Higher dimensional electrical circuits and the matroid dual of a nonplanar graph

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Outline

1. Physical problem and its solution
2. Construction of a basis for mmf space
3. Proofs
4. Conclusion and generalization

A physical problem

- Winding carrying current around a magnetic core.
  Cracks have developed in the magnetic core creating air gaps which the flux cannot avoid.
- Current value *known*.
  Physical location of the entire crack and the local dimensions are *known* so local reluctance can be calculated.
- Permeability of the core *assumed* to be infinite.
  After reluctance calculation is done the cracks may be *assumed* to be of zero thickness.

Compute flux density and mmf everywhere in the cracks.

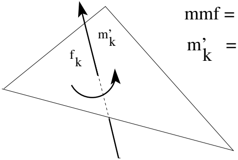
Warning: No PDEs please!

Regions created by cracks

Outline of Solution Procedure

- Cover the crack surfaces with triangles small enough that thickness can be taken to be uniform.
- The effect, of the current in the winding, on the medium is captured by a loop current around every triangle.
- Construct the flux - mmf equation for each triangle.

$$m'_k = H_k d_k = B_k A_k \times \frac{d_k}{\mu_k A_k} = f_k \times \ reluctance.$$



mmf = flux x reluctance

$m'_k = f_k \times$ reluctance

Outline of Solution Procedure

- Whenever triangles come together at an edge convert $\oint_C H.dl = \int_S J.ds$, into an equation of the form

$$m'_{\delta_1} \pm m'_{\delta_2} \pm m'_{\delta_3} = i_{\delta_1} \pm i_{\delta_2} \pm i_{\delta_3} = \pm i_{edge}. \quad KCL$$

- Whenever triangles cover the surface bounding a region convert $\oint B.ds = 0$ into an equation of the form

$$f_{\delta_1} \pm f_{\delta_2} \pm f_{\delta_3} = 0. \quad KVL$$

- Solve the equations simultaneously.

Oriented triangles at an edge and mmf equation

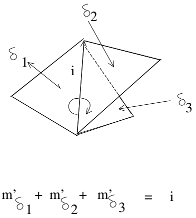


Figure : Triangles around an edge

$m'_{\delta_1} + m'_{\delta_2} + m'_{\delta_3} = i_{\delta_1} + i_{\delta_2} + i_{\delta_3} = i_{edge}$.
(Suppose δ'_2 were oppositely oriented to δ_2. Equation would read
$m'_{\delta_1} - m'_{\delta'_2} + m'_{\delta_3} = i_{\delta_1} - i_{\delta'_2} + i_{\delta_3} = i_{edge}$.)
This equation can be rewritten as
$m_{\delta_1} + m_{\delta_2} + m_{\delta_3} = 0$, where $m_{\delta_j} := m'_{\delta_j} - i_{\delta_j}$.
We call the quantity $m_{\delta_j} := m'_{\delta_j} - i_{\delta_j}$, the **current adjusted mmf** associated with δ_j.

