Theory of linear dynamical systems using generalized minor with applications to electrical networks

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Abstract

In this paper we present an approach to linear dynamical systems which combines the positive features of two well known formulations, namely, standard state space theory (see for e.g., W. M. Wonham 1978) and behavioural systems theory (see J. C. Willems 1997). Our development is also 'geometric' in the tradition of W. M. Wonham and others. But, instead of using maps, we work with subspaces. One of our primary motivations is computational efficiency — all our computations are performed on the system as it is without elimination of variables and further (unlike the 'behaviourists') we work only with real matrices.

Using our formulation we derive the standard vector space results on controlled and conditioned invariant subspaces of linear dynamical systems. Duality, which is a distinctive feature of state space theory but not of the behavioural view point, comes out naturally in our approach too through the use of the adjoint. Dual notions such as state-feedback, and output injection and the characterization theorems through their use are also captured neatly in our formulation. We illustrate our ideas for an important class of dynamical systems viz., electrical networks.

The theory proposed in this paper gives a unified description of both the standard linear dynamical systems and the linear singular systems (or the linear descriptor systems) (see for e.g., F. L. Lewis 1986). Therefore, the algorithms described for the invariant spaces in this paper are also applicable to linear singular systems.

Keywords: Linear Dynamical Systems, State Space Theory, Behavioural Systems Theory, Controlled and Conditioned Invariant Spaces, Duality in Adjoint System, Implicit Duality Theorem

1. Introduction

State variable methods justifiably hold the pride of place in the development of multi-variable control theory. When the number of variables is in the tens or a few hundreds, computations using this formulation are not prohibitively expensive in terms of storage space or time required to complete. Beyond that size putting the system in the state variable form becomes too expensive in comparison with the complexity of the task to be done. Further, some control theorists [3][4] have criticized these methods as not 'natural' to the system since they impose a structure which is not already present in the system. These latter, 'behaviourists', take the constraints as given and try to develop the necessary algorithms. In contrast with state variable based methods which manipulate real entry matrices, the methods used by behaviourists involve more expensive computations with matrices whose entries are polynomials, where the indeterminate 'x' in the polynomial corresponds to the operator 'd/dt'. Additionally, the duality ideas, which are the hallmark of the state variable approach, are not natural to that of the behaviourists.

Our approach, while it also imposes no artificial structure, does not permit the constraints to be as general as in the case of the behaviourists, with the result that, like in the case of state variable methods,

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we too manipulate only real entry matrices. A criticism levelled against the state variable formulation is that it is artificial in insisting on dividing manifest variables into 'input' and 'output' even when the system does not have such a division inherent in it [3][4]. In our case we handle this by specifying a vector space $\mathscr{V}_M$ and speaking of the dynamical system 'relative to' it. So if we wish to partition the manifest variables into 'input' m_u and 'output' m_y, we might work with $\mathscr{V}_M \equiv \mathscr{F}_{mu} \oplus 0_{my}$ or with $\mathscr{V}_M \equiv 0_{mu} \oplus \mathscr{F}_{my}$, where $\mathscr{F}_Z$ denotes the space of all vectors on Z. A consequence of speaking of the dynamical system relative to a space $\mathscr{V}_M$ is that duality ideas are, if anything, even more striking in our theory than in the state variable formulation.

For us the paradigmatic system is an electrical network with inductors, capacitors, static devices and sources. In such a system, the derivative of the dynamical variable is directly accessible through a latent variable. For instance, in a network of the above kind, the derivative of the dynamic variable v_C is 'available' through the latent variable i_C, since $i_C = C\frac{dv_C}{dt}$. We call such latent variables as **additional variables** to avoid confusion with the usual latent variables of the behaviourists. Such notions are applicable also to many chemical and mechanical systems. We start with a generalized dynamical system (GDAS) with 'additional variables' which is simply a vector space $\mathscr{V}(\dot{w}, w, l, m)$ involving the variables $\dot{w}, w, l, m$ where w is the vector of (instantaneous values of) dynamic variables, $\dot{w}$ the derivative of w, l the vector of additional variables and m the manifest variables. (Observe that the solution space to state and output equations

$$\begin{bmatrix} I & 0 & -A & -B \\ 0 & I & -C & -D \end{bmatrix} \begin{pmatrix} \dot{x} \\ y \\ x \\ u \end{pmatrix} = 0$$

is a vector space $\mathscr{V}(\dot{x}, x, m)$, where $m \equiv (u, y)$, and that this space is essentially adequate to discuss most of the control theory based on state variables.) The GDAS is accessed through a vector space $\mathscr{V}_M$ on the manifest variables. This latter space gives us freedom to view the 'input-output' space the way we please and is kept independent of the GDAS.

In our approach we avoid the use of explicit maps and work rather with vector spaces. For instance, instead of saying $y = Kx$, we could look at the solution space of $Iy - Kx = 0$. Instead of saying K is symmetric, we could say the space of all (y, x) is complementary orthogonal to the space of all $(-x, y)$ (see for instance [5]). Our algorithms are all built in terms of vector space operations: sum, intersection and an operation called 'generalized minor' $\mathscr{V}_{AB} \leftrightarrow \mathscr{V}_B$ [6][7]. This latter is the vector space $\mathscr{V}_A$ of vectors f_A obtained from (f_A, f_B) in $\mathscr{V}_{AB}$ provided f_B is in $\mathscr{V}_B$. The token algorithms developed in this paper to illustrate our approach are those that correspond to 'controlled invariant' and 'conditioned invariant' spaces [8] or their special cases, (A, B)-invariant spaces and their duals [9]. The algorithms achieve the same output as state equation based ones without explicitly building the latter. This is essential for large systems. For systems with tens of thousands of variables, building the A, B, C and D matrices would be expensive both in terms of time and space. For electrical networks we show how to simplify the algorithms further using decomposition into 'capacitive', 'inductive' and 'static' multi-ports. We assert, though we do not complete the exercise, that such algorithms can be developed for practically all state variable based algorithms resulting in greater computational efficiency.

The duality structure for our approach is natural since sum and intersection operations are dual to each other and (through the 'implicit duality theorem' [6][7] $(\mathscr{V}_{AB} \leftrightarrow \mathscr{V}_B)^{\perp} = \mathscr{V}_{AB}^{\perp} \leftrightarrow \mathscr{V}_B^{\perp}$) the generalized minor operation is self dual. We show that algorithms related to 'conditioned invariant' are dual to corresponding algorithms for 'controlled invariant'. These mimic the usual controllability-observability duality for state variable formulations.

The representation free formulation of this paper (additionally giving prominence to the first derivative of the dynamic variable) has also been used in hybrid dynamical systems, see, for example, [10]. van der Schaft in [11], defines and characterises (independent of the representation of the system) the bisimulation relation for the systems of the kind

$$E\dot{x} = Ax$$
$$w = Hx$$
$$Kx \leq 0.$$

The bisimulation relation is shown to hold within a subset of the consistent space (maximal controlled invariant subspace) of each such systems.

1.1. *Brief History of Controlled and Conditioned Invariant Spaces*

The terms controlled invariant and conditioned invariant probably first appeared in [12]. These ideas appeared as (A, B) invariance and its dual in [13]. Many applications and their variations have been developed using these invariant subspaces. Applications on observability were developed in [14] [15] [16] [17] [18]. Constructive algorithm for disturbance decoupling is discussed in [19]. Generalization of controllability and observability using invariant spaces is given in [20] [21] [22]. Some computational methods are available in [23]. For linear systems, relation between zero structure and canonical decompositions with the invariant subspaces has been studied in many places, see [24] [25] [22] [26] [27] [28] [29] [30] [31] [32] [33]. Relation between inversion of multivariable linear systems and invariants is discussed in [34]. Applications to regulation problem is discussed in [35]. The invariants spaces are also studied in [36] [37]. The concepts of conditioned and controlled invariants are also covered in the books [9] [8] and [38]. Controlled invariants have appeared in the context of bisimulation of linear dynamical systems and hybrid dynamical systems, cf. [11], [39] [40] [41].

1.2. *Outline*

In Section 2 we define the basic operations which will be used in our formulation. We first define the notion of a vector and of copies of collections of vectors in a manner convenient for our development. For convenience the usual operations of sum and intersection of collections of vectors are defined slightly differently. We introduce the "generalized minor" operation and state and prove the "implicit duality theorem" for this operation. The generalized minor operation and the implicit duality theorem have been used extensively in electrical network theory and other applications and play an important role in the development of our theory also. Since our examples are based on electrical circuits we give a brief list of definitions from graph theory which will be used in this paper. In Appendix A we give techniques to compute the basic operations described in Section 2.

We introduce our definition of a linear dynamical system in Section 3 and then give an example from electrical circuit theory. We also define properties that our dynamical systems may have and relate them to standard state space systems.

The conditioned invariant and controlled invariant spaces are introduced in Sections 4 and 5. We give algorithms to compute the minimal conditioned invariant and maximal controlled invariant subspaces. We also state the special cases under which these spaces turn out to be standard invariant spaces in state space theory. For an important case, we give a computational technique to obtain the route or input trace of a vector in the reachable space. Methods for computing the main steps in the algorithms for maximal controlled and minimal conditioned invariant subspaces are given in Appendix B and Appendix C. In Section 6 we demonstrate how various operations discussed in the previous sections can be computed for electrical networks.

In Section 7 we introduce duality and define adjoint dynamical systems and show how duality arises naturally in our formulation. We illustrate this with the algorithm for maximal controlled invariant subspace which is the dual of minimal conditioned invariant subspace.

Section 8 generalizes the definition of state feedback and output injection to suit our formulation. We generalize the results on existence of feedback for controlled invariant subspaces and injection maps for conditioned invariant subspaces in the state space theory, to our setting.

2. Preliminaries

A **vector** $\boldsymbol{f}$ on A over $\mathbb{F}$ is a function $f : A \to \mathbb{F}$ where $\mathbb{F}$ is a field. In this paper we work primarily with the real field. When A, B are disjoint, a vector $f_{A \uplus B}$ on $A \uplus B$ would be written as f_{AB} and would often be written as (f_A, f_B) during operations dealing with such vectors. The sets on which vectors are defined will invariably be finite. When f is on A over $\mathbb{F}$, $\lambda \in \mathbb{F}$ then $\boldsymbol{\lambda f}$ is on A and is defined by $(\lambda f)(e) \equiv \lambda[f(e)]$, $e \in A$. When f is on A and g on B and both are over $\mathbb{F}$, we define $\boldsymbol{f + g}$ on $A \cup B$ by

$$(f+g)(e) \equiv \begin{cases} f(e)+g(e), & e \in A \cap B \\ f(e), & e \in A \setminus B \\ g(e), & e \in B \setminus A. \end{cases}$$

When $A \cap B = \emptyset$ then $f + g$ is usually written as $\boldsymbol{f \oplus g}$. When f, g are on A over $\mathbb{F}$ the **dot product** $\langle f, g\rangle$ of f and g is defined by

$$\langle f, g\rangle \equiv \sum_{e \in A} f(e)g(e).$$

We say f, g are **orthogonal** if $\langle f, g\rangle = 0$.

An **arbitrary collection** of vectors on A would be denoted by $\mathscr{K}_{\boldsymbol{A}}$. $\mathscr{K}$ on A is a **vector space** if $f, g \in \mathscr{K}$ implies $\lambda f + \sigma g \in \mathscr{K}$ for $\lambda, \sigma \in \mathbb{F}$. We will use the symbol $\mathscr{V}_{\boldsymbol{A}}$ for vector space on A as opposed to $\mathscr{K}_A$ for arbitrary collections of vectors on A. The symbol $\mathscr{F}_{\boldsymbol{A}}$ refers to the **collection of all vectors** on A and $\boldsymbol{0_A}$ to the **zero vector space** on A as well as the **zero vector** on A. When A, B are disjoint we usually write $\mathscr{K}_{AB}$ in place of $\mathscr{K}_{A \uplus B}$.

2.1. Basic operations

The basic operations we use in this paper are as follows:

2.1.1. Building Copies

An operation that is very useful is that of building **copies** of sets and **copies** of collections of vectors on copies of sets. We say set A, A' are **copies of each other** iff there is a bijection, usually clear from the context, mapping $e \in A$ to $e' \in A'$. We need this operation in order to talk, for instance, of the vector (v, i) where v is a voltage vector on the edge set of a graph while i is a current vector on the same set. We handle this by building a copy E' of the edge set E and say v is on E and i is on E'.

The vectors f_A and $f_{A'}$ are said to be copies of each other iff $f_{A'}(e') = f_A(e)$ when A and A' are copies of each other. If the vectors on A and A' are not copies of each other, they would be distinguished using notation, such as, f_A, $\widehat{f_{A'}}$ etc. Vectors without subscripts like, x, $\dot{x}$ and w, $\dot{w}$ are not necessarily copies of each other.

Similarly, the collections $\mathscr{K}_A$, $\mathscr{K}_{A'}$ are **copies of each** other iff

$$\mathscr{K}_{A'} \equiv \{f_{A'} : f_{A'}(e') = f_A(e), f_A \in \mathscr{K}_A\}.$$

When A and A' are copies of each other, the collections of vectors $\mathscr{K}_A$, $\mathscr{K}_{A'}$ (or say $\mathscr{V}_A$, $\mathscr{V}_{A'}$) always represent copies of each other. If they are not copies they would be clearly distinguished from each other by the notation, for instance, $\mathscr{K}_A$, $\widehat{\mathscr{K}_{A'}}$, $\mathscr{V}_A$, $\widehat{\mathscr{V}_{A'}}$, etc.

When A and A' are copies of each other, the notation for interchanging the positions of variables A and A' in a collection $\mathscr{K}_{AA'B}$ is given by $\boldsymbol{(\mathscr{K}_{AA'B})_{\text{swap}(AA')}}$, that is

$$(\mathscr{K}_{AA'B})_{\text{swap}(AA')} = \{(g_A f_{A'} h_B) \mid (f_A g_{A'} h_B) \in \mathscr{K}_{AA'B},\ g_A \text{ being copy of } g_{A'},\ f_{A'} \text{ being copy of } f_A\}.$$

2.1.2. Sum

Let $\mathscr{K}_A$, $\mathscr{K}_B$ be collections of vectors on sets A, B respectively. The **sum** $\boldsymbol{\mathscr{K}_A + \mathscr{K}_B}$ of $\mathscr{K}_A$, $\mathscr{K}_B$ is defined over $A \cup B$ as follows:

$$\mathscr{K}_A + \mathscr{K}_B \equiv \{(f_A, 0_{B\setminus A}) + (0_{A\setminus B}, g_B),\ \text{where } f_A \in \mathscr{K}_A, f_B \in \mathscr{K}_B\},$$

When A, B are disjoint, $\mathscr{K}_A + \mathscr{K}_B$ is usually written as $\boldsymbol{\mathscr{K}_A \oplus \mathscr{K}_B}$ and is called the **direct sum**. Thus,

$$\mathscr{K}_A + \mathscr{K}_B \equiv (\mathscr{K}_A \oplus 0_{B\setminus A}) + (0_{A\setminus B} \oplus \mathscr{K}_B).$$

2.1.3. Intersection

The **intersection** $\boldsymbol{\mathscr{K}_A \cap \mathscr{K}_B}$ of $\mathscr{K}_A$, $\mathscr{K}_B$ is defined over $A \cup B$ as follows:

$$\begin{aligned}\mathscr{K}_A \cap \mathscr{K}_B \equiv \{f_{A\cup B} : f_{A\cup B} = (f_A, x_{B\setminus A}),\qquad & f_{A\cup B} = (y_{A\setminus B}, f_B),\\ \text{where } f_A \in \mathscr{K}_A,\qquad f_B \in \mathscr{K}_B,\qquad & x_{B\setminus A},\ y_{A\setminus B}\\ \text{are arbitrary vectors on } B\setminus A,\ A\setminus B \text{ respectively}\}.&\end{aligned}$$

Thus,

$$\mathscr{K}_A \cap \mathscr{K}_B \equiv (\mathscr{K}_A \oplus \mathscr{F}_{B\setminus A}) \cap (\mathscr{F}_{A\setminus B} \oplus \mathscr{K}_B).$$

2.1.4. Matched Sum

The **matched sum** $\boldsymbol{\mathscr{K}_A \leftrightarrow \mathscr{K}_B}$ is on $(A\setminus B) \uplus (B\setminus A)$ and is defined as follows:

$$\mathscr{K}_A \leftrightarrow \mathscr{K}_B \equiv \{f : g_{A\setminus B} \oplus h_{B\setminus A},\ \text{where } g \in \mathscr{K}_A, h \in \mathscr{K}_B \ \&\ g_{A\cap B} = h_{B\cap A}\}.$$

In the special case where $B \subseteq A$, matched sum is called **generalized minor** operation (generalized minor of $\mathscr{K}_A$ with respect to $\mathscr{K}_B$).

2.1.5. Skewed Sum

The **skewed sum** $\boldsymbol{\mathscr{K}_A \rightleftharpoons \mathscr{K}_B}$ is on $(A\setminus B) \uplus (B\setminus A)$ and is defined as follows:

$$\mathscr{K}_A \rightleftharpoons \mathscr{K}_B \equiv \{f : g_{A\setminus B} \oplus h_{B\setminus A},\ \text{where } g \in \mathscr{K}_A, h \in \mathscr{K}_B \ \&\ g_{A\cap B} = -h_{B\cap A}\}.$$

When A, B are disjoint, the matched and skewed sum both correspond to direct sum.

2.1.6. Vector Space Results

It is clear that $+, \cap, \leftrightarrow, \rightleftharpoons$, all yield vector spaces when they operate on vector spaces. The **generalized minor** of $\mathscr{V}_{AB}$ relative to $\mathscr{V}_B$ is $\boldsymbol{\mathscr{V}_{AB} \leftrightarrow \mathscr{V}_B}$. Observe that $\mathscr{V}_{AB} \leftrightarrow \mathscr{V}_B = \mathscr{V}_{AB} \rightleftharpoons \mathscr{V}_B$ since $\mathscr{V}_{AB}$, $\mathscr{V}_B$ are vector spaces. The operations $\boldsymbol{\mathscr{V}_{AB} \leftrightarrow \mathscr{F}_B}$, $\boldsymbol{\mathscr{V}_{AB} \leftrightarrow 0_B}$ are called the **restriction and contraction** of $\mathscr{V}_{AB}$ and are also denoted by $\boldsymbol{\mathscr{V}_{AB} \circ A}$, $\boldsymbol{\mathscr{V}_{AB} \times A}$, respectively.

If $\mathscr{K}$ is on A, then $\mathscr{K}^{\perp}$ is defined by

$$\mathscr{K}^{\perp} = \{g : \langle f, g\rangle = 0, f \in \mathscr{K}\}.$$

Clearly $\mathscr{K}^{\perp}$ is a vector space even if $\mathscr{K}$ is not. When $\mathscr{V}$ is a vector space on a finite set S, it can be shown that $(\mathscr{V}^{\perp})^{\perp} = \mathscr{V}$. This is fundamental for the results of the present work. The following results are easy to see

$$\begin{aligned}(\mathscr{V}_A + \widehat{\mathscr{V}}_A)^{\perp} &= \mathscr{V}_A^{\perp} \cap \widehat{\mathscr{V}}_A^{\perp}, \qquad A \text{ finite}\\ (\mathscr{V}_A \cap \widehat{\mathscr{V}}_A)^{\perp} &= \mathscr{V}_A^{\perp} + \widehat{\mathscr{V}}_A^{\perp}, \qquad A \text{ finite}\ .\end{aligned}$$

When A, B are disjoint, it is easily verified that $(\mathscr{V}_A \oplus \mathscr{V}_B)^{\perp} = \mathscr{V}_A^{\perp} \oplus \mathscr{V}_B^{\perp}$. When A, B are not disjoint, $\mathscr{V}_A + \mathscr{V}_B \equiv (\mathscr{V}_A \oplus 0_{B\setminus A}) + (\mathscr{V}_B \oplus 0_{A\setminus B})$. So

$$\begin{aligned}(\mathscr{V}_A + \mathscr{V}_B)^{\perp} &= (\mathscr{V}_A^{\perp} \oplus \mathscr{F}_{B\setminus A}) \cap (\mathscr{V}_B^{\perp} \oplus \mathscr{F}_{A\setminus B}) \\ &= \mathscr{V}_A^{\perp} \cap \mathscr{V}_B^{\perp}\end{aligned}$$

by the definition of intersection of vector spaces on two distinct sets. Using $(\mathscr{V}^{\perp})^{\perp} = \mathscr{V}$, we have $(\mathscr{V}_A \cap \mathscr{V}_B)^{\perp} = \mathscr{V}_A^{\perp} + \mathscr{V}_B^{\perp}$. The above pair of equalities will be referred to as the **intersection-sum duality** in subsequent sections.

