Notes on distribution theory

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1 *Why Distributions?*

Consider an ordinary linear differential equation with initial conditions set to zero

$$p(\mathcal{D})x = u,$$

where $x(\cdot)$ and $u(\cdot)$ are real functions over the real line.

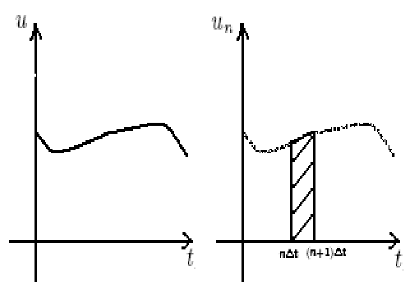

Figure 1:

Suppose we construct the functions $u_1, \ldots, u_n \ldots$ with u_n agreeing with u over the interval $[n\Delta t, (n+1)\Delta t]$ and elsewhere zero.

It is clear that

$$u(\cdot) = \sum_{n=0}^{\infty} u_n(\cdot)$$

Let, as stated earlier, $x(0)$ be zero.

Then the response due to $u(\cdot)$ must be the sum of the responses due to the $u_j(\cdot)$. Let $\tilde{p}_\tau$ represent the pulse of width Δt and a constant height $\frac{1}{\Delta t}$ in the interval $[\tau, \tau + \Delta t]$.

Let us approximate the functions $u_n(\cdot)$ by $\tilde{u}_n(\cdot) = (u(n\Delta t) \cdot \Delta t)\tilde{p}_{n\Delta t}(\cdot)$, essentially making the function constant over $[n\Delta t, (n+1)\Delta t]$.

If Δt is small we expect

$$u(\cdot) \approx \sum_{n=0}^{\infty} \tilde{u}_n(\cdot)$$

and the response to be approximately the sum of the responses due to the $\tilde{u}_n(\cdot)$. As $\Delta t \to 0$, one expects the approximation to become exact.

Suppose the response due to $\tilde{p}_\tau(\cdot)$ is $\tilde{h}_\tau(\cdot)$. Then the response due to $u(\cdot)$ would be $\sum_{n=0}^{\infty} \tilde{h}_\tau(\cdot)u(\tau) \cdot \Delta t$, (approximately) where $\tau = n\Delta t$, i.e.,

$$y(t) \approx \sum_{n=0}^{\infty} (\tilde{h}_\tau(t))u(\tau) \cdot \Delta t.$$

So we expect

$$y(t) = \int_0^\infty h_\tau(t)u(\tau)d\tau,$$

where $h_\tau(t) = \lim_{\Delta t \to 0} \tilde{h}_\tau(t)$. If we assume the system is causal i.e., the response to an input which is non-zero only for $t \geq t_0$ is also non-zero only for $t \geq t_0$, then the above integral would have limits from 0 to t. We remind the reader that $\tilde{h}_\tau(\cdot)$ is the response due to the pulse $\tilde{p}_\tau(\cdot)$ from τ to $\tau + \Delta t$, of width Δt and height $\frac{1}{\Delta t}$. We may think of $h_\tau(\cdot)$ as the response due to the *infinite pulse* $\hat{p}_\tau(\cdot)$ with width Δt and height $\frac{1}{\Delta t}$ and $\Delta t \to 0$.

Of course there exists no such function. But it is nevertheless very convenient to work with this *generalized function*. Distribution theory will justify the use of this construct.

2 Definition of distributions: general and tempered

We begin by generalizing the notion of a function over the real line. The generalization goes through routinely to the functions over $\mathbb{R}^n$ i.e., instead of $f(t)$ we could have $f(t_1, t_2, \ldots, t_n)$ where $t_1, t_2, \ldots, t_n$ are real numbers. We first build a domain that is 'richer' than the $\mathbb{R}$. This is the space $\mathcal{D}$ of *test functions*. A test function $\phi(\cdot)$ is defined over the real line and takes complex values and satisfies the following :

- $\phi(\cdot)$ vanishes outside a finite interval and
- $\frac{d^k\phi}{dt^k}$ exists for every positive integer k (in other words, $\phi(\cdot)$ is infinitely differentiable).

Example of test function:

$$\phi(x) = \begin{cases} e^{\frac{1}{x^2-1}} & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| \geq 0. \end{cases}$$

It is clear that this function is infinitely differentiable at $|x| \neq 1$.

Let us examine what happens at $x = 1_-$ (i.e., $\lim_{x\to 1, x<1}$). We have

$$\frac{d}{dx} e^{\frac{1}{x^2-1}} = \left(\frac{-2x}{(x^2-1)^2} \right) e^{\frac{1}{x^2-1}}$$

Now $y^k e^{-y} \to 0$ as $y \to \infty$ for all k. Hence

$$\lim_{x\to 1_-} \frac{d}{dx} \left(e^{\frac{1}{x^2-1}} \right) = 0.$$

The same idea can be used for all orders of derivatives and it can be proved that

$$\frac{d^k}{dx^k} \left(e^{\frac{1}{x^2-1}} \right) = 0 \text{ at } x = 1_-$$

Thus the given function is indeed an example of a test function. Observe that we can shift this function and change its width quite easily. Also its integral moves from 0 to another constant smoothly from -1 to 1. Therefore one can build test functions with flat tops (or bottoms) of whatever height and widthe we please.

A second class of more general functions, namely the class $\mathcal{S}_1$ of functions of *rapid decay*, is useful in our study. A function of rapid decay $\phi(\cdot)$ is defined over the real line and takes complex values and satisfies the following :

- $\phi(\cdot)$ is infinitely differentiable,
- $\phi(\cdot)$ together with all its derivatives vanishes at $|x| = \infty$ faster than the reciprocal of any polynomial. Thus for each pair of nonnegative integers k and l,

  $$\lim_{|x|\to\infty} \left| x^k \frac{d^l \phi}{dx^l} \right| = 0.$$

  (We may think of ϕ as being similar to e^{-x} in its power to kill polynomials).

Clearly every test function is a function of rapid decay since it vanishes outside a finite interval.

We say a sequence ϕ_n of test functions is a 'null sequence' in $\mathcal{D}$ iff

- all ϕ_n vanish outside a finite interval,
- ϕ_n and all its derivatives approach 0 uniformly in this interval.

Basically the null sequence ϕ_n is analogous to a sequence of numbers tending to zero and we need this idea to talk about small perturbations of a test function $\hat{\phi}$ by studying $\hat{\phi} + \phi_n$.

We define a null sequence ϕ_m of functions of rapid decay in a similar manner for each pair of nonnegative integers k and l,

$$\lim_{m \to \infty} \max_{-\infty < x < \infty} \left| x^k \frac{d^l \phi_m}{dx^l} \right| = 0.$$

Observe that even after being multiplied by a polynomial of any degree the function $\frac{d^l \phi_m}{dx^l}$, must be bounded in the interval $-\infty < x < \infty$ and further this maximum value must tend to zero as $m \to \infty$.

We are now in a position to define a distribution. A *distribution* is a continuous linear functional on the space of test functions, i.e., a distribution q

- takes a complex value on each test function ϕ, usually denoted as $< q, \phi >$
- is linear i.e. $< q, \alpha_1 \phi_1 + \alpha_2 \phi_2 >= \alpha_1 < q, \phi_1 > + \alpha_2 < q, \phi_2 >$,
- is continuous on the space of test functions i.e. $\lim_{n \to \infty} < q, \phi_n >= 0$, whenever ϕ_n is a null sequence in $\mathcal{D}$.

A special class of distributions called 'tempered distributions' or 'distributions of slow growth' are of interest to us. These are linear continuous on the larger class $\mathcal{S}_1$ of functions of rapid decay i.e., $\lim_{n \to \infty} < q, \phi_n >= 0$, for every null sequence of rapidly decaying functions, whenever q is a tempered distribution. Henceforth we use $[q, \phi]$ in place of $< q, \phi >$.

Distributions are generalizations of ordinary functions on the real line. Let f be locally integrable over the real line i.e.,

$$\int_{T_1}^{T_2} |f(x)| \, dx$$

is finite whenever $T_2 - T_1$ is finite.

Define

$$[q_f, \phi] \equiv \int_{-\infty}^{\infty} f(x) \phi(x) \, dx.$$

Then q_f is linear continuous on $\mathcal{D}$. To see this, first note that

$$\int_{-\infty}^{\infty} f(x)\phi(x)\,dx = \int_{T_1}^{T_2} f(x)\phi(x)\,dx,$$

where ϕ vanishes outside $[T_1, T_2]$.

Now since ϕ is differentiable everywhere it must be continuous in $[T_1, T_2]$ and therefore has a maximum value in $[T_1, T_2]$ say M.

So

$$\begin{aligned} \left|\int_{-\infty}^{\infty} f(x)\phi(x)\,dx.\right| &\le \int_{T_1}^{T_2} |f(x)||\phi(x)|\,dx. \\ &\le M\int_{T_1}^{T_2} |f(x)|\,dx. \end{aligned}$$

Hence $\int_{-\infty}^{\infty} f(x)\phi(x)\,dx$ exists.

The linearity is clear since

$$\int_{-\infty}^{\infty} f(x)(\alpha_1\phi_1(x) + \alpha_2\phi_2(x))\,dx = \\ \alpha_1\int_{-\infty}^{\infty} f(x)\phi_1(x)\,dx + \alpha_2\int_{-\infty}^{\infty} f(x)\phi_2(x)\,dx.$$

To prove continuity we need to show $\lim_{m\to\infty}[q_f, \phi_m] \to 0$ when ϕ_m is a null sequence in $\mathcal{D}$.

We have

$$\int_{-\infty}^{\infty} f(x)\phi_m(x)\,dx = \int_{T_1}^{T_2} f(x)\phi_m(x)\,dx$$

for some finite T_1, T_2, by the definition of null sequence.

Let $M_m = \max_{T_1\le x\le T_2}\phi_m(x)$. Hence

$$\left|\int_{-\infty}^{\infty} f(x)\phi_m(x)\,dx.\right| \le M_m\int_{T_1}^{T_2} |f(x)|\,dx$$

Since f is locally integrable, $\int_{T_1}^{T_2} |f(x)|\,dx$ is finite. Hence, since $\lim_{m\to\infty} M_m = 0$, we must have $\lim_{m\to\infty}\int_{-\infty}^{\infty} f(x)\phi_m(x)\,dx = 0$, as needed.