Orientation of $k-$cells

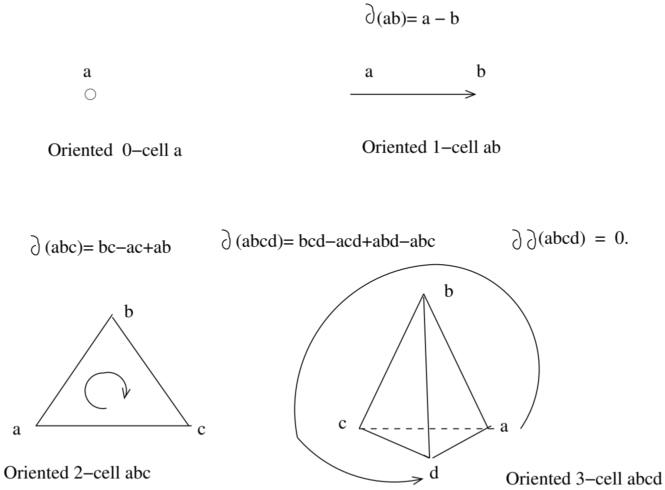


Figure : Oriented cells and their boundaries

Some definitions

- An $n-$complex $\mathcal{C}$, is a collection of oriented $n-$cells, $(n-1)-$cells, $\cdots$, $0-$cells, such that every face of a cell in $\mathcal{C}$ is also in $\mathcal{C}$ and the intersection of any two cells of $\mathcal{C}$ is a face of each of them. Thus, the cells in the boundary of each $j-$cell, $0 < j \leq n$, are also present in the complex.
- **A $2-$complex $\mathcal{C}$, is a collection of oriented triangles, edges and vertices, such that every edge or vertex of a triangle in $\mathcal{C}$ is also in $\mathcal{C}$ and the intersection of any two triangles of $\mathcal{C}$ is an edge or vertex of each of them.**
- A $j-$subcomplex $\mathcal{C}$ of an $n-$complex $\mathcal{C}_{\mathcal{T}}$ with $j \leq n$ has its $j-$cells as subsets of $j-$cells of $\mathcal{C}_{\mathcal{T}}$, $(j-1)-$cells as subsets of $(j-1)-$cells of $\mathcal{C}_{\mathcal{T}}$, and so on with boundary relations consistent with those of $\mathcal{C}_{\mathcal{T}}$.

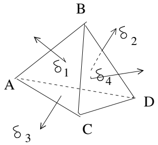
The ‘KCL’ equations for mmf

The KCL equations for the current adjusted mmf vector have the form:

$$A^{(2)}\left[m\right] = \begin{bmatrix} A_1^{(2)} \\ A_2^{(2)} \\ \vdots \\ A_k^{(2)} \end{bmatrix} \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = \left[\mathbf{0}\right]. \qquad (1)$$

The matrix $A^{(2)}$ is called the $2-$coboundary matrix of the complex $\mathcal{C}$ which is made up of triangles, edges, vertices and the incidence relationship between them.

Example



Triangles BAC,ABD,BCD,ADC

Edges AB,AC,AD,BC,CD,DB

current adjusted mmf eqns

$A^{(2)} m = 0.$ i.e.,

$$\begin{array}{c} \\ AB \\ AC \\ AD \\ BC \\ CD \\ DB \end{array} \begin{array}{c} \begin{array}{cccc} BAC & ABD & BCD & ADC \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \end{array} \begin{pmatrix} m\_1 \\ m\_2 \\ m\_4 \\ m\_3 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}.$$

flux eqns

$B f = 0.$

$$\begin{array}{c} \begin{array}{cccc} BAC & ABD & BCD & ADC \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \end{array} = 0.$$

The solution spaces of the current adjusted mmf equations and of the flux equations are complementary orthogonal.

'Nodal' type equations

Let A be the 2−coboundary matrix of the complex $\mathcal{C}$ with S as its set of triangles. We have

$$
\begin{array}{ccc}
A\,\mathbf{m_S} & = & \mathbf{0}. \\
A\,\mathbf{m'_S} = A[\mathbf{m_S} + \mathbf{I_S}] & = & A\mathbf{I_S} \\
G\,\mathbf{f_S} & = & [\mathbf{m_S} + \mathbf{I_S}] \\
 & \textit{Use GTT} & \\
A^T\,\mathbf{y} & = & \mathbf{f_S} \\
AGA^T\mathbf{y} & = & A\mathbf{I_S}.
\end{array} \qquad (2)
$$

where G is a positive diagonal matrix with its (j, j) entry being the reluctance of the triangle δ_j, m_S is the current adjusted mmf vector and f_S is the flux vector.

Problem: AGA^T is singular. How to avoid linear algebra and get a nonsingular coefficient matrix?

What happens in the case of the usual nodal analysis?

In this case A is the incidence matrix of a graph. To obtain a maximally linearly independent set of rows we merely have to omit one row if the graph is connected. The resulting coefficient matrix $\hat{A}\ G\ \hat{A}^T$ would be positive definite and therefore nonsingular.

In the higher dimensional case finding a maximally linearly independent set of rows of the $2-$coboundary matrix A, without use of linear algebra, is not routine.

