THEORY OF MATROIDS AND NETWORK ANALYSIS

THESIS

Submitted in partial fulfilment of the requirements

for the Degree of

DOCTOR OF PHILOSOPHY

by

H. NARAYANAN

Department of Electrical Engineering

INDIAN INSTITUTE OF TECHNOLOGY

BOMBAY

1974

THESIS APPROVAL SHEET

Thesis entitled 'THEORY OF MATROIDS AND NETWORK ANALYSIS' by H. NARAYANAN is approved for the degree of DOCTOR OF PHILOSOPHY.

Guide	Chairman	Examiners
M. Vartak	P. P. Kane	A. Ramachandra Rao
N. S. Bedford (co-guide)		H. B. Kekre

A B S T R A C T

This thesis is concerned with certain fundamental problems in Network Analysis and the problems together with their extensions that these have given rise to in Matroid Theory.

In Chapter 0 we give an introduction to this thesis and outline, mainly without proof, certain preliminary results required in the remainder of the thesis. Chapter 1 contains a fairly rigorous development of Diakoptics for electrical networks. This treatment is carried out conveniently in terms of 'generalized networks' - 'generalized networks' being an abstraction of ordinary networks using matroids instead of graphs. Section 1 of Chapter 1 is concerned with carrying over certain well known results for ordinary networks to 'generalized networks'. In Section 2 we give a description of Kron's methods as well as an extension which should form a natural complement to his theory.

Chapter 1 also serves to highlight a fundamental problem in network theory - that of the determination of the topological degree of freedom of a network and the construction of a partition ('Principal Partition' for instance) of the network corresponding to this quantity.

Chapter 2 is concerned with a partition that is, in a sense, the finest possible refinement of the principal partition.

A key role in the development of this idea is played by the matroid union theorem of Edmonds and Nashwilliams ([Ed 6]), ([Na 3]). In Section 1 we give a detailed discussion of this theorem. In Section 2 we present a preliminary partition for a matroid. This we have termed a P-sequence. For a proper description of this partition we need the idea of the 'density' d (M) of a matroid M on S .($d(M) = \frac{|S|}{r(M)}$). A matroid M on S is said to be molecular (atomic) iff for every $T \subset S$

$$d(M \times T) \leq d(M) \; [\; d(M \times T) < d(M) \;] \; .$$

A P-sequence can now be defined as follows :

Let M be a matroid on S. A sequence $P_1, P_2 \ldots P_n$ of pairwise disjoint subsets of S is a P-sequence of M iff

(1) $\bigcup_{i=1}^{n} P_i = S$. (2) $M \times \bigcup_{i=1}^{j} P_i \cdot P_j$ is molecular

for all $j \in \{1,2 \ldots n\}$ and

(3) $d(M \times \bigcup_{i=1}^{j} P_i \cdot P_j) > d(M \times \bigcup_{i=1}^{k} P_i \cdot P_k)$

iff $j < k$ ($j,k \in \{1,2 \ldots n\}$).

The idea of a P-sequence is based on the following theorem :

Let M be a matroid on S. Then the maximal subset T of S, such that

$d(M \times T) = \max_{R \subseteq S} d(M \times R)$, is unique. We are then able to show that every matroid has a unique P-sequence.

In Section 3 of Chapter 2 we discuss the relationships between the matroid union theorem and our partition. The central result in this section is the following : (Theorem 3.1)

Let M_1 and M_2 be two matroids on a set on S.

If $\max_{Q \subseteq S} d(M_i \times Q) = d(M_i \times P)$ for $i = 1,2$ then

$$\max_{Q \subseteq S} d((M_1 \vee M_2) \times Q) = d[(M_1 \vee M_2) \times P] .$$

We then describe certain functions called 'admissible functions' on the class of all matroids, and study the P-sequence of the image of a matroid under any 'admissible function' in terms of the P-sequence of the original matroid.

The P-sequence of a matroid M represents, essentially, a break up or decomposition of M into molecular matroids in a certain fashion. The natural further step would be to decompose the molecular matroids into atomic matroids. This we do in Section 4 of Chapter 2. In order to describe the decomposition completely we define the set S^1 of atoms of a molecular matroid and a partial order L on S^1. Further we give efficient algorithms for constructing S^1 and L. We also discuss the effect of admissible functions on S^1 and L of a molecular matroid and extend this idea to general matroids. By the end of this section we achieve a partition for any general matroid which can be regarded as the finest refinement of the principal partition since we are left with only atomic matroids on the individual elements of the partition.

In Section 5 we discuss applications of the principal partition to Network Theory. In Section 6 we have given a partial description of the P-sequence of a single element extension of a matroid in terms of the P-sequence of the original matroid.

Since the matroid union theorem plays a key role in Chapter 2, we have studied its applications in other branches of matroid theory in a separate chapter. In Chapter 3, we show that the matroid union theorem has an important part to play in the theory of transversal matroids, base orderable matroids and gammoids. Section 1 and 2 are devoted to applications to transversal theory. In Section 3 of Chapter 3 we use the matroid union theorem to obtain some new results. We show among other things the following :

(1) Gammoids, strongly base orderable matroids and base orderable matroids are closed under matroid unions.

(2) Minimal non base orderable and non strongly base orderable matroids are atomic of kind (2) and 3- connected.

(3) Minimal non gammoids are 3- connected.

(4) Series connection is a special case of matroid union. Dually, parallel connection is a special case of matroid intersection.

(5) The classes of binary gammoids, binary base orderable and binary strongly base orderable matroids, and series parallel networks are identical.

Finally, we give a partial description of the P-sequence of a series or parallel connection of two matroids in terms of the P-sequences of the original matroids.

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NOTATION AND CONVENTIONS

N_1 By $T_1 \dot{\cup} T_2$ we mean the disjoint union of sets T_1 and T_2 (i.e. union of T_1 and T_2 when $T_1 \cap T_2 = \phi$). Similarly, $T_1 \dot{\cup} T_2 \dots \dot{\cup} T_n$ is the union of the sets T_i ($i = 1, 2, \dots, n$) where the T_i are pairwise disjoint.

N_2 By $A \subset B$ we mean A is a subset of B but $A \neq B$, i.e. A is a proper subset of B.

N_3 We use I_μ for the identity matrix of order μ. If the order is clear we just use $[U]$.

N_4 We use $|S|$ for the cardinality of the set S.

N_5 By M_u on S we mean the matroid on S which has S as its set of coloops; i.e. $r[M_u] = |S|$.

N_6 By M_o on S we mean the matroid on S which has S as its set of loops; i.e. $r[M_u] = 0$.

N_7 We write A^t for the transpose of the matrix A.

Conventions

1. Let T be a set and a be an element. Then we use $T \cup a$ instead of the more correct $T \cup \{a\}$. We use $T - a$ instead of $T - \{a\}$.

2. Unless otherwise stated all the matroids are assumed to be defined on finite sets.

3. Let $\underline{v}$ be a column vector of order r x 1. When there is no possibility of confusion instead of $\underline{v} = 0_{r \times 1}$ we write simply $\underline{v} = \underline{0}$.

4. In equations involving matrices, we may partition some of the matrices and vectors while leaving others as they are.

<u>Example</u> : Instead of

$$\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right) \left(\begin{array}{c} \underline{u}_1 \\ \hline \underline{u}_2 \end{array}\right) = \left(\begin{array}{c} \underline{p}_1 \\ \hline \underline{p}_2 \end{array}\right)$$

we may write

$$\left[A \right] \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right) \left(\begin{array}{c} \underline{u}_1 \\ \hline \underline{u}_2 \end{array}\right) = \underline{p}$$

[Our partition of the matrices may be just to indicate the structure of some of the matrices and not to express the product as a partitioned matrix].

5. Any entries omitted in any matrix are always to be taken as zero entries.

6. When it is sufficiently clear that a certain matrix is partitioned we sometimes omit the partitioning lines. Capital letters are always used to indicate the submatrices into which the matrix is partitioned. Block diagonal matrices are similarly represented.

Example :

(1) $$\left[\begin{array}{c|c} R_{11} & R_{12} \\ \hline R_{21} & R_{22} \end{array}\right]$$ is represented simply as :

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

(2) $$\left[\begin{array}{c|c|c} G_1 & & \\ \hline & G_2 & \\ \hline & & G_3 \end{array}\right]$$ is represented as :

$$\begin{bmatrix} G_1 & & \\ & G_2 & \\ & & G_3 \end{bmatrix}.$$

7. Quite often we have to perform a chain of matrix multiplications and are unable to compress this chain into one line. We then proceed as illustrated in the following example :

Instead of : $K = [A]\,[B]\,[C]\,[D]$

we write :

$K = [A]\,[B]\ X$ (R.H.S. contd.) $X\ [C]\,[D]$ **Or simply** $K = [A]\,[B]\ X$ $X\ [C]\,[D]$

We use the symbol (X) to indicate matrix multiplication in such situations.

(8) In describing the standard representative matrix with respect to a base we generally speak of it as if it is unique. It is of course unique only for a particular ordering of the rows and a particular ordering of the columns. We implicitly assume such an ordering to be given so that we may regard it as unique.

(9) With regard to naming of Theorems and Definitions we have used the following convention :

The name given to a theorem or definition indicates the section in which it is included and its number in that section. But the Chapter in which the theorem is present is not indicated in the name of the theorem.

Example : The name Theorem 1.1 is given to three different theorems namely the first theorem in Chapters 1,2 and 3.

In a particular Chapter, if we wish to refer to an earlier theorem or definition we specify its chapter only if it belongs to a different Chapter. If, however, the theorem or definition belongs to the same Chapter its Chapter is not specified.

Example : If in Chapter 2 we wish to refer to Theorem 2.1 of Chapter 1 we would say 'Theorem 2.1 of Chapter 1' but if we wish to refer to Theorem 2.1 of Chapter 2, we would merely say 'Theorem 2.1'.

CHAPTER 0

Section 1

INTRODUCTION

This thesis is concerned with certain fundamental problems in Network Analysis and the problems together with their extensions that these have given rise to in Matroid Theory.

Here, in Network Analysis, we are interested primarily in efficient methods of solving an electrical network. It turns out that the efficiency of any method of analysis is primarily dependent upon the structural properties, i.e. the geometry, of the network. It is, therefore, imperative, for a good understanding of Network Analysis, that a fairly deep study of the geometrical aspects of a network be carried out. Fortunately for us there is a well developed branch of mathematics which can be applied directly to our problems. This branch is the abstract theory of independence otherwise known as 'Matroid Theory'.

Matroid Theory was introduced by Hassler Whitney in a pioneering paper in 1935 [Wh 1]. In this paper Whitney defined matroids for the first time and further gave five equivalent systems of axioms for the theory. He was motivated in his work primarily by a desire to generalise the idea of duality in graphs. Since this is precisely the aspect that is of interest to network analysts it is not surprising that matroid theory is applicable. Extensive and profound work on this aspect has been carried out by Tutte in a series of papers that appeared in the late 50's and early 60's [Tu 4], [Tu 5], [Tu 6], [Tu 10]. Among other things Tutte gave necessary and sufficient 'geometrical conditions' for a given matrix with 0,1, -1 entries to be the cutset matrix of a graph.

Currently, attractive work in Matroid theory is being carried out in two directions. Some of the workers are adding newer examples of matroids to the classical list comprising of vector spaces, chain groups, function spaces, incidence geometries, graphs etc. Prominent among these new additions are transversal matroids and gammoids [Ed 3], [Ma 4], [Brl 7]. The benefits here are two-fold - Matroid Theory develops in a natural way while a deeper insight is gained into the theory of transversals, gammoids etc.

The second section of workers seems to be interested primarily in unification. These workers appear to be motivated by an interest in classical problems such as the 'Four Colour Conjecture' (which they have elevated to a more general setting and study as an example of the 'critical problem' in combinatorial geometries (combinatorial geometries being matroids without 'parallel elements')).

In Chapter 1 of present thesis we have carried out a detailed, self-contained and fairly rigorous discussion of 'Diakoptics of Electrical Networks' or 'Network Analysis by tearing'. 'Diakoptics' is a term coined by G. Kron for 'Piecewise Analysis of large scale systems'. He has given an exposition of his theory in [Kr 1] where he has made use of tensor calculus to increase its range of applicability. For Network Analysis, however, the ideas of vectors and linear transformations are sufficient and we have, therefore, developed the theory using only these concepts.

Although it is common practice to describe networks in terms of graphs, we find that they mask rather than illuminate the fundamental properties of the network.

The fundamental idea in graph theory - the idea of nodes - is not really fundamental in network theory. For a description of networks, sufficient to permit analysis, it is not necessary to know which edges are incident at a particular node; it is however necessary and sufficient to know whether a given set of oriented edges form an oriented circuit. In fact every occurrence of the word 'node' can be replaced in Network Analysis. For instance, in the case of two terminals making up a port, we can add an imaginary oriented edge between these terminals and use it to replace every occurrence of the word 'terminals'. The idea of graph duality does not correspond exactly to that of the electrical variables, whereas matroid duality does. By restricting ourselves to graphs we can very often get an erroneous idea of duality in [See [Br 1] for a discussion] electrical networks. Matroids of vector spaces over ordered fields may therefore be used for a rigorous development of Network Analysis since such an attempt in terms of graphs would prove unwieldy. It is perhaps appropriate at this stage to stress that the model one chooses for a particular physical phenomenon is dependent upon that aspect of the phenomenon in which one is interested. Our model for the electrical network, though eminently justifiable, need not necessarily be the most convenient in order to study a different branch of network theory. We have called the model that we have used in this thesis a 'Generalized Network'. We have here followed, in the main, the line of thinking of Bruno and Weinberg in [Br 1] who have also defined a 'generalized network' for the study of n-ports. Since we are interested in different things our 'generalized network' differs slightly from theirs — for instance we have used matroids of vector spaces over ordered fields where they have used regular matroids.

In Section 1 of Chapter 1 we prove some of the well-known results on ordinary electrical networks for 'Generalised Networks'. This section is added primarily to make the treatment self-contained. In Section 2 of Chapter 1 both the methods of Kron for Network Analysis are described in some detail. Further, a new and important extension of Kron's methods is given. This method should be regarded as a natural complement to Kron's theory.

The first Chapter also highlights a fundamental problem in Network Theory with whose extensions we are concerned in Chapter 2. This is the problem of partitioning the set of edges S of a graph G into two sets S_1, S_2 such that $(r (G . S_1) + \mu (G X S_2))$ is a minimum. This number has been called the 'topological degree of freedom' of a network. An efficient algorithm for finding such a partition (Principal Partition') has been given by Kishi and Kajitani in [Ki 2]. This problem has been discussed in the context of matroids by Bruno and Weinberg in [Br 2] where they have also given an extension of the principal partition in terms of Principal-r-minors and Augmented principal-r-minors.

