On the Duality between Controllability and Observability in Behavioural Systems Theory

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ABSTRACT

The notions of controllability and observability are regarded as duals in the conventional input-state-output formulation of dynamical systems. We examine the duality between these two notions in the case of a Behavioural System by building its 'adjoint', using a result which we call the 'Implicit Duality Theorem (IDT)' and prove in this paper. Using the same result we also prove that this 'adjoint' has some attractive properties. In order to indicate where one might hope to find an IDT like result we present one of its analogues and prove it similarly.

I. INTRODUCTION

In the conventional input-state-output formulation of dynamical systems, the notions of controllability and observability are regarded as duals. The situation is less clear in the more general, representation-free approach taken in Behavioural Systems theory [1],[2]. In this paper we examine the extent to which these notions can be regarded as duals in that theory. Our main tool is a result we call the 'Implicit Duality Theorem' and prove in this paper. This theorem is a modification of a result found in reference [3].

A **behaviour** $\mathcal{B}$ is the solution set of a system of linear constant coefficient differential equations $(R[d/dt])w = 0$, where the entries of w are $\mathcal{C}^\infty$ functions $w_1, \cdots, w_q$. $R[d/dt]$ is a matrix over $\Re[\xi]$ (i.e, the entries are real polynomials in ξ), with ξ substituted by d/dt . A behaviour $\mathcal{B}$ is said to be **controllable** iff for each $w^1, w^2 \in \mathcal{B}$, there exists a $w \in \mathcal{B}$ and a $t' \geq 0$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t) = w_2(t-t')$ for $t \geq t'$. Let a behaviour $\mathcal{B}_{wl}$ be represented by $(R_w[d/dt])w + (R_l[d/dt])l = 0$. We say $\mathcal{B}_{wl}$ is **l-observable** iff $(R_w[d/dt])w + (R_l[d/dt])l_1 = 0$ and $(R_w[d/dt])w + (R_l[d/dt])l_2 = 0$ implies $l_1 = l_2$.It can be shown that when $\mathcal{B}$ has the input-state-output representation

$$dx/dt = Ax + Bu$$
$$y = Cx + Du,$$

that the system is controllable, respectively x-observable, in the 'behaviour sense' iff it is controllable, respectively observable in the conventional 'state sense'. The usual way in which the duality between the two notions is brought out is through the definition of the 'adjoint'. In this paper too we study the duality between controllability and observability for behavioural systems through appropriate definitions of the adjoint.

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II. PRELIMINARIES

We regard (row vectors) over $\Re[\xi]$ as functions on a column set. A row vector $(\overset{e_1}{p_1} \quad \cdots \quad \overset{e_n}{p_n})$ may be thought of as the function $f_S(e_i) \equiv p_i, i = 1, \cdots, n$, the p_i being real polynomials in ξ. A **module** $\mathcal{C}_S$ is a collection of row vectors on the column set S, over the scalar domain $\Re[\xi]$, closed under addition and scalar multiplication. If the rows of the matrix $R[\xi]$ are linearly independent, the module generated by them is said to have $R[\xi]$ as a **representative matrix**. For our purpose it is convenient to characterize a system defined by $(R[d/dt])w = 0$, in an operationally equivalent way, in terms of the collection of all row vectors linearly dependent upon the rows of $R[\xi]$, the scalars being members of $\Re[\xi]$. This **module** is said to be **associated with** $\mathcal{B}$ and is denoted by $\mathcal{C}(\mathcal{B})$. A square matrix $R[\xi]$ is said to be **unimodular** iff its determinant is a real number (i.e., does not involve the transcendental ξ). We say a module is **unimodular** iff we can adjoin additional rows to one of its (therefore any of its) representative matrices $R[\xi]$ to make it unimodular. It is easy to show that this is possible iff $R[\xi]$ has a Smith Canonical Form (SCF) $(D \quad 0)$, where D has only real entries. Such a matrix is said to be **row g-unimodular**. **Column g-unimodular** matrices are defined similarly. If the SCF is real but may have zero rows and columns we say the matrix is **g-unimodular**. It can be shown that a behaviour $\mathcal{B}$ is controllable iff the module $\mathcal{C}(\mathcal{B})$ associated with it is unimodular. Also when behaviour $\mathcal{B}_{wl}$ is represented by $(R_w[d/dt])w + (R_l[d/dt])l = 0$, it is l-observable iff $R_l[\xi]$ is column g-unimodular.

