left(\tfrac{b\lambda_y}{\lambda_q}\right) \le 1 \text{ and } a, b \ge 0 \tag{9}$$

From equation 9 it follows that $\left(1 - \frac{\pi^2}{6}\right) \le r_{xy} \le 0$.

To simplify parameter selection, we could make $\lambda_q = \lambda_p$ and $b = a$.

The LST of the joint distribution of X and Y $\mathcal{XY}(s_1.s_2)$, is obtained as follows.

$$\begin{aligned}
\mathcal{XY}(s_1, s_2) &= E(e^{-s_1X - s_2Y}) = E(e^{-s_1V - s_2W})E(e^{-as_1P - bs_2Q}) \\
&= \left[\left(1 - \frac{a\lambda_x}{\lambda_p}\right)\frac{\lambda_x}{s_1 + \lambda_x} + \left(\frac{a\lambda_x}{\lambda_p}\right)\right] \\
&\quad \left[\left(1 - \frac{b\lambda_y}{\lambda_q}\right)\frac{\lambda_y}{s_2 + \lambda_y} + \left(\frac{b\lambda_y}{\lambda_q}\right)\right] \times \\
&\quad \int_0^1 u^{as_1/\lambda_p}(1-u)^{bs_2/\lambda_q}\, du
\end{aligned} \tag{10}$$

Note that the integral on the RHS of equation 10 corresponds to the beta function.

To minimize the number of parameters to choose, we could have $W = V$, instead of assuming them to be independent. In this case the $\mathtt{cov}(X,Y) = ab\mathtt{cov}(P,Q) + \sigma_v^2$ where σ_v^2 is the variance of V. Also, since P and Q are exponential marginals, we could consider this model to define a 4-variate exponential in which case we could choose not to make P and Q antithetic. In the latter case, $\mathtt{cov}(X,Y) = ab\sigma_v^2$.

Model 2. In the second model to obtain a negative correlation between the exponential marginals of X and Y we make V and W antithetic. Once again we follow Gaver and Lewis (1980), and define V and W as functions of a random variable U that is uniform in $(0,1)$ as follows

$$V = \begin{cases} 0 & \text{if } U \leq c \\ -\frac{1}{\lambda_v}\ln\left(\frac{1-U}{1-c}\right) & \text{if } U > c \end{cases} \qquad W = \begin{cases} 0 & \text{if } U \geq d \\ -\frac{1}{\lambda_w}\ln\left(\frac{U}{d}\right) & \text{if } U < d \end{cases} \tag{11}$$

Here $0 \leq c, d \leq 1$. Note that the random variables V and W as defined above are of the same form as the random variable Z of equation 2. In fact $V = F_1^{-1}(U)$ and $W = F_2^{-1}(1-U)$, where F_1 and F_2 are distribution functions of the same form as the distribution of Z but with different parameter values. Thus we see that for X and Y to be marginally exponential with parameters λ_x and λ_y respectively, $\lambda_v = \lambda_x$, $\lambda_w = \lambda_y$, and P and Q are independent exponential random variables with λ_p and λ_q such that, $c = \frac{a\lambda_x}{\lambda_p}$ and $1 - d = \frac{b\lambda_y}{\lambda_q}$.

Under this model, $\texttt{cov}(X,Y) = \texttt{cov}(V,W)$ and r_{xy}, the correlation between X and Y is given by

$$r_{xy} = \begin{cases} \int_c^d \ln\left(\frac{1-u}{1-c}\right)\ln\left(\frac{u}{d}\right) du - (1-c)d & \text{if } c < d \\ -(1-c)d & \text{if } d \leq c \end{cases} \tag{12}$$

The LST of the joint distribution of X and Y will be given by

$$\begin{aligned} \mathcal{XY}(s_1, s_2) &= E(e^{-s_1X - s_2Y}) = E(e^{-as_1P - bs_2Q})E(e^{-s_1V - s_2W}) \\ &= \begin{cases} \frac{\lambda_p}{\lambda_p + as_1}\frac{\lambda_q}{\lambda_q + bs_2}\int_c^d \left(\frac{1-u}{1-c}\right)^{\frac{s_1}{\lambda_x}}\left(\frac{u}{d}\right)^{\frac{s_2}{\lambda_y}} du & \text{if } c < d, \\ \frac{\lambda_p}{\lambda_p + as_1}\frac{\lambda_q}{\lambda_q + bs_2}\left[d\frac{\lambda_y}{\lambda_y + s_2} + (c-d) + (1-c)\frac{\lambda_x}{\lambda_x + s_1}\right] & \text{if } d \leq c. \end{cases} \end{aligned} \tag{13}$$

The magnitude of negative correlations from Model 2 can exceed $\pi^2/6 - 1$ and hence can be larger than that of Model 1.