We claim that the action $[q_f, \phi]$ is indeed a generalization of the value of the function f. Indeed,we can recover the value of f at points where it is continuous by making f act on suitable ϕ_n and taking limits. Choose ϕ_n as a nonnegative test function with nonnegative values in $[a - \frac{1}{n}, a + \frac{1}{n}]$ and zero outside.

Further let $\int_{a-\frac{1}{n}}^{a+\frac{1}{n}} \phi_n(x)\, dx = 1$. Observe that $\int_{a-\frac{1}{n}}^{a+\frac{1}{n}} f(x)\phi_n(x)\, dx$ gives a 'weighted average' of f in the interval $[a - \frac{1}{n}, a + \frac{1}{n}]$. As $n \to \infty$, provided f is continuous at a, the integral will therefore converge to $f(a)$.

When the context is clear we will denote the distribution $q_f([q_f, \phi] \equiv \int_{-\infty}^{\infty} f(x)\phi(x)\, dx)$ also by f.

A distribution q_f that arises from a locally integrable function through the definition

$$[q_f, \phi] = \int_{-\infty}^{\infty} f(x)\phi(x)dx$$

is said to be *regular*. Not all distributions arise in this manner. These latter distributions are said to be *singular*. The convention even in the case of singular distributions is to write $[q, \phi] = \int_{-\infty}^{\infty} q(x)\phi(x)dx$.

3 δ function and δ-sequences

For our purpose, the most important singular distribution is the delta [*Dirac* delta] functional $\delta(x)$ defined by

$$[\delta, \phi] = \phi(0),$$

where ϕ is a function of rapid decay. (In fact δ functional is linear and continuous even on the space of functions continuous at 0). One cannot expect any function f to have the property $\int_{-\infty}^{\infty} f(x)\phi(x)dx = \phi(0)$. However a sequence of functions, in the limit, can have such property.

Let f_n be a sequence of functions continuous at the origin with the following properties.

1. $\int_{-\infty}^{\infty} f_n(x)dx = 1$ for each n.

2. $\lim_{n\to\infty} \int_a^b f_n(x)dx = 0,\ a,\ b$ both positive or both negative and therefore $\lim_{n\to\infty} \int_a^b f_n(x)dx = 1$, if $a < 0 < b$.

We call such a sequence f_n, a *delta* sequence. Clearly we must have, for every rapidly decaying ϕ,

$$\int_{-\infty}^{\infty} f_n(x)\phi(x)dx = \int_{-\infty}^{-\epsilon} f_n(x)\phi(x)dx + \int_{-\epsilon}^{\epsilon} f_n(x)\phi(x)dx + \int_{\epsilon}^{\infty} f_n(x)\phi(x)dx$$

where $\epsilon > 0$. We then have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\phi(x)dx = \lim_{n\to\infty}\left[\int_{-\infty}^{-\epsilon} + \int_{-\epsilon}^{\epsilon} + \int_{\epsilon}^{\infty}\right]$$
$$= \lim_{n\to\infty}\int_{-\epsilon}^{\epsilon} f_n(x)\phi(x)dx, \text{ for every } \epsilon > 0.$$

Since ϕ is continuous at $\phi(0)$ the right side must be equal to $\phi(0)$ (taking the limit $\epsilon \to 0,\ \epsilon > 0$).

So if we define

$$[q,\phi] \equiv \lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\phi(x)dx,$$

where f has the above properties, we will have $q = \delta$.

It is easy to see by the definition that q is a linear functional. Indeed

$$[q, \alpha_1\phi_1 + \alpha_2\phi_2] = \alpha_1\phi_1(0) + \alpha_2\phi_2(0).$$

To see that q is continuous on the space of rapidly decaying functions we need to show that $\lim_{n\to\infty}[q,\phi_n] = 0$ where $\{\phi_n\}$ is a null sequence of rapidly decaying functions. This is clear since $\lim_{n\to\infty}\phi_n(0) = 0$, by the definition of a null sequence in the space of rapidly decaying functions.

We often write, whenever f_n is a delta sequence

$$\lim_{n\to\infty} f_n(x) = \delta(x).$$

It is to be interpreted as

$$[\delta,\phi] = \lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\phi(x)\,dx.$$

We now give a couple of examples of delta sequences which are available in the literature.

1.
$$s_n(x) \equiv \frac{1}{\pi}\frac{n}{1+n^2x^2}.$$

These functions are continuous at the origin. It is clear that

$$\lim_{n\to\infty}\int_a^b \frac{n}{1+n^2x^2}\,dx = 0$$

whenever a, b are both positive or negative.

We remind the reader that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$ and therefore $\frac{d}{dx}\arctan(kx) = \frac{k}{1+k^2x^2}$. Hence

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{n}{1+n^2x^2}\,dx &= [\arctan(nx)]_{-\infty}^{\infty} \\ &= \pi.\end{aligned}$$

Clearly therefore, $\int_{-\infty}^{\infty} s_n(x)\,dx = 1$. This means that s_n is a delta sequence.

2.
$$s_n(x) \equiv \frac{\sin(nx)}{\pi x}$$

is a delta sequence.

We claim that

(a)
$$\int_{-\infty}^{\infty} \frac{\sin(kx)}{\pi x} dx = 1$$

This result can be proved by using contour integration. Build a closed contour C moving from $-R_1$ to $-R_2$ along the real axis, moving from $-R_2$ to $+R_2$ along a semicircle in the upper half plane, from R_2 to $+R_1$ along the real axis and close the contour by moving from R_1 to $-R_1$ along a semicircle in the upper half plane. Now integrate $\int_C \frac{e^{jkz}}{\pi z}\,dz$, letting R_1 tend to ∞ and R_2 to zero. Since within the contour there is no pole of the integrand, the contour integral will equal zero. The larger semicircle integration can be shown to become zero while the smaller one yields $-j$. Now

$$\int_{-\infty}^{\infty} \frac{\sin kx}{\pi x} dx = \lim_{R_1\to\infty, R_2\to 0} \int_{-R_1}^{R_2} \frac{\sin kx}{\pi x} dx + \int_{R_2}^{R_1} \frac{\sin kx}{\pi x} dx.$$

The RHS is the imaginary part of the portion of the above contour integral along the real axis, which by the above argument equals $+j$. The result follows.

We can also prove this using Fourier transform ideas as follows. Let

$$\begin{aligned} p_k(t) = &1 \quad 0 \le |k| \\ &0 \quad \text{otherwise.}\end{aligned}$$

Fourier transform of this function is

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-j\omega t} p_T(t)dt &= \int_{-T}^{T} e^{-j\omega t}dt \\
&= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-T}^{T} \\
&= \frac{e^{j\omega T} - e^{-j\omega T}}{j\omega} \\
&= \frac{2\sin(\omega T)}{\omega}
\end{aligned}$$

Therefore $\mathbb{F}[\frac{\sin kx}{\pi x}] = \frac{\pi p_k(-\omega)}{\pi} = p_k(\omega)$. Observe that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin(xt)}{\pi t}dt &= \left.\int_{-\infty}^{\infty} \frac{\sin(xt)}{\pi t} e^{-j\omega t}dt\right|_{\omega=0} \\
&= p_k(\omega)|_{\omega=0} = 1.
\end{aligned}$$

(b) We have,

$$\int_a^b \frac{\sin kx}{x}dx = \int_{ka}^{kb} \frac{\sin \upsilon}{\upsilon}d\upsilon,\ \text{taking } \upsilon = kx.$$

Let us consider the case when a, b are both positive or both negative. Using integration by parts as $k \to \infty$, the above integral,

$$\begin{aligned}
&= \lim_{k\to\infty} \left[-\left.\frac{\cos \upsilon}{\upsilon}\right|_{ka}^{kb} + \int_{ka}^{kb} \frac{\sin(\upsilon)}{\upsilon^2}d\upsilon \right] \\
&\leq \quad 0 \quad + \lim_{k\to\infty} \int_{ka}^{kb} \frac{1}{\upsilon^2}d\upsilon \\
&\leq \qquad \lim_{k\to\infty} \left. -\frac{1}{\upsilon}\right|_{ka}^{kb} = 0
\end{aligned}$$

We thus see that the sequence s_n is a δ-sequence.

4 Operations on Distributions

The properties of the integral $\int_{-\infty}^{\infty} f(x)\phi(x)dx$ are used to define various notions related to distributions. In this regard, we will consider the notions of value over intervals, translations, scaling,derivative and Fourier transform. We remind the reader that the support of a function is the closure of the set of points on which it takes non zero values.

4.1 *Equality in an interval:*

If two functions f_1, f_2 have the same value in an interval $[a, b]$ clearly $\int_{-\infty}^{\infty} f_1(x)\phi(x)dx = \int_a^b f_2(x)\phi(x)dx$, whenever the support of ϕ is contained in $[a, b]$. By analogy we define distributions q_1, q_2 to be equal over the interval $[a, b]$ provided $[q_1, \phi] = [q_2, \phi]$, whenever the support of ϕ is contained in $[a, b]$.

4.2 *Translation:*

Let $f_a(x) \equiv f(x - a)$. We say that f_a is a translation of f by a. Clearly

$$\begin{aligned}\int_{-\infty}^{\infty} f_a(x)\phi(x)dx &= \int_{-\infty}^{\infty} f(x-a)\phi(x)dx \\ &= \int_{-\infty}^{\infty} f(y)\phi(y+a)dy\end{aligned}$$

where $y = x - a$.