When a behaviour $\mathcal{B}$ is defined through $(R[d/dt])w + (R[d/dt])l = 0$ its **projection** onto the variables w is the collection $\{w : (w, l) \in \mathcal{B}\}$ and is denoted by $\mathcal{B}/w$. A kernel representation for this projection can be obtained as follows: rewrite the coefficient matrix of the kernel representation of $\mathcal{B}$ through reversible row operations (with entries in $\Re[\xi]$) as

$$\begin{bmatrix} R_{1w} & 0 \\ R_{2w} & R_{2l} \end{bmatrix},$$

where the rows of R_{2l} are linearly independent. Now the equation $(R_{2l}[d/dt])l = f$ always has a solution owing to the linear independence of the rows of the coefficient matrix. It follows that $\mathcal{B}/w$ has the kernel representation $(R_{1w}[d/dt])w = 0$.

Let f_S, g_S be vectors on S. The **dot product** $< f_S, g_S >$ of f_S, g_S is defined by $< f_S, g_S > \equiv \sum_{e_i \in S} f_S(e_i).g_S(e_i)$. We say f_S, g_S are **orthogonal** to each other iff $< f_S, g_S >= 0$. Let $\mathcal{K}_{SP}, \mathcal{K}_P$ denote collections of vectors on $S \uplus P, P$ respectively. The **generalized minor** $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$ of $\mathcal{K}_{SP}$ with respect to $\mathcal{K}_P$ is defined by

$\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P \equiv \{f_S : f_S = f_{SP}/S, \; where \; f_{SP} \in \mathcal{K}_{SP} \; s.t. \; f_{SP}/P \in \mathcal{K}_P\}$.

We denote by $\mathcal{K}_{SP} + \mathcal{K}_P$, the collection of all vectors $f_{SP} + (0_S|f_P)$ on $S \uplus P$, where $f_{SP} \in \mathcal{K}_{SP}$ and $f_P \in \mathcal{K}_P$. Given a collection of vectors $\mathcal{K}_S$ on S, we denote by $\mathcal{K}_S{}^*$, the collection of all vectors orthogonal to vectors in $\mathcal{K}_S$. We say $\mathcal{K}_S$ and $\mathcal{K}_S{}^*$, are **complementary orthogonal** iff $(\mathcal{K}_S{}^*)^* = \mathcal{K}_S$.

III. IMPLICIT DUALITY THEOREM

The following is the main result of this paper:

Theorem III.1: (The Implicit Duality Theorem (IDT)) Let $\mathcal{K}_{SP}, \mathcal{K}_P$ be unimodular modules on $S \uplus P, P$ respectively, such that $\mathcal{K}_{SP} + \mathcal{K}_P$ is also unimodular. Then,
(a) $(\mathcal{K}^*_{SP} \leftrightarrow \mathcal{K}^*_P)^* = \mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$.
(b) $\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P$ is unimodular.
If, in addition, $\mathcal{K}^*_{SP} + \mathcal{K}^*_P$ is unimodular, then
(c) $(\mathcal{K}_{SP} \leftrightarrow \mathcal{K}_P)^* = \mathcal{K}^*_{SP} \leftrightarrow \mathcal{K}^*_P$.
(d) $\mathcal{K}^*_{SP} \leftrightarrow \mathcal{K}^*_P$ is unimodular.
The proof of this theorem requires the use of the following lemmas. of which the first two have their proofs omitted.

Lemma III.1: Let A be a g-unimodular matrix.
Let $\mathcal{C}$ be the module generated by the rows or columns of A. Then $\mathcal{C}^{**} = \mathcal{C}$.