Just like with Model 1, we could choose $Q = P$, instead of assuming them to be independent and minimize the number of parameters. In this case we have $\texttt{cov}(X,Y) = \texttt{cov}(V,W) + ab\sigma_p^2$. Further, just like in Model 1, we could consider this model to describe a trivariate exponential in which case we could also choose not to make V and W antithetic. In the latter case, $\texttt{cov}(X,Y) = ab\sigma_p^2$.

Model 3. In this model we can make both P and Q and V and W antithetic with P and Q being independent of V and W. In this case,

$\mathtt{cov}(X,Y) = ab\mathtt{cov}(P,Q) + \mathtt{cov}(V,W)$. Also, just like in model 1, we could treat this to be a model for a 4-variate exponential.

3. Joint Counting Process

As an application we consider a bivariate counting process (M_t, N_t). Let the interarrival times between the jumps of M and N be distributed as X and Y respectively, which in turn are each marginally exponential and the relation between them is defined as in equation 2. It is easy to see that M_t and N_t are marginally Poisson. Let (X_i, Y_i, Z_i), $i = 1, 2, \ldots$, be an i.i.d. sequence of random vectors with the same joint distribution as (X, Y, Z) of section 2.1. Observe that $\sum_{i=1}^n Z_i$ has the same distribution as $B \times G(B, \lambda_y)$ where B is binomial with parameter (n, p), where $p = 1 - a\rho$, and conditional on B, G is a Erlang (Gamma) with parameters, B and λ_y. Define $g_j(z)$ as the density of a gamma random variable with parameters λ_y and j,

$$g_j(z) = \frac{\lambda_y(\lambda_y z)^{j-1}}{(j-1)!}e^{-\lambda_y z},\ z > 0.$$

In evaluating $f_{m,n} = \Pr[M_t = m, N_t = n], m, n = 0, 1, 2, \ldots$, the joint probability mass function for (M_t, N_t), we have to consider three cases.

Case 1: $a > 1$. In this case $f_{m,n}$ is non zero only if $m \geq n$, since $Y_i = aX_i + Z_i \geq aX_i > X_i$. So we restrict ourselves to the case when $m \geq n$. Thus, we have

$$\begin{aligned}
\Pr[M_t \geq m, N_t \geq n] &= \Pr\left[\sum_{i=1}^m X_i \leq t, \sum_{i=1}^n Y_i \leq t\right] \\
&= (1-p)^n \Pr\left[\sum_{i=1}^m X_i \leq t, \sum_{i=1}^n X_i \leq \frac{t}{a}\right] \\
&\quad + \sum_{j=1}^n \binom{n}{j} p^j (1-p)^{n-j} \\
&\quad \int_0^t \Pr\left[\sum_{i=1}^m X_i \leq t, \sum_{i=1}^n X_i \leq \frac{t-z}{a}\right] g_j(z)dz
\end{aligned}$$

$$\begin{aligned}
\Pr\left[\sum_{i=1}^m X_i \leq t, \sum_{i=1}^n X_i \leq \frac{t-z}{a}\right] &= \Pr\left[M_t \geq m, M_{\left(\frac{t-z}{a}\right)} \geq n\right] \\
= \sum_{k=n}^{m-1} \Pr\left[M_{\left(\frac{t-z}{a}\right)} = k\right] \Pr\left[M_{\left(t-\frac{t-z}{a}\right)} \geq m-k\right] &+ \Pr\left[M_{\left(\frac{t-z}{a}\right)} \geq m\right] \qquad (14)
\end{aligned}$$