Thus the action of f_a on ϕ is the same as the action of f on ϕ_{-a}. In the case of distributions we are thus motivated to define the translation of the distribution q by a as follows:-

$$[q_a, \phi] \equiv [q, \phi_{-a}].$$

4.3 *Linear Combination:*

If we scale a function f by a to yield g, i.e. if $g(x) = af(x)$, we would have

$$\int_{-\infty}^{\infty} g(x)\phi(x)dx = \int_{-\infty}^{\infty} f(x)(a\phi(x))dx.$$

We therefore define, for a distribution q

$$[aq, \phi] = [q, a\phi] = a[q, \phi].$$

Similarly the sum of the distributions q_1 and q_2 is defined to be

$$[q_1 + q_2, \phi] = [q_1, \phi] + [q_2, \phi].$$

4.4 *Scaling the Domain:*

Next suppose we define g by scaling the domain of f, i.e. $g(x) \equiv f(ax)$. In this case

$$\begin{aligned}\int_{-\infty}^{\infty} g(x)\phi(x)dx &= \int_{-\infty}^{\infty} f(ax)\phi(x)dx \\ &= \frac{1}{a}\int_{-\infty}^{\infty} f(y)\phi(\frac{y}{a})dy, a > 0 \\ &= -\frac{1}{a}\int_{-\infty}^{\infty} f(y)\phi(\frac{y}{a})dy, a < 0.\end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} g(x)\phi(x)dx = \frac{1}{|a|}\int_{-\infty}^{\infty} f(y)\phi(\frac{y}{a})dy.$$

We therefore define for a distribution q (writing it as $q(x)$),

$$[q(ax), \phi] \equiv \frac{1}{|a|}[q(x), \phi(\frac{x}{a})].$$

4.5 *Differentiation:*

Distributions were conceived to handle differential equations in a convenient manner, particularly impulse response and its derivatives. The entire theory has been built around the idea that distributions should be differentiable to all orders. Observe that for a differentiable function f,

$$\int_{-\infty}^{\infty} \dot{f}(x)\phi(x)dx = -\int_{-\infty}^{\infty} f(x)\dot{\phi}(x)dx + f(x)\phi(x)]_{-\infty}^{\infty}.$$

Now let f be a function of 'slow growth', i.e., some polynomial grows faster than $f(x)$ as $x \to \infty$, i.e.,

$$\lim_{x\to\infty} \frac{1}{(1+x^2)^p} f(x) = 0$$

for some positive integer p. In the above equation, in the RHS $f(x)\phi(x)]_{-\infty}^{\infty} = 0$, since $\phi(x)$ either has finite support when it belongs to $\mathcal{D}$ or, when it is rapidly decaying, has the property that $\lim_{x\to\infty} p(x)\phi(x) = 0$ for every polynomial $p(x)$.

We thus have

$$\int_{-\infty}^{\infty} \dot{f}(x)\phi(x)dx = -\int_{-\infty}^{\infty} f(x)\dot{\phi}(x)dx$$

whenever f is a function of slow growth and ϕ, a rapidly decaying function. This motivates us to define $\frac{dq}{dx}$, for a distribution as follows:

$$\left[\frac{dq}{dx}, \phi\right] \equiv -\left[q, \dot{\phi}\right].$$

4.6 *Multiplication by a function f:*

If q is a distribution, by the preceding development, we would like to define

$$[fq, \phi] \equiv [q, f\phi].$$

However $f\phi$ would not always be a test function or a rapidly decaying function when ϕ is one. In general, therefore, we require f to be infinitely differentiable for the above definition to work in the case of test functions. For rapidly decaying functions we need additionally that f grow slower than some polynomial, i.e. $f(x) \leq cx^p, |x| \geq x_0$ for some c, p. In the case of special distributions, this rule can be relaxed. For instance

$$[f\delta, \phi] \equiv [\delta, f\phi] \equiv f(0)\phi(0).$$

Here $f\delta$ is clearly defined, provided f is continuous at 0. Similarly we see

$$\left[f\frac{d^k\delta}{dx^k}, \phi\right] \equiv \left[\frac{d^k\delta}{dx^k}, f\phi\right] \equiv (-1)^k \left[\delta, \frac{d^k(f\phi)}{dx^k}\right]$$

which is defined if f has continuous k^{th} derivatives.

4.7 *Fourier transform of distributions:*

Suppose f and ϕ are both Fourier transformable, we have

$$\begin{aligned}\int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} \phi(x)e^{-jyx}dxdy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\phi(x)e^{-jyx}dxdy \\ &= \int_{-\infty}^{\infty} \phi(x) \int_{-\infty}^{\infty} f(y)e^{-jyx}dydx.\end{aligned}$$

Let us denote the Fourier transform $\int_{-\infty}^{\infty} f(x)e^{-jyx}dx$ by $\widehat{f}(y)$. We therefore have

$$\int_{-\infty}^{\infty} f(t)\widehat{\phi}(t)dt = \int_{-\infty}^{\infty} \widehat{f}(t)\phi(t)dt.$$

This motivates our definition of the Fourier transform $\widehat{q}$ of a distribution q:

$$[\widehat{q}, \phi] \equiv [q, \widehat{\phi}],$$

whenever ϕ is a rapidly decaying function. (It is shown below that whenever ϕ is a rapidly decaying function so will $\widehat{\phi}$ be.)

The reader will note that by making the functions on which distributions act very 'well behaved' we are able to define all the above operations on distributions. In particular because rapidly decaying functions are differentiable to any order, distributions also become 'differentiable' to any order. Because rapidly decaying functions are Fourier transformable, tempered distributions become 'Fourier transformable'.

4.8 *Some properties of rapidly decaying functions:*

If ϕ is rapidly decaying then the following holds

1. $x^k\phi$ is rapidly decaying.

2. ϕ is bounded (since ϕ is differentiable in (-∞,∞) and $\lim_{|x|\to\infty}$ ϕ(x) = 0).

3. $\int_{-\infty}^{\infty} |\phi(x)|dx$ exists.

   Proof: $\int_{-\infty}^{\infty}$ | ϕ(x) |dx = $\int_{-\infty}^{\infty}$ $\frac{1+x^2}{1+x^2}$ | ϕ(x) |dx

   $\leq (max_{x\in(-\infty,\infty)}(1+x^2)\phi(x))| \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx|$.

   Since the RHS is finite the result follows.

4. $\frac{d\phi}{dx}$ is rapidly decaying.
   Proof: We need to only show that $\lim_{|x|\to\infty}$ | x^k $\frac{d\phi}{dx}$ | = 0

   We have $\int_{-\infty}^{\infty}$ x^k $\frac{d\phi}{dx}$ dx = -$\int_{-\infty}^{\infty}$ k x^{k-1} ϕ dx + $[x^k\phi]_{-\infty}^{\infty}$

   Of the two terms in the RHS the first integral clearly exists,since x^{k-1} ϕ is rapidly decaying and the second is zero since ϕ is rapidly decaying. Thus $\int_{-\infty}^{\infty}$ x^k $\frac{d\phi}{dx}$ dx exists for all k. But this can only happen if $\lim_{|x|\to\infty}$ | x^k $\frac{d\phi}{dx}$ | = 0.

5. If ϕ is a rapidly decaying function so is $\widehat{\phi}$.

   Proof: We need to show

$$\lim_{x\to\infty} \left| x^k \frac{d^p\widehat{\phi}}{dx^p} \right| \longrightarrow 0$$

for every k and p.

Consider

$$\left| \frac{d^p}{dx^p} \left[\int_{-\infty}^{\infty} e^{-jxt} \phi(t) dt \right] x^k \right|.$$

This expression is equal to

$$\begin{aligned} &\left| \left[\int_{-\infty}^{\infty} (-jt)^p e^{-jxt} \phi(t) dt \right] x^k \right| \\ &= \left| \int_{-\infty}^{\infty} (-jt)^p (-jx)^k e^{-jxt} \phi(t) dt \right| \\ &= \left| \int_{-\infty}^{\infty} (-jt)^p \phi(t) \frac{d^k}{dt^k} e^{-jxt} dt \right| \end{aligned}$$

Integrating by parts this becomes

$$\begin{aligned} &= \left| (-1)^k \int_{-\infty}^{\infty} e^{-jxt} \frac{d^k}{dt^k} [(-jt)^p \phi(t)] dt \right| \\ &\leq \left| \int_{-\infty}^{\infty} \frac{d^k}{dt^k} (t^p \phi(t)) dt \right| \end{aligned}$$

Since ϕ is rapidly decaying so is $t^p \phi(t)$ and therefore so is $\frac{d^k}{dt^k}(t^p\ \phi(\mathrm{t}))$. So the integral exists for all p.

This means $|\ x^k\ \frac{d^p \widehat{\phi}}{dx^p}\ |$ is bounded for all x and all k but this can clearly happen only if for each k, $\lim_{x \to \infty}\ |\ x^k\ \frac{d^p \widehat{\phi}}{dx^p}\ | = 0$, which proves that $\widehat{\phi}$ is a rapidly decaying function.

4.9 *Duality for rapidly decaying functions and tempered distributions*

Suppose $f(\cdot)$ is continuous in $(-\infty, \infty)$ and its Fourier transform exists. Let

$$f_R(x) = \frac{1}{2\pi} \int_{-R}^{R} \widehat{f}(\omega) e^{j\omega x} d\omega,$$

where $\widehat{f}(\omega)$ is the Fourier transform of $f(x)$. We will show that

$$\lim_{R \to \infty} f_R(x) = f(x),$$

or equivalently $\widehat{\widehat{f}}(x) = 2\pi(f(-x))$.

We have

$$f_R(x) = \frac{1}{2\pi}\int_{-R}^{R} e^{j\omega x}\left[\int_{-\infty}^{\infty} e^{-j\omega t} f(t)dt\right]d\omega.$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\left[\frac{e^{j\omega(x-t)}}{j(x-t)}\right]_{-R}^{R}\right] f(t)dt.$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin R(t-x)}{t-x} f(t)dt.$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin Rz}{z} f(x+z)dz.$$

We have already seen that $\frac{\sin Rz}{\pi z}$ is a $\delta-$sequence with $R \to \infty$. Hence

$$\lim_{R\to\infty} f_R(x) = \int_{-\infty}^{\infty}\delta(z)f(x+z)dz = f(x).$$

[Note that RHS involving δ is just short form for $\lim_{R\to\infty}\int_{-\infty}^{\infty}\frac{\sin Rz}{\pi z}f(x+z)dz$]. In particular we see that if ϕ is a rapidly decaying function

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\phi}(\omega)e^{j\omega x}dx = \phi(x).$$

An easy consequence of the duality for rapidly decaying functions is the duality for tempered distributions,i.e., $\widehat{\widehat{q}}(x) = 2\pi q(-x)$. To see this note that $\left[\widehat{\widehat{q}}(x), \phi(x)\right] = \left[\widehat{q}, \widehat{\phi}\right] = \left[q, \widehat{\widehat{\phi}}\right] = [q(x), 2\pi\phi(-x)] = [2\pi q(-x), \phi(x)]$.