Chapter 2 is concerned with a partition of a matroid that is in a sense the finest possible refinement of the principal partition. A key role in the development of this idea is played by the matroid union theorem of Edmonds and Nashwilliams ([Ed 6], [Na 3]). We therefore discuss this theorem in some detail in Section 1 of Chapter 2. Here we give a new algorithmic proof of this theorem. This proof though longer than the known proofs helps to illuminate the structure of the union of two matroids (we indicate the union of M_1 and M_2 by $M_1 V M_2$). Also the methods used in the proof are used repeatedly throughout this Chapter. Among other things we have described

(1) the circuits of $M_1 \vee M_2$ in terms of those of M_1 and M_2.

(2) sufficient conditions to be satisfied by a set $T \subseteq S$ so that the following equation holds :

$$(M_1 \cdot T) \vee (M_2 \cdot T) = (M_1 \vee M_2) \cdot T \quad .$$

(3) a necessary and sufficient condition for the union of matroids to be connected. We show that $T \subseteq S$ is a separator of $M_1 \vee M_2$ iff T is a separator for both M_1 and M_2 (except in one degenerate case).

In Section 2 of Chapter 2 we present a preliminary partition for a matroid. This we have termed a P-sequence. For a proper description of this partition we need the idea of the 'density' $d(M)$ of a matroid M on S . ($d(M) = \frac{|S|}{r(M)}$).

A matroid M on S is said to be molecular (atomic) iff for every $T \subset S$

$$d(M \times T) \leq d(M) \quad (d(M \times T) < d(M)).$$

A P-sequence can now be defined as follows :

Let M be a matroid on S. A sequence $P_1, P_2 \ldots P_n$ of pairwise disjoint subsets of S is a P-sequence of M iff

(1) $\bigcup_{i=1}^{n} P_i = S$

(2) $M \times \bigcup_{i=1}^{j} P_i \cdot P_j$ is molecular for all $j \in \{1,2 \ldots n\}$

and

(3) $d(M \times (\bigcup_{i=1}^{j} P_i) \cdot P_j) > d (M \times (\bigcup_{i=1}^{k} P_i) \cdot P_k)$
iff $j < k$ $(j,k \in \{1,2 \ldots n\})$.

The idea of a P-sequence is based on the following theorem :

Let M be a matroid on S. Then the maximal subset T of S, such that $d(M \times T) = \max_{R \subseteq S} d(M \times R)$, is unique. We are then able to show that every matroid has a unique P-sequence. A matroid whose P-sequence has only one element is obviously molecular.

A recurring theme in matroid theory is the characterization of different classes of matroids in terms of excluded minor conditions. A typical statement of such theorems would run as follows :

'A matroid M belongs to the class $\mathcal{M}$ iff M has no minors isomorphic to the matroids M_1, M_2 ... M_n'.

Examples of matroids so characterized are graphic matroids, cographic matroids, binary matroids, regular matroids, gammoids, base orderable matroids etc. It is an interesting fact that the forbidden minors in each of these cases is an atomic matroid. We give some examples below :

(1) The four point line - forbidden minor for binary matroids.

(2) The seven point seven line geometry of dimension 2 (Fano Matroid),

— forbidden minors for regular matroids, binary matroids which do not have this matroid or its dual as a minor being regular.

(3) The polygon matroids of Kuratowski's graphs K_5 (complete graph on 5 nodes) and $K_{3,3}$ (complete bipartite graph with its nodes being partitioned into two sets of 3 nodes each).

- Forbidden minors for graphic matroids, regular matroids which do not have these matroids as minors being graphic.

(4) The bond matroids of K_5 and $K_{3,3}$ — Forbidden minors for cographic matroids, regular matroids which do not have these matroids as minors being cographic.

(5) The polygon matroid of K_4 (complete graph on 4 nodes) — A forbidden minor for base orderable matroids.

(6) Let $S = \{a_1, a_2, a_3, a_4, d_1, d_2, d_3, d_4\}$. Let M be the matroid on S such that

(1) $r(M) = 4$

(2) Circuits of M of less than 5 elements are the following :

$\{a_1, d_1, d_4\}$, $\{a_2, d_2, d_3, d_4\}$, $\{a_2, a_3, a_4, d_1\}$

and any 4 of $\{a_1, a_3, a_4, d_2, d_3\}$.

This matroid is a forbidden minor for base orderable matroids. (There is no exchange ordering for the bases $\{a_1, a_2, a_3, a_4\}$ and $\{d_1, d_2, d_3, d_4\}$).

(7) Let $S = \{a_1, a_2, a_3, a_4, d_1, d_2, d_3, d_4\}$.

Let M be the matroid on S such that

(1) $r(M) = 4$

(2) Circuits of M of less than 5 elements are the following :

$\{a_1, d_1, d_2, d_4\}$, $\{a_1, a_3, a_4, d_3\}$, $\{a_2, d_1, d_2, d_3\}$

$\{a_2, a_3, a_4, d_4\}$, $\{a_1, a_2, d_3, d_4\}$.

This matroid is base orderable but is a forbidden minor for strongly base orderable matroids.

(There is no 'strong' exchange ordering for the bases $\{a_1, a_2, a_3, a_4\}$ and $\{d_1, d_2, d_3, d_4\}$).

(8) Let $S = \{1, 2, \ldots 7\}$.

Let M be the matroid on S such that

(1) $r(M) = 3$

(2) the bases of M are all 3-sets except $\{1,2,3\}$, $\{4,5,6\}$, $\{1,4,7\}$, $\{2,5,7\}$, $\{3,6,7\}$.

This matroid is base orderable but can be shown to be a forbidden minor for gammoids.

(Examples (6), (7) and (8) are due to Ingleton [Ing 1]).

In Section 3 of Chapter 2 we discuss the relationships between the matroid union theorem and our partition. The central result in this section is the following :-

(Theorem 3.1)

'Let M_1 and M_2 be two matroids on a set S.

If $\max_{Q \subseteq S} d(M_i \times Q) = d(M_i \times P)$ for $i = 1,2$

then $\max_{Q \subseteq S} d((M_1 \vee M_2) \times Q) = d[(M_1 \vee M_2) \times P]$

An interesting consequence of this result is that union of atomic (molecular) matroids is atomic (molecular). We can define functions on the class of all matroids through repeated use of the binary matroid union operator and the unary dualization operator. (Clearly the image of an atomic (molecular) matroid under such functions is atomic (molecular)).

In Section 3, however, we are interested only in a subclass of such functions called 'admissible functions'. It is shown that given the P-sequence of a matroid M, the P-sequence of f(M), where f(M) is an admissible function can be easily obtained. We have also introduced the concept of 'aligned' matroids (depending primarily on the 'relative positions' of the P-sequences) and describe the P-sequence of $M_1 \vee M_2$ in terms of the P-sequences of M_1 and M_2 where M_1 and M_2 are aligned. Finally we show that the P-sequence of a matroid M can be obtained by considering the sets of loops and coloops of certain matroids which are the images of M under certain admissible functions. This enables us to give an efficient algorithm for the construction of the P-sequence of a matroid.

The P-sequence of a matroid M represents, essentially, a break up or decomposition of M into molecular matroids in a certain fashion. The natural further step would be to decompose the molecular matroids into atomic matroids. This we do in Section 4 of Chapter 2. In order to describe the decomposition completely we define the set S^1 of atoms of a molecular matroid and a partial order L on S^1. We give an efficient algorithm for constructing S^1 and the partial order L by using the idea of 'approachability'. We discuss the set of atoms and the corresponding partial order for $M_1 \vee M_2$ in terms of the corresponding entities for M_1 and M_2 when the molecular matroids M_1 and M_2 satisfy certain conditions. We also discuss the effect of admissible functions on S^1 and L of a molecular matroid. Finally we extend these ideas to general matroids. By the end of this section we achieve a partition for any general matroid which can be regarded as the finest refinement of the principal partition, since we are left only with atomic matroids on the individual elements of this partition.

In Section 5 we discuss applications of the principal partition to Network Analysis. We give a new proof for Kishi-Kajitani's theorem on the principal partition. We also show that the partition described by Ohtsuki et al in [Oht 1] is a special case of our partition for molecular matroids in terms of the set of atoms and the partial order on this set.

In Section 6 we have given a partial description of the P-sequence of a single element extension of a matroid in terms of the P-sequence of the original matroid.

The material presented in Chapter 2 represents a detailed study of an important concept in network theory (and its extensions) and therefore such work needs no other justification. However, we should like to point out that our partition, since it is invariant under the automorphisms of the matroid, (and can also be easily 'manipulated' theoretically), can be used as a starting point for the generation (and study) of all invariant sets of a matroid (and in particular, of a graph).

Since the matroid union theorem plays a key role in Chapter 2, we have studied its applications in other branches of matroid theory in a separate chapter. In Chapter 3, we show that the matroid union theorem has an important part to play in the theory of transversal matroids, base orderable matroids and gammoids. In fact practically all the known structural results on finite transversal matroids can be obtained by using this theorem. Sections 1 and 2 are devoted to the illustration of this point.

In Section 3 we use the matroid union theorem to obtain some new results.

We show among other things the following :-

(1) Gammoids, strongly base orderable matroids and base orderable matroids are closed under matroid unions.

(2) Minimal non-base orderable matroids and minimal non-strongly base orderable matroids are atomic of kind (2) and 3-connected.

(3) Minimal non gammoids are 3-connected.

(4) Series connection is a special case of matroid union. Dually, parallel connection is a special case of matroid intersection.

(5) The classes of binary gammoids, binary base orderable and binary strongly base orderable matroids, and series parallel networks are identical.

Finally we give a partial description of the P-sequence of a series or parallel connection of two matroids in terms of the P-sequences of the original matroids.

PRELIMINARIES

Below we have listed the definitions and terms used in Chapters 1 and 2. Additional material required for Chapter 3 is described separately in that Chapter.

Most of the results given here are quite simple. They are generally easy consequences of Tutte's theorems [Tu 8]. The axiom system 1 is taken from [Ed 1]. Axiom system 2 is due to Whitney [Wh 1].

We have used Tutte's notation for the most part (especially the definitions of contraction and reduction). Probably the most significant divergence from his notation is our use of the term 'base' for the 'complement of a dendroid' and the use of the 'rank function' in keeping with the notation of recent authors [Cr 10].

Section 2 : General matroids

Definition 0.1. Matroid. Axiom System 1 :

A matroid is a pair (S,I), where S is a set and I is a class of subsets of S called independent subsets such that

(a) Every subset of an independent set is independent

(b) Maximal independent sets contained in any subset of S are finite and have the same number of elements.

Definition 0.2. Base : A maximal independent subset of a matroid is called a base of the matroid.

Definition 0.3. **Cobase** : Complement of a base of a matroid is a cobase of the matroid.

Definition 0.4. **Circuit** : A circuit of a matroid is a minimal dependent (or non-independent) set of the matroid. It follows that every proper subset of a circuit is independent.

The class of circuits of a matroid can be shown to determine a matroid uniquely. One can give an equivalent definition of a matroid in terms of its circuits as follows :

Definition 0.5. **Axiom System 2** :

A matroid is a pair $(S, \mathcal{C})$ where S is a set and $\mathcal{C}$ is a class of subsets called circuits such that

(a) No proper subset of a circuit is a circuit

(b) If C_1 and C_2 are two circuits and $a \in C_1 \cap C_2$, $b \in C_2 - C_1$, then there exists a circuit C_3 such that $b \in C_3$, $a \notin C_3$, $C_3 \subset C_1 \cup C_2$.

Axiom systems 1 and 2 for a matroid are equivalent under the following substitutions :

Circuit $\equiv$ Minimal non-independent set

Independent set $\equiv$ A set that does not contain a circuit.

In the succeeding pages we will generally speak of a matroid as a 'Matroid M on S', and usually avoid describing it wholly in terms of its circuits or independent sets.

Definition 0.6. **Contraction of a matroid** : Let $M = (S, \mathcal{C})$ be a matroid and let $T \subseteq S$. Let $\mathcal{C} \times T$ be the class of those members of $\mathcal{C}$ which are contained in T. The class $\mathcal{C} \times T$ obviously satisfies (a) and (b) of axiom system 2.

Consequently $(T, \mathcal{C} \times T)$ is a matroid on T. We denote this matroid by $M \times T$. The matroid $M \times T$ is called the contraction of M to T.

Definition 0.7. **Reduction of a matroid** : Let $M = (S, \mathcal{C})$ be a matroid and let $T \subseteq S$. Let L_T be the class of non-null intersections of the members of $\mathcal{C}$ with T. Let $\mathcal{C} . T$ be the class of minimal members of L_T. Tutte [Tu 8] has shown that $\mathcal{C} . T$ satisfies the conditions of axiom system 2 and therefore $(T, \mathcal{C} . T)$ is a matroid on T. We denote this matroid by $M . T$. $M . T$ is called the reduction of M to T.

Definition 0.8. **Minors of a matroid** : Let M be a matroid on S. Let $T \subseteq S$. A matroid M_1 on T is called a minor of M iff $M_1 = (M \times R) . T$ where $T \subseteq R \subseteq S$. We denote $(M \times R) . T$ by $M \times R . T$.

Definition 0.9. **Dual matroid** : Let $M = (S, I)$ be a matroid on a finite set S with I as its class of independent sets. Let I^* be the class of subsets of S which are contained in the cobases of M. It is easy to see that (S, I^*) satisfies the conditions of axiom system 1. Hence $M = (S, I^*)$ is a matroid. M^* is called the dual of M.

We will next define duality in terms of axiom system 2.

<u>Definition 0.10.</u> <u>Orthogonality</u>:

Two sets S and T are said to be orthogonal iff $S \cap T$ is not a singleton.

<u>Definition 0.11.</u> <u>Dual matroid</u> :

Let $M = (S, \mathcal{C})$ be a matroid on a finite set S, with $\mathcal{C}$ as its class of circuits. Let Θ be the class of subsets of S which are orthogonal to every member of $\mathcal{C}$, i.e. $\Theta = \{T \mid T \subseteq S$ and T is orthogonal to every member of $\mathcal{C}\}$.

Let $\mathcal{C}^*$ be the class of minimal members of Θ. Tutte [Tu 8] has shown that $\mathcal{C}^*$ satisfies the conditions of axiom system 2 and therefore $M^* = (S, \mathcal{C}^*)$ is a matroid on S. M^* is called the dual matrod of M.

It can be shown that Definitions 0.9 and 0.11 are equivalent definitions under the substitutions

Circuit $\equiv$ minimal non independent set

Independent set $\equiv$ A set that does not contain a circuit.

<u>Definition 0.12.</u> <u>Bond</u> : Let M be a matroid on S. A bond of M is a circuit of M^*.