The following results can also be easily verified:

$$\begin{aligned}(\mathscr{V}_{AB} \circ A)^{\perp} &= \mathscr{V}_{AB}^{\perp} \times A \\ (\mathscr{V}_{AB} \times A)^{\perp} &= \mathscr{V}_{AB}^{\perp} \circ A \text{ using } (\mathscr{V}^{\perp})^{\perp} = \mathscr{V}.\end{aligned}$$

The above pair of results will be referred to as the **dot-cross duality**.

2.1.7. Results on Generalized Minor

Let A, B and C be disjoint sets, then

$$(\mathscr{V}_{ABC} \leftrightarrow \mathscr{V}_A) \leftrightarrow \mathscr{V}_B = (\mathscr{V}_{ABC} \leftrightarrow \mathscr{V}_B) \leftrightarrow \mathscr{V}_A = \mathscr{V}_{ABC} \leftrightarrow (\mathscr{V}_A \oplus \mathscr{V}_B).$$

We will omit the brackets in such cases.

2.1.8. Implicit Duality Theorem

Let f_Y be a vector on Y and let $X \subseteq Y$. Then the **restriction** of f_Y to X is defined as follows:

$$f_Y/X \equiv g_X, \text{ where } g_X(e) = f_Y(e), e \in X.$$

Let

$$\mathscr{V}_{(-B)C} \equiv \{g_{BC}, \exists f_{BC} \in \mathscr{V}_{BC} \text{ s.t. } g_{BC}/B = -f_{BC}/B, g_{BC}/C = f_{BC}/C\}.$$

From the definition, when (A, B, C) are disjoint,

$$\mathscr{V}_{AB} \leftrightarrow \mathscr{V}_{BC} = (\mathscr{V}_{AB} + \mathscr{V}_{(-B)C}) \times (A \cup C)$$

and also equal to

$$(\mathscr{V}_{AB} \cap \mathscr{V}_{BC}) \circ (A \cup C).$$

Similarly,

$$\mathscr{V}_{AB} \rightleftharpoons \mathscr{V}_{BC} = (\mathscr{V}_{AB} + \mathscr{V}_{BC}) \times (A \cup C)$$

and also equal to

$$(\mathscr{V}_{AB} \cap \mathscr{V}_{(-B)C}) \circ (A \cup C).$$

Hence we have

$$\begin{aligned}(\mathscr{V}_{AB} \leftrightarrow \mathscr{V}_{BC})^{\perp} &= [(\mathscr{V}_{AB} + \mathscr{V}_{(-B)C}) \times (A \cup C)]^{\perp} \\ &= (\mathscr{V}_{AB} + \mathscr{V}_{(-B)C})^{\perp} \circ (A \cup C) \\ &= (\mathscr{V}_{AB}^{\perp} \cap \mathscr{V}_{(-B)C}^{\perp}) \circ (A \cup C) \\ &= \mathscr{V}_{AB}^{\perp} \rightleftharpoons \mathscr{V}_{BC}^{\perp}\end{aligned}$$

In particular,

$$(\mathscr{V}_{AB} \leftrightarrow \mathscr{V}_B)^{\perp} = \mathscr{V}_{AB}^{\perp} \leftrightarrow \mathscr{V}_B^{\perp}$$

since $\mathscr{V}_B = \mathscr{V}_{(-B)}$.

The above pair of results will be referred to as the **implicit duality theorem**.

Observe that the dot-cross duality is also a consequence of the implicit duality theorem since

$$(\mathscr{V}_{AB} \times A)^{\perp} = (\mathscr{V}_{AB} \leftrightarrow 0_A)^{\perp} = \mathscr{V}_{AB}^{\perp} \leftrightarrow \mathscr{F}_A = \mathscr{V}_{AB}^{\perp} \circ A.$$

Implicit duality theorem and its applications are dealt with in detail in [42], [43], [6], [7]. There has been recent interest in the applications of this result in [44], in the context of 'Pontryagin duality'. The above proof is based on the one in [6].

Since we attempt, in this paper, to work with systems as they are, we will avoid the use of explicitly constructed maps, attempting to achieve equivalent effects through vector space constructions and constraints. Consider the space $\mathscr{V}_{ABC}$ of variables (x_A, x_B, x_C). If x_B fully determines x_C without knowledge of x_A we can state this as follows: First we suppress or eliminate x_A. This is done by performing the generalized minor operation $\mathscr{V}_{ABC} \leftrightarrow \mathscr{F}_A = \mathscr{V}_{ABC} \circ (B \cup C)$. Let us say the result is $\mathscr{V}_{BC}$. Now x_B fully determines x_C in $\mathscr{V}_{BC}$ that is, if (x_B, x_C^1), $(x_B, x_C^2) \in \mathscr{V}_{BC}$ then $x_C^1 = x_C^2$. This is the same as saying if $x_B = 0_B$, then $x_C = 0_C$. So we say $\mathscr{V}_{BC} \times C = 0_C$ or equivalently $\mathscr{V}_{BC} \leftrightarrow 0_B = 0_C$. Thus '$x_B$ fully determines x_C' is equivalent to

$$\mathscr{V}_{ABC} \leftrightarrow \mathscr{F}_A \leftrightarrow 0_B = 0_C$$

or

$$\mathscr{V}_{ABC} \circ (B \cup C) \times C = 0_C.$$

Next suppose in $\mathscr{V}_{ABC}$, the knowledge of the projection of x_B onto $\mathscr{V}_B$ is adequate to uniquely determine its projection onto $\mathscr{V}_B^{\perp}$ that is, if $(x_A, (x_B^1 + x_B^2), x_C)$, $(x_A, (x_B^1 + y_B^2), x_C) \in \mathscr{V}_{ABC}$, where $x_B^1 \in \mathscr{V}_B$, x_B^2, $y_B^2 \in \mathscr{V}_B^{\perp}$, then $x_B^2 = y_B^2$. This is captured by stating that if $x_B \in \mathscr{V}_B^{\perp}$, it is equal to 0_B that is,

$$(\mathscr{V}_{ABC} \cap \mathscr{V}_B^{\perp}) \leftrightarrow (\mathscr{F}_A \oplus \mathscr{F}_C) = 0_B.$$

Computational techniques for the basic operations are discussed in Appendix A.

2.2. Graphs

The computational examples in this paper are from electrical circuit theory. So we require few definitions from graph theory to analyze electrical networks.

An **undirected graph** is a triplet (V, E, f^u), where V is the set of **vertices**, E is the set of **edges**, and f^u is the **incidence function** which associates a pair of vertices with each edge. The **incidence function** defines the end vertices of the edges.

Similarly, a **directed graph** is also a triplet (V, E, f^d), where f^d is the **incidence function** which associates with each edge an "ordered pair" of vertices and the sets V, E define the **vertices**, **edges**, respectively of the graph. The **incidence function** defines the end vertices and also the direction of arrow for the edges.

An **undirected path** (of a graph) from vertex v_1 to vertex v_k is a disjoint alternating vertex-edge sequence $v_1, e_{l1}, v_2, e_{l2}, \ldots, e_{l(k-1)}, v_k$, such that every edge e_{lr} is incident on vertices v_r and v_{r+1}. When it is clear from the context, a path is simply denoted by its edge sequence. A graph is said to be **connected** if there exists an undirected path between every pair of nodes. Otherwise it is said to be **disconnected**. A disconnected graph has **connected components** which are individually connected with no edges between the components.

A **loop** (or a **circuit**) (of any graph) is an undirected path in which the starting and ending vertices are the same and no other vertices or edges are repeated.

A **tree** of a graph is a sub-graph of the original graph with no loops. The edges of a tree are called **branches**. A **spanning tree** is a maximal tree with respect to the edges of a graph. Hence, addition of any other edge of the graph to a spanning tree contains a (single) loop. For this reason, a spanning tree is also called **maximal circuit free set**. The loops obtained in this manner are called **fundamental loops** associated with the tree.

It can be verified that every spanning tree of a connected graph with n nodes consists of $n-1$ edges. Since addition of each edge to a spanning tree creates only one loop, the number of fundamental loops (associated with a spanning tree) is $e-n+1$, where e is the number of edges and n the number of vertices of the graph.

A **co-tree** of a graph is defined with respect to a tree. It consists of all the edges of the graph which are not in the tree. The edges of a co-tree are called **links**.

A **forest** of a disconnected graph is a disjoint union of the trees of its connected components. The complement of a forest is called **co-forest**. In this paper, the terms tree and co-tree will be used to mean forest and co-forest when it is clear from the context.

The number of edges in a forest of a graph is called the **rank** of the graph. For a graph $\mathcal{G}$ it is denoted by $r(\mathcal{G})$.

A **crossing edge** set of a graph is the set of all edges which lie between two complementary subsets of the vertex set of the graph.

A **cutset** is a minimal crossing edge set that is, a minimal subset which when deleted from the graph increases the count of connected components by one.

A spanning tree of the graph can be used to systematically generate cutsets. Since in a spanning tree of a connected graph there exists only one path between any two vertices, deletion of a branch from the tree disconnects the tree into two components. The set of all edges between these two components of the tree form a cutset of the graph. Since a spanning tree of a connected graph with n vertices contains $n-1$ edges, one could associate $n-1$ cutsets to it. The cutsets obtained in this manner are called **fundamental cutsets** of the graph with respect to the tree.

In a graph, an edge set E_1 is said to **span** another edge set E_2 if for each $e_2 \in E_2$ there exists a loop (circuit) $L_1 \subseteq e_2 \cup E_1$ with $e_2 \in L_1$. For example, a spanning tree of any graph spans the co-tree of the same graph. Dually, an edge set E_1 is said to **co-span** another edge set E_2 if for each $e_2 \in E_2$ there exists a cutset $C_1 \subseteq e_2 \cup E_1$ with $e_2 \in C_1$. For instance, the co-tree with respect to a spanning tree of any graph co-spans the same spanning tree of the graph.

Let $\mathcal{G}$ be a graph and E be the edge set of $\mathcal{G}$ and $T \subseteq E$. Then $\boldsymbol{\mathcal{G} \times (E-T)}$ denotes the graph obtained by removing the edges T from $\mathcal{G}$ and fusing the end vertices of the removed edges. $\boldsymbol{\mathcal{G} \circ (E-T)}$ denotes the graph obtained by removing the edges T from $\mathcal{G}$ and removing the isolated vertices.

3. Dynamical Systems

In the present paper we recognize the fact that in the usual physical instances of linear systems such as an electrical network, the derivative of a dynamical variable is accessible in terms of some other physical variable which could be thought 'additional' (e.g., in a network, for a capacitor C we have $C\dot{v}_C = i_C$, so that $\dot{v}_C$ is accessible through the additional variable i_C). Secondly, whether or not we can write state equations for the given system, our descriptions of the relevant spaces and our algorithms will not assume that the state equations are explicitly available. This has obvious computational advantages since often building the state equation for the system will be more expensive in terms of time and space than solving a problem associated with the system such as to reach close to a desired state or to observe it. Our definitions of the basic constructs such as that of a dynamical system are in line with this thinking.

A generalized dynamical system with additional variables (GDAS) is a vector space $\mathcal{V}_{\dot{W}WLM}$ on the set $\dot{W} \uplus W \uplus L \uplus M$ where $\dot{W}$, W are copies of each other intended to take care of the dynamical variables and their derivatives, and L, M, the additional variables and the manifest variables respectively. Thus $w(t) : W \rightarrow \mathbb{R}$, $\dot{w}(t) : \dot{W} \rightarrow \mathbb{R}$, $l(t) : L \rightarrow \mathbb{R}$, $m(t) : M \rightarrow \mathbb{R}$. We write $w(t)$ etc., simply as w etc. The manifest variables are the 'external variables' of the system. Usually a GDAS would be available to us in the form of the solution space of constraint equations

$$\begin{bmatrix} S_{\dot{W}} & S_W & S_L & S_M \end{bmatrix} \begin{pmatrix} \dot{w} \\ w \\ l \\ m \end{pmatrix} = 0. \tag{1}$$

We write $\mathscr{V}_{\dot{W}WLM}$ alternatively as $\mathscr{V}(\dot{w}, w, l, m)$.

We define a generalized dynamical system (GDS) as a vector space $\mathscr{V}_{\dot{W}WM}$ on the set $\dot{W} \uplus W \uplus M$ where $\dot{W}$, W are copies of each other. This is obtained from a GDAS by suppressing the additional variables that is,

$$\mathscr{V}_{\dot{W}WM} = \mathscr{V}_{\dot{W}WLM} \circ (\dot{W} \uplus W \uplus M).$$

We study a GDS (or a GDAS) 'relative to' a vector space $\mathscr{V}_M$. This vector space is expected to capture (and is a generalization of) the usual partition of the manifest variables into 'input' variables m_u and 'output' variables m_y. Working with such a partition, as far as the results of this paper are concerned, amounts to working with the space $\mathscr{V}_M \equiv \mathscr{F}_{mu} \oplus 0_{my}$ or $\mathscr{V}_M \equiv 0_{mu} \oplus \mathscr{F}_{my}$.

A prototypical dynamical system with additional variables for us is a linear electrical network with dynamic devices capacitors (C), inductors (L), static devices resistors and controlled sources (r), voltage and current sources (E and J). Let us denote such a network by $rLCEJ$ network. The variables associated with the devices are v_C, i_C, v_L, i_L, v_r, i_r, v_E, i_E, v_J, i_J. In addition we can introduce output variables. Voltage outputs are obtained by introducing open circuit branches across existing nodes and sensing the voltage across them. Current outputs are obtained by splitting an existing node into two, introducing a short circuit between the split nodes and sensing the current through the short circuit. Example of such a network is given in Figure 1.

This, if treated as a dynamical system $\mathscr{V}(w, \dot{w}, l_w, l_r, m)$, has the variables w, $\dot{w}$, l_w, l_r, m as shown below:

$$\begin{aligned}
&w \equiv (v_{C1}, v_{C2}, v_{C3}, i_{L13}, i_{L14}, i_{L15}), \qquad \dot{w} \equiv (\dot{v}_{C1}, \dot{v}_{C2}, \dot{v}_{C3}, \frac{di_{L13}}{dt}, \frac{di_{L14}}{dt}, \frac{di_{L15}}{dt}),\\
&l_r \equiv (i_4, i_5, i_6, i_7, i_9, i_{10}, i_{11}, v_4, v_5, v_6, v_8, v_9, v_{10}, v_{12}),\\
&l_{\dot{w}} \equiv (i_{C1}, i_{C2}, i_{C3}, v_{L13}, v_{L14}, v_{L15}),\\
&m \equiv (m_u, m_y), \qquad m_u \equiv (u_7, u_8), \qquad m_y \equiv (y_{v11}, y_{i12}).
\end{aligned}$$

Note that l_r also includes the variables associated with sources and outputs other than the manifest variables, such as, for instance, currents of voltage sources or voltages of current sensors.

Now

$$\mathscr{V}(w, \dot{w}, l_{\dot{w}}, l_r, m) = \mathscr{V}^T(w, \dot{w}, l_{\dot{w}}, l_r, m) \cap \mathscr{V}^D(\dot{w}, l_{\dot{w}}, l_r)$$

where

$$\mathscr{V}^T(w, \dot{w}, l_{\dot{w}}, l_r, m) \equiv \mathscr{V}_i(\mathcal{G}) \oplus \mathscr{V}_v(\mathcal{G}) \oplus \mathscr{V}^T_{\dot{w}},$$

$\mathscr{V}_i(\mathcal{G})$ and $\mathscr{V}_v(\mathcal{G})$ denote the current and voltage spaces of $\mathcal{G}$ and $\mathscr{V}^T_{\dot{w}}$ is the space of vectors $\dot{w}$ which satisfy the topological conditions on $\dot{w}$. In general the vector $\dot{v}_C$ satisfies KVL conditions of the graph obtained by open circuiting all branches other than capacitor branches and the vector $\frac{di_L}{dt}$ satisfies KCL conditions of the graph obtained by short circuiting all branches other than the inductor branches. The device characteristics constraints are

$$\mathscr{V}^D(\dot{w}, l_{\dot{w}}, l_r) = \mathscr{V}^D(\dot{w}, l_{\dot{w}}) \oplus \mathscr{V}^D(l_r),$$

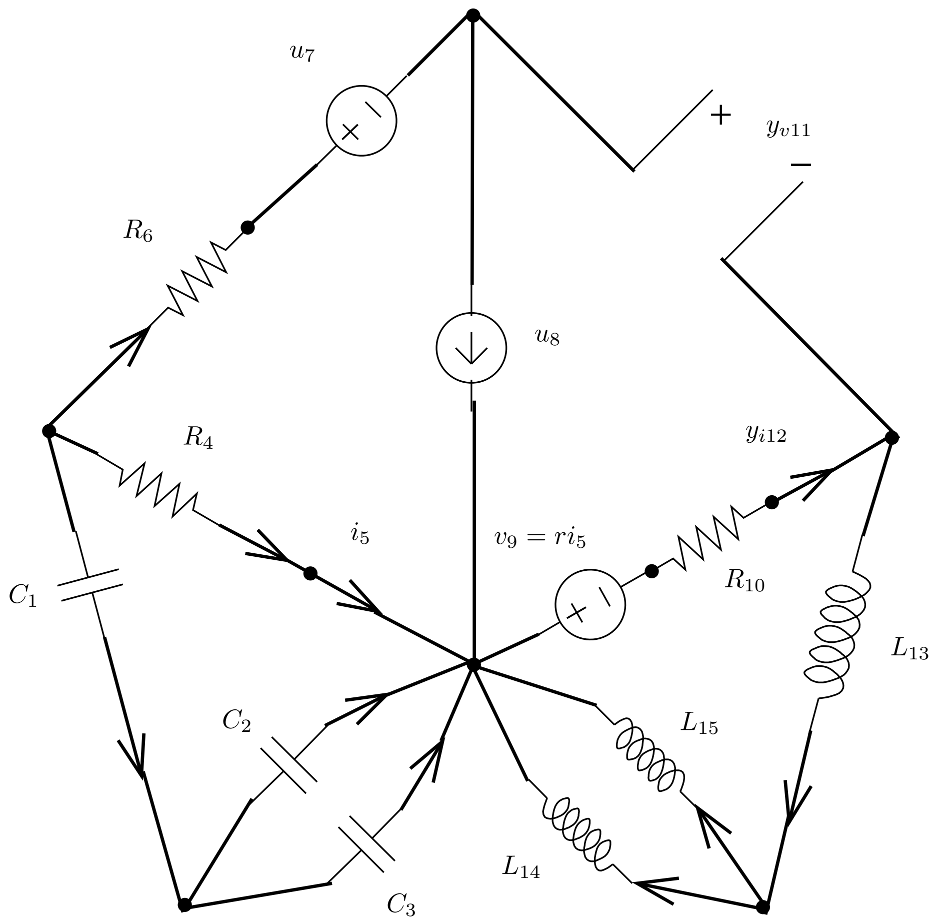

Figure 1: Example $rLCEJ$ Network

where $\mathscr{V}^D(\overset{\bullet}{w}, l_{\overset{\bullet}{w}})$ is the solution space of the equations $i_C = (C)\ \overset{\bullet}{v}_C$, $v_L = (L)\frac{di_L}{dt}$, C, L being positive definite matrices. (Observe w variables are not involved in these equations). $\mathscr{V}^D(l_r)$ is the solution space of the equations (in the case of the present network)

$$v_R = (R)i_R, \qquad v_5 = 0, \qquad v_9 - ri_5 = 0, \qquad i_{11} = 0, \qquad v_{12} = 0.$$

It is clear that the device characteristics constraints on $(\overset{\bullet}{w}, l_{\overset{\bullet}{w}})$ do not involve l_r and vice versa.