Lemma III.2: Let K be a collection of vectors over $\Re[\xi]$ on S. Then K^* is unimodular.

Lemma III.3: Let A be an $m \times n$ g-unimodular matrix over $\Re[\xi], b$ an $m \times 1$ vector over $\Re[\xi]$. Then $Ax = b$ has a solution (with x a vector over $\Re[\xi]$) iff
for every $(1 \times m)$ vector λ^T over $\Re[\xi]$ we have $\lambda^T A = 0 \Rightarrow \lambda^T b = 0$.

Proof :- $Ax = b$ has a solution iff b belongs to the column module $\mathcal{C}$ generated by columns of A,
i.e., iff $b \in \mathcal{C}^{**}$ ($\mathcal{C} = \mathcal{C}^{**}$, since A is g - unimodular),
i.e., iff whenever $\lambda^T \in \mathcal{C}^*$ we also have $\lambda^T b = 0$,
i.e., iff whenever $\lambda^T A = 0$, we also have $\lambda^T b = 0$.
■

Proof of the Implicit Duality Theorem
Let K_{SP} have the representative matrix $[A_S \quad A_P]$ and let K_P have the representative matrix $[\hat{A}_P]$. Then $K_{SP} + K_P$ has the representative matrix

$$\begin{bmatrix} A_S & A_P \\ 0 & \hat{A}_P \end{bmatrix}$$

which is g-unimodular by hypothesis.
Now $x_S \in K_{SP} \leftrightarrow K_S$ iff $\exists \lambda_1, \lambda_2$ over $\Re[\xi]$ s.t.

$$\begin{bmatrix} A_S{}^T & 0 \\ A_P{}^T & \hat{A}^T_P \end{bmatrix} \begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix} = \begin{pmatrix} x_S \\ 0 \end{pmatrix}$$

i.e., iff (since column module of the above matrix is unimodular) whenever, for y_S, y_P over $\Re[\xi]$ we have

$$\begin{bmatrix} y^T_S & y^T_P \end{bmatrix} \begin{bmatrix} A^T_S & 0 \\ A^T_P & \hat{A}^T_P \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

we also have

$$y^T_S x_S = 0;$$

i.e., iff whenever $(y^T_S \; y^T_p) \in K^*_{SP}, \; y^T_p \in K^*_P$, we also have $y^T_S x_S = 0$;
i.e., iff whenever $y_S \in K^*_{SP} \leftrightarrow K^*_P$,
we have $y^T_S x_S = 0$.
Thus, $x_S \in K_{SP} \leftrightarrow K_P$ iff $x_S \in (K^*_{SP} \leftrightarrow K^*_P)^*$.
This proves (a).

The statement in (b) follows from (a) and Lemma III.2.

(c) Suppose $K^*_{SP} + K^*_P$ is unimodular. Since by Lemma III.2 we already have K^*_{SP}, K^*_P as unimodular, we can use (a) above and conclude that

$$(K^{**}_{SP} \leftrightarrow K^{**}_P)^* = K^*_{SP} \leftrightarrow K^*_P$$

But K_{SP}, K_P are unimodular and so by Lemma III.1, $K^{**}_{SP} = K_{SP}$ and $K^{**}_P = K_P$
Hence $(K_{SP} \leftrightarrow K_P)^* = (K^*_{SP} \leftrightarrow K^*_P)$

Statement (d) follows from (c) and Lemma III.2.
■

IV. ADJOINTS FOR BEHAVIOURAL SYSTEMS

The following appear to be desirable while building adjoints
- the notions controllability/observability of the original system should appear to correspond to observability/controllability for the adjoints;
- the adjoint for systems in the i/s/o (input/state/output) form should correspond to the standard construction for systems in this form;
- the computational effort for building the adjoint should be very mild whatever be the original representation for the behaviour.