<u>Definition 0.13.</u> <u>Separator</u> : Let M be a matroid on S. Let $T \subseteq S$. Then T is a separator of M iff there exists no circuit intersecting both T and S-T. We note that if T is a separator so is S-T.

Definition 0.14. **Elementary separator and connectedness :**

A separator that has no separator as its proper subset is an elementary separator. A matroid M on S is said to be connected iff S is an elementary separator of M.

Definition 0.15. **Closure and closed sets :**

Let M be a matroid on S and let $T \subseteq S$. Then the closure of T in M is the set $\bar{T}$, where

$$\bar{T} = \left\{ a \;\middle|\; a \in T \text{ or } \{a\} = C - T \text{ where } C \text{ is a circuit of } M \right\}.$$

If $\bar{T} = T$ we call T a closed set.

Definition 0.16. **Rank and nullity :**

Let M be a matroid on S. Then the rank of M is the cardinality of a base of M. Rank of M is denoted by $r[M]$. Nullity of M is denoted by $\mu[M]$ and defined by $\mu[M] = r[M^*]$.

Definition 0.17. **Rank function :** Let M be a matroid on S. We define a rank function ρ on the subsets of S as follows. Let $T \subseteq S$, then $\rho(T) = r[M \times T]$.

We now state some theorems without proof. These results are either taken directly from [Tu 8] or are easy consequences of results found there.

Theorem T1. Let M be a matroid on S. Let b be a base of M. Let $e \in S - b$. Then $\{e\} \cup b$ contains one and only one circuit C. If $e_1 \in C$, $\{e\} \cup b - \{e_1\}$ is a base of M.

Definition 0.17. **Fundamental circuit :**

Let M be a matroid on S. Let b be a base of M and let $e \in S - b$. Then there exists a unique circuit C such that $e \in C \subseteq \{e\} \cup b$. We call C as the fundamental circuit of e with respect to b.

Theorem T2. Let M be a matroid on S. (1) Let $T \subseteq S$. Let $A \subseteq T$. Then A is independent in $M \times T$ iff A is independent in M. (2) If A is independent in $M.T$, then A is also independent in M. (3) If A is independent in M, and $A \cap (S - T)$ is a base of $M \times (S - T)$, then $A \cap T$ is independent in $M . T$.

Theorem T3. Let M be a matroid on S. Let $T \subseteq S$.

(1) If b is a base of M and $b \cap T$ is a base of $M \times T$ $(M . T)$, $b \cap (S - T)$ is a base of $M . (S - T)$ $(M \times (S - T))$.

(2) If b_1 is a base of (is independent in) $M \times T$, b_2 is a base of (is independent in) $M . (S - T)$, then $b_1 \cup b_2$ is a base of (is independent in) M.

Hence

(3) $r[M] = r[M \times T] + r[M . (S - T)]$.

(4) If b is a base of M, then $b \cap T$ is independent in $M \times T$, and $b \cap T$ contains a base of $M . T$.

Theorem T4. Let M be a matroid on S. Let $A \subseteq S$. If A is not contained in any base (cobase) of M, A contains a circuit (bond).

Theorem T5. Let M be a matroid on S. Let C be any circuit of M and B be any bond of M. Then $C \cap B \neq \{a\}$, where $a \in S$.

Theorem T6. Let M be a matroid on S. Let A and B be disjoint subsets of S such that A can be contained in a cobase of M and B can be contained in a base of M. Then there exists a base b of M such that $A \subseteq S - b$ and $B \subseteq b$.

Theorem T7. Let M be a matroid on S. Let $T \subseteq R \subseteq S$. Then

(1) $M \times R \times T = M \times T$

(2) $M \cdot R \cdot T = M \cdot T$

(3) $M \times R \cdot T = M \cdot (S - (R - T)) \times T$

(4) $M \cdot R \times T = M \times (S-(R - T)) \cdot T$

(5) Every minor of a minor of M is a minor of M.

(6) $(M^*)^* = M$

(7) $r[M] + r[M^*] = |S|$

(8) $(M \times T)^* = M^* \cdot T$

(9) $(M \cdot T)^* = M^* \times T$

(10) $(M \times R \cdot T)^* = M^* \cdot R \times T$

(11) $(M \cdot R \times T)^* = M^* \times R \cdot T$.

Theorem T8. Let M be a matroid on S. Let $R \subseteq T \subseteq S$.

(1) Then R is a separator of M iff $r[M \times R] = r[M . R]$

(2) If R is a separator of M, then

$M \times R = M . R = M \times T . R = M . T \times R$

(3) If R is an elementary separator of M and $a \in R$, $d \in R$, then there exists a circuit C_1 of M such that $a \in C_1$ and $d \in C_1$.

<u>Theorem T9.</u> Let M be a matroid on S and let ρ be its rank function. Let T_1, T_2 be subsets of S. Then

$$\rho(T_1 \cup T_2) + \rho(T_1 \cap T_2) \leq \rho(T_1) + \rho(T_2) .$$

<u>Theorem T10.</u> Let M be a matroid on S. Let $T \subseteq S$. T is a union of circuits of M, iff (S-T) is closed in M^*. T is a circuit of M iff (S - T) is closed in M^* and $r[M^* \times (S-T)] = r[M^*]-1$

<u>Section 3 : Matroids of vector spaces</u>

Let S be a finite set, $S = \{e_1, e_2 \dots e_n\}$ and let F be a field.

<u>Definition 0.18.</u> <u>Vector</u> : By a vector on S over F we mean a mapping f of S into F.

$f(e_i)$ is called the value of f at e_i.

The support $||f||$ of the vector f is the set of all members of S whose values under f are non-zero. If $||f|| = \phi$, then f is the zero vector and it is denoted by $0_{1 \times |S|}$.

Definition 0.19. Sum of vectors : The sum of two vectors f and g on S over F is the vector f + g defined as

$$(f + g)(e_i) = f(e_i) + g(e_i) \quad \text{for all } i \in \{1, \ldots n\}.$$

Definition 0.20. Product : The product of $\lambda \in F$ and a vector f on S over F is a vector λf defined as

$$(\lambda f)(e_i) = \lambda(f(e_i)) \quad \text{for all } i \in \{1, \ldots n\}.$$

Definition 0.21. Vector space : Let V be a collection of vectors on S over F which is closed under the operations of addition of vectors and product of elements of F and vectors. Then V is called a vector space on S over F.

Definition 0.22. Elementary vector : If V is a vector space on S over F, then a vector $f \in V$ is called elementary if it is non-zero and there is no non-zero vector $g \in V$ which satisfies $\|g\| \subset \|f\|$. (We use $\subset$ to denote proper inclusion, that is, $A \subset B$ implies $A \subseteq B$ but $A \neq B$).

The following is a special case of a theorem proved by Tutte [Tu 8] :

Theorem T11. Let V be a vector space on S over F and $\mathcal{C}_V$ the class of supports of elementary vectors in V. Then $\mathcal{C}_V$ satisfies the conditions of Axiom system 2.

Definition 0.23. We denote the matroid $[S, \mathcal{C}_V]$ by M_V. M_V is called the matroid associated with the vector space V.

Definition 0.24. Binary vector space : If V is a vector space on S over F, where F is the field of integers modulo 2, then V is called a binary vector space.

Definition 0.25. Binary matroid : The matroid $M_V = (S, \mathcal{E}_V)$ associated with a binary vector space is called a binary matroid.

Definition 0.26. Primitive vector : Let F be the real number field and V a vector space on S over F. A vector $g \in V$ is called a primitive vector if it is an elementary vector, all of whose values are $\pm$ 1 or 0.

Definition 0.27. Regular vector space : A vector space V on S over F, where F is the field of real numbers, is called regular if corresponding to each elementary vector $f \in V$, there is a primitive vector $g \in V$ satisfying $||f|| = ||g||$.

Definition 0.28. Regular matroid : If V is a regular vector space on S over F, then we call $M_V = (S, \mathcal{E}_V)$ a regular matroid.

Definition 0.29. Linear dependence of vectors :

Let $f_1, f_2 \dots f_n$ be vectors belonging to a vector space V on S over F. Then $f_1, f_2 \dots f_n$ are linearly dependent iff there exist elements $\lambda_1, \lambda_2 \dots \lambda_n$ belonging to F such that

(1) λ_i are not all identically equal to $0 \in F$.

(2) $\lambda_1 f_1 + \lambda_2 f_2 \dots \lambda_n f_n = 0$.

$f_1, f_2 \dots f_n$ are said to be linearly independent iff they are not linearly dependent.

Definition 0.30. Representative vector :

Let V be a vector space on S over F. Then for any $f \in V$ we define R_f, the representative vector of f, as the 1-rowed matrix $R_f = [f(e_1), \ldots, f(e_n)]$.

Definition 0.31. Linear combination : Let V be a vector space on S over F. Let $f_1, f_2 \ldots f_{n-1}$ be vectors of V. A vector f_n is said to be a linear combination of $f_1, f_2 \ldots f_{n-1}$ iff $f_1, f_2 \ldots f_n$ is a set of linearly dependent vectors.

Definition 0.32. Representative matrix : Let V be a vector space on S over F. A matrix R with elements in F is called a representative matrix of V if it satisfies the following :

(1) The rows of R are linearly dependent

(2) Every non-zero vector of V has a representative vector which is a linear combination of the rows of R.

It is clear that a representative matrix of V determines the vector space V completely.

Definition 0.33. Let V be a vector space on S over F, and let R be a representative matrix of V. If $T \subseteq S$, then by R(T) we mean the submatrix of R consisting of those columns of R which correspond to members of T.

From linear algebra we know, for finite S, that all representative matrices for V will have the same number of rows and that the number of rows cannot exceed the number of columns.

It is also a consequence of linear algebra that if R is a representative matrix for V, then any other representative matrix R' for V can be obtained from R by $R' = QR$ where Q is a non-singular matrix. Consequently if R is $\mu \times n$, the columns of R which form non-zero μ-th order minors will also form non-zero μ-th order minors in R' and vice-versa.

<u>Theorem T12.</u> Let the $\mu \times n$ matrix R be a representative matrix for V, a vector space on S over F. Let T be a subset of S. Then $\det [R(T)] \neq 0$ iff T is a cobase of M_V. The dimension of the vector space is equal to μ, the number of elements in a cobase of M_V.

<u>Definition 0.34.</u> <u>Standard representative matrix</u> :

Let R be a $\mu \times n$ representative matrix for the vector space V and let T be a cobase of M_V. By Theorem T12 $\det [R(T)] \neq 0$. Let $R' = [R(T)]^{-1} R$. R' is called the standard representative matrix of V with respect to the cobase T. It follows that $R'(T) = I_\mu$ (The identity matrix of order μ).

It is easy to see that a representative matrix R is a standard representative matrix iff it has a unit submatrix of order equal to number of rows of R.

<u>Theorem T12.</u> Let V be a vector space on S over F and R be a $\mu \times n$ standard representative matrix of V with respect to the cobase T of M_V. Then the rows of R are the representative vectors of elementary vectors in V.

Theorem T13. Let V be a vector space on S over F. Let h and g be two elementary vectors in V satisfying $||h|| = ||g||$. Then $h = \lambda g$ where λ is some member of F.

Definition 0.35. **Orthogonality :** Two vectors f and g on S over F are said to be orthogonal iff

$$\sum_{e_i \in S} f(e_i)\, g(e_i) = 0 \quad .$$

Definition 0.36. **Complementary orthogonal space :**

Let V be a vector space on S over F. Let $V^* = \{ f \mid f$ is a vector on S over F, $f \notin V$ and f is orthogonal to every member of $V \}$.

The set V^* can be shown to be a vector space on S over F under addition of vectors and multiplication of vectors by members of F. V^* is called the complementary orthogonal space of V.

From linear algebra we know that the dimension of the vector space V^* is equal to $|S| - \dim V$.

Theorem T14. Let V be a vector space on S over F and V^* be its complementary orthogonal space. Let R be a representative matrix for V and R^* be a representative matrix for V^*. Then

$$R^* R^t = 0_{\mu \times r} \quad ;$$

where μ = dimension of V^*, r = dimension of V.

Theorem T15. Let V be a vector space on S over F and V^* be its complementary orthogonal space. Let the matroids associated with V and V^* be M_V and M_{V^*} respectively. Then, $(M_V)^* = M_{V^*}$.

<u>Theorem T16.</u> Let V be a vector space on S over F. Let dim $V^* = r$. Then the bases of M_V correspond to the non-zero r-th order minors of R^* (where R^* is the representative matrix of V^*).

Another relationship between the structure of M_V and the matrix R^* follows from Theorem T14. By a minimal dependent set of columns of a matrix we mean a set of columns which are linearly dependent and any proper subset of them is linearly independent.

From Theorem T14 it is clear that the minimal dependent sets of columns of R^* correspond to the representative vectors, in the row space of R, of elementary vectors in V. Consequently the minimal dependent sets of columns of R^* correspond to the members of $\mathcal{C}_V$. A dual statement can be made for R and $\mathcal{C}_{V^*}$.

<u>Theorem T17.</u> Let M be a matroid on S and M^* be its dual. Let b be a base of M and let $e \in S - b$. Let $d \in b$. Let C_1 be the circuit of M such that $C_1 \subseteq \{e\} \cup b$, and let C_2 be the circuit of M^* such that $C_2 \subseteq \{d\} \cup (S - b)$.
Then $d \in C_1$ iff $e \in C_2$.

<u>Theorem T18.</u> Let V be a vector space on S over F and let V^*

be its complementary orthogonal space. Let R be the standard representative matrix of V with respect to a cobase S - b of M_V and R^* be the standard representative matrix of V^* with respect to the base b of M_V (or cobase b of M_V^*).

Let dim of V = r, dim of $V^* = \mu$

Then if $R \equiv [\, I_\mu \mid K \,]$

$$R^* \equiv [\, -K^t \mid I_r \,]$$

(where I_μ is the identity matrix of order μ).

<u>Definition 0.37.</u> <u>V X T</u> , <u>V . T</u> :

Let V be a vector space on S over F. Let $T \subseteq S$. Then we define a vector space V X T on T over F (under the usual vector addition and scalar multiplication) as follows :

$$V \times T = \{ f \mid f \in V \ , \ f(e_i) = 0 \ \text{ if } \ e_i \in S - T \} .$$

We define the vector space V . T on T over F as

$$V \,.\, T = \{ f/T \mid f \in V \} . \ [\, f/T \equiv f \text{ restricted to } T \,].$$

<u>Theorem T19.</u> Let V be a vector space on T over F. Let V X T and V . T be defined as in Definition 0.37. Let M_V, $M_{V \times T}$, $M_{V \,.\, T}$ be the matroids associated with V, V X T, V . T respectively.

Then $M_{V \times T} = (M_V) \times T$

$M_{V \,.\, T} = (M_V) \,.\, T$

Also $(V \cdot T)^* = V^* \times T$, $(V \times T)^* = V^* \cdot T$.