4.10 *Fourier Transform of derivatives*

Let f, g be absolutely integrable in $(-\infty, \infty)$. Suppose $\frac{d}{dx}f(x) = g(x)$. We know that $\widehat{g}(x) = jx\widehat{f}(x)$. We now show that this relation is valid even if f, g are tempered distributions. We first remind the reader that $\frac{d}{dx}\widehat{\phi}(x) = -jx\widehat{\phi}(x)$, when $\phi(x)$ is rapidly decaying and therefore Fourier transformable. We have $[\widehat{g}(x), \phi(x)] = \left[g(x), \widehat{\phi}(x)\right]$, i.e., $\left[\frac{d}{dx}f(x), \widehat{\phi}(x)\right] = \left[f(x), \frac{d}{dx}\widehat{\phi}(x)\right]$ $= -\left[f(x), -\widehat{jx\phi}(x)\right] = \left[\widehat{f}(x), jx\phi(x)\right] = \left[jx\widehat{f}(x), \phi(x)\right]$, as required.

Similarly, or by invoking duality, we can prove that $\frac{d}{dx}\widehat{f}(x) = \widehat{-jxf}(x)$, for any tempered distribution f.

4.11 *Convergent sequences of distributions:*

For distributions we say $\lim_{n\to\infty} q_n = q$, iff for each ϕ $\lim_{n\to\infty} [q_n,\phi] = [q,\phi]$.

Suppose a sequence $\{ q_n \}$ has the property that $\lim_{n\to\infty} [q_n,\phi]$ exists for each ϕ then we can define a functional q on the space of ϕ by

$$[q,\phi] \equiv \lim_{n\to\infty} [q_n,\phi].$$

The linearity of q is clear. Continuity involves showing that $\lim_{n\to\infty} [q,\phi_n] = 0$, whenever ϕ_n is a null sequence. We will skip this sophisticated proof. For practical purposes, the most important such convergent sequences are those that arise from locally integrable functions. For the specific cases of interest, we will give alternative proofs of the fact that the limit is a distribution.

Let f_n converge to f uniformly over every finite interval. We then have

$$\lim_{n\to\infty} [f_n,\phi] = [f,\phi]$$

for every test function. We prove this claim as follows:-

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n \phi dx = \lim_{n\to\infty} \int_a^b f_n \phi dx$$

(where $[a,b]$ contains the support of ϕ).

Since f_n converge uniformly to f over $[a,b]$, for each ϵ we can choose N_ϵ such that

$|f(x) - f_n(x)| \leq \epsilon$ for $x \in [a,b]$, whenever $n \geq N_\epsilon$.

Hence

$$\begin{aligned} |\int_a^b (f-f_n)\phi dx| \leq \int_a^b |f-f_n|\phi dx \\ \leq |(max_{a\leq x\leq b}\phi(x))| \int_a^b |f-f_n| dx \\ \leq |(max_{a\leq x\leq b}\phi(x))\epsilon(b-a)|, n \geq N_\epsilon. \end{aligned}$$

The claim follows.

5 Some special tempered distributions

5.1 *Unit step function*

A function that is very commonly encountered in studying the solution of differential equations is the unit step function

$$\begin{aligned} 1(x) &\equiv 0 \ , \ x < 0 \\ &\equiv 1 \ , \ x \geq 0. \end{aligned}$$

This function is locally integrable. Clearly, the action of $1(\cdot)$ on a rapidly decaying function ϕ is given by

$$[1(x), \phi] \equiv \int_{-\infty}^{\infty} 1(x)\phi(x)dx = \int_{0}^{\infty} \phi(x)dx.$$

We have already shown that this integral exists for a rapidly decaying function ϕ. Thus $1(\cdot)$ is a tempered distribution. In this distributional sense,

$$\begin{aligned} \left[\frac{d}{dx}1(x), \phi(x)\right] &= -[1(x), \dot{\phi}(x)] \\ &= -\int_{0}^{\infty} \dot{\phi}(x)dx \\ &= [\phi(x)]_{\infty}^{0} = \phi(0). \end{aligned}$$

Thus the distributional derivative of $1(x)$ is $\delta(x)$.

We will call a sequence $\{r_n\}$ of functions a 1-sequence if

(a) $r_n(\cdot)$ is locally integrable

(b) $\lim_{n\to\infty} \int_{-\infty}^{\infty} r_n(x)\phi(x)dx = \int_0^{\infty} \phi(x)dx$ for every rapidly decaying function $\phi(x)$.

Suppose the functions r_n are all differentiable over $(-\infty, \infty)$. We then have

$$\begin{aligned} \lim_{n\to\infty} \int_{-\infty}^{\infty} \dot{r_n}(x)\phi(x)dx &= -\int_{-\infty}^{\infty} r_n(x)\dot{\phi}(x)dx \\ &= \int_{0}^{\infty} \dot{\phi}(x)dx \\ &= \phi(0). \end{aligned}$$

It follows therefore that

$$\{\dot{r_n}(x)\} \text{ is a } \delta\text{-sequence.}$$

This gives another interpretation to the expression $\frac{d}{dx}1(x) = \delta(x)$.

5.2 *The tempered distribution* $\frac{1}{x}$

In signal processing applications, the tempered distribution $\frac{1}{x}$ plays an important role. It is related to the unit step function through the Fourier transform. Convolution of a rapidly decaying function by this distribution is called the Hilbert transform and is of use in studying some kinds of modulation of signals.

The tempered distribution $\frac{1}{x}$ is defined by, for $\phi \in \mathcal{S}_1$,

$$\left[\frac{1}{x}, \phi\right] \equiv \lim_{\epsilon \to 0, \epsilon > 0} \int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x} dx + \int_{\epsilon}^{\infty} \frac{\phi(x)}{x} dx.$$

Since ϕ is continuous at 0, the limit in the above expression exists. To see this, define $\psi(x) \equiv \phi(x) - \phi(-x)$. The above expression on the RHS reduces to

$$\lim_{\epsilon \to 0, \epsilon > 0} \int_{\epsilon}^{\infty} \frac{\psi(x)}{x} dx.$$

Observe that $\psi(0) = 0$ and further, since ϕ is rapidly decaying, we need only examine the convergence of the above integral for some positive b in place of ∞.

We therefore need to show that the limit exists in the following expression.

$$lim_{\epsilon \to 0, \epsilon > 0} \int_{\epsilon}^{b} \frac{\psi(x)}{x} dx.$$

In the interval $[\epsilon, b]$, we can bound $\psi(x)$ between $\psi(0) + k_1 x$ and $\psi(0) + k_2 x$ for some k_1, k_2. Noting that $\psi(0) = 0$, we see that the integral $\int_{\epsilon}^{b} \frac{\psi(x)}{x} dx$ lies between $k_1(b - \epsilon)$ and $k_2(b - \epsilon)$ and therefore, as ϵ tends to zero, the integral $\int_{\epsilon}^{b} \frac{\psi(x)}{x} dx$ converges.

Further, $\left[\frac{1}{x}, \phi\right]$ is linear in ϕ and by using the above argument, it can be seen that $\lim_{n \to \infty} \left[\frac{1}{x}, \psi_n\right] = 0$, whenever $\{\psi_n\}$ is a null sequence of rapidly decaying functions (in the limit $\psi(0), k_1$ and k_2 go to zero). Thus $\left[\frac{1}{x}, \phi\right]$ is continuous in ϕ. It follows that $\frac{1}{x}$ is a tempered distribution.

5.3 *Fourier transform of* $1(x)$ *and* $\frac{1}{x}$

In what follows we use 1 for the constant function on the real line which takes value 1 on all real numbers. By $1(\cdot)$ or $1(x)$, we mean the unit step function which takes value 0 for $x < 0$ and 1 for $x > 1$.

Let us first compute the Fourier transform of δ. We have $\left[\widehat{\delta}, \phi\right] = \left[\delta, \widehat{\phi}\right] = \widehat{\phi}(0) = [1, \phi]$ We therefore conclude $\widehat{\delta} = 1$.

We have $\widehat{\frac{d}{dx}1(x)} = \widehat{\delta} = 1$. But in Section4.10 we saw that $\widehat{\frac{d}{dx}q(x)} = jx\widehat{q}(x)$, for any tempered distribution q. So $jx\widehat{1}(x) = 1$. We will show that this means $\widehat{1}(x) = \frac{1}{jx} + c\delta(x)$ for some constant c and later show that the constant must be π.

First observe that the tempered distribution $\frac{1}{jx}$ satisfies $jxq(x) = 1$. We have

$$\begin{aligned}\left[\frac{1}{jx}, jx\phi\right] &= \lim_{\epsilon\to 0,\epsilon>0}\left[\int_{-\infty}^{-\epsilon}\frac{jx}{jx}\phi dx + \int_{\epsilon}^{\infty}\frac{jx}{jx}\phi dx\right] \\ &= \int_{-\infty}^{\infty}\phi(x)dx = \int_{-\infty}^{\infty}1\phi(x)dx.\end{aligned}$$

Consider the equation $jxq(x) = 0$. ($\widehat{1}(x) - \frac{1}{jx}$ satisfies this equation.) We have $[jxq(x), \phi(x)] = 0$ $i.e.$, $[q(x), jx\phi(x)] = 0$.

We first show that a test function $\psi(x)$ has the form $x\phi(x)$ iff $\psi(0) = 0$. Clearly $x\phi(x)$ at $x = 0$ has value 0. Suppose $\psi(x) = 0$. Consider $\frac{\psi(x)}{x}$. Define

$$\begin{aligned}\phi(x) &= \frac{\psi(x)}{x}, x \neq 0 \\ &= \dot{\psi}(0) = \lim_{x\to 0}\frac{\psi(x)}{x}, x = 0.\end{aligned}$$

It is clear that $\phi(x)$ is a test function and $\psi(x) = x\phi(x)$.

We will next show that $q = c\delta$ for some constant c.
We have $[xq(x), \phi(x)] = 0, i.e., [q(x), x\phi(x)] = 0$.
Write $\phi(x)$ as $\phi(0)\phi_0(x) + [\phi(x) - \phi(0)\phi_0(x)]$, where $\phi_0(x)$ is any test function with $\phi_0(x) = 1$.
Now, $[q, \phi] = [q(x), \phi(0)\phi_0(x)] + [q(x), (\phi(x) - \phi(0)\phi_0(x))]$.
But $\phi(x) - \phi(0)\phi_0(x) = 0$ at $x = 0$ since $\phi_0(0) = 1$.
Hence $\phi(x) - \phi(0)\phi_0(x)$ has the form $x\rho(x)$ for some test function $\rho(x)$.
Hence $[q(x), (\phi(x) - \phi(0)\phi_0(x))] = [q(x), x\rho(x)] = 0$.
Hence $[q, \phi] = [q(x), \phi(0)\phi_0(x)] = \phi(0)[q(x), \phi_0(x)]$.
So $[q, \phi] = \phi(0)c$, where $c = [q(x), \phi_0(x)]$. So $q = c\delta$.