Studying this network relative to a specified $\mathscr{V}_M$ can be illustrated as follows: Let $\mathscr{V}_M \equiv 0_{mu} \oplus \mathscr{F}_{my}$. In this case this amounts to setting the sources to zero and the outputs free. If we work with $\mathscr{V}_M^{\perp} \equiv \mathscr{F}_{mu} \oplus 0_{my}$ it amounts to keeping the sources free but the output zero. In this case, note that the output branches have both current and voltage zero ('**nullator**').

Generalized dynamical systems for which state equations can be written say in the form

$$\overset{\bullet}{w} = Aw + Bm_u \tag{2a}$$
$$m_y = Cw + Dm_u \tag{2b}$$

are for us 'regular'. These have the property that given m_u and w, $\overset{\bullet}{w}$ can be determined uniquely and given m_u and w, m_y can be determined uniquely. Further m_u and w can be chosen freely. This motivates the following definitions for which the detailed explanations are given in the succeeding paragraph:

The system $\mathscr{V}_{\dot{W}WM}$ is **$\dot{w}$-zero** (**w-zero**) relative to $\mathscr{V}_M$ iff $(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M)\times \dot{W} = 0_{\dot{W}}$ $((\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \times W = 0_W)$. When $\mathscr{V}_M \equiv 0_{mu} \oplus \mathscr{F}_{my}$, $\dot{w}$-zero means if w and m_u are zero, so will $\dot{w}$ be, that is, $\dot{w}$ is uniquely determined by w and m_u. A similar implication holds for w-zero.

It is **$\dot{w}$-free** (**w-free**) relative to $\mathscr{V}_M$ iff $(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M)\circ \dot{W} = \mathscr{F}_{\dot{W}}$ $((\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \circ W = \mathscr{F}_W)$. With $\mathscr{V}_M$ as above, w-free means whatever be the input, w can be chosen independently.

It is **m-zero** relative to $\mathscr{V}_M$ iff $(\mathscr{V}_{\dot{W}WM} \circ (W \uplus M) \times M) \cap \mathscr{V}_M = 0_M$. With $\mathscr{V}_M$ as above, m-zero means that when w and m_u are zero, m_y must be zero.

It is **m-free** relative to $\mathscr{V}_M$ iff $(\mathscr{V}_{\dot{W}WM} \circ (W \uplus M) \times M) + \mathscr{V}_M = \mathscr{F}_M$. Equivalently, iff the projection of $\mathscr{V}_{\dot{W}WM} \circ (W \uplus M) \times M$ onto $\mathscr{V}_M^{\perp}$ is equal to $\mathscr{V}_M^{\perp}$. With $\mathscr{V}_M \equiv 0_{mu} \oplus \mathscr{F}_{my}$, m-free relative to $\mathscr{V}_M$ means every input can be chosen.

The GDS $\mathscr{V}_{\dot{W}WM}$ is said to be **regular** relative to $\mathscr{V}_M$ iff it is w-free, $\dot{w}$-zero and m-zero, m-free relative to $\mathscr{V}_M$. The GDAS $\mathscr{V}_{\dot{W}WLM}$ is regular iff the GDS $\mathscr{V}_{\dot{W}WLM} \circ (\dot{W} \uplus W \uplus M)$ is regular.

A vector $f_{\dot{W}WM} = (\dot{w}, w, m)$ can be thought of as a possible instantaneous vector in the dynamical system with w as the instantaneous value of the dynamical variable $w(t)$, $\dot{w}$ as the instantaneous value of the dynamical variable $\dot{w}\,(t)$ and m the instantaneous value of the manifest variable $m(t)$.

The picture we have of a dynamical system $\mathscr{V}_{\dot{W}WM}$ which is $\dot{w}$-zero, m-zero and m-free relative to $\mathscr{V}_M$ is as follows: Let us take $\mathscr{V}_M = 0_{mu} \oplus \mathscr{F}_{my}$. Since

$$(\mathscr{V}_{\dot{W}WM} \circ (W \uplus M) \times M) + \mathscr{V}_M = \mathscr{F}_M,$$

m_u can be chosen to be arbitrary even when w is zero, that is, m_u can be chosen to be arbitrary independent of w. Since

$$(\mathscr{V}_{\dot{W}WM} \circ (W \uplus M) \times M) \cap \mathscr{V}_M = 0_M,$$

that is, m-zero relative to $\mathscr{V}_M$, once w is set to zero and m is forced to be in $0_{mu} \oplus \mathscr{F}_{my}$, m_y must go to zero, that is, m_y is a unique function of w and m_u.

At time 0 some of the m variables say m_u, the input variables, are specified (equivalently the projection of m onto $\mathscr{V}_M^{\perp}$ is specified) and also the vector w. If $\mathscr{V}_{\dot{W}WM}$ is $\dot{w}$-zero relative to $\mathscr{V}_M$, then w and m (as long as $m \in \mathscr{V}_M$) fix $\dot{w}$ uniquely.

This fixes $\dot{w}$ (that is, $\frac{dw}{dt}$), and since the system is m-zero relative to $\mathscr{V}_M$, the remaining m variables m_y, the 'output', (equivalently the projection of m onto $\mathscr{V}_M$). If we can allow the w variable to be arbitrary even when m_u is set to zero (in general, forced to be in the space $\mathscr{V}_M$) we have the w-free situation.

If the system is not $\dot{w}$-zero, then $(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M)\times \dot{W} \supsetneq 0_{\dot{W}}$ and the evolution of the system is only defined 'modulo' the above space.

When the system is regular relative to $\mathscr{V}_M = 0_{mu} \oplus \mathscr{F}_{my}$ we thus see that m_u can be chosen independently of w, w can be chosen independently of m_u, and m_u, w fully determine $\dot{w}$ and m_y, that is, the system is governed by (2) above.

4. Conditioned Invariant spaces

Let $\mathscr{V}_{\dot{W}WM}$ be a generalized dynamical system and let $\mathscr{V}_M$ be a vector space. Let $\widehat{\mathscr{V}}_W, \widehat{\mathscr{V}}_{\dot{W}}$ be copies of each other on W, $\dot{W}$ respectively. We say $\widehat{\mathscr{V}}_W$ is **conditioned invariant** in $\mathscr{V}_{\dot{W}WM}$ relative to $\mathscr{V}_M$ iff

$$(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow \widehat{\mathscr{V}}_W \subseteq \widehat{\mathscr{V}}_{\dot{W}}. \tag{3}$$

Motivation for the Definition of Conditioned Invariant Space

Let us take $\mathscr{V}_{\dot{W}WM}$ to be $\dot{w}$-zero relative to $\mathscr{V}_M$. This means that whenever $(\dot{w}\ (t), w(t), m(t)) \in \mathscr{V}_{\dot{W}WM}$ if $m(t) \in \mathscr{V}_M$ then $m(t)$, $w(t)$ uniquely determine $\dot{w}\ (t)$.

Consider the vector $(\dot{w}\ (t), w(t), m(t)) \in \mathscr{V}_{\dot{W}WM}$. Think of it as existing at time t. Now $(\dot{w}\ (t), w(t)) \in (\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M)$ provided $m(t) \in \mathscr{V}_M$ and $\dot{w}\ (t)$ belongs to LHS of (3) provided $w(t) \in \widehat{\mathscr{V}}_W$. The condition (3) states that $\dot{w}\ (t)$ must then belong to the copy $\widehat{\mathscr{V}}_{\dot{W}}$ of $\widehat{\mathscr{V}}_W$. As the system evolves, if $w(t)$ and $\dot{w}\ (t)$ are in a fixed vector space for $t \leq t_0$, with $m(t) \in \mathscr{V}_M$ then $w(t)$ will continue to remain in that space, that is, for $t > t_0$ $\lim_{t \to t_0} w(t)$ will be in that space.

Consider the case where $\dot{w} = Aw + Bm_u$ and $m_y = Cw + Dm_u$. Here we assume that m has been partitioned into (m_u, m_y). $\widehat{\mathscr{V}}_W$ would be conditioned invariant (according to conventional definition in [8]) if (3) is satisfied with $\mathscr{V}_M = \mathscr{F}_{mu} \oplus 0_{my}$. $\widehat{\mathscr{V}}_W$ would be A-invariant if (3) is satisfied with $\mathscr{V}_M = 0_{mu} \oplus \mathscr{F}_{my}$. Thus (3) captures our usual picture of an invariant space of states where the system remains if it starts there and if the manifest variables remain in a predetermined space.

Often there would be a specific $\mathscr{V}_W^{\text{small}}$ and one would be asked to find a 'minimal subspace' $\widehat{\mathscr{V}}_W$ which is conditioned invariant in $\mathscr{V}_{\dot{W}WM}$ relative to $\mathscr{V}_M$ and which contains $\mathscr{V}_W^{\text{small}}$. For instance, when m is partitioned into (m_u, m_y) and we have $\dot{w} = Aw + Bm_u$, $m_y = Cw + Dm_u$, the **strongly reachable space** is the minimal conditioned invariant subspace with $\mathscr{V}_W^{\text{small}}$ as a copy of $(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \times \dot{W}$ on W and $\mathscr{V}_M = \mathscr{F}_{mu} \oplus 0_{my}$, that is, $\mathscr{V}_W^{\text{small}} = B\ker(D)$ and the **reachability space** is the minimal conditioned invariant subspace with $\mathscr{V}_W^{\text{small}}$ as a copy of $(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \times \dot{W}$ on W and $\mathscr{V}_M = \mathscr{F}_{mu} \oplus \mathscr{F}_{my}$, that is $\mathscr{V}_W^{\text{small}} = \text{im}(B)$.

We first observe that one solution to (3) containing $\mathscr{V}_W^{\text{small}}$ is guaranteed, namely the space $\mathscr{F}_W$. There is a unique minimal solution $\widetilde{\mathscr{V}}_W$ of (3) which satisfies the condition $\widetilde{\mathscr{V}}_W \supseteq \mathscr{V}_W^{\text{small}}$. For, if $\widetilde{\mathscr{V}}_W^1$ and $\widetilde{\mathscr{V}}_W^2$ are two such solutions, we have

$$(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow \widetilde{\mathscr{V}}_W^i \subseteq \widetilde{\mathscr{V}}_{\dot{W}}^i \qquad i = 1, 2.$$

Clearly, by the definition of the 'generalized minor' operation

$$\begin{aligned}(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow (\widetilde{\mathscr{V}}_W^1 \cap \widetilde{\mathscr{V}}_W^2) &\subseteq (\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow \widetilde{\mathscr{V}}_W^i \\ &\subseteq \widetilde{\mathscr{V}}_{\dot{W}}^i \qquad i = 1, 2.\end{aligned}$$

Thus

$$(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow (\widetilde{\mathscr{V}}_W^1 \cap \widetilde{\mathscr{V}}_W^2) \subseteq (\widetilde{\mathscr{V}}_{\dot{W}}^1 \cap \widetilde{\mathscr{V}}_{\dot{W}}^2)$$

Hence there must be a unique minimal space containing $\mathscr{V}_W^{\text{small}}$ which is a solution to (3).

Algorithm I

Algorithm for computation of unique minimal space containing $\mathscr{V}_W^{small}$ and satisfying the condition $(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow \widehat{\mathscr{V}}_W \subseteq \widehat{\mathscr{V}}_{\dot{W}}$

Let us denote $(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M)$ by $\mathscr{V}_{\dot{W}W}$. Let $\mathscr{V}_W^1 = \mathscr{V}_W^{\text{small}}$.

1. Check if $\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_W^1 \subseteq \mathscr{V}_{\dot{W}}^1$. If so, this is the desired subspace.
2. If not, let $\mathscr{V}_{\dot{W}}^2 = (\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_W^1) + \mathscr{V}_{\dot{W}}^1$ and $\mathscr{V}_W^2$ be the copy of $\mathscr{V}_{\dot{W}}^2$ on W. (In this case observe that $\dim(\mathscr{V}_W^2) > \dim(\mathscr{V}_W^1)$).

3. Repeat taking $\mathcal{V}_W^{j+1}$ to be the W-copy of $(\mathcal{V}_{\dot{W}W} \leftrightarrow \mathcal{V}_W^{j})+\mathcal{V}^{j}_{\dot{W}}$ and stop when $\mathcal{V}_W^{n+1} = \mathcal{V}_W^{n}$. $\mathcal{V}_W^{n}$ is the desired minimal space.

Justification for Algorithm I

We have $\mathcal{V}_W^1 = \mathcal{V}_W^{\text{small}}$, $\mathcal{V}_W^{j+1} \supseteq \mathcal{V}_W^{j}$ and $(\mathcal{V}_{\dot{W}W} \leftrightarrow \mathcal{V}_W^{n})+\mathcal{V}^{n}_{\dot{W}} = \mathcal{V}^{n+1}_{\dot{W}} = \mathcal{V}^{n}_{\dot{W}}$. Thus $(\mathcal{V}_{\dot{W}W} \leftrightarrow \mathcal{V}_W^{n})+\mathcal{V}^{n}_{\dot{W}} \subseteq \mathcal{V}^{n}_{\dot{W}}$ and therefore, $(\mathcal{V}_{\dot{W}W} \leftrightarrow \mathcal{V}_W^{n}) \subseteq \mathcal{V}^{n}_{\dot{W}}$. Hence $\mathcal{V}^{n}_{\dot{W}}$ is a solution to (3) that contains $\mathcal{V}_W^{\text{small}}$.

Next observe that if $\widehat{\mathcal{V}}_W \supseteq \mathcal{V}_W^{j}$, we have $\widehat{\mathcal{V}}_{\dot{W}} \supseteq \mathcal{V}^{j}_{\dot{W}}$ and further, if $\widehat{\mathcal{V}}_W$ satisfies (3) then

$$\widehat{\mathcal{V}}_{\dot{W}} \supseteq (\mathcal{V}_{\dot{W}W} \leftrightarrow \widehat{\mathcal{V}}_W) \supseteq (\mathcal{V}_{\dot{W}W} \leftrightarrow \mathcal{V}_W^{j}).$$

Hence

$$\begin{aligned}\widehat{\mathcal{V}}_{\dot{W}} &\supseteq (\mathcal{V}_{\dot{W}W} \leftrightarrow \mathcal{V}_W^{j}) + \mathcal{V}^{j}_{\dot{W}} \\ &\supseteq \mathcal{V}^{j+1}_{\dot{W}}.\end{aligned}$$

If follows that if $\widehat{\mathcal{V}}_W \supseteq \mathcal{V}_W^{\text{small}}$ and $\widehat{\mathcal{V}}_W$ satisfies (3) we must have $\widehat{\mathcal{V}}_W$ containing all $\mathcal{V}^{j}_{\dot{W}}$ and therefore also $\mathcal{V}_W^{n+1} = \mathcal{V}_W^{n}$. It thus follows that $\mathcal{V}_W^{n}$ is a minimal subspace containing $\mathcal{V}_W^{\text{small}}$ and satisfying (3).

Computation of the main step in the algorithm is described in the Appendix B.

4.1. Tracing the route to a given vector w^n

We next consider the problem of tracing the route to a given vector w^n in the minimal conditioned invariant space $\mathcal{V}_W^{\text{final}}$ containing $\mathcal{V}_W^{\text{small}}$ from the latter subspace. For this purpose we first recast the conditioned invariant Algorithm I in a more convenient form in the following theorem.

Theorem 1. *Let $\mathcal{V}_{\dot{W}W} \circ \dot{W} \supseteq \mathcal{V}^{1}_{\dot{W}}$, where $\mathcal{V}^{1}_{\dot{W}}$ is a copy of $\mathcal{V}_W^{1}$. Let the copy of $\mathcal{V}_{\dot{W}W} \circ \dot{W}$ on W be contained in $\mathcal{V}_{\dot{W}W} \circ W$. Also let*

$$\begin{aligned}\mathcal{V}^{j+1}_{\dot{W}} &\equiv (\mathcal{V}_{\dot{W}W} \leftrightarrow \mathcal{V}_W^{j}) + \mathcal{V}^{j}_{\dot{W}}, \qquad \widetilde{\mathcal{V}}_W^{1} = \mathcal{V}_W^{1} \\ \widetilde{\mathcal{V}}^{j+1}_{\dot{W}} &\equiv (\mathcal{V}_{\dot{W}W} \leftrightarrow \widetilde{\mathcal{V}}_W^{j}).\end{aligned}$$

Then,

$$\sum_{j=1}^{k} \widetilde{\mathcal{V}}_W^{j} \equiv \mathcal{V}_W^{k}.$$

Remark 1. *The extra assumption, that a copy of $\mathcal{V}_{\dot{W}W} \circ \dot{W}$ on W is contained in $\mathcal{V}_{\dot{W}W} \circ W$, made at the beginning of Theorem 1, is usually satisfied by many dynamical systems, for instance, a regular dynamical system (or a system in state space form). No such assumption is made in Algorithm I because, there, we are concerned about computing the minimal subspace and here we are looking to find the sequence of vectors to reach a given state in the minimal subspace. The assumptions made in this theorem are meant to handle the computation of controllable subspace.*

We state the modified algorithm, whose output has the 'Krylov space' form, explicitly below.

Algorithm I′

Algorithm for computation of unique minimal space containing $\mathscr{V}_W^{small}$ and satisfying the condition $(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow \widehat{\mathscr{V}}_W \subseteq \widehat{\mathscr{V}}_{\dot{W}}$.

`Assumption:` `Let` $\mathscr{V}_{\dot{W}W} \circ \dot{W} \supseteq \mathscr{V}_{\dot{W}}^{\text{small}}$, `where` $\mathscr{V}_{\dot{W}}^{\text{small}}$ `is a copy of` $\mathscr{V}_W^{\text{small}}$. `Let the copy of` $\mathscr{V}_{\dot{W}W} \circ \dot{W}$ `on` W `be contained in` $\mathscr{V}_{\dot{W}W} \circ W$.

1. `Set` $\widetilde{\mathscr{V}}_W^1 = \mathscr{V}_W^{\text{small}}$
2. $\widetilde{\mathscr{V}}_{\dot{W}}^{j+1} \equiv (\mathscr{V}_{\dot{W}W} \leftrightarrow \widetilde{\mathscr{V}}_W^j)$
3. `Stop when`

$$\sum_{j=1}^{k} \widetilde{\mathscr{V}}_{\dot{W}}^j = \sum_{j=1}^{k+1} \widetilde{\mathscr{V}}_{\dot{W}}^j .$$

4. `The desired space is` $\sum_{j=1}^{k} \widetilde{\mathscr{V}}_W^j$.

We need a couple of lemmas for the proof of this theorem.