In addition we note that if the adjoint is in some way related to the module complementary orthogonal to that asscociated with the orginal behaviour, there are technical advantages. Let behaviour $\mathcal{B}$ have the kernel representation

$$\begin{pmatrix} R_w & R_l \end{pmatrix} \begin{matrix} w \\ l \end{matrix} = 0$$

Define the adjoint $\mathcal{B}^{adj1}$ as having the kernel representation

$$\begin{bmatrix} I & -R^T_w \\ 0 & -R^T_l \end{bmatrix} \begin{matrix} \hat{w} \\ \hat{l} \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Observe that, while $w, \hat{w}$ have the same number of entries $l, \hat{l}$ may not have. It is easy to see that
$\mathcal{B}$ is controllable $\equiv \mathcal{B}^{adj1}$ is $\hat{l}$ - observable
$\mathcal{B}$ is l- observable $\equiv \mathcal{B}^{adj1}$ is controllable.
We further have (through the use of Implicit Duality Theorem)

Theorem IV.1: :- If behaviour $\mathcal{B}$ is controllable

$$\mathcal{C}(\mathcal{B}/w) = (\mathcal{C}(\mathcal{B}^{adj1}/\hat{w}))^*.$$

If $\mathcal{B}$ is l-observable then

$$(\mathcal{C}(\mathcal{B}/w))^* = \mathcal{C}(\mathcal{B}^{adj1}/\hat{w})$$

This theorem is a restricted version of Theorem IV.2 proved below. Its proof is therefore omitted.

Note that the adjoint is essentially, but not exactly, unique. If $(R_w R_l)$ is replaced by a matrix obtained by unimodular row transformation, then in the adjoint $\hat{w}$ would change through the inverse transformation. Similarly the adjoint of the adjoint would be essentially the original behaviour but would not be identical to it. We show now that this definition captures the usual i/s/o adjoint nicely.

If the original system is in the i/s/o form we have

(taking $w^T = (u^T \quad y^T), x = l$),

$$\begin{bmatrix} -B & 0 & (\xi I - A) \\ -D & I & -C \end{bmatrix} \begin{matrix} u \\ y \\ x \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The adjoint then is

$$\begin{bmatrix} I & 0 & B^T & +D^T \\ 0 & I & 0 & -I \\ 0 & 0 & -(\xi I - A)^T & C^T \end{bmatrix} \begin{matrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{l}_1 \\ \hat{l}_2 \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The variables $\hat{w}_2, \hat{l}_2$ are seen to be identical using the second row. The constraints on $\hat{w}_1, \hat{w}_2, \hat{l}_1$ are

$$\begin{bmatrix} I & D^T & B^T \\ 0 & C^T & -(\xi I - A)^T \end{bmatrix} \begin{matrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{l}_1 \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Clearly this is the usual adjoint of a system in i/s/o form.

Next let us consider the behaviour given in the input, output, latent variable form.
Let the behaviour $\mathcal{B}$ have the kernel representation with linearly independent rows

$$\begin{bmatrix} R_{1u} & R_{1l} & 0 \\ R_{2u} & R_{2l} & I \end{bmatrix} \begin{matrix} u \\ l \\ y \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Observe that practical systems, when they have clearly defined inputs and outputs, very often have this form. The latent variables need not however correspond to the state variables.

Let the adjoint $\mathcal{B}^{adj2}$ be defined through the representation

$$\begin{bmatrix} I & -R_{1u}^T & -{R_{2u}}^T \\ 0 & -R_{1l}^T & -R_{2l}^T \end{bmatrix} \begin{matrix} \hat{u} \\ \hat{l} \\ \hat{y} \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Observe that, while $u, \hat{y}$, and $y, \hat{u}$ have the same number of entries, $l, \hat{l}$ may not have.