It is thus clear that $\widehat{1}(x) = \frac{1}{jx} + c\delta(x)$.

We next evaluate the constant c. Consider

$$1(x) + 1(-x) = 1.$$

By duality we know that $\widehat{1} = 2\pi\delta$, i.e., $(1(x)\widehat{+1(}-x)) = 2\pi\delta$,
i.e., $\widehat{1}(x) + \widehat{1}(-x) = 2\pi\delta(x)$. But $\widehat{q(-x)} = \widehat{q}(-x)$. So $\widehat{1}(x) + \widehat{1}(-x) = \frac{1}{jx} + \frac{1}{-jx} + c\delta(x) + c\delta(-x) = 2c\delta(x)$. It follows that $c = \pi$. Thus $\widehat{1}(x) = \frac{1}{jx} + \pi\delta(x)$.

By duality, we must have $\widehat{\frac{1}{jx}} + \widehat{\pi\delta} = 2\pi 1(-x) = 2\pi - 2\pi 1(x)$. Since $\widehat{\pi\delta} = \pi$, it follows that $\widehat{\frac{1}{jx}} = \pi - 2\pi 1(x)$, i.e., $\widehat{\frac{1}{x}} = -j\pi sgn(x)$, where $sgn(x)$ is -1 for negative x and 1 for positive x.

6 Multiplication Rule for Distributional Derivative

As we saw before, under certain conditions, fq would be defined when f is a function and q is a distribution. For instance, if f has derivatives of all orders fq is always defined.

$$[fq, \phi] \equiv [q, f\phi].$$

In special cases, the harsh conditions on f can be relaxed. When q is say the δ-function, fq is defined if f is continuous at the origin. Let us examine if the usual multiplication rule $\frac{d}{dx}(fq) = q\frac{df}{dx} + f\frac{dq}{dx}$ works in the case of distributions when f is a function whose derivatives of all orders exist, and q, a distribution. We have

$$\begin{aligned}
\left[\frac{d}{dx}(fq), \phi\right] &\equiv -\left[fq, \frac{d\phi}{dx}\right] = -\left[q, f\frac{d\phi}{dx}\right] \\
&= -\left[q, \frac{d(f\phi)}{dx} - \phi\frac{df}{dx}\right] \\
&= -\left[q, \frac{d(f\phi)}{dx}\right] + \left[q, \phi\frac{df}{dx}\right] \\
&= \left[\frac{dq}{dx}, f\phi\right] + \left[q, \phi\frac{df}{dx}\right] \\
&= \left[f\frac{dq}{dx}, \phi\right] + \left[q\frac{df}{dx}, \phi\right]
\end{aligned}$$

Therefore it follows that

$$\frac{d(fq)}{dx} = f\frac{dq}{dx} + q\frac{df}{dx}$$

as is the case with functions.

Example:

Consider the function in Figure 2 below

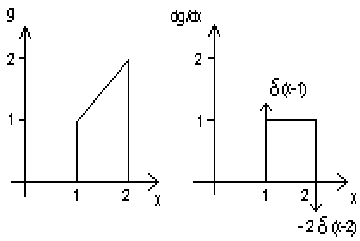

Figure 2:

$$
\begin{aligned}
g(x) &= x(1(x-1) - 1(x-2)) \\
\frac{dg}{dx} &= 1(1(x-1) - 1(x-2)) + x(\delta(x-1) - \delta(x-2)) \\
&= 1(x-1) - 1(x-2) + \delta(x-1) - 2\delta(x-2).
\end{aligned}
$$

$$
\begin{aligned}
\text{(Observe that } [f(x)\delta(x-a), \phi(x)] &= [\delta(x-a), f(x)\phi(x)] \\
= [\delta(x), f(x-a)\phi(x-a)] &= [f(x-a)\delta(x), \phi(x-a)] \\
= [f(a)\delta(x), \phi(x-a)] &= [f(a)\delta(x-a), \phi(x)].\text{)}
\end{aligned}
$$

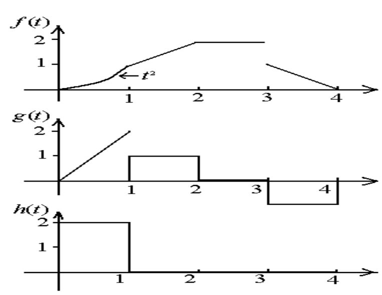

Figure 3:

Consider the function f in Figure 3. This function is piecewise continuous being made up of polynomials in the open intervals $(0, 1), (1, 2), (2, 3), (3, 4)$. The function has derivatives of all orders (of course the third derivative is the zero function). However, from the first or second derivative functions

$(g = \dot{f}, h = \ddot{f})$, we cannot recapture $f(t)$ back again if we work with ordinary functions and use the ordinary notion of derivatives. On the other hand, if $f(\cdot)$ is treated as a distribution and the derivatives are taken in the distributional sense, $f(\cdot)$ can be fully recovered. Let us denote the distributional derivative of f by $\frac{df}{dt}$.

$$\begin{aligned}
\frac{df}{dt} &= \frac{d}{dt}\left[t^2(1(t) - 1(t-1)) + t(1(t-1) - 1(t-2))\right] \\
&\quad + [2(1(t-2) - 1(t-3)) - (t-4)(1(t-3) - 1(t-4))] \\
&= 2t(1(t) - 1(t-1)) + 1(1(t-1) - 1(t-2)) \\
&\quad +0(1(t-2) - 1(t-3)) - 1(1(t-3) - 1(t-4)) \\
&\quad -\delta(t-1) + \delta(t-1) - 2\delta(t-2) + 2\delta(t-2) - 2\delta(t-3) + \delta(t-3) \\
&= g(t) - \delta(t-3) \\
\text{similarly, } \frac{d^2 f}{dt^2} &= \frac{dg}{dt} - \dot{\delta}(t-3) \\
&= h(t) - \delta(t-1) - \delta(t-2) - \delta(t-3) + \delta(t-4) - \dot{\delta}(t-3)
\end{aligned}$$

Observe that from the distributional derivative of any order of the function, we can recover the original function - the δs and the $\dot{\delta} s$ do the bookkeeping for us.

7 Convolution of Distributions

7.1 $q_1 * q_2$ *when* q_2 *has finite support*

In order to define the notion of convolution for distributions we first examine the concept in the case of absolutely integrable functions. Let f_1, f_2 be absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |f_i(x)|\, dx,\ i = 1, 2\ ,$$

be finite. We define the convolution $f_1 * f_2$ by

$$\begin{aligned}
f_1 * f_2(y) &\equiv \int_{-\infty}^{\infty} f_1(x) f_2(y-x)\, dx \\
&= \int_{-\infty}^{\infty} f_1(y-x) f_2(x)\, dx \\
&= f_2 * f_1(y)
\end{aligned}$$

We see that $\int_{-\infty}^{\infty} |f_1 * f_2(y)|\, dy$ exists, since

$$\begin{aligned}
\int_{-\infty}^{\infty} |f_1 * f_2(y)|\, dy &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f_1(x) f_2(y-x)\, dx \right| dy \\
&\leq \int_{-\infty}^{\infty} |f_1(x)| \left[\int_{-\infty}^{\infty} |f_2(y-x)|\, dy \right] dx \\
&\leq \left[\int_{-\infty}^{\infty} |f_1(x)|\, dx \right] \left[\int_{-\infty}^{\infty} |f_2(z)|\, dz \right], \text{ taking } z = (y-x).
\end{aligned}$$

Let us examine the distribution $q_{f_1 * f_2}$ defined by

$$[q_{f_1 * f_2}, \phi] \equiv \int_{-\infty}^{\infty} f_1 * f_2(y) \phi(y)\, dy.$$

We can write the above integral as

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(x) f_2(y-x)\, dy \right] \phi(y)\, dy = \int_{-\infty}^{\infty} f_1(x) \left[\int_{-\infty}^{\infty} f_2(y-x) \phi(y)\, dy \right] dx.$$

This has the form

$$\int_{-\infty}^{\infty} f_1(x) \psi(x)\, dx,$$

$$\begin{aligned}
\text{where } \psi(x) &\equiv \int_{-\infty}^{\infty} f_2(y-x) \phi(y)\, dy \\
&= \int_{-\infty}^{\infty} f_2(z) \phi(x+z)\, dz \text{ (taking } z = y-x).
\end{aligned}$$

In order to make this appear as the action of a distribution on a test function we would require $\psi(x)$ to be a test function. It is clear that it is infinitely differentiable since $\frac{d\psi}{dx} = \int_{-\infty}^{\infty} f_2(z) \frac{d(\phi(x+z))}{dx}\, dz$ and $\phi(x+z)$ is infinitely differentiable. But if f_2 doesn't have finite support then $\psi(x)$ can not have finite support. The above discussion brings out the difficulties in defining convolution for distributions and suggests we attempt to $q_1 * q_2$ only when q_2 has finite support or when q_1, q_2 have some other special properties.

When q_1, q_2 are distributions and q_2 has finite support $[a, b]$ (ie $[q_2, \phi] = 0$ whenever support of ϕ doesn't intersect $[a, b]$), we define $q_1 * q_2$ as follows.

$$[q_1 * q_2(x), \phi(x)] = [q_1(x), [q_2(z), \phi(x+z)]]\,.$$

For this notion to be well defined we need to verify that $\psi(x) = [q_2(z), \phi(x+$

$z)]$ is a test function. First we observe that

$$\begin{aligned}
\frac{d\psi}{dx} &= \frac{d[q_2(z), \phi(x+z)]}{dx} \\
&= \lim_{\triangle x \to 0} \frac{[[q_2(z), \phi(x+z+\triangle x)] - [q_2(z), \phi(x+z)]]}{\triangle x} \\
&= \lim_{\triangle x \to 0} \left[q_2(z), \frac{\phi(x+z+\triangle x) - \phi(x+z)}{\triangle x} \right] \\
&= \left[q_2(z), \lim_{\triangle x \to 0} \left[\frac{\phi(x+z+\triangle x) - \phi(x+z)}{\triangle x} \right] \right] \\
& \quad \text{(by the continuity of } q_2) \\
&= \left[q_2(z), \frac{d\phi(x+z)}{dx} \right]
\end{aligned}$$

Since ϕ is infinitely differentiable it will follow that ψ will also be. Next let q_2 have support $[-T_1, T_2]$ and let ϕ have support $[-T_3, T_4]$; T_1, T_2, T_3, T_4 being positive. Suppose $x \notin [-T_1 - T_4, T_2 + T_3]$. We have $\psi(x) = [q_2(z), \phi(x+z)]$.