In the above we have used the homogeneity and independent increment properties of the Poisson process. Substitute for $Pr[M_s = k] = \exp(-\lambda_x s)(\lambda_x s)^k/k!$ in the above expression and simplify using

$$\begin{aligned}\Pr[M_t = m, N_t = n] &= \Pr[M_t \geq m, N_t \geq n] - \Pr[M_t \geq m+1, N_t \geq n] - \\ &\quad -\Pr[M_t \geq m, N_t \geq n+1] \\ &\quad +\Pr[M_t \geq m+1, N_t \geq n+1]\end{aligned}$$

to get, for $m \geq n$,

$$\begin{aligned}f_{m,n} = &\left[(1-p)^n \left(\frac{\lambda_x t}{a}\right)^n \left(\lambda_x \left(t - \frac{t}{a}\right)\right)^{m-n} + \sum_{j=1}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \right. \\ &\left. \times \int_0^t \left(\lambda_x \frac{t-z}{a}\right)^n \left(\lambda_x \left(t - \frac{t-z}{a}\right)\right)^{m-n} g_j(z) dz\right] \frac{e^{-\lambda_x t}}{n!(m-n)!} \qquad (15)\end{aligned}$$

The calculations are similar for the case $a \leq 1$. We only write the expression for $f_{m,n}$ in the remaining two cases.

Case 2: $a \leq 1$, $m > n$. In this case, it is easy to see that

$$\begin{aligned}f_{m,n} &= \left[\sum_{j=1}^{n} \binom{n}{j} p^j (1-p)^{n-j} \int_{t(1-a)}^{t} \left(\lambda_x \frac{t-z}{a}\right)^n \right. \\ &\quad \left. \left(\lambda_x \left(t - \frac{t-z}{a}\right)\right)^{m-n} g_j(z) dz\right] \times \frac{e^{-\lambda_x t}}{n!(m-n)!} \qquad (16)\end{aligned}$$

Case 3: $a \leq 1$, $m \leq n$. As before, we will just write down the corresponding equations for $m < n$ and $m = n$ respectively. If $m < n$ then,

$$\begin{aligned}f_{m,n} &= (1-p)^n (\lambda_x t)^m \left(\lambda_x \left(\frac{t}{a} - t\right)\right)^{n-m} \frac{e^{-\lambda_x \frac{t}{a}}}{m!(n-m)!} + \sum_{j=1}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\ &\quad \times \int_0^{t(1-a)} (\lambda_x t)^m \left(\lambda_x \left(\frac{t-z}{a} - t\right)\right)^{n-m} \frac{e^{-\lambda_x \frac{t-z}{a}}}{m!(n-m)!} g_j(z) dz \qquad (17)\end{aligned}$$

If $m = n$ then,

$$\begin{aligned}f_{m,n} &= \left[(1-p)^n (\lambda_x t)^n + \sum_{j=1}^{n} \binom{n}{j} p^j (1-p)^{n-j} \int_{t(1-a)}^{t} \left(\lambda_x \left(\frac{t-z}{a}\right)\right)^n g_j(z) dz\right] \\ &\quad \times \frac{e^{-\lambda_x \frac{t}{a}}}{n!} + \sum_{j=1}^{n} \binom{n}{j} p^j (1-p)^{n-j} \int_0^{t(1-a)} (\lambda_x t)^n g_j(z) \frac{e^{-\lambda_x \frac{t-z}{a}}}{n!} dz \qquad (18)\end{aligned}$$

The joint counting processes for the models with negative cross-correlation between the interarrivals of the processes M and N do not yield compact formulas as above. In this case it is easier to write an expression for $Pr[M_t \geq m, N_t \geq n]$ than in the positive correlation case. This is due to the fact that the interarrivals X_i and Y_i satisfy two different linear equations. The difficulty however stems from the link between X and Y via antithetic variables which makes subsequent calculations very messy. However, numerical simulation still remains straightforward.

4. **Discussion and Conclusion**

We have presented bivariate distributions that have exponential marginal distributions and a positive or a negative correlation between the variates. When the correlation is positive, a linear relation between the variates can be used. To obtain a negative correlation between the variates, four more variables, two of which are exponential, are used and the concept of antithetic variables has been used.

A major advantage of our description of X and Y according to equations (2) and (7) is that they lend themselves to easy simulation. Consider for example generating a sequence (X_n, Y_n) in which the X_n and Y_n are marginally exponential with a positive cross correlation. We use equation 2 and generate X_n as iid exponential random variables and to obtain the corresponding Y_n, we need to generate a Bernoulli variable, say B_n with mean $(1 - a\rho)$, and another exponential random variable Z_n with mean λ_y and obtain $Y_n = aX_n + B_n Z_n$. Negatively correlated X and Y can be similarly generated using equation 7.

We wish to emphasize that the distributions derived here are different from the bivariate exponential distribution of Marshall and Olkin (1967a,b) and other bivariate exponential models described in Basu (1988). In particular, the bivariate exponential described here does not satisfy the lack of memory property even though the marginals are exponentials.

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