Lemma 2. *Let $f_B \in \mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A$. Then $f_B + v_B \in \mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A$, whenever $v_B \in \mathscr{V}_{AB} \times B$.*

Proof. If $f_B \in \mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A$, we have $(f_A, f_B) \in \mathscr{V}_{AB}$ for some $f_A \in \mathscr{K}_A$. Now since $v_B \in \mathscr{V}_{AB} \times B$, we have $(0, v_B) \in \mathscr{V}_{AB}$. Hence $(f_A, f_B + v_B) \in \mathscr{V}_{AB}$ and $f_B + v_B \in \mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A$. □

Lemma 3. *Let $\mathscr{K}_A, \mathscr{K}_A^1, \mathscr{K}_A^2 \subseteq \mathscr{V}_{AB} \circ A$ and let $\mathscr{K}_A = \mathscr{K}_A^1 + \mathscr{K}_A^2$. Then*

$$(\mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A^1) + (\mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A^2) = (\mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A).$$

Proof. First, it is clear that the following containment is true:

$$(\mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A^1) + (\mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A^2) \subseteq (\mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A). \tag{4}$$

We will now show the reverse containment

$$(\mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A) \subseteq (\mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A^1) + (\mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A^2). \tag{5}$$

Let $f_B \in \mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A$. Then, there exists $f_A \in \mathscr{K}_A$ such that $(f_A, f_B) \in \mathscr{V}_{AB}$. Since $\mathscr{K}_A = \mathscr{K}_A^1 + \mathscr{K}_A^2$, we have

$$f_A = f_A^1 + f_A^2,$$

for some $f_A^1 \in \mathscr{K}_A^1$, $f_A^2 \in \mathscr{K}_A^2$. So there exists f_B^1, f_B^2 such that

$$(f_A^1, f_B^1) \in \mathscr{V}_{AB}, \qquad (f_A^2, f_B^2) \in \mathscr{V}_{AB}$$

since $\mathscr{K}_A^1 \subseteq \mathscr{V}_{AB} \circ A$, $\mathscr{K}_A^2 \subseteq \mathscr{V}_{AB} \circ A$. As $\mathscr{V}_{AB}$ is a vector space, we also have

$$(f_A^1 + f_A^2, f_B^1 + f_B^2) = (f_A, f_B^1 + f_B^2) \in \mathscr{V}_{AB}.$$

Since $(f_A, f_B) \in \mathscr{V}_{AB}$, we have $f_B^1 + f_B^2 - f_B \in \mathscr{V}_{AB} \times B$. So there exists some $v_B \in \mathscr{V}_{AB} \times B$ such that

$$f_B^1 + f_B^2 = v_B + f_B.$$

Now by Lemma 2, $f_B^2 - v_B \in \mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A^2$. Hence

$$f_B = f_B^1 + (f_B^2 - v_B) \in (\mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A^1) + (\mathscr{V}_{AB} \leftrightarrow \mathscr{K}_A^2).$$

□

Proof of Theorem 1

Let

$$\mathscr{V}_W^{k-1} \equiv \sum_{j=1}^{k-1} \widetilde{\mathscr{V}}_W^j.$$

We will show the equality for k. Now,

$$\begin{aligned}
\mathscr{V}_{\dot{W}}^k &= \left(\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_W^{k-1}\right) + \mathscr{V}_{\dot{W}}^{k-1} \\
&= \left[\mathscr{V}_{\dot{W}W} \leftrightarrow \left(\widetilde{\mathscr{V}}_W^{k-1} + \mathscr{V}_W^{k-2}\right)\right] + \mathscr{V}_{\dot{W}}^{k-1} \quad (\text{ follows from definition of } \mathscr{V}_W^{k-1}) \\
&= \left(\mathscr{V}_{\dot{W}W} \leftrightarrow \widetilde{\mathscr{V}}_W^{k-1}\right) + \left(\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_W^{k-2}\right) + \mathscr{V}_{\dot{W}}^{k-1} \quad (\text{using Lemma 3}) \\
&= \left(\mathscr{V}_{\dot{W}W} \leftrightarrow \widetilde{\mathscr{V}}_W^{k-1}\right) + \mathscr{V}_{\dot{W}}^{k-1} \quad \left(\text{noting that } (\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_W^{k-2}) + \mathscr{V}_{\dot{W}}^{k-2} = \mathscr{V}_{\dot{W}}^{k-1}\right) \\
&= \widetilde{\mathscr{V}}_{\dot{W}}^k + \mathscr{V}_{\dot{W}}^{k-1}
\end{aligned}$$

The theorem follows since it is given to be true for $k = 1$. □

We will now illustrate the above idea for the usual 'input-state-output' dynamical systems. For the discussions that follows $\mathscr{V}_{\dot{W}WM}$ allows a description in terms of state and output equations. Our discussion will not however need the computations of state and output equations explicitly. Unless otherwise stated we take $\mathscr{V}_M = \mathscr{F}_M$.

Let us now pick a basis for $\mathscr{V}_W^n = \mathscr{V}_W^{n+1} = \sum_{j=1}^n \widetilde{\mathscr{V}}_W^j$ as follows:

Let the rows of $B_1 = \widetilde{B}_1$ be a basis for $\widetilde{\mathscr{V}}_W^1 = \mathscr{V}_W^1$

$$B_2 = \begin{pmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \end{pmatrix}, \qquad \text{be a basis for } \widetilde{\mathscr{V}}_W^1 + \widetilde{\mathscr{V}}_W^2 = \mathscr{V}_W^2$$

$$\vdots$$

$$B_j = \begin{pmatrix} \widetilde{B}_1 \\ \vdots \\ \widetilde{B}_j \end{pmatrix}, \qquad \text{be a basis for } \mathscr{V}_W^j, \widetilde{B}_i \subseteq \widetilde{\mathscr{V}}_W^i$$

$$\vdots$$

$$B_n = \begin{pmatrix} \widetilde{B}_1 \\ \vdots \\ \widetilde{B}_n \end{pmatrix} \qquad \text{be a basis for } \mathscr{V}_W^n = \sum_{j=1}^n \widetilde{\mathscr{V}}_W^j$$

Let w be a specified vector. To check if $w \in \mathscr{V}_W^n$ we merely check if the equation $B_n^T \lambda = w$ can be solved (by checking if $w =$ projection of w on $\mathscr{V}_W^n$, for instance). Suppose the equation has a solution. Then $w \in \mathscr{V}_W^n$ and we can write $w = w^1 + \ldots + w^n$ where $w^j = \widetilde{B}_j^T \lambda_j$. Now $w^j \in \widetilde{\mathscr{V}}_W^j$, $j = 1, \ldots, n$. For each w^j compute the sequence

$$w^{j1}, \dot{w}^{j2}, w^{j2}, \dot{w}^{j3}, \ldots, \dot{w}^{jj}, w^{jj} = w^j,$$

where

$$\left.\begin{array}{c}
(\dot{w}^{ji}, w^{j(i-1)}) \in \mathscr{V}_{\dot{W}W} = (\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \\
w^{j(i-1)} \in \widetilde{\mathscr{V}}_W^{(i-1)} \\
w^{ji} \text{ is a copy of } \dot{w}^{ji}
\end{array}\right\} \text{ for } i = 2, \ldots, j.$$

We associate with the vector w the set of initial vectors

$$\{w^{11}, w^{21}, \ldots, w^{n1}\}.$$

Appropriately interpreted this set contains the information required to reach w starting from $\mathscr{V}^1_{\dot{W}}$.

Let us consider the case of discrete systems of the kind governed by $w^{k+1} = Aw^k + Bm_u^k$. In this case $(\dot{w}, w)$ plays the role of (w^{k+1}, w^k). Note that if $\dot{w}$ occurs at $t = k-1$, it becomes its copy w at $t = k$.

In such a system let us suppose that at any discrete instant k we can somehow add a state $\widehat{w} \in \mathscr{V}^1_{W}$ to the current state. To reach w at $t = n$ we start with w^{n1} at $t = 1$. Let the system evolve to some state w^2 at $t = 2$. Now add $w^{(n-1)1}$ at $t = 2$ to this state. As a result let the system evolve to w^3 at $t = 3$. Now add $w^{(n-2)1}$ at $t = 3$ to this state and so on finally adding w^{11} at $t = n$ to w^n.

Suppose $\mathscr{V}^1_{\dot{W}} \subseteq \mathscr{V}_{\dot{W}WM} \times (\dot{W} \uplus M) \circ \dot{W}$. (The space on the RHS corresponds to the space of all $\dot{w}$ vectors which can be 'reached' immediately starting from $w = 0$ and using some m. This is the same as the space of all w vectors which can be reached in one time instant starting from $w = 0$ and using some m.) Adding a state w^{j1} at time instant $t = k$ can be achieved by imposing a suitable manifest variable m^{j1} at $t = k-1$ since there is a vector $(\dot{w}^{j1}, 0, m^{j1}) \in \mathscr{V}_{\dot{W}WM}$. Thus adding state vectors at various time instants, in this case, is the same as adding suitable manifest variable vectors at the previous time instant.

For continuous systems of the kind governed by $\dot{w} = Aw + Bm_u$ and $m_y = Cw + Dm_u$, the effect is achieved 'instantaneously' by the use of impulses and their derivatives. Let us suppose that the manifest variable m is partitioned into (m_u, m_y) as usual. As before, let us assume $\mathscr{V}^1_{\dot{W}} \subseteq \mathscr{V}_{\dot{W}WM} \times (\dot{W} \uplus M) \circ \dot{W}$. Consider w^{11}. Let $\dot{w}^{11}$ be a copy of w^{11}. There is a vector $(\dot{w}^{11}, 0, m^{11}) \in \mathscr{V}_{\dot{W}WM}$. Let $m^{11} = (m_u^{11}, m_y^{11})$. We impose the input $(m_u^{11})\delta$ on the system. This will cause the state to jump by w^{11} (copy of $\dot{w}^{11}$). Similarly for adding w^{n1} one computes m^{n1} and imposes the input $m_u^{n1}\delta^n$. Thus we can reach w 'instantaneously' by imposing $(m_u^{11}\delta + \ldots + m_u^{n1}\delta^n)$ as the input. The corresponding output m_y can be computed as follows:

To compute m_y, we observe that it is the sum of the two terms: that due to input with state zero (zero state response) and that due to state with input zero (zero input response).

The zero state response is obtained by finding, for each $j \neq 1$, m_y such that

$$(m_u^{j1}, m_y^{j1}) \in \mathscr{V}_{\dot{W}WM} \times (\dot{W} \uplus M) \circ M$$

and setting

$$y_{\text{zero state}} = m_y^{11}\delta + \ldots + m_y^{n1}\delta^n.$$

The zero input response is obtained as follows: Let $\mathscr{V}_M$ be the space of vectors $(0_u, m_y)$ where m_y is free. We compute the vector $(0_u, y_{\text{zero input}})$ corresponding to w in the space $\mathscr{V}_{\dot{W}WM} \circ (\dot{W} \uplus M) \leftrightarrow \mathscr{V}_M$.

The complete output is $y = y_{\text{zero input}} + y_{\text{zero state}}$.

5. Controlled Invariant spaces

We say $\widehat{\mathscr{V}}_W$ is **controlled invariant** in $\mathscr{V}_{\dot{W}WM}$ relative to $\mathscr{V}_M$ iff

$$(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow \widehat{\mathscr{V}}_{\dot{W}} \supseteq \widehat{\mathscr{V}}_W. \tag{6}$$

Motivation for the Definition of Controlled Invariant Space

Consider the vector $(\dot{w}(t), w(t), m(t)) \in \mathscr{V}_{\dot{W}WM}$. Think of it as existing at time t. Now $(\dot{w}(t), w(t)) \in (\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M)$ provided $m(t) \in \mathscr{V}_M$ and if $\dot{w}(t) \in \widehat{\mathscr{V}}_{\dot{W}}$, $w(t)$ belongs to LHS of (6). The LHS of (6) contains RHS which is the space $\widehat{\mathscr{V}}_W$. So at time t if we choose $w(t) \in \widehat{\mathscr{V}}_W$ we can always find a vector

$\overset{\bullet}{w}(t) \in \widehat{\mathscr{V}}_{\overset{\bullet}{W}}$ and a vector $m(t) \in \mathscr{V}_M$ such that $(\overset{\bullet}{w}(t), w(t), m(t)) \in \mathscr{V}_{\overset{\bullet}{W}WM}$. Now if the dynamical system $\mathscr{V}_{\overset{\bullet}{W}WM}$ is $\overset{\bullet}{w}$-zero relative to $\mathscr{V}_M$ it follows $w(t)$ and $m(t)$ fix $\overset{\bullet}{w}(t)$ uniquely. Thus if we start from a state $w(t) \in \widehat{\mathscr{V}}_W$ we can always find an $m(t) \in \mathscr{V}_M$ which keeps $\overset{\bullet}{w}(t)$ in $\widehat{\mathscr{V}}_{\overset{\bullet}{W}}$ and therefore $w(t)$ continues in $\widehat{\mathscr{V}}_W$. As the system evolves, if $w(t)$ and $\overset{\bullet}{w}(t)$ are in a fixed vector space for $t \leq t_0$, with suitable $m(t) \in \mathscr{V}_M$, $w(t)$ will continue to remain in that space, that is, for $t > t_0$, $\lim_{t \to t_0} w(t)$ will be in that space.

Consider the case where $\overset{\bullet}{w}= Aw + Bm_u$ and $m_y = Cw + Dm_u$. Here we assume that m has been partitioned into (m_u, m_y). $\widehat{\mathscr{V}}_W$ would be controlled invariant (according to the definition in [8]) if (6) is satisfied with $\mathscr{V}_M = \mathscr{F}_{mu} \oplus 0_{my}$. $\widehat{\mathscr{V}}_W$ would be (A, B) invariant if (6) is satisfied with $\mathscr{V}_M = \mathscr{F}_M$. Thus (6) captures our usual picture of a subspace of states where the system can be forced to remain by suitable choice (at each instant) of the manifest variable $m(t)$.

Often there would be a specific $\mathscr{V}_W^{\text{big}}$ and one would be asked to find a 'maximal subspace' $\widehat{\mathscr{V}}_W$ which is controlled invariant and a subspace of $\mathscr{V}_W^{\text{big}}$. For instance, when m is partitioned into (m_u, m_y) and we have $\overset{\bullet}{w}= Aw + Bm_u$ and $m_y = Cw + Dm_u$. Then the **weakly unobservable space** is the maximal controlled invariant space with $\mathscr{V}_W^{\text{big}} \equiv (\mathscr{V}_{\overset{\bullet}{W}WM} \leftrightarrow \mathscr{V}_M) \circ W$ and $\mathscr{V}_M = \mathscr{F}_{mu} \oplus 0_{my}$ and the **unobservable space** has the same $\mathscr{V}_W^{\text{big}}$ but $\mathscr{V}_M = 0_{mu} \oplus 0_{my}$.

We first observe that there is a unique maximal solution $\widetilde{\mathscr{V}}_W$ of (6) which satisfies the condition $\widetilde{\mathscr{V}}_W \subseteq \mathscr{V}_W^{\text{big}}$. For, if $\widetilde{\mathscr{V}}_W^1$ and $\widetilde{\mathscr{V}}_W^2$ are two such solutions, we have

$$(\mathscr{V}_{\overset{\bullet}{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow \widetilde{\mathscr{V}}^i_{\overset{\bullet}{W}} \supseteq \widetilde{\mathscr{V}}^i_W, \qquad i = 1, 2.$$

Clearly by the definition of the 'generalized minor' operation

$$\begin{aligned}(\mathscr{V}_{\overset{\bullet}{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow (\widetilde{\mathscr{V}}^1_{\overset{\bullet}{W}} + \widetilde{\mathscr{V}}^2_{\overset{\bullet}{W}}) &\supseteq (\mathscr{V}_{\overset{\bullet}{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow \widetilde{\mathscr{V}}^i_{\overset{\bullet}{W}}, \qquad && i = 1, 2 \\ &\supseteq \widetilde{\mathscr{V}}^i_W, && i = 1, 2.\end{aligned}$$

Thus

$$(\mathscr{V}_{\overset{\bullet}{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow (\widetilde{\mathscr{V}}^1_{\overset{\bullet}{W}} + \widetilde{\mathscr{V}}^2_{\overset{\bullet}{W}}) \supseteq (\widetilde{\mathscr{V}}^1_W + \widetilde{\mathscr{V}}^2_W).$$

One solution to (6) is guaranteed, namely the space 0_W. Hence there must be a unique maximal space contained in $\mathscr{V}_W^{\text{big}}$ which is a solution to (6).

Algorithm II

Algorithm for computation of unique maximal space contained in $\mathscr{V}_W^{big}$ and satisfying the condition $(\mathscr{V}_{\overset{\bullet}{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow \widetilde{\mathscr{V}}_{\overset{\bullet}{W}} \supseteq \widetilde{\mathscr{V}}_W$.

Let us denote $(\mathscr{V}_{\overset{\bullet}{W}WM} \leftrightarrow \mathscr{V}_M)$ by $\mathscr{V}_{\overset{\bullet}{W}W}$. Let $\mathscr{V}_W^1 = \mathscr{V}_W^{\text{big}}$.

1. Check if $\mathscr{V}_{\overset{\bullet}{W}W} \leftrightarrow \mathscr{V}^1_{\overset{\bullet}{W}} \supseteq \mathscr{V}^1_W$. If so this is the desired subspace.
2. If not let $\mathscr{V}^2_W = (\mathscr{V}_{\overset{\bullet}{W}W} \leftrightarrow \mathscr{V}^1_{\overset{\bullet}{W}}) \cap \mathscr{V}^1_W$ and $\mathscr{V}^2_{\overset{\bullet}{W}}$ be the copy of $\mathscr{V}^2_W$ on $\overset{\bullet}{W}$. In this case observe that $\dim(\mathscr{V}^2_W) < \dim(\mathscr{V}^1_W)$.
3. Repeat taking $\mathscr{V}^{j+1}_{\overset{\bullet}{W}}$ to be the $\overset{\bullet}{W}$-copy of $(\mathscr{V}_{\overset{\bullet}{W}W} \leftrightarrow \mathscr{V}^j_{\overset{\bullet}{W}}) \cap \mathscr{V}^j_W$ and stop when $\mathscr{V}^{n+1}_W = \mathscr{V}^n_W$. $\mathscr{V}^n_W$ is the desired maximal space.

Justification for Algorithm II

We have $\mathscr{V}_W^1 = \mathscr{V}_W^{\rm big}$, $\mathscr{V}_W^{j+1} \subseteq \mathscr{V}_W^j$ and $(\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_{\dot{W}}^n) \cap \mathscr{V}_W^n = \mathscr{V}_W^{n+1} = \mathscr{V}_W^n$. Thus $(\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_{\dot{W}}^n) \cap \mathscr{V}_W^n \supseteq \mathscr{V}_W^n$. Hence $\mathscr{V}_W^n$ is a solution to (6) that is contained in $\mathscr{V}_W^{\rm big}$. Next observe that if $\widehat{\mathscr{V}}_W \subseteq \mathscr{V}_W^j$

$$(\mathscr{V}_{\dot{W}W} \leftrightarrow \widehat{\mathscr{V}}_W) \subseteq (\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_W^j).$$

Now if $\widehat{\mathscr{V}}_W$ satisfies (6) then

$$\widehat{\mathscr{V}}_W \subseteq (\mathscr{V}_{\dot{W}W} \leftrightarrow \widehat{\mathscr{V}}_{\dot{W}}) \subseteq (\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_{\dot{W}}^j)$$

Therefore, since $\widehat{\mathscr{V}}_W \subseteq \mathscr{V}_W^j$

$$\widehat{\mathscr{V}}_W \subseteq (\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_{\dot{W}}^j) \cap \mathscr{V}_W^j = \mathscr{V}_{\dot{W}}^{j+1}.$$

It follows that if $\widehat{\mathscr{V}}_W \subseteq \mathscr{V}_W^{\rm big}$ and $\widehat{\mathscr{V}}_W$ satisfies (6) we must have $\widehat{\mathscr{V}}_W$ contained in all $\mathscr{V}_W^j$ and therefore, also $\mathscr{V}_W^n = \mathscr{V}_W^{n+1}$. It thus follows that $\mathscr{V}_W^n$ is a maximal subspace contained in $\mathscr{V}_W^{\rm big}$ and satisfying (6).

Computation of the main step in the algorithm is given in Appendix C

6. Computations for Electrical Networks

We illustrate the computations associated with conditioned and controlled invariant spaces for the case of electrical networks. One primary aim is to show that no explicit construction of state and output equations are required.