'Loop' type equations

Let P be a representative matrix of the $2-$cycle space of $\mathcal{C}$, i.e., of the solution space of the equations $A\ \mathbf{m_S} = \mathbf{0}$. Using GTT, we can restate the constraints of theghter dimensional network $\mathcal{N}$ as follows.

$$\begin{aligned} P\ \mathbf{f_S} &= \mathbf{0}. \\ P^T\ \mathbf{q} &= \mathbf{m_S} \\ \mathbf{f_S} &= R\ [\mathbf{m_S} + \mathbf{l_S}]. \end{aligned} \tag{3}$$

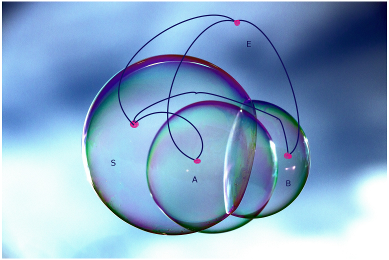
(Here R is a positive diagonal matrix whose entries are the inverse reluctances of the triangles.)

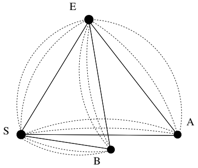
$$P\ R\ P^T\ \mathbf{q} \ = \ -P\ R\ \mathbf{l_S}. \tag{4}$$

When R is positive diagonal, since the rows of P are linearly independent, the matrix PRP^T is positive definite and so invertible so that a unique solution is guaranteed.

P is easy to build.

The Dual Graph





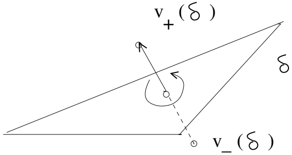
The dotted lines are parallel edges corresponding to triangles between the same regions.

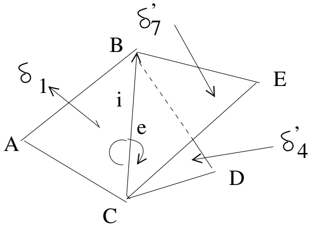
The dual graph through triangle adjacency graph

Directly building the dual graph using embedding information of the complex is not easy since determining connected regions in $3-$D is cumbersome.

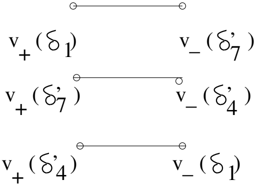
An indirect but efficient method is through building the triangle adjacency graph.

Construction of $tag(\mathcal{C})$



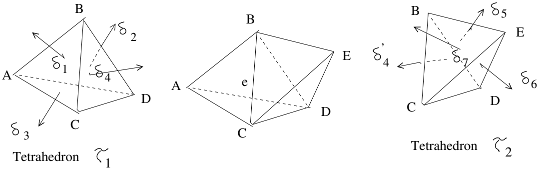


Triangles incident at edge e



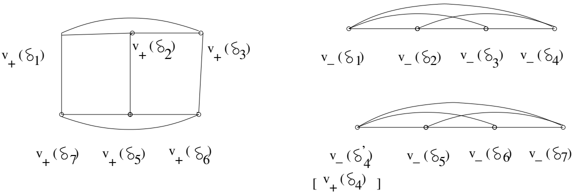
Corresponding edges in tag($\mathcal{C}$)

Example for $tag(\mathcal{C})$



The 3−complex $\mathcal{C}^3$ obtained by pasting τ_1 , τ_2 actoss δ_4

$\mathcal{C}$ is the 2−subcomplex on its triangles



Triangle adjacency graph $tag(\mathcal{C})$ for the complex $\mathcal{C}$

The graph $\mathcal{G}_{comptag(\mathcal{C})}$ for the complex $\mathcal{C}$

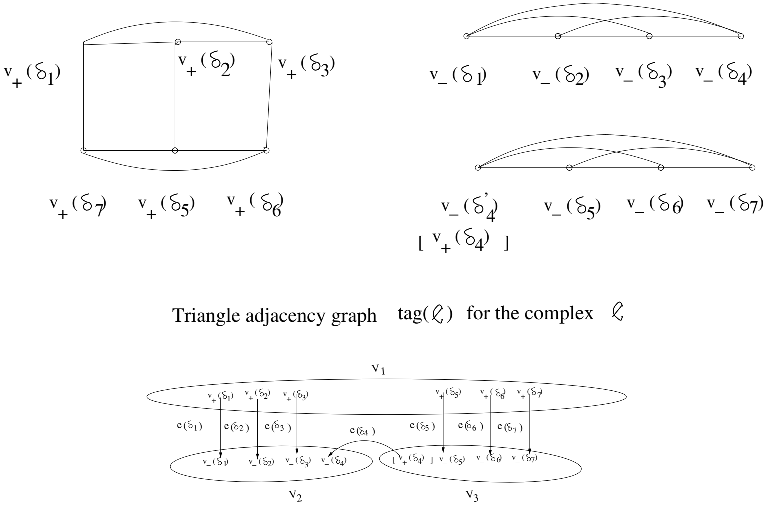


Figure : The graph $\mathcal{G}_{comptag(\mathcal{C})}$ for the complex $\mathcal{C}$