Now $\phi(x+z)$ has support (in terms of variable z) $[-T_3 + x, T_4 + x]$. We must have $[-T_3 + x, T_4 + x] \cap [-T_1, T_2] \neq \emptyset$, in order that $\psi(x+z)$ is nonzero. If

$$x > T_2 + T_3$$

or if

$$x < -T_1 - T_4$$

the above intersection is null, i.e., the support of $\phi(x+z)$ and $q_2(z)$ do not intersect so that $[q_2(z), \phi(x+z)] = 0$. Thus $\psi(x)$ has finite support and is infinitely differentiable and is therefore a test function. Hence,

$$[q_1(x), [q_2(z), \phi(x+z)]]$$

is well defined.
Linearity of $q_1 * q_2$ is clear, since,

$$\begin{aligned}
[q_1 * q_2(x), (\alpha\phi_1 + \beta\phi_2)(x)] &= [q_1(x), [q_2(z), (\alpha\phi_1 + \beta\phi_2)(x+z)]] \\
&= [q_1(x), \alpha[q_2(z), \phi_1(x+z)] + \beta[q_2(z), \phi_2(x+z)]] \\
&= \alpha[q_1 * q_2(x), \phi_1(x)] + \beta[q_1 * q_2(x), \phi_2(x)].
\end{aligned}$$

Continuity of $q_1 * q_2$ can be shown as follows. Let ϕ_n be a null sequence of test functions.

$$\lim_{n \to \infty} [q_1 * q_2(x), \phi_n(x)] = \lim_{n \to \infty} [q_1(x), [q_2(z), \phi_n(x+z)].$$

Since q_2 is continuous and has finite support, it is clear that $\psi_n(x) \equiv [q_2(z), \phi_n(x+z)]$ is a null sequence of test functions so that the limit is zero as required for continuity.

We next examine the derivative of the convolution of distributions. We will show that

$$\frac{d(q_1 * q_2)}{dx} = q_1 * \frac{dq_2}{dx}.$$

We have

$$\begin{aligned}
\left[\frac{d(q_1 * q_2)}{dx}, \phi(x)\right] &= -[q_1 * q_2, \dot{\phi}(x)] \\
& \quad \text{to avoid notational confusion we denote } \dot{\phi}(x) \text{ by } \psi(x). \\
&= -[q_1(x), [q_2(z), \psi(x+z)]] \\
&= -\left[q_1(x), -\left[\frac{dq_2}{dz}, \phi(x+z)\right]\right] \\
&= \left[q_1(x), \left[\frac{dq_2}{dz}, \phi(x+z)\right]\right]
\end{aligned}$$

Let us denote the distribution $\frac{dq_2}{dz}$ by $q_3(z)$. It is defined by

$$[q_3(z), \phi(z)] = -[q_2(z), \dot{\phi}(z)].$$

Thus,

$$\begin{aligned}
\left[\frac{d(q_1 * q_2)}{dx}, \phi(x)\right] &= [q_1(x), [q_3(z), \phi(x+z)]] \\
&= [q_1 * q_3(x), \phi(x)].
\end{aligned}$$

Hence, $\frac{d(q_1 * q_2)}{dx} = q_1 * \frac{dq_2}{dx}$.

The way convolution has been defined for distributions does not make the operation naturally commutative. In special cases, however, this would be true. For instance, if q_1 is δ or its derivative of some order and q_2 is of finite support or regular, it can verified that the operation is indeed commutative.

So, $\phi * \delta = \delta * \phi = \phi$.

7.2 *Convolution of a distribution with $\phi \in S_1$.*

The special case of a distribution with a rapidly decaying function is of importance in signal processing- the Hilbert transform, for instance, is of this kind.

We will show that $q * \phi$ is a tempered distribution when $\phi \in S_1$ and q is tempered. We need the following preliminary lemma.

Lemma: Let $\phi, \psi \in S_1$. Then $\phi * \psi \in S_1$.

Proof: We have $\widehat{\phi}, \widehat{\psi} \in S_1$ and therefore by the definition of rapidly decaying functions $\widehat{\phi}.\widehat{\psi} \in S_1$. But we know that $\widehat{\phi * \psi} = \widehat{\phi}.\widehat{\psi}$ and $\phi * \psi$ is continuous in $(-\infty, \infty)$.

By duality we know that,

$$(\widehat{\widehat{\phi}.\widehat{\psi}})(x) = 2\pi\phi * \psi(-x)$$

Thus $\phi * \psi(-x) \in S_1$ and therefore $\phi * \psi(x) \in S_1$.
QED

Theorem: If q is a tempered distribution and $\phi \in S_1$, $q * \phi$ is a tempered distribution.

Proof: We have

$$[q * \phi, \psi] \equiv [q(x), [\phi(z), \psi(x + z)]]$$

when $\psi \in S_1$. We have,

$$\begin{aligned}
[\phi(z), \psi(x + z)] &= [\phi(y - x), \psi(y)] \\
&= [\tilde{\phi}(x - y), \psi(y)] \\
&= \psi * \tilde{\phi}(x), \text{ denoting } \phi(-t) \text{ by } \tilde{\phi}(t)
\end{aligned}$$

Since $\psi, \tilde{\phi} \in S_1$, $\psi * \tilde{\phi}(x) \in S_1$. Hence, $[q(x), \psi * \tilde{\phi}(x)]$ and therefore $[q(x), [\phi(z), \psi(x + z)]]$ is well defined. Hence, $[q * \phi, \psi]$ is well defined for all $\psi \in S_1$. Linearity of $q * \phi$ over S_1 is clear. We need to verify continuity, i.e., that,

$$\lim_{n\to\infty} [q * \phi, \psi_n] = 0$$

whenever ψ_n is a null sequence in S_1.

Clearly this would follow if $\{\psi_n * \tilde{\phi}\}$ is a null sequence in S_1, whenever $\phi \in S_1$ and $\{\psi_n\}$ is a null sequence in S_1. We need to verify that,

$$\lim_{n\to\infty} \max_{-\infty<x<\infty} \left| x^k \frac{d^l(\psi_n * \tilde{\phi}(x))}{dx^l} \right| = 0$$

for every pair of integers k, l.

Now,

$$x^k \frac{d^l(\psi_n * \tilde{\phi})(x)}{dx^l} = \psi_n * \frac{d^l\tilde{\phi}(x)}{dx^l}$$

Since $\frac{d^l\tilde{\phi}(x)}{dx^l} \in S_1$ whenever $\phi \in S_1$, it is adequate to verify that,

$$\lim_{n\to\infty} \max_{-\infty<x<\infty} \left| x^k(\psi_n * \phi(x)) \right| = 0$$

for every $\phi \in S_1$.

We have,

$$\begin{aligned}\left|x^k(\psi_n * \phi(x))\right| &= \left|\int_{-\infty}^{\infty} |x^k \psi_n(x-\tau) * \phi(\tau)\, d\tau\right| \\ &\le M_n \int_{-\infty}^{\infty} \phi(\tau)\, d\tau, \\ &\quad \text{where } M_n = \max_{-\infty<\tau<\infty} \left|x^k \psi_n(x-\tau)\right|\end{aligned}$$

Since $\lim_{n\to\infty} M_n = 0$, we have

$$\lim_{n\to\infty} \max_{-\infty<x<\infty} \left|x^k(\psi_n * \phi(x))\right| = 0.$$

Thus, $\psi_n * \tilde{\phi}(x)$ is a null sequence whenever $\psi_n \in S_1$ and $\phi \in S_1$ and the continuity of $q * \phi$ on S_1 follows. Thus $q * \phi$ is a tempered distribution as required.
QED

7.3 *Convolution of a distribution with δ and its derivatives*

Let us first examine the convolution of δ with $\phi \in \mathcal{S}_1$. We have
$[\delta * \phi(x), \psi(x)] = [\delta(x), [\phi(z), \psi(x+z)] = [\phi(z), \psi(z)]$. So, $\delta * \phi = \phi$.
On the other hand, $[\phi * \delta(x), \psi(x)] = [\phi(x), [\delta(z), \psi(x+z)]] = [\phi(x), \psi(x)]$.
So, $\phi * \delta = \delta * \phi = \phi$.

Next, $\frac{d\phi}{dx} = \frac{d(\phi*\delta)}{dx} = \phi * \frac{d\delta}{dx}$.
On the other hand, consider the distribution $\frac{d\delta}{dx} * \phi$. We have,
$[\frac{d\delta}{dx} * \phi, \psi] = [\frac{d\delta}{dx}, [\phi(z), \psi(x+z)]] = [\frac{d\delta}{dx}, \tilde{\phi} * \psi(x)]$, where $\tilde{\phi}(x) \equiv \phi(-x)$. Now
$[\frac{d\delta}{dx}, \tilde{\phi} * \psi(x)] = -[\delta, \frac{d(\tilde{\phi}*\psi(x))}{dx}] = [\delta, \frac{d\tilde{\phi}}{dx} * \psi] = [\delta * \frac{d\phi}{dx}, \psi]$.
Thus, $\frac{d\delta}{dx} * \phi = \phi * \frac{d\delta}{dx} = \frac{d\phi}{dx}$.

By induction it will follow that the convolution of the k^{th} derivative of δ with $\phi \in \mathcal{S}_1$ will yield the k^{th} derivative of ϕ and the order of convolution is immaterial.

The convolution of a distribution q with δ yields the same distribution.
We have, $[q * q_2, \phi] \equiv [q(x), [q_2(x), \phi(x+z)]]$.
So, $[q * \delta, \phi] = [q(x), [\delta(z), \phi(x+z)]] = [q(x), \phi(x)]$.
Next we have $\frac{d(q_1*q_2)}{dx} = q_1 * \frac{dq_2}{dx}$, when q_2 has finite support. Hence if q_2 is the k^{th} derivative of δ, $q * q_2$ would be the k^{th} derivative of $q * \delta = q$.