6.1. Multi-port Decomposition

For electrical networks, a convenient way of handling computations is to first decompose the network into suitable 'multi-ports'. While this is intuitive, it has been formalized in [42] [7]. This formalization is topological that is, does not depend on the device characteristics. Let $\mathcal{G}$ be a directed graph and let E_1, E_2 be a partition of edges of the graph. Then it can be shown that we can build three graphs $\mathcal{G}_{E_1P_1}$, $\mathcal{G}_{P_1P_2}$, $\mathcal{G}_{E_2P_2}$ such that

$$\mathscr{V}_i(\mathcal{G}) = \Big(\mathscr{V}_i(\mathcal{G}_{E_1P_1}) \oplus \mathscr{V}_i(\mathcal{G}_{E_2P_2})\Big) \leftrightarrow \mathscr{V}_i(\mathcal{G}_{P_1P_2})$$

and (therefore through Tellegen's theorem and the implicit duality theorem)

$$\mathscr{V}_v(\mathcal{G}) = \Big(\mathscr{V}_v(\mathcal{G}_{E_1P_1}) \oplus \mathscr{V}_v(\mathcal{G}_{E_2P_2})\Big) \leftrightarrow \mathscr{V}_v(\mathcal{G}_{P_1P_2}),$$

where P_1, P_2 are additional sets of branches. This process is called multi-port decomposition.

Let $v_E = (v_{E_1}, v_{E_2})$ be a voltage vector (that is, a vector that satisfies KVL conditions) of $\mathcal{G}$. Then there exists (v_{P_1}, v_{P_2}) such that (v_{E_1}, v_{P_1}), (v_{P_1}, v_{P_2}), (v_{P_2}, v_{E_2}) are voltage vectors of $\mathcal{G}_{E_1P_1}$, $\mathcal{G}_{P_1P_2}$, $\mathcal{G}_{P_2E_2}$ respectively. Conversely, if (v_{E_1}, v_{P_1}), (v_{P_1}, v_{P_2}), (v_{P_2}, v_{E_2}) are voltage vectors of $\mathcal{G}_{E_1P_1}$, $\mathcal{G}_{P_1P_2}$, $\mathcal{G}_{P_2E_2}$ then (v_{E_1}, v_{E_2}) is a voltage vector of $\mathcal{G}$. A similar statement is true for the current vectors.

There is a linear time algorithm available for decomposing minimally, that is, in such a way that $|P_1|$, $|P_2|$ have minimum size. We can show that this size is

$$\begin{aligned}|P_1| = |P_2| &= r(\mathcal{G} \circ E_1) - r(\mathcal{G} \times E_1)\\ &= r(\mathcal{G} \circ E_2) - r(\mathcal{G} \times E_2).\end{aligned}$$

(This decomposition agrees with intuition when both $\mathcal{G} \circ E_1$ and $\mathcal{G} \circ E_2$ are connected but not when both are disconnected.) When the decomposition is minimal the sets P_i will not contain circuits or cutsets in $\mathcal{G}_{E_1P_1}$, $\mathcal{G}_{P_1P_2}$, $\mathcal{G}_{P_2E_2}$.

Theorem 4. *Let the decomposition of the directed graph $\mathcal{G}$ into $\mathcal{G}_{E_1P_1}$, $\mathcal{G}_{E_2P_2}$, $\mathcal{G}_{P_1P_2}$ be minimal. Let i_E, v_E be current and voltage vectors of $\mathcal{G}$. Let (i_{E_1}, i_{P_1}), (i_{E_2}, i_{P_2}), (i_{P_1}, i_{P_2}) be current vectors and let (v_{E_1}, v_{P_1}), (v_{E_2}, v_{P_2}), (v_{P_1}, v_{P_2}) be voltage vectors of $\mathcal{G}_{E_1P_1}$, $\mathcal{G}_{E_2P_2}$, $\mathcal{G}_{P_1P_2}$, respectively. Then i_{P_1}, i_{P_2} can be written in terms of i_{E_1} as well as i_{E_2} and v_{P_1}, v_{P_2} can be written in terms of v_{E_1} as well as v_{E_2}.*

Proof. In any directed graph it is well known (and easy to show) that the currents in a tree can be uniquely expressed in terms of the currents in the co-tree and the voltages in a co-tree can be uniquely expressed in terms of the voltages in the tree. In a graph $\mathcal{G}$, if a set P of branches does not contain a loop, it can be extended into a tree by adding edges and therefore the currents of P can be expressed in terms of the currents of its complement. If P does not contain a cutset, its deletion (open circuiting) will not disconnect the graph. Hence the complement of P contains a tree of $\mathcal{G}$. Hence the voltages of P can be expressed in terms of voltages of its complement.

Since P_1, P_2 do not contain cutsets in any of the three graphs of the decomposition, we can write v_{P_1} in terms of v_{E_1} in $\mathcal{G}_{E_1P_1}$, v_{P_2} in terms of v_{P_1} in $\mathcal{G}_{P_1P_2}$ and therefore v_{P_2} in terms of v_{E_1} and similarly v_{P_1} in terms of v_{E_2}. Similarly, using the fact that P_1, P_2 do not contain loops in any of the graphs of the decomposition we can write i_{P_2} in terms of i_{E_1} and i_{P_1} in terms of i_{E_2}. □

For an $rLCEJ$ network such as the one in Figure 1, we do minimal multi-port decomposition twice as in Figure 2. First we partition E into E_C, $E_r \uplus E_m \uplus E_L$. Next we minimally decompose $\mathcal{G}$ into $(\mathcal{G}_{E_CP_C}, \mathcal{G}_{E_rE_mE_LP'_C}; \mathcal{G}_{P_CP'_C})$. Then we decompose $\mathcal{G}_{E_rE_mE_LP'_C}$ minimally into $(\mathcal{G}_{E_rE_mP'_CP'_L}, \mathcal{G}_{E_LP_L}, \mathcal{G}_{P_LP'_L})$. Now we may think of the original network as being decomposed into three multi-ports: (a) the capacitive multi-port $\mathcal{N}_{E_CP_C}$ on the graph $\mathcal{G}_{E_CP_C}$ and device characteristics for E_C being $i_C = (C)\overset{\bullet}{v}_C$ and no constraints on i_{P_C}, v_{P_C} (b) the inductive multi-port $\mathcal{N}_{E_LP_L}$ on the graph $\mathcal{G}_{E_LP_L}$ and device characteristics for E_L being $v_L = (L)\frac{di_L}{dt}$ and no constraints on i_{P_L}, v_{P_L} (c) the static multi-port $\mathcal{N}_{E_rE_mP'_CP'_L}$ on graph $\mathcal{G}_{E_rE_mP'_CP'_L}$ and device characteristics for E_r as in the original network with no constraints on $i_{P'_L}$, $i_{P'_C}$, $v_{P'_L}$, $v_{P'_C}$.

The port variables i_{P_C}, $i_{P'_C}$, v_{P_C}, $v_{P'_C}$ are linked through the KCL and KVL conditions of $\mathcal{G}_{P_CP'_C}$ and the port variables i_{P_L}, $i_{P'_L}$, v_{P_L}, $v_{P'_L}$ are similarly linked through the KCL and KVL conditions of $\mathcal{G}_{P_LP'_L}$ and each member of the primed, unprimed quantities can be written in terms of other. Formally,

$$\mathscr{V}_v(\mathcal{G}) = \Big(\mathscr{V}_v(\mathcal{G}_{E_CP_C}) \oplus \mathscr{V}_v(\mathcal{G}_{E_LP_L}) \oplus \mathscr{V}_v(\mathcal{G}_{E_rE_mP'_CP'_L})\Big) \leftrightarrow \Big(\mathscr{V}_v(\mathcal{G}_{P_CP'_C}) \oplus \mathscr{V}_v(\mathcal{G}_{P_LP'_L})\Big).$$

We will assume that in $\mathcal{G}_{E_mE_rP'_CP'_L}$ there are no loops containing only branches from P'_C, voltage sources and current output branches and no cutsets containing only branches from P'_L, current sources and voltage output branches. This is true in the present example. It would also be true if all voltage sources (including current output branches which are short circuits) occur in series with (small) resistors and all current sources (including voltage output branches which are open circuits) in parallel with (large) resistors. Practical circuits satisfy such conditions.

The sets of branches P_C, P'_C, P_L, P'_L do not contain loops or cutsets in any of the graphs of the decomposition where they occur. Hence $v_{P'_C}$ can be written in terms of v_C, $i_{P'_C}$ can be written in terms of i_C, $i_{P'_L}$ can be written in terms of i_L and $v_{P'_L}$ can be written in terms of v_L. Using the fact that $i_C = C\overset{\bullet}{v}_C$ and $v_L = L\frac{di_L}{dt}$ it follows that $i_{P'_C}$ can be written in terms of $\overset{\bullet}{v}_C$ and $v_{P'_L}$ in terms of $\frac{di_L}{dt}$.

In the discussion that follows immediately we have derived explicit relations for the sake of clarity. We will however perform the computations of this paper on the network working with multi-ports but without explicitly computing (8).

If the above loop-free, cutset free condition is satisfied by the static multi-port, then under mild genericity assumptions we can write

$$\begin{pmatrix} i_{P'_C} \\ v_{P'_L} \end{pmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{pmatrix} v_{P'_C} \\ i_{P'_L} \end{pmatrix} + \begin{bmatrix} H_{13} \\ H_{23} \end{bmatrix} u \tag{7a}$$

$$y = [H_{31}\ H_{32}] \begin{pmatrix} v_{P'_C} \\ i_{P'_L} \end{pmatrix} + [H_{33}]u. \tag{7b}$$

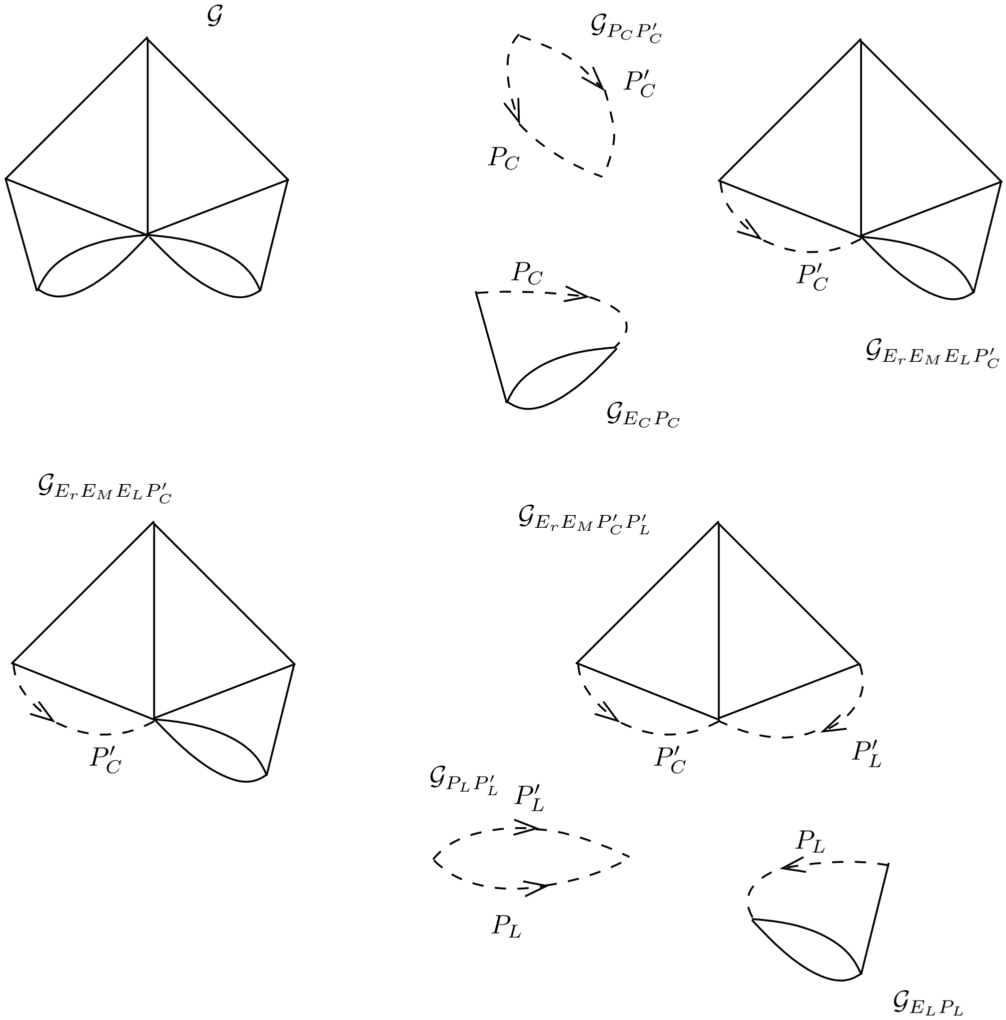

Figure 2: rLC$\mathcal{E}$J network graph

Using the constraints of the capacitive multi-port we can write, for a suitable $\tilde{C}$, K_C,

$$
\begin{aligned}
i_{P_C} &= \tilde{C}\ \dot{v}_{P_C} \ \text{ and } \ \dot{v}_C = K_C\ \dot{v}_{P_C} \\
v_C &= v_C(0) + K_C \int_0^t \dot{v}_{P_C}\ dt \\
&= v_C(0) + K_C(v_{P_C}(t) - v_{P_C}(0)).
\end{aligned}
$$

This can be done for instance by solving the 'static' network on $\mathcal{G}_{E_C P_C}$, i_C, i_{P_C} satisfying KCL, $\dot{v}_C, \dot{v}_{P_C}$ satisfying KVL, $i_C = C\ \dot{v}_C$ as device characteristics and $\dot{v}_{P_C}$ as 'voltage sources'.

Using the constraints of the inductive multi-port we can write, for a suitable $\tilde{L}$, K_L,

$$
\begin{aligned}
v_{P_L} &= \tilde{L}\frac{di_{P_L}}{dt} \ \text{and} \ \frac{di_L}{dt} = K_L \frac{di_{P_L}}{dt} \\
i_L &= i_L(0) + K_L \int_0^t \frac{di_{P_L}}{dt} dt \\
&= i_L(0) + K_L(i_{P_L}(t) - i_{P_L}(0)).
\end{aligned}
$$

Using the port connection diagram $\mathcal{G}_{P_C P'_C}$ we get $v_{P'_C} = T_C v_{P_C}$, $i_{P_C} = -T_C^T i_{P'_C}$ and using the port connection diagram $\mathcal{G}_{P_L P'_L}$ we get $i_{P'_L} = T_L i_{P_L}$, $v_{P_L} = -T_L^T v_{P'_L}$.

We therefore have from (7) and the foregoing discussion

$$\dot{v}_{P_C} = \left[-\widetilde{C}^{-1}T_C^T\right][H_{11}\ H_{12}]\begin{pmatrix}T_C v_{P_C}\\ T_L i_{P_L}\end{pmatrix} + \left[-\widetilde{C}^{-1}T_C^T\right][H_{13}]u \tag{8a}$$

$$\frac{di_{P_L}}{dt} = \left[-\widetilde{L}^{-1}T_L^T\right][H_{21}\ H_{22}]\begin{pmatrix}T_C v_{P_C}\\ T_L i_{P_L}\end{pmatrix} + \left[-\widetilde{L}^{-1}T_L^T\right][H_{23}]u. \tag{8b}$$

Essentially (8) captures the dynamics of the network. However, v_{P_C}, i_{P_L} are not all the state variables. But knowing $v_C(0)$, $i_L(0)$, $v_{P_C}(t)$, $i_{P_L}(t)$ all dynamic variables can be computed instantaneously as we showed above. As mentioned before, we will perform the computations of this paper on the network working with the multi-ports without explicitly computing (8). The multi-port decomposition itself is linear time and therefore computationally inexpensive.

6.2. Computing $\mathcal{V}_{\dot{W}W} \leftrightarrow \mathcal{V}_W$, $\mathcal{V}_{\dot{W}W} \leftrightarrow \mathcal{V}_{\dot{W}}$ for Networks

In all the algorithms of this paper the key step is the computation of $\mathcal{V}_{\dot{W}W} \leftrightarrow \mathcal{V}_W$ or $\mathcal{V}_{\dot{W}W} \leftrightarrow \mathcal{V}_{\dot{W}}$. We describe how to perform this computation for a $rLCEJ$ network for the usual instances of $\mathcal{V}_M$. The essential idea is to transform the problem to one of **solving** the three multi-ports under appropriate conditions.

6.2.1. $\mathcal{V}_{\dot{W}W} \leftrightarrow \mathcal{V}_W$, $\mathcal{V}_M = 0_{mu} \oplus \mathcal{F}_{my}$

We will only show, given a vector $w \in \mathcal{V}_W$ how to find all $\dot{w}$ such that $(\dot{w}, w) \in \mathcal{V}_{\dot{W}W}$. In the generic situation, this will be unique since for $w = 0$, $m_u = 0$ the only $\dot{w}$ consistent will be the zero vector. The space $\mathcal{V}_{\dot{W}WLM} \leftrightarrow \mathcal{V}_M$ is obtained by setting all input sources to zero (in Figure 1, u_7, u_8 are short circuited and open circuited respectively). The space $\mathcal{V}_{\dot{W}W}$ is equal to $\mathcal{V}_{\dot{W}WLM} \leftrightarrow \mathcal{V}_M \circ (\dot{W} \uplus W)$.

Let $w \in \mathcal{V}_W$. This is a vector v_C, i_L in $\mathcal{N}$. There is a unique vector $v_{P'_C}$, $i_{P'_L}$ (through topological constraints in $\mathcal{G}_{E_C P_C}$, $\mathcal{G}_{E_L P_L}$, $\mathcal{G}_{P_C P'_C}$, $\mathcal{G}_{P_L P'_L}$ since P_C, P'_C, P_L, P'_L do not form loops or cutsets in any of the graphs) corresponding to v_C, i_L. Impose $v_{P'_C}$, $i_{P'_L}$ onto the ports of $\mathcal{N}_{E_r E_m P'_C P'_L}$ with the sources set to zero. As mentioned before, under mild genericity assumptions on the static characteristics, there will be a unique $i_{P'_C}$, $v_{P'_L}$ corresponding to this port condition. By **solving** this (static) multi-port we would get $i_{P'_C}$, $v_{P'_L}$ from which we can get i_{P_C}, v_{P_L} through the topological constraints in $\mathcal{G}_{P_C P'_C}$ and $\mathcal{G}_{P_L P'_L}$.

We now **solve** the multi-port $\mathcal{N}_{E_C P_C}$ with currents i_{P_C} in the ports P_C using the device characteristics constraints $i_C = C\ \dot{v}_C$, KCL conditions of $\mathcal{G}_{E_C P_C}$ on (i_C, i_{P_C}), KVL conditions of $\mathcal{G}_{E_C P_C}$ on $(\dot{v}_C, \dot{v}_{P_C})$. This yields $\dot{v}_C$.

Similarly, we **solve** the multi-port $\mathcal{N}_{E_L P_L}$ with voltages v_{P_L} in the ports P_L using the device characteristics $v_L = L\frac{di_L}{dt}$, KVL conditions of $\mathcal{G}_{E_L P_L}$ on (v_L, v_{P_L}), KCL conditions of $\mathcal{G}_{E_L P_L}$ on $(\frac{di_L}{dt}, \frac{di_{P_L}}{dt})$. This yields $\frac{di_L}{dt}$. We then get the unique vector $\dot{w} = (\dot{v}_C, \frac{di_L}{dt})$ corresponding to the vector $w = (v_C, i_L)$. Observe that the computationally significant steps are the solutions of the three multi-ports.

6.2.2. $\mathcal{V}_{\dot{W}W} \leftrightarrow \mathcal{V}_{\dot{W}}$, $\mathcal{V}_M = 0_{mu} \oplus \mathcal{F}_{my}$

To compute $\mathcal{V}_{\dot{W}W} \leftrightarrow \mathcal{V}_{\dot{W}}$, we begin once again by setting all sources to zero. Let $\dot{w} = (\dot{v}_C, \frac{di_L}{dt}) \in \mathcal{V}_{\dot{W}}$. We will show how to find all w such that $(w, \dot{w}) \in \mathcal{V}_{\dot{W}W}$.