Dual graph of $\mathcal{C}$ is $\mathcal{G}_{comptag(\mathcal{C})}$ for connected $\mathcal{C}$

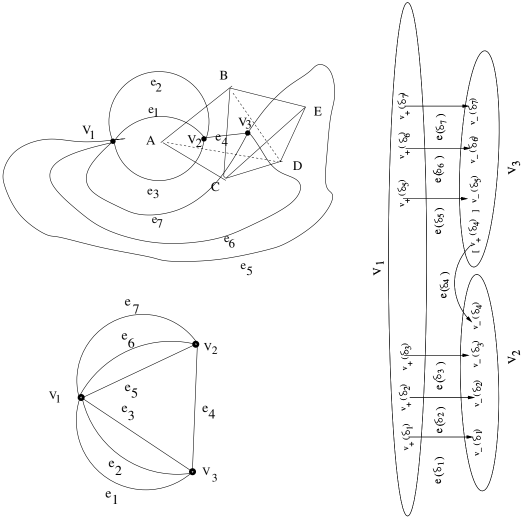


Figure : Dual graph of $\mathcal{C}$ and $\mathcal{G}_{comptag(\mathcal{C})}$

Dual of a nonplanar graph

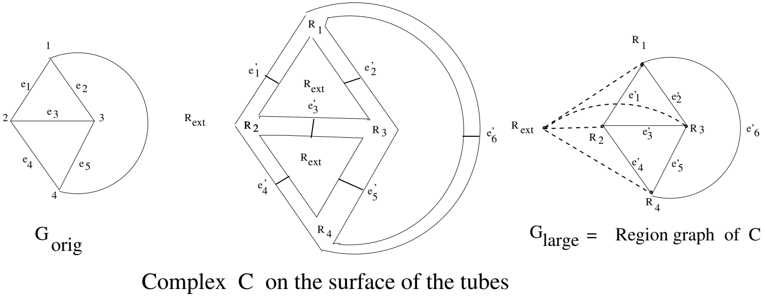


Figure : Dual of a general graph

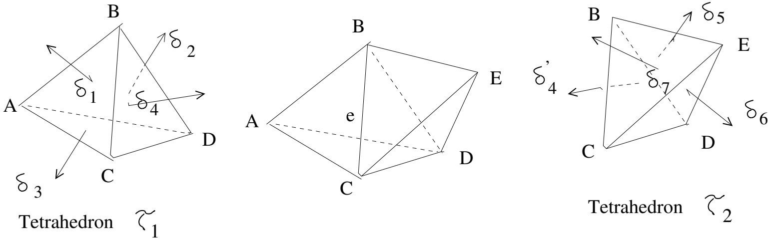
Basic facts

- Boundary of a boundary of a $k-$cell c_i is zero.
  For any $k-$chain $\sum_1^m \alpha_i c_i$, we take $\partial(\sum_1^m \alpha_i c_i) := \sum_1^m \alpha_i \partial(c_i)$.
  Boundary of a boundary of a $k-$chain is zero.
- The boundary of the $k-$chain $\sum_1^n x_j c_j^{(k)}$ is the $(k-1)-$chain $\sum_1^m y_j c_j^{(k-1)}$, where $\mathbf{y} = \left[A^{(k)}(\mathcal{C})\right] [\mathbf{x}]$.
  $\left[A^{(k-1)}(\mathcal{C})\right] \left[A^{(k)}(\mathcal{C})\right] = [\mathbf{0}]$, $k \geq 2$, i.e.,
  row space of $\left[A^{(k-1)}(\mathcal{C})\right]$ is **orthogonal** to column space of $\left[A^{(k)}(\mathcal{C})\right]$.
- If an $n-$complex $\mathcal{C}_{\mathcal{T}}$ is embedded in $\mathbb{R}^n$, and the space covered by it is contractible to a point, then the above spaces are actually **complementary orthogonal** for $0 < (k-1) < n$, i.e.,
  if y is a $(k-1)$-cycle, equivalently, satisfies $\left[A^{(k-1)}(\mathcal{C}_{\mathcal{T}})\right] \mathbf{y} = 0$, it can be written as the boundary of a $k-$chain x.