<u>Theorem T20.</u> Let V be a vector space on S over F. Let R be a representative matrix of V. If $T \subseteq S$, R(T) ~~is~~ contains a representative matrix of V . T. as a submatrix

<u>Theorem T21.</u> Let V be a vector space on S over F. Let $T \subseteq S$. Let L_1 be a base of $(M_V)^* \times (S - T)$ and $L_2 \subseteq T$ be a base of $(M_V)^* \cdot T$.

Let $r[(M_V)^* \times (S - T)] = \mu_1$

$\mu[(M_V)^* \times (S - T)] = r_1$

$r[(M_V)^* \cdot T] = \mu_2$

$\mu[(M_V)^* \cdot T] = r_2$.

Let R be a standard representative matrix of V chosen with respect to the cobase $L_1 \cup L_2$ of M_V .

$$\text{Let } R = \begin{array}{c} \begin{array}{cc} \quad T \quad & \quad S - T \quad \end{array} \\ \left[\begin{array}{cc|cc} R_{11} & 0 & I_{\mu_1} & R_{14} \\ \hline R_{21} & I_{\mu_2} & 0 & R_{24} \end{array}\right] \end{array}$$

Then (1) $R_{24} = 0_{\mu_2 \times r_1}$

(2) The standard representative matrix of $V \times T$ with respect to the cobase L_2 of $M_{V \times T}$ is $R_{V \times T} = [\, R_{21} \mid I_{\mu_2} \,]$.

Section 4 : Graphs

Definition 0.38. Graph : A graph G is defined by.

(1) E (G), a finite set of edges

(2) V (G), a finite set of vertices and

(3) a relation of incidence which associates with each edge a pair of vertices, not necessarily distinct, called its ends.

Definition 0.39. Loop : An edge with coincident ends is called a loop.

Definition 0.40. Subgraph : A graph H is called a subgraph of G if $E(H) \subseteq E(G)$, $V(H) \subseteq V(G)$ and the ends of the edges in H are same as in G.

Definition 0.41. Reduction of a graph : If $S \subseteq E(G)$ we denote by G . S that subgraph of G whose edges are the members of S and whose vertices are the ends in G of the members of S. G . S is called the reduction of G to S.

Definition 0.42. Valence of a vertex : Let $v \in V(G)$. The valence of v is equal to the number of edges incident to v, where loops are counted twice.

Definition 0.43. Polygon graph : A connected graph with each vertex having valence 2 is called a polygon graph.

Definition 0.44. Polygon and loop : A set $S \subseteq E(G)$ is called a polygon of G if G . S is a polygon graph. A single edged polygon of G is called a loop of G.

Theorem T22 (Whitney) [Wh 1]. Let G be any graph and let $\mathcal{C}_G$ denote the class of polygons of G. Then $\mathcal{C}_G$ satisfies the conditions of axiom system 2 and thus defines a matroid on the set E(G).

Definition 0.45. We will call the matroid (E(G), $\mathcal{C}_G$) as the polygon matrod of the graph G and denote it by Pol (G).

Definition 0.46. Cographic matroid : A matroid which can be represented as the polygon matroid of some graph G is said to be cographic.

Let G be a graph and $S \subseteq E(G)$. Define H to be the subgraph of G with vertices V(G) and edges (E(G) - S). Let H_i for $i \in \{1, \ldots p\}$ denote the connected components of H. (A graph G is said to be connected if between nodes $v_i, v_j \in V(G)$ one can find a sequence of edges such that —

(1) v_i is an end of the first edge, v_j is an end of the last edge.

(2) If e_n, e_{n+1} are edges in the sequence they have a vertex in common.

A maximal connected subgraph of a graph is said to be a connected component of the graph). The graph G ctr. S has the vertex set $\{H_1, H_2 \ldots H_p\}$ and the edge set S. The ends of a member $e \in S$ in G ctr. S are those components H_{i_1} and H_{i_2} which contain the ends of e in G.

Definition 0.47. Contraction of a graph : Let G be a graph. Let $S \subseteq E(G)$. The contraction of G to S is denoted by $G \times S$ and is defined by $G \times S = (G \text{ ctr. } S) \cdot S$.

(G ctr. S may have isolated vertices. $G \times S$ is obtained from G ctr. S by deleting the isolated vertices).

Definition 0.48. Bond graph : A graph G is called a bond graph if $V(G) = \{v_1, v_2\}$, $E(G) \neq \phi$ and the ends of each member of E(G) are v_1 and v_2.

Definition 0.49. Bond and coloop : Let G be any graph and $S \subseteq E(G)$. S is called a bond of G iff $G \times S$ is a bond graph. A single edge bond of G is called a coloop.

Theorem T23. Let G be a graph and $\mathcal{C}_{G'}$ be its class of bonds. $\mathcal{C}_{G'}$ satisfies the conditions of Axiom system 2, and thus defines a matroid on the set E(G).

Definition 0.50. Bond matroid : $(E(G), \mathcal{C}_{G'})$ is called the bond matroid of G, and denoted by Bon(G).

Definition 0.51. Tree : Let G be a graph. A tree of G is a maximal set $S \subseteq E(G)$ which contains no polygon of G.

Obviously a tree of G is a base of the matroid Pol (G) and vice versa.

Definition 0.52. Cotree :

Let G be a graph. A maximal set $T \subseteq E(G)$ which contains no bond of G is called a cotree of G.

Clearly every cotree of G is a base of Bon(G) and vice versa.

<u>Definition 0.53.</u> <u>Subtree and subcotree</u> : A subset of a tree is called a subtree and a subset of a cotree is called a subcotree.

Subtree and subcotree are the graph theoretic counterparts of an independent set (of M and M^* respy.).

The number of edges in a tree of G is the rank of G denoted by $\gamma(G)$.

The number of edges in a cotree of G is the nullity of G denoted by $\mu(G)$.

<u>Theorem T24.</u> Let G be a graph and $S \subseteq E(G)$. Then

$$\text{Pol}(G) \times S = \text{Pol}(G \cdot S).$$

$$\text{Pol}(G) \cdot S = \text{Pol}(G \times S).$$

$$\text{Bon}(G) \times S = \text{Bon}(G \times S).$$

$$\text{Bon}(G) \cdot S = \text{Bon}(G \cdot S).$$

<u>Theorem T25.</u> Graphic and cographic matroids are regular matroids.

<u>Theorem T26.</u> Every regular matroid is a binary matroid and hence graphic and cographic matroids are binary matroids.

Section 5 : Matrices

Definition 0.54. Positive definite and semidefinite matrices :

Let P be a real square matrix. Then P is said to be a positive definite matrix iff

$Y^t P Y > 0$ for all non-null real column matrices of the same number of rows as P.

P is said to be a positive semidefinite matrix iff

$Y^t P Y \geq 0$ for all non-null real column matrices Y of the same number of rows as P.

Theorem T27. Every principal submatrix [submatrix whose principal diagonal is a subset of the principal diagonal of the original matrix] of a positive-definite (positive semi-definite) matrix is positive definite (positive-semi-definite).

Theorem T28. Every positive definite matrix is non-singular. Inverse of a positive definite matrix is positive definite.

Theorem T29. (1) If P is a positive-semi-definite matrix the matrix TPT^t where T is a real matrix, is positive semidefinite. (2) If P is positive definite and T has linearly independent rows TPT^t is positive definite.

Theorem T30. Sum of a positive definite and a positive semi-definite matrix is positive definite.

Theorem T31. Let P be a symmetric positive semi-definite matrix.

Then there exists an orthogonal matrix T such that

$$T P T^t = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_k & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

where the λ_i are positive and k = rank of P.

Section 6 : Partial order and partitions

Definition 0.55. Partial order and its dual : Let S be a set. By a partial order L on S we mean a pair (S, >) where (>) is a relation defined on S such that

(1) $a > b$ implies $b > a$ is not possible (we write briefly $b \not> a$).

(2) $a > b$, $b > d$ implies $a > d$.

Let L be the partial order (S, >). The dual L' of L is the partial order (S, >'), where $a >' b$ ($a, b \in S$) iff $b > a$ in L. It is clear that the dual of L' is L.

We write $a \geq b$ iff $a > b$ or $a = b$.

Definition 0.56. Partial order on the set of partitions :

Let S be a set. Let $\mathcal{P}$ be the set of all partitions of S. Then we define a partial order on $\mathcal{P}$ as follows :

Let $p, q \in \mathcal{P}$

$p \equiv (P_1, P_2 \ldots P_n)$ $\left[\bigcup_{i=1}^{n} P_i = S, P_i \cap P_j = \phi \ (i \neq j) \right]$.

$q \equiv (Q_1, Q_2 \dots Q_m) \left[\bigcup_{i=1}^{m} Q_i = S, \; Q_i \cap Q_j = \phi \; (i \neq j) \right]$

Then $q > p$ iff p is a refinement of q i.e.

(1) $n > m$

(2) For every $i \in \{1,2 \dots n\}$, we can find a $j \in \{1,2 \dots m\}$ such that

$$P_i \subseteq Q_j \quad .$$

If $q \geq p$ we say p is finer than q or q is coarser than p.

<u>Definition 0.57.</u> Let $T \subseteq \mathcal{P}$. Then $p \in \mathcal{P}$ is said to be the supremum of T iff

(1) $p > q$ for all $q \in T$ and

(2) $s > q$ for all $q \in T$ implies $s > p$ or $s = p$.

$p \in \mathcal{P}$ is said to be the infimum of T iff

(1) $q > p$ for all $q \in T$

(2) $p > s$ or $p = s$ ($p \geq s$) is implied by $q > s$ for all $q \in T$.

For any $p, q \in \mathcal{P}$ it can be shown that $\{p,q\}$ has both a supremum and an infimum.

(If $p \equiv (P_1, P_2 \dots P_n)$

$q \equiv (Q_1, Q_2 \dots Q_m)$

the infimum is formed from

$P_1 \cap Q_1, \; P_1 \cap Q_2 \dots, P_1 \cap Q_m, \dots P_n \cap Q_1, \dots, P_n \cap Q_m$

by dropping the void sets).

The above order on partitions actually defines a geometric lattice.

Section 7 : Automorphisms of matroids and graphs

Definition 0.58. Let M be a matroid on a finite set S. A mapping σ from S onto S is an automorphism of M iff

(1) σ is one to one

(2) If b is a base of M, then $\sigma(b)$ is a base of M (By $\sigma(P)$, $P \subseteq S$ we mean $\bigcup_{a \in P} \{\sigma(a)\}$).

The following simple theorem from [Gal 1] is stated without proof.

Theorem T32. Let M be a matroid on a finite set S. Let σ be a one to one mapping from S onto S. Then the following are equivalent.

(1) σ is an automorphism of M

(2) σ^{-1} is an automorphism of M

(3) If C is any circuit of M, $\sigma(C)$ is a circuit of M

(4) If b is independent in M, $\sigma(b)$ is independent in M

(5) σ is an automorphism of M^*.

Definition 0.59. Let M be a matroid on S.

Then $A(M) = \{ a \mid a \in S,$ a can be included in some base of M and some cobase of M$\}$.

$B(M) = \{ a \mid a \in S,$ a cannot be included in any cobase of M.$\}$

i.e. $a \in B(M)$ iff $\{a\}$ is a bond of M (a is a coloop of M).
B(M) would be referred to as the set of coloops of M.

$C(M) = \{a \mid a \in S,$ a cannot be included in any base of M $\}$

i.e. $a \in C(M)$ iff $\{a\}$ is a circuit of M

(a is a loop of M).

C(M) would be referred to as the set of loops of M.

<u>Theorem T33.</u> Let M be a matroid on S. Then A(M), B(M), C(M) are invariant under the automorphisms of M.

<u>Theorem T34.</u> Let M be a matroid on S. Let $B_1, B_2 \ldots B_n$ be its elementary separators. Then if σ is an automorphism of M,

$$\sigma(B_i) = B_j \text{ , for some } j \in \{1,2, \ldots n\} .$$

<u>Proof</u> : Suppose $\sigma(B_i) \cap B_j \neq \phi$ and $\sigma(B_i) \cap B_k \neq \phi$ where $B_j \neq B_k$. Then there exists $a \in B_i$ and $d \in B_i$ such that $\sigma(a) \in B_j$ and $\sigma(d) \in B_k$. Now there exists a circuit C of M such that $a \in C$ and $d \in C$ by Theorem T8. But there exists no circuit C_1 of M such that $\sigma(a) \in C_1$ and $\sigma(d) \in C_1$. Hence $\sigma(C)$ is not a circuit, which contradicts Theorem T32.

<u>Definition 0.60.</u> <u>Automorphisms of a graph</u> : Let G be a graph and V(G), E(G) be its set of vertices and set of edges respectively. A mapping σ from E(G) onto E(G) is an edge automorphism of the graph G iff

(1) σ is one to one
and

(2) If edges $e_1, e_2 \ldots e_k$ are incident at a single node, so are $\sigma(e_1) \ldots \sigma(e_k)$.
and

(3) If e_1 is a loop of G (coloop of G) $\sigma(e_1)$ is a loop of G (coloop of G).

A mapping α from V(G) onto V(G) is a vertex automorphism of the graph G iff

(1) α is one to one and

(2) If $v_1, v_2 \in V(G)$ and k edges ($k \geq 0$) have v_1 as one end and v_2 as the other end, then k edges have $\alpha(v_1)$ as one end and $\alpha(v_2)$ as the other end.

<u>Theorem T35.</u> Given a vertex automorphism α it is always possible to find an edge automorphism σ (and vice versa) such that if edge e is incident on vertices v_1 and v_2, then $\sigma(e)$ is incident on vertices $\alpha(v_1)$ and $\alpha(v_2)$.

It is easy to see that every edge automorphism of a graph is an automorphism of Pol(G) and Bon(G). Hence sets which are invariant under the automorphisms of Pol(G) (or Bon(G)) are invariant under the edge automorphisms of the graph G. [However unless the graph is 3-connected every automorphism of Pol(G) is not necessarily an edge automorphism of the graph G]. From Theorem T35 it follows that edge sets invariant under the edge automorphisms of G 'induce' vertex sets invariant under vertex automorphisms of G and vertex sets invariant under vertex automorphisms of G 'induce' edge sets invariant under edge automorphisms of G.

Definition 0.61. Let G_1 and G_2 be two graphs. A mapping α from $V(G_1)$ onto $V(G_2)$ is said to be a vertex isomorphism from G_1 to G_2 iff

(1) α is one to one and

(2) If $v_1, v_2 \in V(G_1)$ and k edges ($k \geq 0$) of G_1 have v_1 as one end and v_2 as the other end, then k edges of G_2 have $\alpha(v_1)$ as one end and $\alpha(v_2)$ as the other end.