7.4 *Fourier transform of convolution of distributions*

We define Fourier transform of convolution of distributions only in the special case where one of them is tempered and other a rapidly decaying function. Let q be tempered and let $\phi \in \mathcal{S}_1$. We have seen that $q * \phi$ is tempered and therefore has a Fourier transform. We verify below that $\widehat{q * \phi} = \widehat{q}.\widehat{\phi}$. (We have denoted $\phi(-x)$ by $\tilde{\phi}$.)

$$\begin{aligned}
[(\widehat{q*\phi}), \psi] &\equiv \left[q * \phi, \widehat{\psi}\right] \\
&= \left[q(x), \left[\phi(z), \widehat{\psi}(z+x)\right]\right] \\
&= \left[q(x), \left(\tilde{\phi} * \widehat{\psi}\right)(x)\right] \\
&= \left[q(x), 2\pi \left(\widehat{\frac{1}{2\pi}\widehat{\phi}.\psi}\right)(x)\right] \\
&= \left[\widehat{q}(x), \left(\widehat{\phi}.\psi\right)(x)\right] \\
&= \left[\widehat{q}(x), \widehat{\phi}(x).\psi(x)\right] \\
&= \left[\widehat{q}(x).\widehat{\phi}(x), \psi(x)\right].
\end{aligned}$$

By using duality, or directly, we can verify, when q is tempered and ϕ is rapidly decaying, that

$$\widehat{q.\phi} = \frac{1}{2\pi}\widehat{q} * \widehat{\phi}.$$

We remind the reader that the tempered distribution $\frac{1}{x}$ is defined by,for $\phi \in \mathcal{S}_1$,

$$\left[\frac{1}{x}, \phi\right] \equiv \lim_{\epsilon \to 0, \epsilon > 0} \int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x} dx + \int_{\epsilon}^{\infty} \frac{\phi(x)}{x} dx.$$

We have shown earlier that $\widehat{\frac{1}{x}} = -j\pi sgn(x)$.

The distribution $\frac{1}{x} * \phi$ where $\phi \in \mathcal{S}_1$ is tempered and has the Fourier transform $-j\pi sgn(x)\widehat{\phi}(x)$.

In signal processing literature $\frac{1}{\pi t} * \phi(t)$ is called the Hilbert transform of $\phi(t)$ and has important applications in the study of modulation. We give an illustration below.

7.5 *An application of the Hilbert transform*

In the discussion to follow, we follow the convention of signal processing literature and take signals to be lower case functions of 't' and their Fourier transform to be the corresponding upper case functions of '$j\omega'$. Thus $\widehat{a}(t)$ is denoted $A(j\omega)$.

Consider an amplitude modulated wave $a(t)\cos(\omega_0 t) = \alpha(t)$. We assume that $A(j\omega)$ is zero for $|\omega| \geq \omega_0$ and is infinitely differentiable. As a consequence, we have that $a(t)$ and $\alpha(t)$ are rapidly decaying. Given $\alpha(t)$, the problem is to recover the signal $a(t)$. For simplicity let us take $a(t)$ to be real. We have

$$\widehat{\alpha}(j\omega) = A(j\omega) * \left[\frac{\delta(j\omega - j\omega_0) + \delta(j\omega + j\omega_0)}{2}\right] = \frac{1}{2}(A(j\omega - j\omega_0) + A(j\omega + j\omega_0)).$$

Hence,

$$A(j\omega - j\omega_0) = 2\widehat{\alpha}(j\omega).1(j\omega)$$

and therefore

$$A(j\omega) = \left[(2\widehat{\alpha}(j\omega).1(j\omega)) * \delta(j\omega + j\omega_0)\right].$$

We then have

$$\begin{aligned} a(t) = \mathcal{F}^{-1}\left[(2\widehat{\alpha}(j\omega).1(j\omega)) * \delta(j\omega + j\omega_0)\right] &= \left[(\alpha(t) * \left(\frac{j}{\pi t} + \delta(t)\right)\right] e^{-j\omega_0 t.} \\ &= \left[\alpha(t) + j\alpha^H(t)\right] e^{-j\omega_0 t}, \end{aligned}$$

where

$$\alpha^H(t) = \frac{1}{\pi t} * \alpha(t).$$

Since we assumed $a(t)$ to be real, we must have

$$a(t) = \alpha(t)\cos(\omega_0 t) + \alpha^H(t)\sin(\omega_0 t).$$

$\alpha^H(t)$ is the Hilbert transform of $\alpha(t)$.

As we noted before $\alpha(t)$ is a rapidly decaying function. $\frac{1}{\pi t}$ is a tempered distribution. $\frac{1}{\pi t} * \alpha(t)$ is therefore a tempered distribution. In the present case it turns out to be a regular distribution.

$$\frac{1}{\pi t} * \alpha(t) = \int_{-\infty}^{\infty} \frac{1}{\pi\tau}\alpha(t-\tau)d\tau.$$

$$\lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon} \frac{1}{\pi t}\alpha(t-\tau)d\tau + \int_{\epsilon}^{\infty} \frac{1}{\pi t}\alpha(t-\tau)d\tau\right].$$

8 Summary of properties of Fourier Transforms for tempered distributions

For notational convenience, we use the convention of signal processing literature: a tempered distribution is written as $q(t)$ and its Fourier Transform is written as $\widehat{q}(j\omega)$.

1. Linearity
$$\widehat{\alpha q_1 + \beta q_2} = \alpha \widehat{q_1} + \beta \widehat{q_2}.$$

2. Time shifting:
$$\widehat{q(t - t_o)} = e^{-j\omega t_o} \widehat{q}(j\omega).$$

3. Frequency shifting:
$$\widehat{e^{j\omega_0 t} q(t)} = q(\widehat{j\omega - j}\omega_0).$$

4. Time scaling:
$$\widehat{q(at)} = \frac{1}{|a|} \widehat{q(\frac{j\omega}{|a|})}.$$

5. Time differentiation:
$$\widehat{\frac{dq}{dt}} = j\omega \widehat{q}(j\omega).$$

6. Frequency Differentiation:
$$\widehat{tq(t)} = j \frac{d\widehat{q}}{d\omega}.$$

7. Convolution:
$$\widehat{q * f(t)} = \widehat{q}.\widehat{f},$$
where q is tempered and f rapidly decaying.

8. Multiplication:
$$\widehat{q.f(t)} = \frac{1}{2\pi} \widehat{q} * \widehat{f}(j\omega),$$
where q is tempered and f rapidly decaying.

9. Duality:

$$\widehat{\widehat{q}} = 2\pi\tilde{q},$$

where $q(\tilde{t}) \equiv q(-t)$.

The proofs are routine with the starting point $[\widehat{q}, \phi] \equiv [q, \widehat{\phi}]$ and proceed by using the corresponding property for Fourier transformable functions.

9 Periodic Distributions

In the discussion to follow, we follow the convention of signal processing literature and take signals to be lower case functions of 't' and their Fourier transform to be the corresponding upper case functions of '$j\omega'$. Thus $\widehat{a}(t)$ is denoted $A(j\omega)$.

A distribution q is said to be *periodic* with period T iff

$$[q, \phi] = [q, \phi_T].$$

In signal processing applications it is quite common to encounter the situation described in Figure 4 over one period.

We wish to show that we can obtain the *Fourier series expansion* (a distributional equation that has the same form as the usual Fourier series expansion) of this *generalized function* by the usual process valid for the functions satisfying Dirichlet conditions:

$$q(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 nt}$$

where

$$c_n = \frac{\int_0^T q(t)e^{-j\omega_0 nt}\, dt}{< e^{j\omega_0 nt}, e^{j\omega_0 nt} >}.$$

The integral on the numerator is to be interpreted appropriately for the singular distributions which are the constituent parts of $q(t)$. The technique that we describe is valid for periodic distributions whose 'average value' over one period is zero. Essentially this means that $q(t)$ must be composed of regular distributions, δ functions and derivatives. In this case the action $[q, 1]$ would be defined over one period and this can be subtracted out before we seek an expansion.

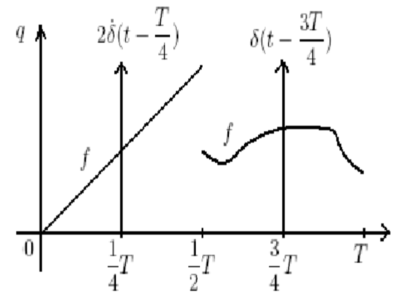

Figure 4:

While we usually treat the above construct as a function it is clearly a distribution q which is the sum of f, $\delta(t-\frac{3T}{4})$ and $2\dot{\delta}(t-\frac{T}{4})$. This may be regarded as periodic with period T in which case we would be working with

$$f_p + \sum_{n=-\infty}^{\infty} \delta(t - \frac{3T}{4} - nT) + 2\sum_{n=-\infty}^{\infty} \dot{\delta}\ (t - \frac{T}{4} - nT).$$

f_p being a periodic function agreeing with f over the period $[0, T]$.

Its action on a rapidly decaying function ϕ is given by

$$[q, \phi] = \int_{-\infty}^{\infty} f(t)\phi(t)dt + \sum_{n=-\infty}^{\infty} \phi(\frac{3T}{4} + nT) - 2\sum_{n=-\infty}^{\infty} \dot{\phi}\ (\frac{T}{4} + nT).$$

Let us examine whether we can obtain a Fourier series expansion of this periodic distribution. By such an expansion we mean that we should be able to write the following distributional equation

$$q = \sum_{n=-\infty_{n\neq 0}}^{\infty} c_n e^{j\omega_0 nt} + c_0, \quad \omega_0 = \frac{2\pi}{T}.$$

We will now assume that f in $[0, T]$ is made up of polynomials over some subintervals say $[0, T_1], [T_1, T_2]...[T_{k-1}, T]$.

Clearly there exist functions $f_2(t)$, $g_\delta(t)$, $g_{\delta 1}(t)$ such that

(a.) $f_2(t)$ is made up of polynomials over $[0, T_1], \ldots, [T_{k-1}, T]$,

(b.) $\frac{d^2 f_2(t)}{dt^2} = f(t)$ in $(0, T_1), \ldots, (T_{k-1}, T)$,

(c.) $\frac{d^2 g_\delta(t)}{dt^2} = \delta(t - \frac{3T}{4})$,

(d.) $\frac{d^2 g_{\delta 1}(t)}{dt^2} = +2\delta(t - \frac{T}{4})$.

The Fourier series expansion of $m(t) = f_2(t) + g_\delta(t) + g_{\delta 1}(t)$ over $[0, T]$ is say $m(t) = \sum_{n=-\infty}^{\infty} a_n e^{j\omega_0 nt}$. The second derivative of the periodic function $m_p(t)$ would differ from $q(t)$ by a constant.