We observe that $\mathcal{B}/ul$ has the kernel representation

$$\begin{bmatrix} R_{1u} & R_{1l} \end{bmatrix} \begin{matrix} u \\ l \end{matrix} = 0$$

and that $\mathcal{B}^{adj2}/\hat{y}\hat{l}$ has the kernel representation

$$\begin{bmatrix} -R_{1l}^T & -R_{2l}^T \end{bmatrix} \begin{matrix} \hat{y} \\ \hat{l} \end{matrix} = 0.$$

We note that the original behaviour is invariant if in the kernel representation, a linear combination of the first set of rows is added to the second set of rows. It follows that the adjoint is not uniquely defined but depends on the original representation.
However, the adjoint does have the following attractive properties which can be proved through the Implicit Duality Theorem.

Theorem IV.2: 1. $\mathcal{B}/ul$ is controllable $\equiv$ $\mathcal{B}^{adj2}$ is $\hat{l}$-observable.
2. $\mathcal{B}$ is l-observable $\equiv \mathcal{B}^{adj2}/\hat{y}\hat{l}$ is controllable.
3. If $\mathcal{B}/ul$ is controllable then $(\mathcal{C}(\mathcal{B}^{adj2}/\hat{u}\hat{y}))^* = \mathcal{C}(\mathcal{B}/uy)$.
4. If $\mathcal{B}$ is l-observable then $(\mathcal{C}(\mathcal{B}/uy))^* = \mathcal{C}(\mathcal{B}^{adj2}/\hat{u}\hat{y})$.
Proof:- LHS and RHS of part (1) are equivalent to the condition that $(R_{1u}R_{1l})$ is g-unimodular and LHS and RHS of part (2) are equivalent to the condition that $(R_{1l}^T R_{2l}^T)$ is g-unimodular. We now prove parts (3) and (4).
Using additional variables l' the kernel representation of $\mathcal{B}$ may be re-written as follows:

$$\begin{pmatrix} R_{1u} & 0 & I & 0 \\ R_{2u} & R_{2l} & 0 & I \end{pmatrix} \begin{matrix} u \\ l \\ l' \\ y \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (1)$$

$$\begin{bmatrix} 0 & -R_{1l} & I & 0 \end{bmatrix} \begin{matrix} u \\ l \\ l' \\ y \end{matrix} = 0 \qquad (2)$$

The kernel representation of $\mathcal{B}^{adj2}$ may be rewritten as follows:-

$$\begin{pmatrix} I & 0 & -R_{1u}^T & -R_{2u}^T \\ 0 & I & 0 & -R_{2l}^T \end{pmatrix} \begin{matrix} \hat{u} \\ \hat{l} \\ \hat{l}' \\ \hat{y} \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (3)$$

$$\begin{bmatrix} 0 & I & R_{1l}^T & 0 \end{bmatrix} \begin{matrix} \hat{u} \\ \hat{l} \\ \hat{l}' \\ \hat{y} \end{matrix} = 0 \qquad (4)$$

Let $K_{VLL'Y}$, be the module spanned by the rows of the coefficient matrix in Equation (1), $K_{LL'}$ be the module spanned by the rows of $[-R_{1l}I]$ in Equation (2).

Clearly ${K_{VLL'Y}}^*$ is spanned by the rows of the coefficient matrix in Equation (3) and $K_{LL'}^*$ is spanned by the rows of (IR_{1l}^T) in Equation (4).

Clearly $K_{LL'}, K_{VLL'Y}, {K_{LL'}}^*, {K_{VLL'Y}}^*$ are unimodular modules since they have representative matrices with full rank identity submatrices.
The coefficient matrix for Equations (1), (2) taken together is g-unimodular if $(R_{1u}R_{1l})$ is g-unimodular. If this holds, by IDT

$$({K^*}_{VLL'Y} \leftrightarrow {K^*}_{LL'})^* = K_{VLL'Y} \leftrightarrow K_{LL'}$$

Since $K_{VLL'Y} \leftrightarrow K_{LL'}$ is the module corresponding to $\mathcal{B}/uy$ and ${K_{VLL'Y}}^* \leftrightarrow {K_{LL'}}^*$ is the module corresponding to $\mathcal{B}^{adj2}/\hat{u}\hat{y}$, it follows

$$(\mathcal{C}(\mathcal{B}^{adj2}/\hat{u}\hat{y}))^* = \mathcal{C}(\mathcal{B}/uy)$$