A mapping σ from $E(G_1)$ onto $E(G_2)$ is an edge isomorphism from G_1 to G_2 iff

(1) σ is one to one and

(2) If $e_1, e_2 \ldots e_k$ are incident at a single node in G_1, then $\sigma(e_1), \sigma(e_2) \ldots \sigma(e_k)$ are incident at a single node in G_2, and

(3) If e_1 is a loop (coloop) in G_1, then $\sigma(e_1)$ is a loop (coloop) in G_2.

Two graphs G_1 and G_2 are said to be isomorphic iff there exists a vertex automorphism from G_1 to G_2. We can see that this is equivalent to the following :

Two graphs G_1 and G_2 are said to be isomorphic iff there exists an edge automorphism from G_1 to G_2.

Section 8 : Graphs and vector spaces

Definition 0.62. Oriented graph : A graph G, each of whose edges has a positive and a negative end, is an oriented graph.

Definition 0.63. Let G be an oriented graph. We define the integer $\eta(e,v)$ for each $e \in E(G)$ and $v \in V(G)$ as follows :

$$\eta(e,v) = \begin{cases} 0 & \text{if } v \text{ is not an end of } e \text{ or } e \text{ has coincident ends} \\ 1 & \text{if } v \text{ is the positive end of } e \\ -1 & \text{if } v \text{ is the negative end of } e. \end{cases}$$

Definition 0.64. Let G be an oriented graph and let $E(G) = \{e_1, \ldots, e_n\}$ and let f be a vector on E(G) over the field of real numbers F. We call f a 1-cycle of G over F iff

$$\sum_{i=1}^{n} \eta(e_i, v)\, f(e_i) = 0 \qquad \text{for all } v \in V(G).$$

Definition 0.65. Let G be an oriented graph and V the class of 1-cycles of G over F. It is clear that V is a vector space on E(G) over F.

V is called the 1-cycle space of G.

Theorem T36. Let G be an oriented graph on E(G) = S and V be its 1-cycle space. Then if $\mathcal{E}_V$ is the class of elementary vectors of V,

$$\text{Pol}\ (G) \cong (S, \mathcal{E}_V) .$$

Definition 0.66. Any representative matrix of the 1-cycle space V of the oriented graph G, is called a circuit matrix. A standard representative matrix of V is called as a fundamental circuit matrix.

<u>Definition 0.67.</u> Let G be an oriented graph. Let g be a vector defined on V(G) over F, where $V(G) = \{v_1, \dots v_m\}$. We define a vector f on E(G) over F by

$$f(e_i) = \sum_{j=1}^{m} \eta(e_i, v_j)\, g(v_j) \quad \text{for all} \quad e_i \in E(G).$$ f is

called the coboundary of g.

<u>Definition 0.68.</u> Let V' be the collection of all coboundaries of all vectors g on V(G) over F. It is clear that V' is a vector space on E(G) over F. V' is called the coboundary space of G.

<u>Theorem T37.</u> Let G be an oriented graph on E(G) = S and V' be its coboundary space. Then if $\mathcal{C}_{V'}$ is the class of elementary vectors of V',

$$\mathrm{Bon}(G) \cong (S, \mathcal{C}_{V'}) .$$

<u>Definition 0.69.</u> Any representative matrix of the coboundary space V' of an oriented graph G is called a cutset matrix. A standard representative matrix of V' is called a fundamental cutset matrix.

<u>Theorem T38.</u> Let V be the 1-cycle space of the oriented graph G and V' be the coboundary space of G. Then $V' = (V)^*$.

CHAPTER 1

DIAKOPTICS FOR ELECTRICAL NETWORKS

In this Chapter we give a self-contained and fairly rigorous account of 'Network Analysis by Tearing' in the slightly abstract setting of 'generalised networks'.

'Diakoptics' is a term coined by G. Kron for piecewise analysis of large scale systems [Kr 1]. Kron has claimed that his theory is applicable to numerous fields such as linear programming, hydrodynamics, elasticity, crystal optics etc.

We have, however, confined our interest only to electrical networks. We, therefore, do not need to use tensors; vectors and linear transformations being sufficient for our purpose. We have permitted the presence of R, L, C elements and mutual inductances.

In [Br 1] Bruno and Weinberg have used regular matroid theory to examine n-port resistance networks. (A consequence of such a study is that one obtains a clear and accurate picture of duality in such networks). This is done through the use of an interesting abstraction of ordinary networks which they have termed 'generalized networks'. In this Chapter, in the main, we follow their line of thinking. However, our interest is primarily in methods of network analysis. For this study one basically needs the following ideas :

(1) The coboundary space and the 1-cycle space are vector spaces over the field of real numbers.

(2) These spaces are orthogonal complements of each other.

(3) It is convenient to imagine each edge to be a composite entity being composed of a voltage source, a current source and a passive element in the usual manner (as in Fig. 1.1). Of course one or two of these elements might be absent in a particular edge. Actual analysis is carried out conveniently in a network which has no edge composed wholly of sources. For this we need the idea of transportation of sources.

(4) There is no need to restrict oneself to resistance networks. The methods that we discuss are equally valid with mutual inductances. We, therefore, assume a symmetric positive definite immittance matrix.

(5) We do not need the fact that the fundamental circuit or cutset matrix is unimodular.

(6) We do need the idea of matroids of vector spaces in order to bring out the duality of the methods of analysis. We deal with duality in a fairly simple manner. We develop the theory in terms of certain vectors, matrices and matroids obeying certain conditions. At no stage in our analysis do we use properties which are possessed by current vectors but not by voltage vectors or vice versa. Hence from this theory one can obtain the dual methods of analysis by making substitution of

dual entities in our vectors, matrices, matroids etc. Consideration of the above leads us to the conclusion that we can afford to be more general than in [Br 1]. Our model of an electrical network, therefore, differs from that of Bruno and Weinberg. Hence our definition of a 'generalised network' is different.

It must be emphasized that the abstraction carried out is not just for the 'sake of abstraction'. Our abstraction has two main aims :-

(1) to define basic concepts precisely and

(2) to develop a theory that brings out the duality aspects clearly.

A justification of our approach is yielded by the fact that the insight we gain by it has enabled us to develop a simple and new method of analysis by tearing (see Case III) without which Kron's theory cannot be regarded as complete.

In Section 1 we define and study generalized networks. We develop certain results which enable us to confine our attention to 'canonical generalised networks' which correspond to networks in which each source is 'accompanied'. We also show that solving for a certain mixture of the network variables is equivalent to obtaining the solution of the network. In Section 2 we develop and justify Kron's method of Diakoptics and in addition, give a new extension which, in a sense, completes his theory.

Section 1

Generalized Networks

Let V be a vector space on a finite set S over the field of real numbers F. Let M_V be the matroid associated with V. The set S is partitioned into three sub-sets S_o, S_u and S_w i.e. $S = S_o \cup S_u \cup S_w$.

Let $S_o = \{e_1, e_2, \ldots e_p\}$, $S_o \equiv$ 'The set of ordinary elements'

$$S_u = \{e_{p+1}, \ldots\ldots e_q\}$$

$$S_w = \{e_{q+1}, \ldots\ldots e_n\} .$$

Definition 1.1. We now define the solution vectors $\underline{u}$, $\underline{v}$ and the source vectors $\bar{\underline{u}}$, $\bar{\underline{v}}$, with $\underline{u}_i = \underline{u}(e_i)$, $\underline{v}_i = \underline{v}(e_i)$, $\bar{\underline{u}}_i = \bar{\underline{u}}(e_i)$, $\bar{\underline{v}}_i = \bar{\underline{v}}(e_i)$, $(e_i \in S)$. We partition the vectors $\underline{u}$, $\underline{v}$, $\bar{\underline{u}}$, $\bar{\underline{v}}$ in conformity with S_o, S_u, S_w as follows :

$$\underline{u} = \begin{bmatrix} \underline{u}_o \\ --- \\ \underline{u}_u \\ --- \\ \underline{u}_w \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} \underline{v}_o \\ --- \\ \underline{v}_u \\ --- \\ \underline{v}_w \end{bmatrix}, \quad \bar{\underline{u}} = \begin{bmatrix} \bar{\underline{u}}_o \\ --- \\ \bar{\underline{u}}_u \\ --- \\ \bar{\underline{u}}_w \end{bmatrix}, \quad \bar{\underline{v}} = \begin{bmatrix} \bar{\underline{v}}_o \\ --- \\ \bar{\underline{v}}_u \\ --- \\ \bar{\underline{v}}_w \end{bmatrix}.$$

where

$\underline{u}_o^t = (u_1, u_2 \ldots u_p)$, $\underline{u}_u^t = (u_{p+1}, \ldots u_q)$

$\underline{u}_w^t = (u_{q+1}, \ldots u_n)$ and $\underline{v}_o^t$, $\underline{v}_u^t$, $\underline{v}_w^t$; $\bar{\underline{u}}_o^t$, $\bar{\underline{u}}_u^t$, $\bar{\underline{u}}_w^t$, $\bar{\underline{v}}_o^t$, $\bar{\underline{v}}_u^t$, $\bar{\underline{v}}_w^t$ etc. have similar meanings.

We further stipulate

$$\underline{u}_u = \bar{\underline{u}}_u \quad , \quad \bar{\underline{u}}_w = \underline{0} \ ;$$

$$\underline{v}_w = \bar{\underline{v}}_w \quad , \quad \bar{\underline{v}}_u = \underline{0} \ .$$

Our aim here is merely to introduce analogous concepts to the three types of edges in an ordinary electrical network — (1) the set S_o of edges each of which is a composite edge associated with a passive element, a voltage source and a current source as in Figure 1.1. [The direction of positive current flow is in the direction of the current arrow. (The arrow head is not darkened). The direction of the voltage rise is opposite to the direction of the current arrow but in the direction of the voltage arrow. (The arrow head is darkened)]. (2) The other two sets of edges are each composed wholly of one type of source i.e. if one takes $\bar{\underline{v}}$ ($\bar{\underline{u}}$) to be voltage sources and $\bar{\underline{u}}$($\bar{\underline{v}}$) to be current sources, then S_u is composed wholly of current sources (voltage sources) and S_w is composed wholly of voltage sources (current sources).

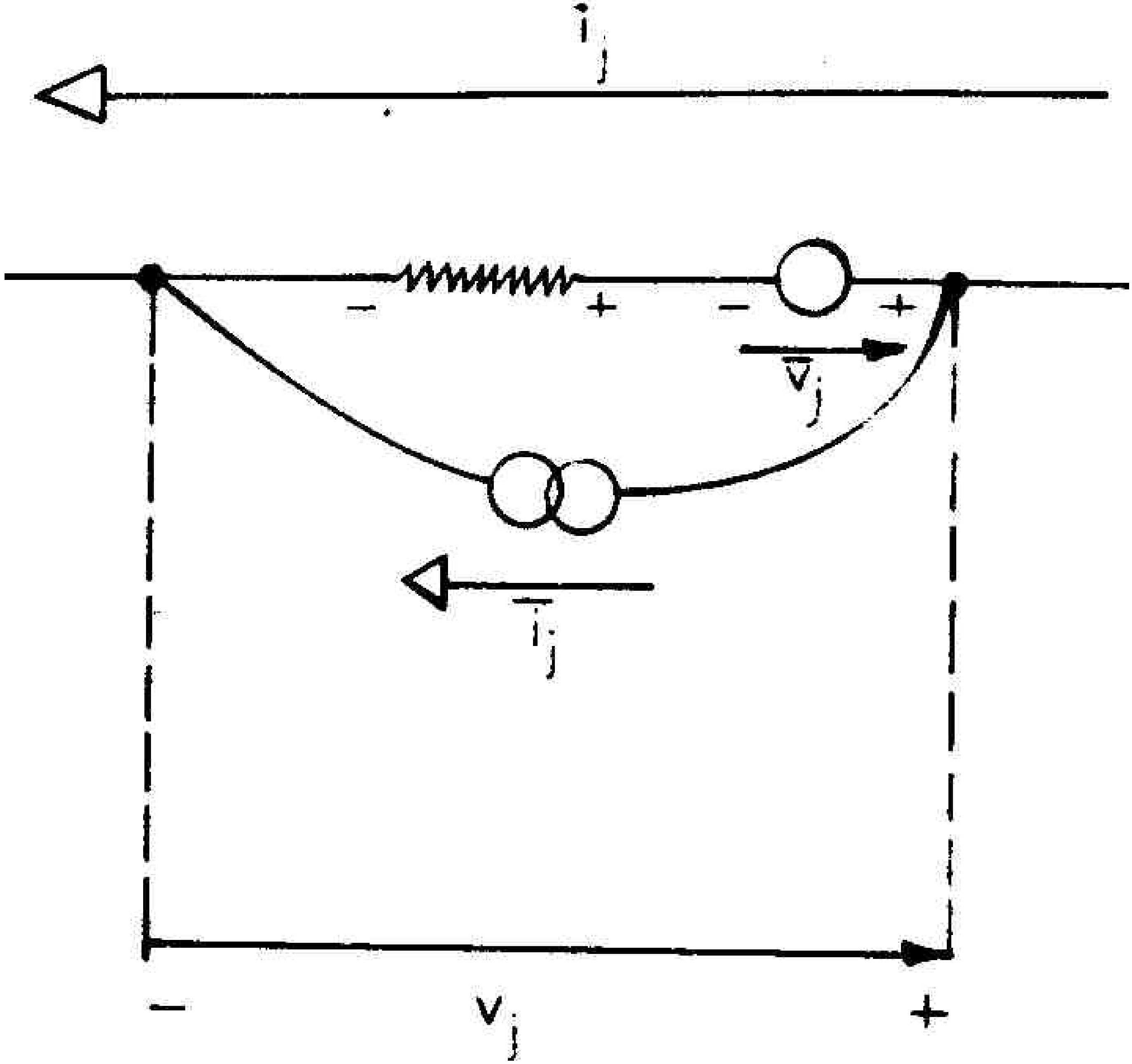


FIG.1.1.

THE 'ORDINARY' EDGE

Next we stipulate

$$[\underline{v}_o - \underline{\hat{v}}_o] = G[\underline{u}_o - \underline{\hat{u}}_o]$$

where G is a real symmetric positive definite matrix.

G serves as our 'immittance matrix'. There is some possibility of confusion here. In actual practice, when one works in the frequency domain, G could be composed of three types of elements —

(1) R_i a positive element

(2) sL_{jj} and sL_{jk}, where L_{jk}, L_{jj} are positive and s is a complex variable

(3) $\frac{1}{sC_{pp}}$, C_{pp} a positive element.

One then obtains certain matrices, involving s, to be inverted. In order to justify the Diakoptics procedure, one has to show that these matrices are non-singular at least for some values of s. What we are doing here is to show this for all positive real values of s.