Let $p(t)$ be such that $\frac{d^k p(t)}{dt^k} = m(t)$. then the Fourier series expansion of $p(t)$ would be

$$p(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a_n}{(j\omega_0 n)^k} e^{j\omega_0 nt} + b_0. \qquad (**)$$

Since $m(t)$ is made up of polynomials, for k sufficiently large (actually 2) we can take the series to be absolutely convergent. It would follow that in the equation (**) the series on the right converges to $p(t)$ uniformly. Hence (**) is an equation valid distributionally. Differentiating term by term, we get,

$$\frac{d^{k+2}p(t)}{dt^{k+2}} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a_n (j\omega_0 n)^{k+2}}{(j\omega_0 n)^k} e^{j\omega_0 nt} \qquad (*)$$

to be a valid distributional equation. But the left side would be the distribution q (within a constant) that we began with. Thus ** is a distributionally valid Fourier expansion for q minus a constant.

We will now show that the coefficients could have directly been obtained as

$$c_n = \frac{< q(t), e^{j\omega_0 nt} >}{T} = \frac{\int_0^T q(t) e^{-j\omega_0 nt} dt}{T}$$

as though $q(t)$ is an *ordinary* function satisfying Dirichlet conditions. In the present case this term would be

$$\int_0^T f(t) e^{-j\omega_0 nt} dt + \int_0^T \delta(t - \frac{3T}{4}) e^{-j\omega_0 nt} dt + \int_0^T 2\ \dot{\delta}\ (t - \frac{T}{4}) e^{-j\omega_0 nt} dt$$

where we interpret the second and third terms above as

$$\int_{-\infty}^{\infty} \delta(t - \frac{3T}{4}) e^{-j\omega_0 nt} dt + 2\int_{-\infty}^{\infty} \dot{\delta}\ (t - \frac{T}{4}) e^{-j\omega_0 nt} dt = e^{-\frac{3}{4}(j\omega_0 nT)} + 2(j\omega_0 n) e^{-\frac{1}{4}(j\omega_0 nT)}$$

Consider the Fourier series expansion for $m(t)$

$$m(t) = \sum_{n=-\infty}^{\infty} a_n e^{j\omega_0 nt}.$$

Hence

$$a_n = \frac{< m(t), e^{j\omega_0 nt} >}{< e^{j\omega_0 nt}, e^{j\omega_0 nt} >} = \frac{\int m(t)e^{-j\omega_0 nt}dt}{T}$$

We saw that this was also valid distributionally. The Fourier series expansion $q(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 nt}$ could be obtained by taking $c_n = (j\omega_0 n)^2 a_n$.

But this is exactly the same as

$$(-1)^2 \int_0^T \frac{d^2}{dt^2}(e^{-j\omega_0 nt})dt.$$

If $m(t)$ had a second derivative in the *ordinary function* sense by integrating by parts we would get (using $m(0) = m(T)$, ????) the above to be equal to $\int_0^T \frac{d^2 m(t)}{dt^2} e^{-j\omega_0 nt}dt$.

When $\frac{d^2 m(t)}{dt^2} = q$, the meaning of $\int_0^T q(t)e^{-j\omega_0 nt}dt$ would be the same as $(-1)^2 \int_0^T \frac{d^2}{dt^2}(e^{-j\omega_0 nt})dt$

The above discussion may be summarized as:

If a periodic distribution q is composed of polynomials, delta functions and its derivatives, the Fourier series coefficient can be obtained directly as

$$c_n = \frac{[q, e^{-j\omega_0 nt}]}{T}$$

just as though $q(t)$ is a regular function and the integral is over $[0, T]$.

Let us consider an important special case of periodic distributions.

$$q(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

In the interval $(0_-, T_-)$ we have the single delta function $\delta(t)$. The Fourier series expansion is therefore $q(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 nt}$ where

$$\begin{aligned} c_n &= \frac{[q(t), e^{-j\omega_0 nt}]}{< e^{j\omega_0 nt}, e^{j\omega_0 nt} >} \\ &= \frac{\int_{0_-}^{T_-} \delta(t)e^{-j\omega_0 nt}dt}{T} \\ &= \frac{1}{T} \end{aligned}$$

Thus $q(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j\omega_0 nt}$.

Distributionally

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j\omega_0 nt}$$

Consider the infinite series of distributions on the right. We know that if

$$\sum_{i=-\infty}^{\infty} q_i = q \text{ then } \hat{q} = \sum_{i=-\infty}^{\infty} \hat{q}_i.$$

The Fourier transform of $e^{j\omega_0 nt}$ is $2\pi\delta(\omega-\omega_0)$ (by duality). Hence

$$\hat{q} = \frac{2\pi}{T}\sum_{i=-\infty}^{\infty} \delta(\omega-\omega_0).$$

Thus the Fourier transform of the train of impulses $\sum_{n=-\infty}^{\infty}\delta(t-nT)$ is the train of impulses $\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-\omega_0)$

10 Infinite series of tempered distributions

Our primary interest in infinite series of distributions is in dealing with Fourier transforms of such series. Therefore we confine ourselves only to tempered distributions.

We say $\sum_{i=1}^{\infty} q_i = q$ iff the sequence of partial sums $s_n = \sum_{i=1}^{n} q_i$ converges to q,i.e.,

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n} q_i, \phi\right] = [q,\phi], \phi \in \mathcal{S}_1.$$

We say $\sum_{i=-\infty}^{\infty} q_i = q$ when the same thing happens to the partial sum $s_n = \sum_{i=-n}^{n} q_i$

If a sequence $\sum_{i=-n}^{n} f_i$ of locally integrable functions converges to another such function f uniformly within $(-\infty,\infty)$, we know that

$$\lim_{n\to+\infty}\left[\sum_{i=-n}^{n} f_i, \phi\right] = [f,\phi]\,, \phi \in \mathcal{S}_1.$$

We thus have the distributional equation

$$\sum_{i=-\infty}^{\infty} f_i = f.$$

Distributional equations involving infinite series have a very convenient property (which is not shared in general by uniformly convergent series of functions) viz. the equations remain valid even if we differentiate both sides term by term.

The proof is easy to see

$$\begin{aligned}
\text{Suppose} \sum_{i=-\infty}^{\infty} q_i &= q \\
\text{Then} \left[\sum_{i=-\infty}^{\infty} \frac{dq_i}{dx}, \phi\right] &= -\left[\sum_{i=-\infty}^{\infty} q_i, \dot{\phi}\right] \\
-\left[q, \dot{\phi}\right] &= \left[\frac{dq}{dx}, \phi\right]
\end{aligned}$$

This fact is of great importance in signal processing. Suppose the periodic function f is expanded into Fourier series as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jw_0 nt}$$

When f satisfies Dirichlet conditions we expect pointwise convergence at t provided f is continuous at t . Sometimes however, the series on the right converges uniformly to f in $(-\infty, \infty)$. More often the following situation occurs.

We consider the sequence,

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{c_n}{(j\omega_0 n)^k} e^{j\omega_0 nt}, n \neq 0.$$

For k sufficiently large, the series $\sum_{n=-\infty}^{\infty} \frac{|c_n|}{(\omega_0 n)^k} e^{j\omega_0 nt}, n \neq 0$, would often be convergent (i.e. $\sum_{n=-\infty}^{\infty} \frac{c_n}{(j\omega_0 n)^k}, n \neq 0$, is absolutely convergent).

In such a case, $\sum_{n=-\infty}^{\infty} \frac{c_n}{(j\omega_0 n)^k} e^{j\omega_0 nt}, n \neq 0$, converges to $g(t)$ uniformly in the interval $(-\infty, \infty)$. We then have the distributional equation

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{c_n}{(j\omega_0 n)^k} e^{j\omega_0 nt}, n \neq 0.$$

We know that the distribution $\frac{d^k g}{dt^k}$ is then obtained by differentiating the right side term by term k times.

We therefore have the distributional equation

$$f_1(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 nt}, n \neq 0,$$

where $f_1 = \frac{d^k g}{dt^k}$, even though regarded as functions the RHS does not converge uniformly over $(-\infty, \infty)$ to $f_1(t)$. On the other hand, the series

$$\sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 nt}, n \neq 0,$$

converges pointwise (at points of continuity) in the interval $(-\infty, \infty)$ to $f(t) - c_0$. The function $f(t) - c_0$ and the distribution $f_1(t)$ are therefore equal distributionally. Therefore $f(t)$ has the Fourier series expansion

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 nt},$$

that is also distributionally valid. Once this happens we can take derivatives on either side any number of times and get valid distributional equations that will have the form of a Fourier series expansion. This situation occurs for instance when $0 < T_1 < ..T_m = T$ and f equals some polynomial in $[T_i, T_j]$ and the function is periodic with period T.

These polynomials could be different in different subintervals and f could be discontinuous at the T_i.

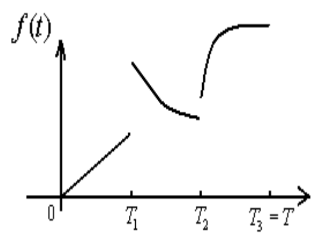

Figure 5:

In particular, consider the function $f(t) = t, t \in [0, T]$. Let f_p be the periodic function which agrees with f over $[0, T]$ (see Fig 6).

We can show that

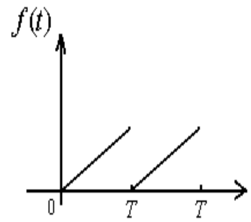

Figure 6:

$$f_p(t) = \sum_{n=-\infty}^{\infty} \frac{1}{-j\omega_0 n} e^{j\omega_0 nt} + \frac{1}{2}.(***)$$

We have,

$$g_p(t) = \sum_{n=-\infty}^{\infty} \frac{c_n}{(j\omega_0 n)^k} e^{j\omega_0 nt} \ , -\infty < t < \infty$$

with the right side converging uniformly to the left side over $(-\infty, \infty)$. We can obtain equation $(***)$ from the latter equation by differentiating term by term k times and adding the term c_0. Therefore equation $(***)$ is valid distributionally.

Therefore, we have the distributional equation,

$$\frac{df_p}{dt} = 1 - \sum_{n=-\infty}^{\infty} \delta(t - nT) = -\sum_{n=-\infty}^{\infty} e^{j\omega_0 nt}, n \neq 0.$$

Thus

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) = 1 + \sum_{n=-\infty}^{\infty} e^{j\omega_0 nt}, n \neq 0 = \sum_{n=-\infty}^{\infty} e^{j\omega_0 nt}.$$