Similarly, the coefficient matrix for Equations (3), (4) taken together is g-unimodular if $(-R_{1l}^T - R_{2l}^T)$ is g-unimodular. If this holds, by IDT, we have

$$(K_{VLL'Y} \leftrightarrow K_{LL'})^* = K_{VLL'Y}^* \leftrightarrow K_{LL'}^*$$

i.e. we have

$$(\mathcal{C}(\mathcal{B}/uy))^* = \mathcal{C}(\mathcal{B}^{adj2}/\hat{u}\hat{y})$$

■

Finally we verify that the i/s/o adjoint does fit into the present definition. Let the original behaviour $\mathcal{B}$ have the representation.

$$\begin{bmatrix} -B & (\xi I - A) & 0 \\ -D & -C & I \end{bmatrix} \begin{matrix} u \\ x \\ y \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then $\mathcal{B}^{adj2}$ has the representation

$$\begin{bmatrix} I & B^T & D^T \\ 0 & -(\xi I - A)^T & C^T \end{bmatrix} \begin{matrix} \hat{u} \\ \hat{x} \\ \hat{y} \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is identical to the usual i/s/o adjoint.

V. Implicit Duality Theorem 2

In this section we present an analogue of IDT. Our primary aim is to show that such a theorem may be found whenever the 'duality' is strong enough that a result like Lemma III.3 holds. We use the following notation: $[f[\xi], b]$, where $f[\xi]$ is a vector on S over $\Re[\xi]$ and b a vector on S whose entries are $\mathcal{C}^\infty$ functions, denotes $\sum_{e_i \in S}(f_{e_i}[d/dt])(b_{e_i})$.
The **generalized minor** $\mathcal{B}_{SP} \leftrightarrow \mathcal{B}_P$ of behaviour $\mathcal{B}_{SP}$ with respect to behaviour $\mathcal{B}_P$ is defined by
$\mathcal{B}_{SP} \leftrightarrow \mathcal{B}_P \equiv \{w_S : w_S = w_{SP}/S, \ where \ w_{SP} \in \mathcal{B}_{SP} \ s.t. \ w_{SP}/P \in \mathcal{B}_P\}$.

Theorem V.1: (The Implicit Duality Theorem 2 (IDT2)) Let $\mathcal{C}_{SP}, \mathcal{C}_P$ be modules on $S \uplus P, P$ associated with behaviours $\mathcal{B}_{SP}, \mathcal{B}_P$, respectively. Then $\mathcal{C}(\mathcal{B}_{SP} \leftrightarrow \mathcal{B}_P) = \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$.

The proof of this theorem requires the use of the Lemma V.2 which in turn can be proved using Lemma V.1.

Lemma V.1: Let $A[\xi]$ be a matrix over $\Re[\xi]$ with linearly independent rows, b, a vector whose entries are $\mathcal{C}^\infty$ functions. Then $(A[d/dt])x = b$, has a solution.
The routine proof (for instance by transforming to SCF) is omitted.

Lemma V.2: Let $A[\xi]$ be an $m \times n$ matrix over $\Re[\xi]$, b an $m \times 1$ vector whose entries are $\mathcal{C}^\infty$ functions. Then $(A[d/dt])x = b$ has a solution (with x a vector whose entries are $\mathcal{C}^\infty$ functions) iff for every $(1 \times m)$ vector $\lambda^T[\xi]$ over $\Re[\xi]$ we have
$\lambda^T[\xi](A[\xi]) = 0 \Rightarrow [\lambda^T[\xi], b] = 0$.
Proof :- The necessity of the condition is obvious. We prove the sufficiency.