G is non-singular by Theorem T 28. We denote G^{-1} by H.

We now introduce conditions equivalent to KCL and KVL. This we do by stipulating that $\underline{u}^t$ be the representative vector of some member of V, and $\underline{v}^t$ be the representative vector of some member of V^*. As there is no possibility of confusion, in the interest of simplicity we write the generalized KCL and KVL as

$\underline{u} \in V$ (instead of 'there exists a vector $f \in V$ such that $\underline{u}^t$ is a representative vector for f')

$\underline{w} \in V^*$.

We are now in a position to define a generalized network.

<u>Definition 1.2.</u> A generalized network N is a quadruple (M_V, V, G, S), where S is a finite set, V is a vector space on S over the real field F, M_V is the matroid associated with V and G is a real symmetric positive definite matrix. We may often call N just a 'network', since only generalized networks are considered in this Chapter.

The generalized network equations are

$$\underline{u} \in V \qquad .. \quad (1)$$

$$\underline{w} \in V^* \qquad .. \quad (2)$$

$$\underline{w}_o - \bar{\underline{w}}_o = G(\underline{u}_o - \bar{\underline{u}}_o) \qquad .. \quad (3)$$

The set S_o is the set of ordinary elements while S_u, S_w are sets of source elements. (Equations (1) and (2) are the 'topological' constraints on $\underline{u}$ and $\underline{w}$ while matrix G corresponds to the immittance matrix).

<u>Definition 1.3.</u> A pair of vectors ($\underline{u}, \underline{w}$), satisfying the generalized network equations (1), (2) and (3), we will call a 'solution' of the generalized network for the source vector pair ($\bar{\underline{u}}, \bar{\underline{w}}$). When the source vectors are clear from the context we will call ($\underline{u}, \underline{w}$) merely as the 'solution'.

Analysis of Generalized Networks

We will now consider the problem of obtaining the solution ($\underline{u}$, $\underline{v}$) when the source vector pair ($\bar{\underline{u}}$, $\bar{\underline{v}}$) is given. This procedure we will term 'solving' the generalized network.

Definition 1.4. A generalized network $N = (M_V, V, G, S)$ is said to be a U.S.G. (uniquely solvable generalized network) network iff

(1) S_u is contained in a cobase of M_V, and
(2) S_w is contained in a cobase of $(M_V)^* = M_{V^*}$.

One of our aims in the succeeding pages is to show that this name is justified for such networks i.e. if N is a U.S.G. network, for arbitrary source vectors $\bar{\underline{u}}$, $\bar{\underline{v}}$ there exists a unique solution ($\underline{u}$, $\underline{v}$).

We note here that if $f \in V$ and x^t is a representative vector of f, we write $x \in V$; also we define $\|x\| = \|f\|$.

The following theorem is well known for ordinary networks. In [Br 1] this theorem is proved for a slightly different kind of 'generalized network'. The proof however is essontially the same.

Theorem 1.1. Let N be a generalized network that is not U.S.G. Then there exist source vectors $\bar{\underline{u}}$, $\bar{\underline{v}}$ such that N has no solution.

Proof. Let N be not U.S.G. Since the network equations are obviously symmetric with respect to V and V^* (and therefore M_V and M_V^*) we need merely consider the case where condition (1) of Definition 1.4 is violated.

Suppose S_u can be contained in no cobase of M_V. Then it is clear from Theorem T4, that there exists a circuit C of $(M_V)^*$ such that $C \subseteq S_u$. Then there exists a vector $\underline{x} \in V^*$ such that $\|x\| = C$ by the definition of M_V .

By the generalized network equations

$$\underline{x}^t \underline{u} = 0 \quad \text{i.e.} \quad \underline{x}^t \underline{u}_u = 0 \; .$$

This shows that there exists a linear relationship between the coordinates of $\underline{u}_u$. But by definition of source vectors we have $\underline{u}_u = \bar{\underline{u}}_u$.

Hence there exists a linear relationship between the coordinates of $\bar{\underline{u}}_u$ which means that if we choose $\bar{\underline{u}}_u$ violating this linear relationship, the network would have no solution.

Q.E.D.

The reader may note that U.S.G. networks correspond to networks which do not have a cutset composed wholly of current sources or a circuit composed wholly of voltage sources.

<u>Definition 1.5.</u> Two generalized networks $N_1 = (M_V, V, G, S)$ and $N_2 = (M_{V_1}, V_1, G, S_1)$ are said to be similar iff there exists a bijection $\tau : S_o \rightarrow S_{1o}$ (from the set of ordinary elements of N_1 to the set of ordinary elements of N_2) such that

for any specification of sources ($\bar{\underline{u}}$, $\bar{\underline{v}}$) for network N_1, there exist sources ($\bar{\underline{u}}'$, $\bar{\underline{v}}'$) for N_2 such that if ($\underline{u}$, $\underline{v}$), ($\underline{u}'$, $\underline{v}'$) are any corresponding solutions for N_1 and N_2, the following conditions hold :

when $e_i \in S_o$, $\tau(e_i) \in S_{10}$, then

(1) $\underline{u}_i - \bar{\underline{u}}_i = \underline{u}'_{\tau(i)} - \bar{\underline{u}}'_{\tau(i)}$ and

(2) $\underline{v}_i - \bar{\underline{v}}_i = \underline{v}'_{\tau(i)} - \bar{\underline{v}}'_{\tau(i)}$.

It is easy to see that similarity is an equivalence relation.

<u>Definition 1.6.</u> A generalized network $N_1 = (M_V, V, G, S)$ is said to be in canonical form iff

$S_u = S_v = \phi$. We will call N_1 a canonical network.

We will find it convenient to handle canonical networks. (Theorem 1.2 indicates this). If we could transform any network into a canonical network whose solutions corresponds to the solutions of the original network, we could be satisfied with considering methods of analysis for canonical networks alone. Theorem 1.3 describes such a transformation.

<u>Theorem 1.2.</u> Let $N = (M_V, V, G, S)$ be a canonical network. Let $\bar{\underline{u}}$, $\bar{\underline{v}}$ be its source vectors. Then N has a unique solution ($\underline{u}$, $\underline{v}$).

Proof. Let R^{*} be a standard representative matrix of V^{*} with respect to a base b of M_V. We then have the following condition on $\underline{u}$.

$$R^{*}\,\underline{u} = \underline{0} \;.$$

Let R^{*} be partitioned as follows :

$$R^{*} = \begin{array}{c|c} b & S-b \\ \end{array} \left(\begin{array}{c|c} U & R_{12}^{*} \end{array}\right)$$; let $\underline{u}$ be partitioned in conformity with R^{*} as

$$\underline{u} = \begin{pmatrix} \underline{u}_1 \\ \hline \underline{u}_2 \end{pmatrix} \;.$$

Then

$$\left(\begin{array}{c|c} U & R_{12}^{*} \end{array}\right) \begin{pmatrix} \underline{u}_1 \\ \hline \underline{u}_2 \end{pmatrix} = \underline{0} \;.$$

Hence

$$\left(\begin{array}{c|c} U & R_{12}^{*} \end{array}\right) \begin{pmatrix} \underline{u}_1 - \bar{\underline{u}}_1 \\ \hline \underline{u}_2 - \bar{\underline{u}}_2 \end{pmatrix} = - \left(\begin{array}{c|c} U & R_{12}^{*} \end{array}\right) \begin{pmatrix} \bar{\underline{u}}_1 \\ \hline \bar{\underline{u}}_2 \end{pmatrix} .$$

Since $\underline{u} - \bar{\underline{u}} = H(\underline{v} - \bar{\underline{v}})$ for a canonical network by (3) in Definition 1.2, we impose this condition next.

$$\left[U \;\middle|\; R_{12}^{*}\right] H \begin{pmatrix} \underline{v}_1 - \bar{\underline{v}}_1 \\ \hline \underline{v}_2 - \bar{\underline{v}}_2 \end{pmatrix} = - \left[U \;\middle|\; R_{12}^{*}\right] \begin{pmatrix} \bar{\underline{u}}_1 \\ \hline \bar{\underline{u}}_2 \end{pmatrix} .$$

Hence

$$\left[U \;\middle|\; R_{12}^{*}\right] H \begin{pmatrix} \underline{v}_1 \\ \hline \underline{v}_2 \end{pmatrix} = \left[U \;\middle|\; R_{12}^{*}\right] \left[H \begin{pmatrix} \bar{\underline{v}}_1 \\ \hline \bar{\underline{v}}_2 \end{pmatrix} - \begin{pmatrix} \bar{\underline{u}}_1 \\ \hline \bar{\underline{u}}_2 \end{pmatrix} \right] \quad .. \quad (A)$$

$$= \bar{\underline{u}}' \text{ say.}$$

But by (2) of Definition 1.2, $\underline{v} \in V^{*}$. Hence we must have

$$\begin{pmatrix} \underline{v}_1 \\ \hline \underline{v}_2 \end{pmatrix} = \begin{pmatrix} U \\ \hline (R_{12}^{*})^t \end{pmatrix} \left[\underline{v}_1 \right] .$$

Hence

$$\left[U \;\middle|\; R_{12}^{*}\right] H \begin{pmatrix} U \\ \hline (R_{12}^{*})^t \end{pmatrix} \left[\underline{v}_1 \right] = \bar{\underline{u}}' .$$

Let us call the coefficient matrix on the L.H.S. as $\widetilde{H}$. Now by definition G is positive definite. Hence, by Theorem T28 H is positive definite. Hence

$$\widetilde{H} = \left[U \;\middle|\; R_{12}^{*}\right] H \begin{pmatrix} U \\ \hline (R_{12}^{*})^t \end{pmatrix} \quad \text{is positive definite}$$

by Theorem T29 and is therefore non-singular.

$$\widetilde{\hat{H}}\ \underline{v}_1 = \bar{\underline{u}}' \quad .$$

Hence $$\underline{v}_1 = (\ \widetilde{\hat{H}}\ )^{-1}\ \bar{\underline{u}}' \quad .$$

Clearly $\underline{v}_1$ is uniquely determined for given $\bar{\underline{u}}$ and $\bar{\underline{v}}$. Now

$$\underline{v}_2 = (\ R_{12}^{*}\ )^{t}\ \underline{v}_1 \quad .$$ Hence $\underline{v}$ is uniquely determined.

Also, since N is a canonical network,

$$\underline{u} - \bar{\underline{u}} = H\ (\ \underline{v} - \bar{\underline{v}}\ ) \quad .$$

Thus $\underline{u}$ is uniquely determined. It is easy to verify that the vectors $\underline{u}$ and $\underline{v}$ that we have obtained above do indeed satisfy conditions (1), (2) and (3) of Definition 1.2. Hence ($\underline{u}$, $\underline{v}$) is the unique solution of the network N for the given source pair ($\bar{\underline{u}}$, $\bar{\underline{v}}$).

Q.E.D.

<u>Theorem 1.3.</u> Let N = (M_V, V, G, S) be a U.S.G. network. Then there exists a canonical network N_1 = (M_{V_1}, V_1, G, S_1) such that N and N_1 are similar.

<u>Proof.</u> Let R^* be the standard representative matrix for V^* with respect to a base b of M_V. Let R be the standard representative matrix for V with respect to the cobase S - b of M_V. R, R^* are partitioned according to $\underline{u}$, $\underline{v}$ as follows :

$$R = \begin{array}{c|c|c} S_o & S_u & S_v \\ \end{array} \quad\quad R^{*} = \begin{array}{c|c|c} S_o & S_u & S_v \\ \end{array}$$

$$R = \left[\begin{array}{c|c|c} R_o & R_u & R_v \end{array}\right] \quad , \quad R^{*} = \left[\begin{array}{c|c|c} R_o^{*} & R_u^{*} & R_v^{*} \end{array}\right]$$

Let us take b to be such that $b \supseteq S_w$ and $b \cap S_u = \phi$.
This is possible by Theorem T6. We can then further partition R as

$$R = \begin{array}{c} \\ \left[\begin{array}{c|c|c|c} S_o \cap (S-b) & S_o \cap b & S_u & S_w \\ \hline U & R_{12} & 0 & R_{14} \\ 0 & R_{22} & U & R_{24} \end{array} \right] \end{array}$$

By Theorem T18, R^{*} can then be expressed as

$$R^{*} = \left[\begin{array}{c|c|c|c} S_o \cap (S-b) & S_o \cap b & S_u & S_w \\ \hline -R_{12}^{\,t} & U & -R_{22}^{\,t} & 0 \\ -R_{14}^{\,t} & 0 & -R_{24}^{\,t} & U \end{array} \right]$$

Consider the matroid $M_V \times (S_o \cup S_w)$. $S_o = M_V .(S_o \cup S_u) \times S_o = M$
$(V' = (V \times (S_o \cup S_w)) . S_o)$. By the use of Theorem T21, V' has the standard representative matrix R' with respect to $S_o \cap$ (S-

$$R' = \left[\begin{array}{c|c} S_o \cap (S-b) & S_o \cap b \\ \hline U & R_{12} \end{array} \right]$$

By Theorem T18, $(R')^{*}$, the standard representative matrix of $(V')^{*}$ with respect to the base $b \cap S_o$ of $M_{V'}$, has the following form :

$$(R')^{*} = \begin{pmatrix} \overset{S_o \cap (S-b)}{-R_{12}^{t}} & \overset{S_o \cap b}{U} \end{pmatrix} .$$

Now suppose N has the source vectors

$$\bar{\underline{u}} = \begin{pmatrix} \bar{\underline{u}}_{10} \\ --- \\ \bar{\underline{u}}_{20} \\ --- \\ \bar{\underline{u}}_{u} \\ --- \\ \bar{\underline{u}}_{v} \end{pmatrix}, \quad \bar{\underline{v}} = \begin{pmatrix} \bar{\underline{v}}_{10} \\ --- \\ \bar{\underline{v}}_{20} \\ --- \\ \bar{\underline{v}}_{u} \\ --- \\ \bar{\underline{v}}_{v} \end{pmatrix} ,$$

the partitions of $\bar{\underline{u}}$ and $\bar{\underline{v}}$ being carried out in conformity with the partitioning of matrices R and R^{*}.