Consequences for a $2-$complex $\mathcal{C}$ embedded in $\mathbb{R}^3$

- Let $\mathcal{C}_{\mathcal{T}}$ be the $3-$complex obtained by decomposing a large tetrahedron, containing the $2-$complex $\mathcal{C}$ in question, into smaller tetrahedra whose set of faces contain the triangles of $\mathcal{C}$.
  Orient the tetrahedra so that the boundary triangles appear outward oriented.
  Call a sum of such tetrahedra a **region**.
  The boundary of such a region is a $2-$chain composed of only the boundary triangles of the region assigned values ± 1 depending on the orientation of the triangle relative to the outward orientation.
- Boundary vectors of $\mathcal{C}_{\mathcal{T}}$ which have non zero entries only on triangles of $\mathcal{C}$ span $2-$cycles of $\mathcal{C}$.

Boundary of a region is a 2−chain composed of only the boundary triangles of the region assigned values ±1



The region $\zeta_1 + \zeta_2$

The boundary of $\zeta_1 + \zeta_2 = \delta_1 + \delta_2 + \delta_3 + \delta_4 - \delta_4 + \delta_5 + \delta_6 + \delta_7$

$= \delta_1 + \delta_2 + \delta_3 + \delta_5 + \delta_6 + \delta_7$

Figure : Boundary of a region

Proof of GTT:The solution spaces of the current adjusted mmf equations and of the flux equations are complementary orthogonal.

The complex that we are dealing with can be covered by a tetrahedron.
A tetrahedron is contractible to a point.
The solution space of current adjusted mmf equations is the space of boundary vectors of $3-$chains.
These latter can be spanned by boundary vectors of regions.
Flux equations state that flux vectors are orthogonal to boundary vectors of regions.
Flux vector being orthogonal to boundary vectors of regions is equivalent to being orthogonal to the solution space of current adjusted mmf equations.

The cell dual of $\mathcal{C}_{\mathcal{T}}$ and the dual graph $\mathcal{G}_{region(\mathcal{C})}$ of $\mathcal{C}$

Let $\mathcal{C}_{\mathcal{T}}$ be the 3−complex obtained by decomposing a large tetrahedron, containing the 2−complex $\mathcal{C}$ in question, into smaller tetrahedra whose set of faces contain the triangles of $\mathcal{C}$.

- The cell dual of $\mathcal{C}_{\mathcal{T}}$ is the graph $\mathcal{G}_{\mathcal{T}}$ obtained by putting a vertex inside each tetrahedron and connecting two vertices inside adjacent (i.e., share a common triangle) tetrahedron by an edge. The direction of the edge is determined by the orientation of the triangle that it crosses. There is a vertex for the entire external region to $\mathcal{C}_{\mathcal{T}}$. The reduced incidence incidence matrix (removing row corresponding to external node) of $\mathcal{G}_{\mathcal{T}}$ is the transpose of $A^{(3)}(\mathcal{C}_{\mathcal{T}})$.
- The dual graph $\mathcal{G}_{region(\mathcal{C})}$ to $\mathcal{C}$ can be shown to be obtained by fusing endpoints of edges corresponding to non-$\mathcal{C}$ triangles and then deleting them (i.e., short circuiting them) from $\mathcal{G}_{\mathcal{T}}$.
- Because row space of $A^{(2)}(\mathcal{C}_{\mathcal{T}})$ is complementary orthogonal to column space of $A^{(3)}(\mathcal{C}_{\mathcal{T}})$, row space of $A^{(2)}(\mathcal{C})$ is complementary orthogonal to row space of incidence matrix of $\mathcal{G}_{region(\mathcal{C})}$.

Why is $\mathcal{G}_{region(\mathcal{C})} = \mathcal{G}_{comp(\mathcal{C})}$ for connected $\mathcal{C}$?

Here we only show row spaces of their incidence matrices are equal.