The first step is to compute all $w = (v_c, i_L)$ corresponding to $\dot{w} = 0$. We will show that generically the collection of all such v_C is the voltage space of $\mathcal{G}_{E_C P_C} \times E_C$ and the collection of all such i_L is the current space of $\mathcal{G}_{E_L P_L} \circ E_L$.

$\dot{w} = 0$ implies $\dot{v}_C = 0$ and $\frac{di_L}{dt} = 0$. This implies $C\ \dot{v}_C = 0$ and $L\frac{di_L}{dt} = 0$. Hence $i_C = 0$, $v_L = 0$. Therefore in the multi-ports $\mathcal{N}_{E_C P_C}$, $\mathcal{N}_{E_L P_L}$, since P_C, P_L do not contain cutsets, $i_{P_C} = 0$, $v_{P_L} = 0$. Using KCL, KVL of $\mathcal{G}_{P_C P'_C}$, $\mathcal{G}_{P_L P'_L}$ this implies $i_{P'_C} = 0$, $v_{P'_L} = 0$. In the multi-port $\mathcal{N}_{E_r E_m P'_C P'_L}$, the sources have already been set to zero (since $\mathscr{V}_M = 0_{mu} \oplus \mathscr{F}_{my}$). In this network P'_C, P'_L contain no loops or cutsets even when all sources are set to zero. Under mild genericity conditions on the static devices, this network can be shown to have a unique solution for arbitrary values of $i_{P'_C}$, $v_{P'_L}$. So in particular if $i_{P'_C} = 0$, $v_{P'_L} = 0$ we would have only the zero solution. Hence $v_{P'_C}$, $i_{P'_L}$ are zero vectors. This means, through KCL, KVL of $\mathcal{G}_{P_C P'_C}$, $\mathcal{G}_{P_L P'_L}$ that $v_{P_C} = 0$, $i_{P_L} = 0$. The corresponding space of v_C (with $v_{P_C} = 0$) would therefore be the space of solutions of KVL constraints of $\mathcal{G}_{E_C P_C}$ with P_C short circuited, that is, the voltage space of $\mathcal{G}_{E_C P_C} \times E_C$. (Under minimal multi-port decomposition this is the space of solutions of KVL constraints of $\mathcal{G}$ with all branches other than the C branches short circuited). The corresponding space of i_L (with $i_{P_L} = 0$) can similarly be shown to be the space of KCL constraints of $\mathcal{G}_{E_L P_L}$ with P_L open circuited, that is, the current space of $\mathcal{G}_{E_L P_L} \circ E_L$. (Under minimal multi-port decomposition this is the space of solutions of KCL constraints of $\mathcal{G}$ with all branches of $\mathcal{G}$ other than L branches open circuited).

Next we compute a single $w_1 \in \mathscr{V}_W$ corresponding to a $\dot{w}_1 \in \mathscr{V}_{\dot{W}}$. Let $\dot{w}_1 \in \mathscr{V}_{\dot{W}}$. Now let $\dot{w}_1 = (\dot{v}^1_C, \frac{di^1_L}{dt})$. Using $i^1_C = C\, \dot{v}^1_C$, $v^1_L = L\frac{di^1_L}{dt}$ we get i^1_C, v^1_L. Using KVL, KCL of $\mathcal{G}_{E_C P_C}$, $\mathcal{G}_{E_L P_L}$, since P_C, P_L contain no loops or cutsets in these graphs we can compute $i^1_{P_C}$, $v^1_{P_L}$ (uniquely). Using KCL, KVL of $\mathcal{G}_{P_C P'_C}$, $\mathcal{G}_{P_L P'_L}$ we can compute $i^1_{P'_C}$, $v^1_{P'_L}$. In the multi-port $\mathcal{N}_{E_r E_m P'_C P'_L}$, the sources have already been set to zero (since $\mathscr{V}_M = 0_{mu} \oplus \mathscr{F}_{my}$). As before we can compute $v^1_{P'_C}$, $i^1_{P'_L}$ uniquely. Through KCL, KVL of $\mathcal{G}_{P_L P'_L}$, $\mathcal{G}_{P_C P'_C}$ $v^1_{P_C}$, $i^1_{P_L}$ can be computed uniquely.

One solution v^1_C can be obtained by extending P_C to a tree t_C in $\mathcal{G}_{E_C P_C}$ assigning arbitrary voltages to branches in $t_C - P_C$ and hence by KVL of $\mathcal{G}_{E_C P_C}$, computing voltages for $E_C - t_C$. Similarly, one solution i^1_L corresponding to $i^1_{P_L}$ can be obtained by extending P_L to a co-tree $\bar{t}_L$ in $\mathcal{G}_{E_L P_L}$ assigning arbitrary currents to branches in $\bar{t}_L - P_L$ and hence by KCL of $\mathcal{G}_{E_L P_L}$ computing currents for $E_L - \bar{t}_L$. v^1_C, i^1_L together constitute the desired $w_1 \in \mathscr{V}_W$.

The collection of all vectors $\mathscr{K}^1_C \oplus \mathscr{K}^1_L$ corresponding to $v^1_{P_C}$, $i^1_{P_L}$ (and therefore also corresponding to $\dot{w}_1 = (\dot{v}^1_C, \frac{di^1_L}{dt})$) is obtained as follows: $\mathscr{K}^1_C = v^1_C + \mathscr{V}^0_C$, $\mathscr{K}^1_L = i^1_L + \mathscr{V}^0_L$, where $\mathscr{V}^0_C$, $\mathscr{V}^0_L$ correspond to $\dot{v}_C = 0$, $\frac{di_L}{dt} = 0$. In the above, the computations which involve only KCL, KVL are, being linear time, inexpensive. The significant computation is the one that involves solution of the multi-port $\mathcal{N}_{E_r E_m P'_C P'_L}$ for given $i^1_{P'_C}$, $v^1_{P'_L}$.

6.2.3. $\mathscr{V}_M = \mathscr{F}_{mu} \oplus \mathscr{F}_{my}$, $\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_W$, $\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_{\dot{W}}$

In this case we have to compute first as in Sub-Sections 6.2.1 and 6.2.2 above with $\mathscr{V}_M = 0_{mu} \oplus \mathscr{F}_{my}$. Next we make the computation with $w = 0$ ($\dot{w} = 0$) as the case may be but with $m_u = (1, 0, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots (0, 0, 0, \ldots, 1)$ and get a collection of vectors $B^u_{\dot{w}}$ (B^u_w). Let the space spanned by these vectors be $\mathscr{V}^u_{\dot{W}}$ ($\mathscr{V}^u_W$). Suppose $\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_W$ is $\widetilde{\mathscr{V}_{\dot{W}}}$ when $\mathscr{V}_M = 0_{mu} \oplus \mathscr{F}_{my}$. Then in the present case it would be $\widetilde{\mathscr{V}_{\dot{W}}} + \mathscr{V}^u_{\dot{W}}$. A similar sum of spaces has to be computed also in the case $\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_{\dot{W}}$.

6.2.4. $\mathscr{V}_M = \mathscr{F}_{mu} \oplus 0_{my}$

We will only examine how to compute all $\dot{w}$ in $\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}_W$ corresponding to a given $w \in \mathscr{V}_W$ and a given m_u. Let $w \in \mathscr{V}_W$, $m \in \mathscr{F}_{mu}$. Then $w = (v_C, i_L)$. As before we can compute v_{P_C}, i_{P_L} (by using the constraints of $\mathcal{G}_{E_C P_C}$, $\mathcal{G}_{E_L P_L}$) and therefore (by using the constraints of $\mathcal{G}_{P_C P'_C}$, $\mathcal{G}_{P_L P'_L}$) $v_{P'_C}$, $i_{P'_L}$.

Now we move to the network $\mathcal{N}_{E_r E_m P'_C P'_L}$ with all output branches treated as nullators. We obtain one solution of this network corresponding to input source values m_u, and to $v_{P'_C}$, $i_{P'_L}$ which in turn will yield $i_{P'_C}$, $v_{P'_L}$ and therefore i_{P_C}, v_{P_L} from which working with $\mathcal{G}_{E_C P_C}$, $\mathcal{G}_{E_L P_L}$, $(\widehat{\dot{v}}_C, \frac{d\widehat{i}_L}{dt})$ can be computed.

We next compute $(\dot{v}_C, \frac{di_L}{dt})$ when (v_C, i_L) are zero and $m_u = 0$, that is, find all $i_{P'_C}$, $v_{P'_L}$ when $(v_{P'_C}, i_{P'_L})$ are zero and $m_u = 0$ in $\mathcal{N}_{E_r E_m P'_C P'_L}$. Each such $(v^j_{P'_C}, i^j_{P'_L})$ will yield a unique $(\dot{v}^j_C, \frac{di^j_L}{dt})$ in $\mathcal{N}_{E_C P_C}$, $\mathcal{N}_{E_L P_L}$.

Let the space of all $(\overset{\bullet}{v}^j_C, \frac{di^j_L}{dt})$ be denoted by $\mathscr{V}^0_W$. All $\overset{\bullet}{w}$ corresponding to m_u, w is obtained as

$$(\widehat{\overset{\bullet}{v}}_C, \frac{d\widehat{i}_L}{dt}) + \mathscr{V}^0_{\overset{\bullet}{W}}.$$

It can be seen that the above computations on large systems involve 'solving' the system under specific conditions. In our approach, as has been seen, the computatins do not require first casting the system in the state variable form. However computations such as $\mathscr{V}_{\overset{\bullet}{W}W} \leftrightarrow \widehat{\mathscr{V}}_W$ for large systems involve solving the system typically $O(dim\widehat{\mathscr{V}}_W)$ times. While this is cheaper than going through state equation (both in terms of time and space) it is still very expensive for large systems and may not be a convincing reason for adopting our approach.

However, if one is trying to construct a reduced order model of a large dynamical system our approach would clearly be superior to first constructing the state equation. Suppose the model order reduction is to be performed on the network decomposed as in Section 6.1. If this system had the state equation

$$\overset{\bullet}{x} = Ax + Bu$$

a typical model reduction procedure would use the Arnoldi algorithm. This would involve picking a 'generic' vector $b \in \text{im}(B)$ building the sequence $b, Ab, \ldots, A^{k-1}b$ (upto a suitable value k) and orthonormalizing the vectors. Suppose the resulting set of vectors are the columns of W, the reduced order system would be

$$\overset{\bullet}{z} = W^T AWz + W^T bu.$$

In our approach building $\mathscr{B}, A\mathscr{B}, \ldots$ correspond to using Algorithm I$'$ on a copy $\widehat{\mathscr{V}}_W$ of $\mathscr{V}_{\overset{\bullet}{W}W} \times \overset{\bullet}{W}$ on W, where $\mathscr{V}_{\overset{\bullet}{W}W} = \mathscr{V}_{\overset{\bullet}{W}WM} \leftrightarrow (0_{m_u} \oplus \mathscr{F}_{m_y})$.

The key step in the Arnoldi algorithm is the computation of Ax for a given x. In Section 6.2.1 we have shown how to make this computation for a given large scale network when we computed $\overset{\bullet}{W}$ corresponding to a given W. Computing the matrix W with k columns requires that the process be repeated k times. We know how to compute AW (as above). Hence $W^T AW$ and $W^T B$ can be computed.

7. Duality

One of the attractive features of classical multi-variable control theory is the duality that underlies the development. If anything, duality ideas are even more natural in our way of handling dynamical systems. The key idea for us is the following: Let

$$\epsilon(\mathscr{V}_1, \ldots, \mathscr{V}_k, +, \cap, \leftrightarrow, \supseteq, \subseteq)$$

be a statement. We assume that the connective '$\leftrightarrow$' is between spaces of the form $\mathscr{V}_S$, $\mathscr{V}_T$ where $S \supset T$. The statement

$$\epsilon(\mathscr{V}_1^\perp, \ldots, \mathscr{V}_k^\perp, \cap, +, \leftrightarrow, \subseteq, \supseteq)$$

is obtained by replacing $\mathscr{V}_i$ by $\mathscr{V}_i^\perp$, $+$ by $\cap$, $\cap$ by $+$ and interchanging $\supseteq, \subseteq$. Then $\epsilon(\mathscr{V}_1, \ldots, \mathscr{V}_k, +, \cap, \leftrightarrow, \supseteq, \subseteq)$ is true iff $\epsilon(\mathscr{V}_1^\perp, \ldots, \mathscr{V}_k^\perp, \cap, +, \leftrightarrow, \subseteq, \supseteq)$ is true. This fact is a routine consequence of the basic results $(\mathscr{V}^\perp)^\perp = \mathscr{V}$, $(\mathscr{V}_1 + \mathscr{V}_2)^\perp = \mathscr{V}_1^\perp \cap \mathscr{V}_2^\perp$, $(\mathscr{V}_1 \leftrightarrow \mathscr{V}_2)^\perp = \mathscr{V}_1^\perp \leftrightarrow \mathscr{V}_2^\perp$.

7.1. Adjoint System

For dynamical systems, we need a natural notion of adjoint. In the usual state, input, output representation, suppose the original system is defined by

$$\overset{\bullet}{x} = Ax + Bu$$
$$y = Cx + Du$$

that is,

$$\begin{bmatrix} I & -A & -B & 0 \\ 0 & -C & -D & I \end{bmatrix} \begin{pmatrix} \dot{x} \\ x \\ u \\ y \end{pmatrix} = 0.$$

Then the adjoint is defined by

$$\begin{aligned} \dot{\widehat{x}} &= A^T\widehat{x} + C^T\widehat{u} \\ \widehat{y} &= B^T\widehat{x} + D^T\widehat{u} \end{aligned}$$

that is,

$$\begin{bmatrix} -A^T & I & 0 & -C^T \\ -B^T & 0 & I & -D^T \end{bmatrix} \begin{pmatrix} \widehat{x} \\ \dot{\widehat{x}} \\ \widehat{y} \\ \widehat{u} \end{pmatrix} = 0.$$

Thus if the original dynamical system is $\mathscr{V}_{\dot{x}xuy}$, then the adjoint is $\mathscr{V}^{\perp}_{-\widehat{x}-\dot{\widehat{x}}\widehat{y}-\widehat{u}}$. For our purposes the change of sign for $\widehat{x}$, $\widehat{u}$ is not critical. So we will work with $\mathscr{V}^{\perp}_{\widehat{x}\dot{\widehat{x}}\widehat{y}\widehat{u}}$ or its equivalent. Note that the positions of x, $\dot{x}$ and y, u are interchanged in the adjoint.

We recall the notation for interchanging the variables in a collection of vectors. When A and A' are copies of each other, the notation for interchanging the positions of variables A and A' in a collection $\mathscr{K}_{AA'B}$ is given by $\boldsymbol{(\mathscr{K}_{AA'B})_{\mathbf{swap}(AA')}}$, that is

$$(\mathscr{K}_{AA'B})_{\text{swap}(AA')} = \{(g_A f_{A'} h_B) \mid (f_A g_{A'} h_B) \in \mathscr{K}_{AA'B},\ g_A \text{ being copy of } g_{A'},\ f_{A'} \text{ being copy of } f_A\}.$$

In our framework the partition of the manifest variable into u, y is achieved by the artifice of using the space $\mathscr{V}_M$ (and therefore also $\mathscr{V}_M^{\perp}$) and using the projection of the manifest variable m on $\mathscr{V}_M$ and $\mathscr{V}_M^{\perp}$.

Let $\mathscr{V}_{\dot{W}WM}$ be a generalized dynamical system and let $\mathscr{V}_M$ be a vector space on M. Then the adjoint of the pair

$$(\mathscr{V}_{\dot{W}WM}, \mathscr{V}_M)$$

is

$$\left((\mathscr{V}^{\perp}_{\dot{W}WM})_{\text{swap}(W\dot{W})}, \mathscr{V}_M^{\perp}\right).$$

Again note that the positions of W and $\dot{W}$ are interchanged in the adjoint. The dynamical system $\mathscr{V}_{\dot{W}WM}$ with reference to the subspace $\mathscr{V}_M$ will often be referred to by the pair $(\mathscr{V}_{\dot{W}WM}, \mathscr{V}_M)$. We will interchangeably use both these terms.

7.2. Duality in Adjoint Systems

To illustrate the use of duality we will now dualize some of the standard notions in Sections 3, 4 and 5. Let the system $\mathscr{V}_{\dot{W}WM}$ be $\dot{w}$-zero relative to $\mathscr{V}_M$ that is, $(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow 0_W = 0_{\dot{W}}$. Replace each occurrence of a vector space $\mathscr{V}$ by $\mathscr{V}^{\perp}$ and $\leftrightarrow$ by $\leftrightarrow$. Further, interchange W with $\dot{W}$. We then have $\left((\mathscr{V}^{\perp}_{\dot{W}WM})_{\text{swap}\,\dot{W}W} \leftrightarrow \mathscr{V}_M^{\perp}\right) \leftrightarrow \mathscr{F}_{\dot{W}} = \mathscr{F}_W$ that is, $(\mathscr{V}_{\dot{W}WM}, \mathscr{V}_M)$ is $\dot{w}$-zero iff $\left((\mathscr{V}^{\perp}_{\dot{W}WM})_{\text{swap}(W\dot{W})}, \mathscr{V}_M^{\perp}\right)$ is w-free. Similarly, $(\mathscr{V}_{\dot{W}WM}, \mathscr{V}_M)$ would be $\dot{w}$-free iff $\left((\mathscr{V}^{\perp}_{\dot{W}WM})_{\text{swap}(W\dot{W})}, \mathscr{V}_M^{\perp}\right)$ is w-zero.

Next let $\mathscr{V}_{\dot{W}WM}$ be m-zero relative to $\mathscr{V}_M$, that is, $(\mathscr{V}_{\dot{W}WM} \circ (W \uplus M) \times M) \cap \mathscr{V}_M = 0_M$. Dualizing we get

$$(\mathscr{V}^{\perp}_{W\dot{W}M} \times (\dot{W} \uplus M) \circ M) + \mathscr{V}^{\perp}_M = \mathscr{F}_M$$

that is,

$$(\mathscr{V}^{\perp}_{W\dot{W}M} \circ (W \uplus M) \times M) + \mathscr{V}^{\perp}_M = \mathscr{F}_M.$$

Thus $\mathscr{V}_{\dot{W}WM}$ is m-zero relative to $\mathscr{V}_M$ iff $\mathscr{V}^{\perp}_{\dot{W}WM}$ is m-free relative to $\mathscr{V}^{\perp}_M$.

Let $\widehat{\mathscr{V}}_W$ be conditioned invariant in $\mathscr{V}_{\dot{W}WM}$ relative to $\mathscr{V}_M$, that is,

$$(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow \widehat{\mathscr{V}}_W \subseteq \widehat{\mathscr{V}}_{\dot{W}}. \tag{9}$$

Dualizing we conclude that this is true iff

$$(\mathscr{V}^{\perp}_{W\dot{W}M} \leftrightarrow \mathscr{V}^{\perp}_M) \leftrightarrow \widehat{\mathscr{V}}^{\perp}_{\dot{W}} \supseteq \widehat{\mathscr{V}}^{\perp}_W. \tag{10}$$

that is, $\widehat{\mathscr{V}}^{\perp}_W$ is controlled invariant in $\mathscr{V}^{\perp}_{W\dot{W}M}$ relative to $\mathscr{V}^{\perp}_M$.

Consider the Algorithm I for building the unique minimal space containing $\mathscr{V}^{\text{small}}_W$ and satisfying (9). The key step in the algorithm is

$$\mathscr{V}^{j+1}_{\dot{W}} = \left((\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow \mathscr{V}^j_W\right) + \mathscr{V}^j_{\dot{W}}.$$

The dual to Algorithm I would be to build the unique maximal space contained in $\mathscr{V}^{\text{big}}_W$ and satisfying (10). This of course is the Algorithm II whose key step is the dual

$$(\mathscr{V}^{j+1}_W)^{\perp} = \left((\mathscr{V}^{\perp}_{W\dot{W}M} \leftrightarrow \mathscr{V}^{\perp}_M) \leftrightarrow (\mathscr{V}^j_{\dot{W}})^{\perp}\right) \cap (\mathscr{V}^j_W)^{\perp}$$

of the above mentioned key step of Algorithm I.