Let the corresponding solution of N be ($\underline{u}$, $\underline{v}$) such that

$$\underline{u}^{t} = (\underline{u}_{10}^{t} \mid \underline{u}_{20}^{t} \mid \underline{u}_{u}^{t} \mid \underline{u}_{v}^{t}) ,$$

$$\underline{v}^{t} = (\underline{v}_{10}^{t} \mid \underline{v}_{20}^{t} \mid \underline{v}_{u}^{t} \mid \underline{v}_{v}^{t}) .$$

Consider the canonical network $N_1 = (M_{V'}, V', G, S_o)$ with the sources

$$\bar{\underline{u}}' = \begin{pmatrix} \bar{\underline{u}}_{10} \\ ---------- \\ \bar{\underline{u}}_{20} - R_{22}^{t} \bar{\underline{u}}_{u} \end{pmatrix} \begin{matrix} S_o \cap (S-b) \\ \\ S_o \cap b \end{matrix} , \quad \bar{\underline{v}}' = \begin{pmatrix} \bar{\underline{v}}_{10} + R_{14} \bar{\underline{v}}_{v} \\ ---------- \\ \bar{\underline{v}}_{20} \end{pmatrix} \begin{matrix} S_o \cap (S-b) \\ \\ S_o \cap b \end{matrix}$$

Then we claim that the pair ($\underline{u}'$, $\underline{v}'$) defined below forms a solution (and therefore by Theorem 1.2, a unique solution) for N_1. We have to show that

(1) $R' \underline{v}' = \underline{0}$, (2) $(R')^* \underline{u}' = \underline{0}$ and

(3) $(\underline{v}_o' - \bar{\underline{v}}_o') = G (\underline{u}_o' - \bar{\underline{u}}_o')$.

From the generalized network equations for N we have

(1) $R \underline{v} = \underline{0}$.

Hence

$$\left(U \;\vdots\; R_{12} \;\vdots\; 0 \;\vdots\; R_{14} \right) \begin{pmatrix} \underline{v}_{10} \\ --- \\ \underline{v}_{20} \\ --- \\ \underline{v}_u \\ --- \\ \underline{v}_v \end{pmatrix} = \underline{0} \qquad \text{But } \underline{v}_v = \bar{\underline{v}}_v .$$

Therefore

$$\left(U \;\vdots\; R_{12} \right) \begin{pmatrix} \underline{v}_{10} + R_{14} \bar{\underline{v}}_v \\ \text{---------} \\ \underline{v}_{20} \end{pmatrix} = \underline{0}$$

Again, from the equations of N,

$$R^* \underline{u} = \underline{0} .$$

Hence

$$\left[-R_{12}^{t} \;\middle|\; U \;\middle|\; -R_{22}^{t} \;\middle|\; 0 \right] \begin{bmatrix} \underline{u}_{10} \\ --- \\ \underline{u}_{20} \\ --- \\ \underline{u}_{u} \\ --- \\ \underline{u}_{w} \end{bmatrix} = \underline{0} \; .$$

But $\underline{u}_u = \bar{\underline{u}}_u$. Therefore

$$\left[-R_{12}^{t} \;\middle|\; U \right] \begin{bmatrix} \underline{u}_{10} \\ ---------- \\ \underline{u}_{20} - R_{22}^{t} \, \bar{\underline{u}}_{u} \end{bmatrix} = \underline{0} \; .$$

We now define

$$\underline{u}' = \begin{bmatrix} \underline{u}_{10} \\ ---------- \\ \underline{u}_{20} - R_{22}^{t} \, \bar{\underline{u}}_{u} \end{bmatrix} \; , \qquad \underline{v}' = \begin{bmatrix} \underline{v}_{10} + R_{14} \, \bar{\underline{v}}_{w} \\ ---------- \\ \underline{v}_{20} \end{bmatrix} \; .$$

It is easy to see that

$$\underline{u}_o - \bar{\underline{u}}_o = \underline{u}' - \bar{\underline{u}}' = \underline{u}_o' - \bar{\underline{u}}_o' \quad (S_u = S_v = \phi \text{ for } N_1)$$

$$\underline{v}_o - \bar{\underline{v}}_o = \underline{v}' - \bar{\underline{v}}' = \underline{v}_o' - \bar{\underline{v}}_o' \; .$$

Since $(\underline{v}_o - \bar{\underline{v}}_o) = G (\underline{u}_o - \bar{\underline{u}}_o)$, it follows that

$$(\underline{v}_o' - \bar{\underline{v}}_o') = G (\underline{u}_o' - \bar{\underline{u}}_o') \; .$$

Thus the pair ($\underline{u}'$, $\underline{v}'$) satisfies the generalized network equations of N_1 and also the conditions of similarity with N.

We have shown that for any source pair ($\bar{\underline{u}}$, $\bar{\underline{v}}$) of N and a corresponding solution ($\underline{u}$, $\underline{v}$) we can specify a source pair ($\bar{\underline{u}}'$, $\bar{\underline{v}}'$) of N_1 such that the corresponding unique solution of N_1 satisfies the conditions of similarity of N with N_1. Clearly this implies that N and N_1 are similar. Since N_1 is canonical, this proves the theorem.

Q.E.D.

<u>Corollary.</u> Let N = (M_V, V, G, S) be a U.S.G. network. Then if $\bar{\underline{u}}$, $\bar{\underline{v}}$ are a pair of source vectors for N, there exists a unique solution ($\underline{u}$, $\underline{v}$) of N for this source pair.

<u>Proof.</u> We will first show that if a solution for N exists it is unique. Choose a base b of M_V such that $b \supseteq S_v$, $b \cap S_u = \Phi$. As in Theorem 1.3, take R as follows :

$$R = \begin{array}{c|c|c|c} S_o \cap (S-b) & S_o \cap b & S_u & S_v \\ \hline U & R_{12} & 0 & R_{14} \\ 0 & R_{22} & U & R_{24} \end{array}$$

Take V' to be the vector space, $V' = V \times (S_o \cup S_v) \cdot S_o$.

Then $M_{V'} = M_V \times (S_o \cup S_v) \cdot S_o$.

V' has the standard representative matrix R' with respect to $(S_o \cap (S-b))$.

$$R' = \left[\begin{array}{c|c} S_o \cap (S-b) & S_o \cap b \\ U & R_{12} \end{array} \right]$$

$(V')^*$ has the standard representative matrix $(R')^*$ with respect to $S_o \cap b$,

$$(R')^* = \left[\begin{array}{c|c} S_o \cap (S-b) & S_o \cap b \\ -R_{12}^t & U \end{array} \right]$$

The network $N_1 = (M_{V'}, V', G, S_o)$ is canonical and has a unique solution $(\underline{u}', \underline{v}')$ for the source pair $(\bar{\underline{u}}', \bar{\underline{v}}')$

$$\bar{\underline{u}}' = \left(\begin{array}{c} \bar{\underline{u}}_{10} \\ \hline \bar{\underline{u}}_{20} - R_{22}^t \bar{\underline{u}}_u \end{array} \right) , \quad \bar{\underline{v}}' = \left(\begin{array}{c} \bar{\underline{v}}_{10} + R_{14} \bar{\underline{v}}_v \\ \hline \bar{\underline{v}}_{20} \end{array} \right) .$$

(We have taken $\bar{\underline{u}}, \bar{\underline{v}}$ to be partitioned in accordance with R so that

$$\bar{\underline{u}}^t = (\bar{\underline{u}}_{10}^t \mid \bar{\underline{u}}_{20}^t \mid \bar{\underline{u}}_u^t \mid \bar{\underline{u}}_v^t) ; \quad \bar{\underline{v}}^t = (\bar{\underline{v}}_{10}^t \mid \bar{\underline{v}}_{20}^t \mid \bar{\underline{v}}_u^t \mid \bar{\underline{v}}_v^t)).$$

If now $(\underline{u}, \underline{v})$ is any solution of N for the source pair $(\bar{\underline{u}}, \bar{\underline{v}})$ we have by Theorem 1.3,

$$(\underline{u}' - \bar{\underline{u}}') = (\underline{u}_o - \bar{\underline{u}}_o)$$

$$(\underline{v}' - \bar{\underline{v}}') = (\underline{v}_o - \bar{\underline{v}}_o)$$

(where

$$\underline{u}^t = (\underline{u}_{10}{}^t \mid \underline{u}_{20}{}^t \mid \underline{u}_u{}^t \mid \underline{u}_w{}^t) ; \quad \underline{v}^t = (\underline{v}_{10}{}^t \mid \underline{v}_{20}{}^t \mid \underline{v}_u{}^t \mid \underline{v}_w{}^t)$$

$$= (\quad \underline{u}_o{}^t \quad \mid \underline{u}_u{}^t \mid \underline{u}_w{}^t) ; \quad = (\quad \underline{v}_o{}^t \quad \mid \underline{v}_u{}^t \mid \underline{v}_w{}^t)).$$

Hence $\underline{u}_o$, $\underline{v}_o$ are uniquely determined.

Now, if ($\underline{u}$, $\underline{v}$) is a solution of N, then $\underline{u} \in V$, and $\underline{v} \in V^*$.

Hence

$$\begin{pmatrix} \underline{u}_{10} \\ \hline \underline{u}_{20} \\ \hline \underline{u}_u \\ \hline \underline{u}_w \end{pmatrix} = \left(\begin{array}{c|c} U & 0 \\ \hline R_{12}{}^t & R_{22}{}^t \\ \hline 0 & U \\ \hline R_{14}{}^t & R_{24}{}^t \end{array}\right) \begin{pmatrix} \underline{u}_{10} \\ \hline \underline{u}_u \end{pmatrix}$$

But by definition $\underline{u}_u = \bar{\underline{u}}_u$. Since $\underline{u}_{10}$ and $\underline{u}_u$ are uniquely determined it follows that $\underline{u}$ is uniquely determined. Similarly we can show that $\underline{v}$ is uniquely determined by

$$\begin{pmatrix} \underline{v}_{10} \\ \hline \underline{v}_{20} \\ \hline \underline{v}_u \\ \hline \underline{v}_w \end{pmatrix} = \left(\begin{array}{c|c} -R_{12} & -R_{14} \\ \hline U & 0 \\ \hline -R_{22} & -R_{24} \\ \hline 0 & U \end{array}\right) \begin{pmatrix} \underline{v}_{20} \\ \hline \underline{v}_w \end{pmatrix}$$

Thus if a solution of N exists for ($\bar{\underline{u}}$, $\bar{\underline{v}}$), it is unique. We will now give one solution of N.

$$\underline{u} = \left(\begin{array}{c} \underline{u}_1' \\ \hline \underline{u}_2' + R_{22}^t \, \bar{\underline{u}}_u \\ \hline \bar{\underline{u}}_u \\ \hline R_{14}^t \, \underline{u}_{10} + R_{24}^t \, \bar{\underline{u}}_u \end{array} \right) , \quad \underline{v} = \left(\begin{array}{c} \underline{v}_1' - R_{14} \, \bar{\underline{v}}_w \\ \hline \underline{v}_2' \\ \hline -R_{22} \underline{v}_2' - R_{24} \, \bar{\underline{v}}_w \\ \hline \bar{\underline{v}}_w \end{array} \right)$$

It is easy to see that ($\underline{v}_o - \bar{\underline{v}}_o$) = G ($\underline{u}_o - \bar{\underline{u}}_o$) , since ($\underline{v}_o - \bar{\underline{v}}_o$) = ($\underline{v}' - \bar{\underline{v}}'$) , ($\underline{u}_o - \bar{\underline{u}}_o$) = ($\underline{u}' - \bar{\underline{u}}'$) .

$$R \, \underline{v} = \left(\begin{array}{c|c|c|c} U & R_{12} & 0 & R_{14} \\ \hline 0 & R_{22} & U & R_{24} \end{array} \right) \left(\begin{array}{c} \underline{v}_1' - R_{14} \, \bar{\underline{v}}_w \\ \hline \underline{v}_2' \\ \hline -R_{22} \underline{v}_2' - R_{24} \, \bar{\underline{v}}_w \\ \hline \bar{\underline{v}}_w \end{array} \right)$$

$$= \left(\begin{array}{c} \underline{v}_1' - R_{14} \, \bar{\underline{v}}_w + R_{12} \, \underline{v}_2' + R_{14} \, \bar{\underline{v}}_w \\ \hline \underline{0} \end{array} \right)$$

$$= \left(\begin{array}{c} \underline{v}_1' + R_{12} \, \underline{v}_2' \\ \hline \underline{0} \end{array} \right) .$$

Since $(\underline{u}', \underline{v}')$ is a solution of N_1

$$\left(U \;\middle|\; R_{12} \right) \begin{pmatrix} \underline{v}_1' \\ --- \\ \underline{v}_2' \end{pmatrix} = \begin{pmatrix} \underline{0} \\ --- \\ \underline{0} \end{pmatrix} .$$

Hence

$$R \; \underline{v} = \underline{0}$$

Hence $\underline{v} \in V^*$.

In a similar manner one can verify that $\underline{u} \in V$.

Thus $(\underline{u}, \underline{v})$ satisfies conditions (1), (2) and (3) of Definition 1.2. Hence N has a unique solution $(\underline{u}, \underline{v})$ for the source pair $(\bar{\underline{u}}, \bar{\underline{v}})$.

Q.E.D.

The procedure that we have adopted for showing that a U.S.G. network is similar to a canonical network is known as 'source transportation' in ordinary network theory terminology. We will now use this technique to arrange the sources about the network in a convenient manner. The next theorem corresponds to the fact that the current sources can all be taken to be in parallel with some tree branches while the voltage sources can be taken to be in series with some cotree branches.

<u>Theorem 1.4.</u> Let $N = (M_V, V, G, S)$ be a U.S.G. network. There exists a network $N_2 = (M_{V'}, V', G, S_2)$ similar to N such

that, if $\bar{\underline{u}}$, $\bar{\underline{v}}$ are the source vectors for N, a corresponding set of source vectors for N_2 are $\bar{\underline{u}}'$, $\bar{\underline{v}}'$ (so that the resulting solutions of N_2 and N satisfy the condition for similarity of the networks in Definition 1.5) having the following characterist

There exists a base b of M_V, such that

$$\bar{\underline{u}}_i{}' = 0 \text{ if } e_i \notin b \; ; \quad \bar{\underline{v}}_i{}' = 0 \text{ if } e_i \in b \; .$$

<u>Proof.</u> There exists a canonical network N_1 similar to N by Theorem 1.3. Since similarity is an equivalence relation we need prove the theorem just for the case when N is canonical. Let b be a base of M_V.

Let R be the standard representative matrix for V associated with S - b. Let R^* be the standard representative matrix for V^* associated with b.

We could then partition R, R^*, $\underline{u}$, $\underline{v}$, $\bar{\underline{u}}$, $\bar{\underline{v}}$ according to b, S - b as follows :

$$R = \begin{array}{c} \begin{array}{c|c} b & S-b \end{array} \\ \left(\begin{array}{c|c} R_{11} & U \end{array}\right) \end{array} , \qquad R^* = \begin{array}{c} \begin{array}{c|c} b & S-b \end{array} \\ \left(\begin{array}{c|c} U & -R_{11}{}^t \end{array}\right) \end{array}$$

$$\underline{u} = \begin{pmatrix} \underline{u}_1 \\ --- \\ \underline{u}_2 \end{pmatrix} \begin{array}{c} b \\ \\ S-b \end{array} , \quad \underline{v} = \begin{pmatrix} \underline{v}_1 \\ --- \\ \underline{v}_2 \end{pmatrix}$$

$$\bar{\underline{u}} = \begin{pmatrix} \bar{\underline{u}}_1 \\ --- \\ \bar{\underline{u}}_2 \end{pmatrix} , \qquad \bar{\underline{v}} = \begin{pmatrix} \bar{\underline{v}}_1 \\ --- \\ \bar{\underline{v}}_2 \end{pmatrix} .$$

(($\underline{u}$, $\underline{v}$) being the solution for N when the source vectors are $\bar{\underline{u}}$, $\bar{\underline{v}}$).