Assume for every $(1 \times m)$ vector λ^T over $\Re[\xi]$ we have $\lambda^T A = 0 \Rightarrow [\lambda^T, b] = 0$. By reversible row operations A can be transformed to

$$\begin{bmatrix} A_1 \\ 0 \end{bmatrix}$$

where the rows of A_1 are linearly independent and the equation $Ax = b$ can be transformed through the same operations to

$$\begin{bmatrix} A_1 \\ 0 \end{bmatrix} x = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Now, the assumption made, implies that every entry of b_2 is zero. So if we show $(A_1[d/dt])x = b_1$ has a solution we are done. But this holds by Lemma V.1 since A_1 has linearly independent rows. ■

Proof of the Implicit Duality Theorem 2
Let C_{SP} have the representative matrix $[A_S \quad A_P]$ and let C_P have the representative matrix $[\hat{A}_P]$. Then $C_{SP} + C_P$ has the representative matrix

$$\begin{bmatrix} A_S & A_P \\ 0 & \hat{A}_P \end{bmatrix}.$$

Let $\mathcal{B}_+$ be associated with the module $C_{SP} + C_P$. By definition, the behaviour $\mathcal{B}_{SP} \leftrightarrow \mathcal{B}_P = \mathcal{B}_+/w_S$. In other words, $w'_S \in \mathcal{B}_{SP} \leftrightarrow \mathcal{B}_P$
iff $\exists w'_P$, s.t.

$$\begin{bmatrix} A_S & A_P \\ 0 & \hat{A}_P \end{bmatrix} \begin{matrix} w'_S \\ w'_P \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e., iff $\exists w'_P$, s.t.

$$\begin{bmatrix} A_P \\ \hat{A}_P \end{bmatrix} w'_P = \begin{pmatrix} -A_S w'_S \\ 0 \end{pmatrix}$$

i.e., (by Lemma V.2) iff whenever

$$\begin{bmatrix} \lambda_1[\xi] & \lambda_2[\xi] \end{bmatrix} \begin{bmatrix} A_P \\ \hat{A}_P \end{bmatrix} = \begin{pmatrix} 0_P \end{pmatrix}$$

(equivalently whenever $\lambda_1[\xi]A_S \in \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$),
we also have $[\lambda_1[\xi], A_S w'_S] = 0$ (equivalently $[\lambda_1[\xi]A_S, w'_S] = 0$), i.e., iff whenever $\lambda_1[\xi]A_S \in \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$, we also have $[\lambda_1[\xi]A_S, w'_S] = 0$, i.e., iff $f_S \in \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ implies $[f_S, w'_S] = 0$ (since every vector in $\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ is a linear combination of the rows of A_S). Thus $\mathcal{C}(\mathcal{B}_{SP} \leftrightarrow \mathcal{B}_P) = \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$. ■

Two special cases of the above result may be mentioned. Consider the case where $\mathcal{C}_P$ is the zero module on P. In this case $\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ is obtained by restricting to S all those vectors in $\mathcal{C}_{SP}$ which take zero value on P. This is usually called the contraction of $\mathcal{C}_{SP}$ on P. The corresponding behaviour is merely the projection $\mathcal{B}_{SP}/w_S$, since $\mathcal{B}_P$ has no constraints on it and is therefore 'free'. Next let $\mathcal{C}_P$ be the 'free' module on P spanned by the rows of the unit matrix.In this case $\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ is obtained by restricting to S all vectors in $\mathcal{C}_{SP}$. The corresponding behaviour is the set $\{w_S : (w_S, 0_P) \in \mathcal{B}_{SP}\}$.

Finally we remark that Theorem IDT2 would hold also for those N-D systems for which Lemma V.2 holds since the proof depends only on that lemma.

VI. Conclusion

In this paper we have presented a useful result for building implicit 'duals' of Behavioural Systems which are themselves specified implicitly, for instance, by including latent variables in the description of the system. Using this Implicit Duality Theorem we build two 'adjoints' for Behavioural Systems, in each of which controllability appears as a dual notion to observability. Both the definitions of adjoints are shown to be compatible with the usual adjoint when the system is specified through a state space description. Finally, in order to indicate where one might hope to find an IDT like result we have presented one of its analogues and proved it similarly.

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