- Every row of the incidence matrix of $\mathcal{G}_{comp(\mathcal{C})}$ is orthogonal to every row of $A^{(2)}(\mathcal{C})$. So row space of incidence matrix of $\mathcal{G}_{comp(\mathcal{C})}$ $\subseteq$ row space of incidence matrix of $\mathcal{G}_{region(\mathcal{C})}$.
  So rank of $\mathcal{G}_{comp(\mathcal{C})} \leq$ rank of $\mathcal{G}_{region(\mathcal{C})}$.
- Rank of a connected graph = number of nodes -1.
- Each node of $\mathcal{G}_{region(\mathcal{C})}$ corresponds to a connected region when $\mathcal{C}$ is deleted from the large tetrahedron covered by $\mathcal{C}_{\mathcal{T}}$.
- Nonzero entries of a row of the incidence matrix of $\mathcal{G}_{comp(\mathcal{C})}$ correspond to $v_{\pm}(\delta)$ which can be reached from each other by sliding along the surface of $\mathcal{C}$ embedded in $\mathbb{R}$. So they must belong to the same connected region mentioned above.
- So number of nodes of $\mathcal{G}_{comp(\mathcal{C})} \geq$ that of $\mathcal{G}_{region(\mathcal{C})}$.
- When $\mathcal{C}$ is connected row space of incidence matrix of $\mathcal{G}_{comp(\mathcal{C})}$ is equal to that of $\mathcal{G}_{region(\mathcal{C})}$.

Reminder: 'Loop' type equations

Let P be a representative matrix of the $2-$cycle space of $\mathcal{C}$, i.e., of the solution space of the equations $A\ \mathbf{m_S} = \mathbf{0}$.

$$\begin{array}{ccc} P\ \mathbf{f_S} & = & \mathbf{0}. \\ P^T\ \mathbf{q} & = & \mathbf{m_S} \\ \mathbf{f_S} & = & R\ [\mathbf{m_S} + \mathbf{l_S}] \\ P\ R\ P^T\ \mathbf{q} & = & -P\ R\ \mathbf{l_S}. \end{array} \quad (5)$$

When R is positive diagonal, since the rows of P are linearly independent, the matrix PRP^T is positive definite and so invertible so that a unique solution is guaranteed.

P is easy to build.
Take it to be the reduced incidence matrix of $\mathcal{G}_{comptag(\mathcal{C})}$!

Past Work

- The well known magnetic circuit techniques can be regarded as dealing with networks based on $3-$complexes - through the use of the cell dual, replacing a $3-$dimensional object by a node and connecting adjacent objects by an edge.
- Branin and coworker have dealt with $3-$dimensional complexes in
  1. F. H. Branin Jr., The network concept as a unifying principle in engineering and the physical sciences, Problem Analysis in Science and Engineering, Academic Press, (1977) 41–111.
  2. F. H. Branin Jr, N. Y Kingston, The Relation Between Network Theory, Vector Calculus, and Theoretical Physics, Proceedings of the International Symposium on Operator Theory of Networks and Systems, vol. 2 (1977) 97–102.
  They handle both electric and magnetic fields through their approach. They do not consider computational aspects.

Past Work

- Van der Schaft and coworker have considered 2−dimensional electrical circuits in
  3.A. J. van der Schaft, B.M. Maschke, Conservation laws and open systems on higher-dimensional networks, Proc. 47th IEEE Conf. on Decision and Control, Cancun, Mexico, December 9-11, (2008) 799–804.
  4. A. J. van der Schaft, B.M. Maschke, Conservation Laws and Lumped System Dynamics, Model-Based Control; Bridging Rigorous Theory and Advanced Technology, P.M.J. Van den Hof, C. Scherer, P.S.C. Heuberger, eds., Springer, ISBN 978-1-4419-0894-0, (2009) 31–48.
  In their definition, the complementary orthogonality of spaces analogous to those of flux and mmf vectors is **assumed**. So it is not clear what underlies these ideas and under what conditions the result will be true. They do not consider computational aspects.

Conclusion and generalization

- We gave a 'natural' definition for $2-$dimensional electrical circuits embedded in $\mathbb{R}^3$ and proved the Generalized Tellegen's Theorem for such circuits.
- We reduced the computations concerning such circuits to those on a 'dual' electrical circuit through linear time graph theoretic algorithms.
- Our definition and algorithms are valid for an electrical circuit based on $(n-1)-$dimensional complexes embedded in $\mathbb{R}^n$.
- Given any non planar graph, we can build a graph that 'contains' it and for which there is a dual $2-$complex. This dual agrees with the matroid dual.