Now we had recast Algorithm I as Algorithm I$'$ using Theorem 1. The dual of this theorem is

Theorem 5. *Let* $\mathscr{V}_{\dot{W}W} \times W \subseteq \mathscr{V}^1_W$. *Let the copy of* $\mathscr{V}_{\dot{W}W} \times W$ *on* $\dot{W}$ *contain* $\mathscr{V}_{\dot{W}W} \times \dot{W}$. *Also let*

$$\mathscr{V}^{j+1}_W \equiv (\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}^j_{\dot{W}}) \cap \mathscr{V}^j_W, \qquad \widetilde{\mathscr{V}}^1_W = \mathscr{V}^1_W$$
$$\widetilde{\mathscr{V}}^{j+1}_W \equiv (\mathscr{V}_{\dot{W}W} \leftrightarrow \widetilde{\mathscr{V}}^j_{\dot{W}})$$

then

$$\bigcap_{j=1}^{k} \widetilde{\mathscr{V}}^j_W \equiv \mathscr{V}^k_W.$$

We can then recast Algorithm II as Algorithm II$'$, using Theorem 5, for finding the maximal controlled invariant space contained in $\mathscr{V}^{\text{Big}}_W$ where we assume $\mathscr{V}^{\text{Big}}_W \supseteq \mathscr{V}_{\dot{W}W} \times W$.

Algorithm II$'$

Algorithm for computation of unique maximal space contained in $\mathscr{V}^{big}_W$ *and satisfying the condition* $(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow \widetilde{\mathscr{V}}_{\dot{W}} \supseteq \widetilde{\mathscr{V}}_W$.

`Assumption:` `Let` $\mathscr{V}^{\text{Big}}_W \supseteq \mathscr{V}_{\dot{W}W} \times W$. `Let the copy of` $\mathscr{V}_{\dot{W}W} \times W$ `on` $\dot{W}$ `contain` $\mathscr{V}_{\dot{W}W} \times \dot{W}$.

1. `Set` $\widetilde{\mathscr{V}}^1_W = \mathscr{V}^{\text{Big}}_W$
2. $\widetilde{\mathscr{V}}^{j+1}_W \equiv (\mathscr{V}_{\dot{W}W} \leftrightarrow \widetilde{\mathscr{V}}^j_{\dot{W}})$
3. `Stop when`

$$\bigcap_{j=1}^{k} \widetilde{\mathscr{V}}^j_W = \bigcap_{j=1}^{k+1} \widetilde{\mathscr{V}}^j_W.$$

4. `The desired space is` $\bigcap_{j=1}^{k} \widetilde{\mathscr{V}}^j_W$.

We illustrated Algorithm I$'$ using a space $\mathscr{V}_{\dot{W}WM}$ which permitted state and output descriptions and with $\mathscr{V}_M = \mathscr{F}_M$. Here we could take $\mathscr{V}_{\dot{W}WM}$, $\mathscr{V}_M$ dually that is, $\mathscr{V}_{\dot{W}WM}$ should permit state and output descriptions and $\mathscr{V}_M = 0_{mu} \oplus 0_{my}$. The basis of $\bigcap_{j=1}^{k} \widetilde{\mathscr{V}}^j_W$ (as in Algorithm II$'$) can be computed routinely.

Let us first interpret Algorithm II$'$ taking $\mathscr{V}_M = (0_u \oplus \mathscr{F}_y)$. This would better clarify the case $\mathscr{V}_M = 0_{mu} \oplus 0_{my}$. Let the system be in some state w and let the corresponding m^1_y be specified. Let $\widetilde{\mathscr{K}^1_W} \equiv (\mathscr{V}_{\dot{W}WM} \leftrightarrow \{0 \oplus m^1_y\}) \circ W$. This is an affine space and may be available as the set of solutions to, say

$$[S^1_W]\, w = [S^1_y]\, m^1_y.$$

Let

$$\widetilde{\mathscr{K}^2_{\dot{W}}} \equiv (\mathscr{V}_{\dot{W}WM} \leftrightarrow \{0 \oplus m^1_y\}) \leftrightarrow \widetilde{\mathscr{K}^1_W}$$

and let

$$(0 \oplus m^2_y) \in ((\mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_M) \leftrightarrow \widetilde{\mathscr{K}^2_W}) \circ M,$$

where $\widetilde{\mathscr{K}^2_W}$ is the W-copy of $\widetilde{\mathscr{K}^2_{\dot{W}}}$. Let $\widetilde{\mathscr{K}^2_W}$ be available as the set of solutions to

$$[S^2_W]\, w = [S^2_y]\, m^2_y.$$

We can similarly compute $\widetilde{\mathscr{K}^j_W}$ in general. Let

$$\mathscr{K}^{\text{final}}_W = \bigcap_{j=1}^{k} \widetilde{\mathscr{K}^j_W},$$

where $\widetilde{\mathscr{K}^{k+1}_W} = \widetilde{\mathscr{K}^k_W}$.

This space is the set of solutions to

$$\begin{aligned}
[S^1_W]\, w &= [S^1_y]\, m^1_y \\
&\vdots \\
[S^j_W]\, w &= [S^j_y]\, m^j_y \\
&\vdots \\
[S^k_W]\, w &= [S^k_y]\, m^k_y.
\end{aligned}$$

If two vectors x_W and $\widetilde{x}_W$ have the same values for $m_y^1, \ldots, m_y^j, \ldots, m_y^k$, then $x_W - \widetilde{x}_W$ must have $m_y = \ldots = m_y^j = \ldots = m_y^k = 0$. The space $\widetilde{\mathscr{K}_W^1}$ corresponding to $x_W - \widetilde{x}_W$ is

$$\left(\mathscr{V}_{\dot{W}WM} \leftrightarrow \{0_u \oplus 0_y\}\right) \circ W.$$

This would be a vector space $\widetilde{\mathscr{V}}_W^1$. Let

$$\widetilde{\mathscr{V}}_{\dot{W}}^2 = \left(\mathscr{V}_{\dot{W}WM} \leftrightarrow \{0_u \oplus 0_y\}\right) \leftrightarrow \widetilde{\mathscr{V}}_W^1,$$

and in general

$$\widetilde{\mathscr{V}}_{\dot{W}}^{j+1} = \left(\mathscr{V}_{\dot{W}WM} \leftrightarrow \{0_u \oplus 0_y\}\right) \leftrightarrow \widetilde{\mathscr{V}}_W^j,$$

$\widetilde{\mathscr{V}}_W^j$ being the W-copy of $\widetilde{\mathscr{V}}_{\dot{W}}^j$. Now let

$$\mathscr{V}_W^{\text{final}} = \bigcap_{j=1}^{k} \widetilde{\mathscr{V}}_W^j.$$

This is the unobservable space.

In the usual A, B, C, D notation for continuous systems x satisfies

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} x = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(k-1)} \end{bmatrix}$$

and the unobservable space $\mathscr{V}_W^{\text{final}}$ is the solution space of the above equation with right side zero.

8. Equivalence of Controlled and Conditioned Invariants

When is a conditioned (controlled) invariant space also controlled (conditioned) invariant? We have seen that the two notions are duals of each other. A space $\widehat{\mathscr{V}}_W$ is conditioned invariant if and only if whenever x is a state in it, for all m consistent with x, the corresponding $\dot{x}$ will be in $\widehat{\mathscr{V}}_{\dot{W}}$. But note that there may be no m consistent with a given $x \in \widehat{\mathscr{V}}_W$. The space $\widehat{\mathscr{V}}_W$ is therefore controlled invariant if and only if whenever x is a state in it, for some m, a consistent $\dot{x}$ will be in $\widehat{\mathscr{V}}_{\dot{W}}$.

A conditioned invariant space is therefore also controlled invariant, if 'for every' $x \in \widehat{\mathscr{V}}_W$ we have 'some' $\dot{x} \in \widehat{\mathscr{V}}_{\dot{W}}$ consistent with it. Similarly, a controlled invariant space is conditioned invariant, if 'for every' $x \in \widehat{\mathscr{V}}_W$ 'all' $\dot{x}$ consistent with it belong to $\widehat{\mathscr{V}}_{\dot{W}}$.

As we know, the defining relations for controlled and conditioned invariants are given by

$$\mathscr{V}_{\dot{W}W} \leftrightarrow \widehat{\mathscr{V}}_{\dot{W}} \supseteq \widehat{\mathscr{V}}_W, \tag{11}$$

$$\mathscr{V}_{\dot{W}W} \leftrightarrow \widehat{\mathscr{V}}_W \subseteq \widehat{\mathscr{V}}_{\dot{W}}, \tag{12}$$

where $\mathscr{V}_{\dot{W}W} = \mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M$ in (11) and (12).

A necessary and sufficient condition for a conditioned invariant space to be controlled invariant is given in the next lemma.

Lemma 6. *Let $\widehat{\mathscr{V}}_W$ satisfy* (12). *Then it satisfies* (11) *if and only if $\mathscr{V}_{\dot{W}W} \circ W \supseteq \widehat{\mathscr{V}}_W$.*

Proof. **(only if):** Suppose $\widehat{\mathscr{V}}_W$ also satisfies (11). Then for each $w \in \widehat{\mathscr{V}}_W$ we must have some $(w, \dot{w}) \in \mathscr{V}_{\dot{W}W}$. That is, $\mathscr{V}_{\dot{W}W} \circ W \supseteq \widehat{\mathscr{V}}_W$.

(if): Next suppose $\mathscr{V}_{\dot{W}W} \circ W \supseteq \widehat{\mathscr{V}}_W$. Then for each $w \in \widehat{\mathscr{V}}_W$ we must have some $(\dot{w}, w) \in \mathscr{V}_{\dot{W}W}$. Since $\widehat{\mathscr{V}}_W$ also satisfies (12), we have $\dot{w} \in \widehat{\mathscr{V}}_{\dot{W}}$. Hence $\widehat{\mathscr{V}}_W$ satisfies (11). □

The dual lemma below gives necessary and sufficient conditions for a controlled invariant space to be conditioned invariant. We give a direct proof.

Lemma 7. *Let* $\widehat{\mathscr{V}}_W$ *satisfy* (11). *Then it satisfies* (12) *if and only if* $\mathscr{V}_{\dot{W}W} \times \dot{W} \subseteq \widehat{\mathscr{V}}_{\dot{W}}$.

Proof. **(only if):** Suppose it also satisfies (12). Since $\widehat{\mathscr{V}}_W$ is a vector space $0_W \in \widehat{\mathscr{V}}_W$. It follows that $\dot{w} \in \widehat{\mathscr{V}}_{\dot{W}}$ whenever $(\dot{w}, 0_W) \in \mathscr{V}_{\dot{W}W}$, that is, $\mathscr{V}_{\dot{W}W} \times \dot{W} \subseteq \widehat{\mathscr{V}}_{\dot{W}}$.

(if): On the other hand suppose $\mathscr{V}_{\dot{W}W} \times \dot{W} \subseteq \widehat{\mathscr{V}}_{\dot{W}}$. Let $\dot{w} \in \mathscr{V}_{\dot{W}W} \leftrightarrow \widehat{\mathscr{V}}_W$. Then there exists w such that $(\dot{w}, w) \in \mathscr{V}_{\dot{W}W}$. Since $\widehat{\mathscr{V}}_W$ satisfies (11), there exists $\dot{w}' \in \widehat{\mathscr{V}}_{\dot{W}}$ such that $(\dot{w}', w) \in \mathscr{V}_{\dot{W}W}$. Thus, $(\dot{w}' - \dot{w}, 0_W) \in \mathscr{V}_{\dot{W}W}$ and therefore $(\dot{w}' - \dot{w}) \in \mathscr{V}_{\dot{W}W} \times \dot{W} \subseteq \widehat{\mathscr{V}}_{\dot{W}}$. But then $\dot{w} \in \widehat{\mathscr{V}}_{\dot{W}}$ because $\widehat{\mathscr{V}}_{\dot{W}}$ is a vector space. Therefore, $\widehat{\mathscr{V}}_W$ satisfies (12). □

Results of the kind in Lemmas 7 and 6 have already appeared in connection with multi-port decomposition problem, see for instance Theorem 11 and Theorem 18 of [43].

8.1. (W, M) *Feedback and* $(M, \dot{W})$ *Injection*

Usually in control systems we make a controlled invariant space to behave also like a conditioned invariant space by using 'state feedback' and a conditioned invariant space to behave also like a controlled invariant space by using 'output injection'. We introduce notions generalizing these next and use them for the above mentioned 'conversions'.

We say we are using **(W, M) feedback** on $\mathscr{V}_{\dot{W}WM}$, when the dynamical system $\mathscr{V}_{\dot{W}WM}$ is replaced by $\mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_{WM}$, where $\mathscr{V}_{WM} \circ W = \mathscr{F}_W$. The usual feedback situation is as follows:

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du \\
u &= Fx.
\end{aligned}
$$

Here for each x there is a unique u. But the manifest variables are already partitioned as u and y. We would not like a partition of the m variables into input and output (except through $\mathscr{V}_M$). So we permit many m values for the same w value. As in the usual feedback we restrict values of m using w. This in turn may, in the dynamical system, result in a restriction of w values. But the feedback does not directly put a restriction on w values. So we keep $\mathscr{V}_{WM} \circ W = \mathscr{F}_W$.

We are using **$(M, \dot{W})$ injection** on $\mathscr{V}_{\dot{W}WM}$, when $\mathscr{V}_{\dot{W}WM}$ is replaced by $\mathscr{V}_{\dot{W}WM} + \mathscr{V}_{M\dot{W}}$ where $\mathscr{V}_{M\dot{W}} \times \dot{W} = 0_{\dot{W}}$. The usual output injection situation is:

$$
\begin{aligned}
\dot{x} &= Ax + Bu + Fy \\
y &= Cx + Du.
\end{aligned}
$$

So, in this case the value of $\dot{w}$ is restricted by the manifest variable m. Also, the output injection map F produces a unique value of $\dot{x}$ for each y. Hence we require $\mathscr{V}_{M\dot{W}} \times \dot{W} = 0_{\dot{W}}$.

The key ideas of state feedback and output injection appear to be captured by the fact that, in one case we are performing an intersection with $\mathscr{V}_{WM}$ and in the other, the sum operation with $\mathscr{V}_{M\dot{W}}$. Usually state feedback is used in control systems to limit the inputs in order to force the states of the dynamical system to be within a prescribed space $\widehat{\mathscr{V}_X}$. Output injection is used to provide suitable 'inputs' (obtained using the output) such that starting from a state in $\widehat{\mathscr{V}_X}$ we can continue to remain inside this space no matter what the input is.

Feedback and injection can be used to alter the 'eigen-structure' of the system (corresponding to input zero condition). This can also be captured in our approach but will take us too far afield [45].

We now show that using (W, M) feedback we can make a controlled invariant space into a conditioned invariant space without losing its controlled invariance.

Theorem 8. *Let $\mathscr{V}_{\dot{W}WM}$, $\widehat{\mathscr{V}}_W$ be such that $\mathscr{V}_{\dot{W}WM} \times \dot{W} \subseteq \widehat{\mathscr{V}}_{\dot{W}}$. Then the space $\widehat{\mathscr{V}}_W$ is controlled invariant in $\mathscr{V}_{\dot{W}WM}$ relative to $\mathscr{V}_M$ if and only if there exists a space $\mathscr{V}_{WM}$ such that $\widehat{\mathscr{V}}_W$ is both controlled invariant and conditioned invariant in $\mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_{WM}$ relative to $\mathscr{V}_M$.*

Proof. **(only if)**: Define

$$\mathscr{V}_{WM} = \widetilde{\mathscr{V}}_{WM} + (\mathscr{C}_W \oplus 0_M)$$

where

$$\widetilde{\mathscr{V}}_{WM} = (\mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_M) \leftrightarrow \widehat{\mathscr{V}}_{\dot{W}}$$

and $\mathscr{C}_W$ is complementary to $\widetilde{\mathscr{V}}_{WM} \circ W$ in $\mathscr{F}_W$, that is,

$$\widetilde{\mathscr{V}}_{WM} \circ W + \mathscr{C}_W = \mathscr{F}_W \text{ and}$$
$$\widetilde{\mathscr{V}}_{WM} \circ W \cap \mathscr{C}_W = \{0\}.$$

We claim that $\widehat{\mathscr{V}}_W$ is both controlled invariant and conditioned invariant in $\mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_{WM}$ relative to $\mathscr{V}_M$.

(i) **Claim:** $\widehat{\mathscr{V}}_W$ is controlled invariant in $\mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_{WM}$ relative to $\mathscr{V}_M$
We know that

$$(\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M) \leftrightarrow \widehat{\mathscr{V}}_{\dot{W}} \supseteq \widehat{\mathscr{V}}_W.$$

Suppose $w \in \widehat{\mathscr{V}}_W$ then there exist $(\dot{w}, w, m) \in \mathscr{V}_{\dot{W}WM}$ with $\dot{w} \in \widehat{\mathscr{V}}_{\dot{W}}$, $m \in \mathscr{V}_M$. But then by the definition of $\widetilde{\mathscr{V}}_{WM}$, $(w, m) \in \widetilde{\mathscr{V}}_{WM} \subseteq \mathscr{V}_{WM}$. Hence $w \in (\mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_{WM}) \leftrightarrow \mathscr{V}_M \leftrightarrow \widehat{\mathscr{V}}_{\dot{W}}$, that is, $\widehat{\mathscr{V}}_W$ is controlled invariant in $\mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_{WM}$ relative to $\mathscr{V}_M$.

(ii) **Claim:** $\widehat{\mathscr{V}}_W$ is conditioned invariant in $\mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_{WM}$ relative to $\mathscr{V}_M$
By Lemma 7, we need to show that

$$((\mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_{WM}) \leftrightarrow \mathscr{V}_M) \times \dot{W} \subseteq \widehat{\mathscr{V}}_{\dot{W}}$$

Let $\dot{w} \in$ LHS. Then there exists $(\dot{w}, 0, m) \in \mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_{WM}$ such that $m \in \mathscr{V}_M$. Now $(0, m) \in \mathscr{V}_{WM}$. Since $\mathscr{V}_{WM} = \widetilde{\mathscr{V}}_{WM} + (\mathscr{C}_W \oplus 0_M)$ and $\mathscr{C}_W$ is complementary to $\widetilde{\mathscr{V}}_{WM} \circ W$, we have $(0, m) \in \widetilde{\mathscr{V}}_{WM}$, that is, $(0, m) \in (\mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_{WM}) \leftrightarrow \widehat{\mathscr{V}}_{\dot{W}}$. Hence there exists $\dot{w}_1 \in \widehat{\mathscr{V}}_{\dot{W}}$ such that $(\dot{w}_1, 0, m) \in \mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_{WM}$. It follows that $(\dot{w} - \dot{w}_1, 0, 0) \in \mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_{WM} \subseteq \mathscr{V}_{\dot{W}WM}$. Hence $\dot{w} - \dot{w}_1 \in \mathscr{V}_{\dot{W}WM} \times \dot{W} \subseteq \widehat{\mathscr{V}}_{\dot{W}}$. We conclude that $\dot{w} \in \widehat{\mathscr{V}}_{\dot{W}}$ since $\dot{w}_1 \in \widehat{\mathscr{V}}_{\dot{W}}$.

(if): This is trivial since

$$(\mathscr{V}_{\dot{W}WM} \cap \mathscr{V}_{WM}) \leftrightarrow \mathscr{V}_M \leftrightarrow \widehat{\mathscr{V}}_{\dot{W}} \supseteq \widehat{\mathscr{V}}_W$$

implies

$$\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M \leftrightarrow \widehat{\mathscr{V}}_{\dot{W}} \supseteq \widehat{\mathscr{V}}_W,$$

which is the definition of controlled invariant. □

The result on $(M, \dot{W})$ injection is the dual of Theorem 8 and follows by duality. We thus have the theorem for 'output injection'.

Theorem 9. *Let $\mathscr{V}_{\dot{W}WM}$, $\widehat{\mathscr{V}}_W$ be such that $\mathscr{V}_{\dot{W}WM} \circ W \supseteq \widehat{\mathscr{V}}_W$. Then the space $\widehat{\mathscr{V}}_W$ is conditioned invariant in $\mathscr{V}_{\dot{W}WM}$ relative to $\mathscr{V}_M$ if and only if there exists a space $\mathscr{V}_{M\dot{W}}$ such that $\widehat{\mathscr{V}}_W$ is both conditioned invariant and controlled invariant in $\mathscr{V}_{\dot{W}WM} + \mathscr{V}_{M\dot{W}}$ relative to $\mathscr{V}_M$.*

9. Conclusions

In this paper, we developed an alternative formulation for linear dynamical systems which combined the advantages of two major approaches to dynamical systems, namely, the state space and behavioural systems theories. We defined the conditioned and controlled invariant subspaces of linear systems theory using our approach and gave algorithms to compute the minimal conditioned invariant and maximal controlled invariant subspaces. We showed that the computations of the fundamental invariant spaces could be carried out without transforming the system to the state space representation. We also showed that by appropriately choosing a subspace on manifest variables, the input-output partition could be avoided as in the case of behavioural systems theory.

For an important subclass of dynamical systems, namely electrical networks, we demonstrated how the operations involving 'generalized minor' could be carried out. This eliminated the need for computing the A, B, C, and D matrices of the state space theory. The main idea was to reduce most of the computations of the full network to ones on the static multi-port subnetwork. This was achieved by using the computationally inexpensive multi-port decomposition of electrical networks.

We showed how duality is natural to our formulation. We defined the adjoint of a dynamical system and established its properties using the "implicit duality theorem" and the duality of sum and intersection operations on vector spaces. Specifically, we demonstrated the duality between the definitions and algorithms of controlled and conditioned invariant spaces.

We derived the circumstances under which the conditioned and controlled invariant subspaces were equivalent. We also generalized the notion of state feedback and output injection. Finally, we characterized the controlled invariant and conditioned invariant subspaces using (W, M) feedback $(M, \dot{W})$ injection subspaces, respectively.

Appendix A. Computing the basic operations

Appendix A.1. Representation

A vector space $\mathscr{V}_A$ may be specified in one of the two following ways:

1. Through generator set for $\mathscr{V}_A$:

$$f_A = \lambda^T \left[R_A\right],$$

where the rows of R_A generate $\mathscr{V}_A$ by linear combination. When the rows of R_A are linearly independent R_A is called a representative matrix of $\mathscr{V}_A$.

2. Through constraint equations:

$$\left[S_A\right] f_A = 0,$$

where the rows of S_A generate $\mathscr{V}_A^{\perp}$. The solution space of this equation is $(\mathscr{V}_A^{\perp})^{\perp} = \mathscr{V}_A$.

One or the other of these representations might be convenient depending upon the context.

Appendix A.2. Computing $\mathscr{V}_{AB} \circ A$ and $\mathscr{V}_{AB} \times B$ explicitly

Let $\mathscr{V}_{AB}$ be defined through

$$\begin{bmatrix} S_A & S_B \end{bmatrix} \begin{pmatrix} f_A \\ f_B \end{pmatrix} = 0.$$

We do row operations so that the coefficient matrix is in the form

$$\begin{bmatrix} S_{1A} & 0 \\ S_{2A} & S_{2B} \end{bmatrix},$$

where the rows of S_{2B} are linearly independent (or, if more convenient, columns of S_{2A} are dependent on columns of S_{2B}). Then $[S_{1A}]f_A = 0$ defines $\mathscr{V}_{AB} \circ A$ and $[S_{2B}]f_B = 0$ defines $\mathscr{V}_{AB} \times B$. The statement about $\mathscr{V}_{AB} \circ A$ may be seen as follows: Clearly if any $(f_A, f_B) \in \mathscr{V}_{AB}$, then f_A satisfies $[S_{1A}]f_A = 0$ so that $\mathscr{V}_{AB} \circ A$ is contained in the solution space of the latter. Next let f_A satisfy $[S_{1A}]f_A = 0$. Since columns of S_{2A} are linearly dependent on columns of S_{2B}, the equation

$$\left[S_{2B}\right] f_B = -\left[S_{2A}\right] f_A$$

has a solution $\widehat{f}_B$. Thus for any solution of $[S_{1A}]f_A = 0$, we can always find $\widehat{f}_B$ such that $(f_A, \widehat{f}_B) \in \mathscr{V}_{AB}$. This completes the proof that $\mathscr{V}_{AB} \circ A$ is the solution space of $[S_{1A}]f_A = 0$.

The statement about $\mathscr{V}_{AB} \times B$ is routine.

Appendix A.3. Computing $\mathscr{V}_A^1 \cap \mathscr{V}_A^2$ Explicitly

Suppose $[S_A^1]f_A = 0$, $[S_A^2]f_A = 0$ define $\mathscr{V}_A^1$, $\mathscr{V}_A^2$ respectively. Let A', A'' be copies of A with A, A', A'' disjoint. Build the space $\mathscr{V}_{AA'A''}$ defined through

$$\begin{bmatrix} S_A^1 & & \\ & S_A^2 & \\ 0 & I & -I \\ I & 0 & -I \end{bmatrix} \begin{pmatrix} f_{A'}^1 \\ f_{A''}^2 \\ f_A \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{A.1}$$

It is clear that the vectors f_A are precisely those in both $\mathscr{V}_A^1$ and $\mathscr{V}_A^2$. Thus $\mathscr{V}_{AA'A''} \circ A$ is $\mathscr{V}_A^1 \cap \mathscr{V}_A^2$. The explicit construction of the dot operation has been discussed above.

Appendix A.4. Computing $\mathscr{V}_A^1 + \mathscr{V}_A^2$ Explicitly

Let $\mathscr{V}_A^1$ and $\mathscr{V}_A^2$ be defined as above. We define $\widehat{\mathscr{V}}_{AA'A''}$ through

$$\begin{bmatrix} S_A^1 & & \\ & S_A^2 & \\ -I & -I & I \end{bmatrix} \begin{pmatrix} f_{A'}^1 \\ f_{A''}^2 \\ f_A \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{A.2}$$

In this case again it is easy to see that $\widehat{\mathscr{V}}_{AA'A''} \circ A = \mathscr{V}_A^1 + \mathscr{V}_A^2$ and we complete the explicit construction as above.

Often the spaces $\mathscr{V}_A^1$, $\mathscr{V}_A^2$ may themselves be available to us only implicitly — usually in the form of a generalized minor $\mathscr{V}_{AB} \leftrightarrow \mathscr{V}_B$. We discuss this situation below after we discuss the computation of the generalized minor.

Appendix A.5. Computing $\mathscr{V}_{AB} \leftrightarrow \mathscr{V}_B$ Implicitly

Let $\mathscr{V}_{AB}$ be defined by

$$\begin{bmatrix} S_A & S_B \end{bmatrix} \begin{pmatrix} f_A \\ f_B \end{pmatrix} = 0$$

and $\mathscr{V}_B$ by

$$\begin{bmatrix} \widehat{S}_B \end{bmatrix} f_B = 0.$$

Let $\widehat{\mathscr{V}}_{AB}$ be defined by

$$\begin{bmatrix} S_A & S_B \\ 0 & \widehat{S}_B \end{bmatrix} \begin{pmatrix} f_A \\ f_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{A.3}$$

Then

$$\begin{aligned} \widehat{\mathscr{V}}_{AB} \circ A &\equiv \{f_A \mid (f_A, f_B) \in \mathscr{V}_{AB}, f_B \in \mathscr{V}_B\} \\ &= \mathscr{V}_{AB} \leftrightarrow \mathscr{V}_B. \end{aligned}$$

Appendix A.6. Computing $\mathscr{V}_{AB} \leftrightarrow \mathscr{V}_B$ Explicitly

In order to explicitly represent $\mathscr{V}_{AB} \leftrightarrow \mathscr{V}_B$ as the solution space of an equation

$$\begin{bmatrix} \widetilde{S}_A \end{bmatrix} f_A = 0,$$

one has to do row operations and recast (A.3) as

$$\begin{bmatrix} S'_{1A} & S'_{1B} \\ S'_{2A} & 0 \\ 0 & \widehat{S}_B \end{bmatrix} \begin{pmatrix} f_A \\ f_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{A.4}$$

where the rows of S'_{1B} together with those of $\widehat{S}_B$ are independent. The solution space of

$$\begin{bmatrix} S'_{2A} \end{bmatrix} f_A = 0,$$

is clearly $\widehat{\mathscr{V}}_{AB} \circ A$ that is, $\mathscr{V}_{AB} \leftrightarrow \mathscr{V}_B$ ($\widehat{\mathscr{V}}_{AB}$ as in A.3).

In general when one is performing operations such as intersection or sum with vector spaces, their representation may not be explicitly available but in the form $\mathscr{V}_{AB} \leftrightarrow \mathscr{V}_B$.

Let $\widehat{f}_B \in (\mathscr{V}_{AB} \cap \mathscr{V}_B) \circ B$. To find $\mathscr{V}_{AB} \leftrightarrow \{\widehat{f}_B\}$ we first find one solution $\widehat{f}_A$ to the equation using the first rows of (A.4)

$$\begin{bmatrix} S'_{1A} \\ S'_{2A} \end{bmatrix} \widehat{f}_A = \begin{bmatrix} S'_{1B}\widehat{f}_B \\ 0 \end{bmatrix}.$$

Then

$$\mathscr{V}_{AB} \leftrightarrow \{\widehat{f}_B\} = \widehat{f}_A + \mathscr{V}_{AB} \times A.$$

$\mathscr{V}_{AB} \times A$ can be seen to be the solution to

$$\begin{bmatrix} S'_{1A} \\ S'_{2A} \end{bmatrix} \widetilde{f}_A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and $\mathscr{V}_{AB} \leftrightarrow \{\widehat{f}_B\}$ as the collection of all solutions to

$$\begin{bmatrix} S'_{1A} \\ S'_{2A} \end{bmatrix} f_A = \begin{bmatrix} S'_{1B}\widehat{f}_B \\ 0 \end{bmatrix}.$$

Appendix A.7. Computing $\mathscr{V}_A^1 \cap \mathscr{V}_A^2$ Implicitly

Suppose $\mathscr{V}_A^1 \cap \mathscr{V}_A^2$ is to be computed where

$$\mathscr{V}_A^1 = \mathscr{V}_{AB}^1 \leftrightarrow \mathscr{V}_B^1$$
$$\mathscr{V}_A^2 = \mathscr{V}_{AB}^2 \leftrightarrow \mathscr{V}_B^2.$$

Let $\mathscr{V}_{AB}^1$, $\mathscr{V}_{AB}^2$ be defined respectively through

$$\begin{bmatrix} S_A^1 & S_B^1 \end{bmatrix} \begin{pmatrix} f_A^1 \\ f_B^1 \end{pmatrix} = 0,$$
$$\begin{bmatrix} S_A^2 & S_B^2 \end{bmatrix} \begin{pmatrix} f_A^2 \\ f_B^2 \end{pmatrix} = 0.$$

Let $\mathscr{V}_B^1$, $\mathscr{V}_B^2$ be defined respectively through

$$\begin{bmatrix} \widehat{S}_B^1 \end{bmatrix} f_B = 0,$$
$$\begin{bmatrix} \widehat{S}_B^2 \end{bmatrix} f_B = 0.$$

We proceed now as in (A.1) by first copying set A into A', A'' and B into B', B'' and write the final equations as

$$\begin{bmatrix} S_A^1 & S_B^1 & & & \\ & \widehat{S}_B^1 & & & \\ & & S_A^2 & S_B^2 & \\ & & & \widehat{S}_B^2 & \\ & & +I & & -I \\ I & & & & -I \end{bmatrix} \begin{pmatrix} f_{A'}^1 \\ f_{B'}^1 \\ f_{A''}^2 \\ f_{B''}^2 \\ f_A \end{pmatrix} = 0 \tag{A.5}$$

If the above equation has the solution space $\mathscr{V}_{AA'A''B'B''}$ then we can show that

$$\mathscr{V}_{AA'A''B'B''} \circ A = \mathscr{V}_A^1 \cap \mathscr{V}_A^2.$$

To see this, note that $\mathscr{V}_{A'}^1$ is the collection of all $f_{A'}^1$ which can be part of a solution vector $(f_{A'}^1, f_{B'}^1)$ of the first two rows of (A.5). $\mathscr{V}_{A''}^2$ is the space of all vectors $f_{A''}^2$ which can be a part of the solution vector $(f_{A''}^2, f_{B''}^2)$ of the next two rows of (A.5). Now f_A, by the last two rows of the equation, is seen to a copy of both $f_{A'}^1$ as well as $f_{A''}^2$. Any such vector therefore belongs to $\mathscr{V}_A^1 \cap \mathscr{V}_A^2$. Conversely any vector in $\mathscr{V}_A^1 \cap \mathscr{V}_A^2$ can clearly be a copy of a part of some solution vector of the first two rows and therefore of the middle two rows also — which means it is a part of a solution vector of (A.5).

Appendix A.8. Computing $\mathscr{V}_A^1 + \mathscr{V}_A^2$ Implicitly

By similar arguments we can show that we can write the constraints for $\mathscr{V}_A^1 + \mathscr{V}_A^2$ as an implicit version (A.2) as follows:

$$\begin{bmatrix} S_A^1 & S_B^1 & & & \\ & \widehat{S}_B^1 & & & \\ & & S_A^2 & S_B^2 & \\ & & & \widehat{S}_B^2 & \\ -I & & -I & & I \end{bmatrix} \begin{pmatrix} f_{A'}^1 \\ f_{B'}^1 \\ f_{A''}^2 \\ f_{B''}^2 \\ f_A \end{pmatrix} = 0 \tag{A.6}$$

If $\widetilde{\mathscr{V}}_{AA'A''B'B''}$ is the solution space of the above equation then

$$\mathscr{V}_A^1 + \mathscr{V}_A^2 = \widetilde{\mathscr{V}}_{AA'A''B'B''} \circ A.$$

Appendix B. Computation of the main step in Algorithm I

1. Computing $\mathscr{V}_{\dot{W}W}$
   Let $\mathscr{V}_{\dot{W}WM}$ be defined by:

$$\begin{bmatrix}\widetilde{S}^1_{\dot{W}} & \widetilde{S}^2_W & \widetilde{S}^3_M\end{bmatrix}\begin{pmatrix}\dot{w}\\ w\\ m\end{pmatrix} = 0$$

   and let $\mathscr{V}_M$ be defined by

$$\begin{bmatrix}\widehat{S}_M\end{bmatrix} m = 0.$$

   Then $\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M$ is defined implicitly by

$$\begin{bmatrix}\widetilde{S}^1_{\dot{W}} & \widetilde{S}^2_W & \widetilde{S}^3_M\\ & & \widehat{S}_M\end{bmatrix}\begin{pmatrix}\dot{w}\\ w\\ m\end{pmatrix} = 0. \tag{B.1}$$

   More explicitly we could perform row operations on the coefficient matrix of (B.1) and get an explicit representation of $\mathscr{V}_{\dot{W}WM} \leftrightarrow \mathscr{V}_M$ as in Appendix A.6

$$\begin{bmatrix}S^2_{\dot{W}} & S^1_W\end{bmatrix}\begin{pmatrix}\dot{w}\\ w\end{pmatrix} = 0 \tag{B.2}$$

   where (B.2) is equivalent in the variables $(\dot{w}, w)$ to (B.1).
2. Computing $(\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}^j_W) + \mathscr{V}^j_{\dot{W}}$
   Next let $\mathscr{V}^j_{\dot{W}}$ be defined by

$$\begin{bmatrix}\widehat{S}^j_{\dot{W}}\end{bmatrix}\dot{w} = 0.$$

   Then $\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}^j_W$ can be represented implicitly by

$$\begin{bmatrix}S^1_W & S^2_{\dot{W}}\\ \widehat{S}^j_W & \end{bmatrix}\begin{pmatrix}w\\ \dot{w}\end{pmatrix} = 0,$$

   where $\widehat{S}^j_W$ is a copy of $\widehat{S}^j_{\dot{W}}$, and $(\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}^j_W) + \mathscr{V}^j_{\dot{W}}$ be represented implicitly by

$$\begin{bmatrix}S^1_W & S^2_{\dot{W}} & & \\ \widehat{S}^j_W & & & \\ & & \widehat{S}^j_{\dot{W}} & \\ & -I & -I & I\end{bmatrix}\begin{pmatrix}w\\ \dot{w}'\\ \dot{w}''\\ \dot{w}\end{pmatrix} = 0. \tag{B.3}$$

   If we represent the solution space of (B.3) by $\widehat{\mathscr{V}}^j_{W\dot{W}'\dot{W}''\dot{W}}$, then $\mathscr{V}^{j+1}_{\dot{W}} = \widehat{\mathscr{V}}^j_{W\dot{W}'\dot{W}''\dot{W}} \circ \dot{W}$ represented explicitly as, say, the solution space to $[\widehat{S}^{j+1}_{\dot{W}}]\ \dot{w} = 0$. When we move on to compute $(\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}^{j+1}_W) + \mathscr{V}^{j+1}_{\dot{W}}$ it may be usually worth while to have $\mathscr{V}^{j+1}_{\dot{W}}$ explicitly as a solution space.

Appendix C. Computation of the main step in Algorithm II

1. Computing $(\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}^j_{\dot{W}}) \cap \mathscr{V}^j_W$. Let $\mathscr{V}_{\dot{W}W}$ be defined by

$$\begin{bmatrix} S^1_{\dot{W}} & S^2_W \end{bmatrix} \begin{pmatrix} \dot{w} \\ w \end{pmatrix} = 0.$$

Let $\mathscr{V}^j_W$ be defined by

$$\begin{bmatrix} \widetilde{S}^j_W \end{bmatrix} w = 0.$$

As we had noted earlier $(\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}^j_{\dot{W}}) \cap \mathscr{V}^j_W$ can be represented implicitly by

$$\begin{bmatrix} S^1_{\dot{W}} & S^2_W \\ \widetilde{S}^j_W & \\ & \widetilde{S}^j_W \end{bmatrix} \begin{pmatrix} \dot{w} \\ w \end{pmatrix} = 0. \tag{C.1}$$

If we represent the solution space of (C.1) by $\widetilde{\mathscr{V}}^j_{\dot{W}W}$, then $\mathscr{V}^{j+1}_W = \widetilde{\mathscr{V}}^j_{\dot{W}W} \circ W$, represented explicitly by $[\widetilde{S}^{j+1}_W]w = 0$.
Observe that the above could also have been achieved by dualizing (B.3) as below:

$$\begin{bmatrix} S^1_{\dot{W}} & S^2_W & & \\ \widetilde{S}^j_W & & & \\ & & \widetilde{S}^j_W & \\ & I & & I \\ & & I & I \end{bmatrix} \begin{pmatrix} \dot{w} \\ w' \\ w'' \\ w \end{pmatrix} = 0. \tag{C.2}$$

If we represent the solution space of (C.2) by $\widetilde{\mathscr{V}}^j_{\dot{W}W'W''W}$, then $\mathscr{V}^{j+1}_W = \widetilde{\mathscr{V}}^j_{\dot{W}W'W''W} \circ W$ represented explicitly by $[\widetilde{S}^{j+1}_W]w = 0$. When we move onto compute $(\mathscr{V}_{\dot{W}W} \leftrightarrow \mathscr{V}^{j+1}_{\dot{W}}) + \mathscr{V}^{j+1}_W$ it may be usually worth while to have $\mathscr{V}^{j+1}_W$ explicitly as a solution space to $[\widetilde{S}^{j+1}_W]w = 0$.

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