Let N_2 be the network N itself,(Note that N is canonical), i.e.

$$N_2 = N = (M_V, V, G, S) .$$

But let the new source vectors for N_2 be

$$\bar{\underline{u}}' = \begin{pmatrix} \bar{\underline{u}}_1 - R_{11}{}^t \bar{\underline{u}}_2 \\ \hline \underline{0} \end{pmatrix}, \quad \bar{\underline{v}}' = \begin{pmatrix} \underline{0} \\ \hline \bar{\underline{v}}_2 + R_{11} \bar{\underline{v}}_1 \end{pmatrix} .$$

It is easy to verify that the corresponding solution is

$$\underline{u}' = \begin{pmatrix} \underline{u}_1 - R_{11}{}^t \bar{\underline{u}}_2 \\ \hline \underline{u}_2 - \bar{\underline{u}}_2 \end{pmatrix}, \quad \underline{v}' = \begin{pmatrix} \underline{v}_1 - \bar{\underline{v}}_1 \\ \hline \underline{v}_2 + R_{11} \bar{\underline{v}}_1 \end{pmatrix} ,$$

and that

$$\underline{u}' - \bar{\underline{u}}' = \underline{u} - \bar{\underline{u}}$$

$$\underline{v}' - \bar{\underline{v}}' = \underline{v} - \bar{\underline{v}} .$$

This proves the theorem.

Q.E.D.

While obtaining the solution of a network it is often convenient to solve for some current variables and some voltage variables. The next theorem indicates the kind of mixtures that are equivalent to the solution of the network.

<u>Theorem 1.5.</u> Let $N = (M_V, V, G, S)$ be a canonical network. Let $S = S_1 \cup S_2$ and let b_1 be a base of $M_V \times S_1$, C_2 be a cobase of $M_V \cdot S_2$. Let $\bar{\underline{u}}, \bar{\underline{v}}$ be the source vectors. If G can be partitioned as :

$$G = \left(\begin{array}{c|c} G_{S_1} & 0 \\ \hline 0 & G_{S_2} \end{array}\right) \begin{array}{c} S_1 \\ \\ S_2 \end{array} ,$$

then the determination of the solution ($\underline{u}, \underline{v}$) is equivalent to the determination of $\underline{v}_{b_1}$ and $\underline{u}_{C_2}$ (in the sense that no matrix other than G has to be inverted to arrive at ($\underline{u}, \underline{v}$) from $\underline{u}_{C_2}$ and $\underline{v}_{b_1}$).

<u>Proof.</u> Let b be a base of M_V such that $b \cap S_1 = b_1$. Let R be the standard representative matrix of V associated with S - b, and let R^* be the standard representative matrix of V^* associated with b.

$$\text{Let } S_1 - b_1 = C_1 \;, \quad S_2 - C_2 = b_2 .$$

Then we can partition R, R^*, $\underline{u}$, $\underline{v}$, $\bar{\underline{u}}$, $\bar{\underline{v}}$, G as follows :

$$R = \begin{array}{c} \begin{array}{cccc} C_1 & b_1 & C_2 & b_2 \end{array} \\ \left(\begin{array}{c|c|c|c} U & R_{12} & 0 & R_{14} \\ 0 & R_{22} & U & R_{24} \end{array}\right) \end{array}$$

$$R^* = \begin{array}{c} \\ \\ \end{array}\begin{pmatrix} -R_{12}^t & U & -R_{22}^t & 0 \\ -R_{14}^t & 0 & -R_{24}^t & U \end{pmatrix}$$

(column blocks: C_1 | b_1 | C_2 | b_2)

$$\underline{u}^t = (\underline{u}_{C_1}^t \mid \underline{u}_{b_1}^t \mid \underline{u}_{C_2}^t \mid \underline{u}_{b_2}^t) \quad ,$$

$$\underline{v}^t = (\underline{v}_{C_1}^t \mid \underline{v}_{b_1}^t \mid \underline{v}_{C_2}^t \mid \underline{v}_{b_2}^t) \quad ,$$

$$\bar{\underline{u}}^t = (\bar{\underline{u}}_{C_1}^t \mid \bar{\underline{u}}_{b_1}^t \mid \bar{\underline{u}}_{C_2}^t \mid \bar{\underline{u}}_{b_2}^t) \quad ,$$

$$\bar{\underline{v}}^t = (\bar{\underline{v}}_{C_1}^t \mid \bar{\underline{v}}_{b_1}^t \mid \bar{\underline{v}}_{C_2}^t \mid \bar{\underline{v}}_{b_2}^t) \quad .$$

$$G = \begin{array}{c} C_1 \\ b_1 \\ C_2 \\ b_2 \end{array} \begin{pmatrix} G_{11} & G_{12} & & \\ G_{21} & G_{22} & & \\ & & G_{33} & G_{34} \\ & & G_{43} & G_{44} \end{pmatrix}$$

Since $\underline{v} \in V^*$ we have

$$\begin{bmatrix} \underline{v}_{C_1} \\ --- \\ \underline{v}_{b_1} \\ --- \\ \underline{v}_{C_2} \\ --- \\ \underline{v}_{b_2} \end{bmatrix} = \left[\begin{array}{c|c} -R_{12} & -R_{14} \\ \hline U & 0 \\ \hline -R_{22} & -R_{24} \\ \hline 0 & U \end{array}\right] \begin{bmatrix} \underline{v}_{b_1} \\ --- \\ \underline{v}_{b_2} \end{bmatrix}$$

Hence if $\underline{v}_{b_1}$ and $\underline{v}_{b_2}$ were known, $\underline{v}$ is known, and therefore, since

$$G^{-1} (\underline{v} - \bar{\underline{v}}) = (\underline{u} - \bar{\underline{u}}) \quad ,$$

$\underline{u}$ is known.

Hence to prove the theorem we need merely show that $\underline{v}_{b_1}$ and $\underline{u}_{C_2}$ determine $\underline{v}_{b_2}$.

Since u ε V, we have

$$\begin{bmatrix} \underline{u}_{C_1} \\ --- \\ \underline{u}_{b_1} \\ --- \\ \underline{u}_{C_2} \\ --- \\ \underline{u}_{b_2} \end{bmatrix} = \left[\begin{array}{c|c} U & 0 \\ \hline R_{12}^t & R_{22}^t \\ \hline 0 & U \\ \hline 0 & R_{24}^t \end{array}\right] \begin{bmatrix} \underline{u}_{C_1} \\ --- \\ \underline{u}_{C_2} \end{bmatrix}$$

(since by the Theorem T21, $R_{14} = 0$, $R_{14}^t = 0$).

Hence $\underline{u}_{b_2} = \left(R_{24}^t\right) \quad \underline{u}_{c_2}$

$$\underline{v}_{b_2} - \bar{\underline{v}}_{b_2} = \left(G_{43} \;\middle|\; G_{44}\right) \begin{pmatrix} \underline{u}_{c_2} - \bar{\underline{u}}_{c_2} \\ \hline \underline{u}_{b_2} - \bar{\underline{u}}_{b_2} \end{pmatrix}$$

$$= \left(G_{43} \;\middle|\; G_{44}\right) \begin{pmatrix} \underline{u}_{c_2} - \bar{\underline{u}}_{c_2} \\ \hline R_{24}^t \, \underline{u}_{c_2} - \bar{\underline{u}}_{b_2} \end{pmatrix} .$$

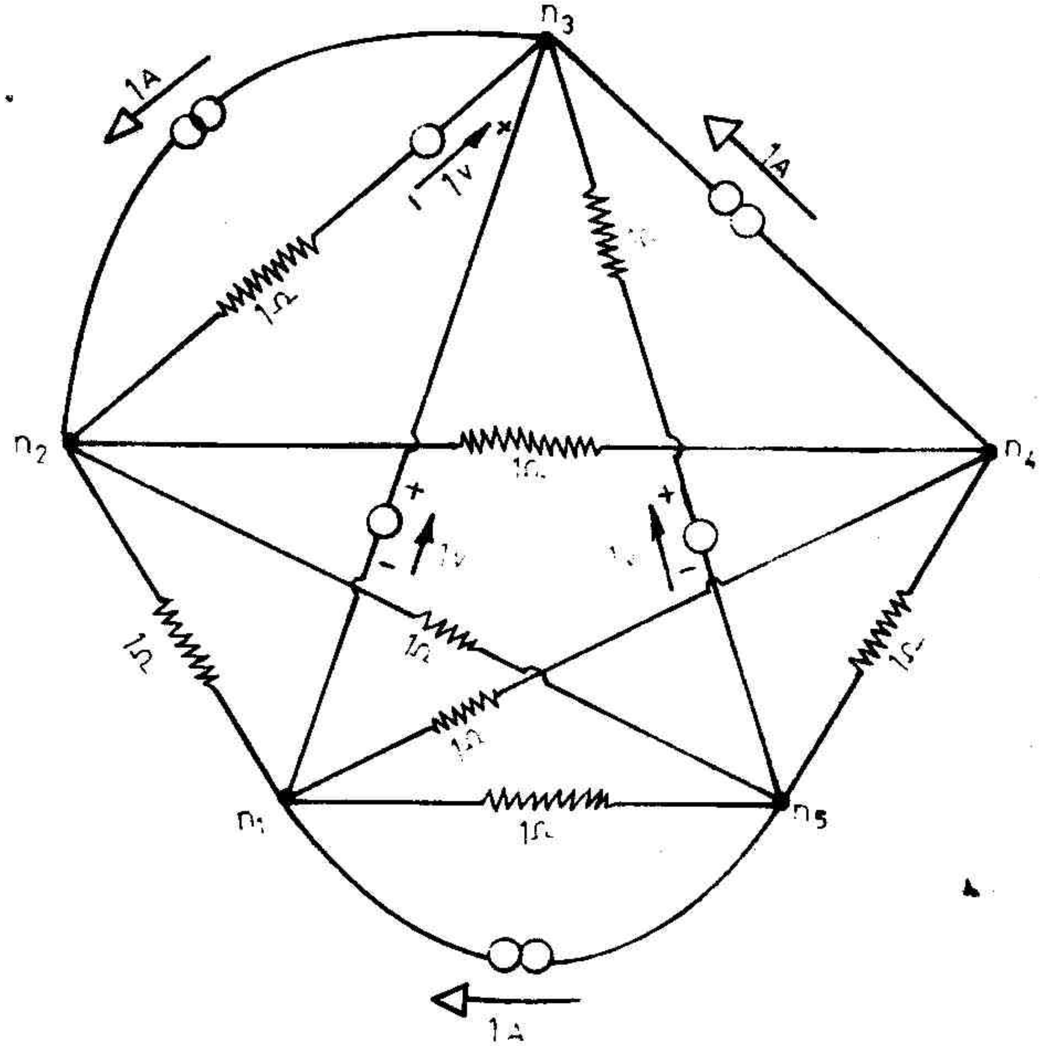
Thus if $\underline{v}_{b_1}$ and $\underline{u}_{c_2}$ were known we can determine $\underline{v}_{b_1}$, $\underline{v}_{b_2}$ and therefore the solution ($\underline{u}$, $\underline{v}$) of N , for the given source vectors $\bar{\underline{u}}$, $\bar{\underline{v}}$.

Q.E.D.

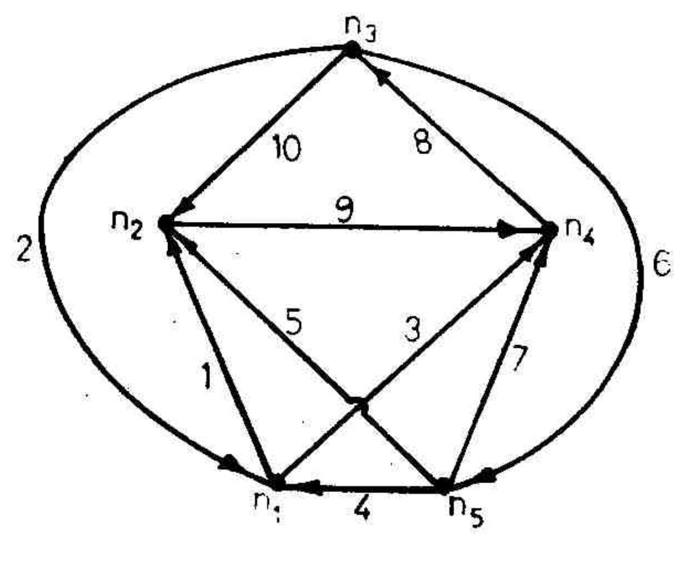
Example 1.1. We give an example of a U.S.G. network and a canonical network similar to it. Our example corresponds to a realistic network. If G is the oriented graph in Figure 1.3, V is its 1-cycle space, S its set of edges, Z the impedance matrix, then the U.S.G. network we are considering is

$$N = (\text{Pol} . G, V, Z, S) .$$

In Figure 1.2 we have given the 'diagram' of this network and in Figure 1.4 the 'diagram' of a canonical network N_1 similar to it. The diagrams contain enough information for constructing the



FIG·1·2· THE 'DIAGRAM' OF N.



FIG·1·3. THE 'GRAPH' OF N

networks N and N_1 and also to obtain the source vectors for these networks. The procedure for obtaining N_1 from N is given below.

We first interpret the vectors $\underline{u}$, $\underline{v}$ and $\bar{\underline{u}}$, $\bar{\underline{v}}$ in conformity with our definition of N.

Put $\quad \underline{u} = \underline{i} \qquad \underline{v} = \underline{v}$

$\qquad \bar{\underline{u}} = \bar{\underline{i}} \qquad \bar{\underline{v}} = \bar{\underline{v}}$.

$\underline{i}$, $\underline{v}$ being the current and voltage vectors respectively and $\bar{\underline{i}}$, $\bar{\underline{v}}$ the current source and voltage source vectors respectively.

Then $\quad R$ = fundamental circuit matrix

$\qquad R^*$ = fundamental cutset matrix.

We will take R^* with respect to the tree $t = \{1,2,3,4\}$. Note that $2 \in t$ and $8 \notin t$ which proves the network is U.S.C.

We partition R^* and the circuit matrix R as in the proof of Theorem 1.3.

Noting that

$$S_o \cap t = \{1,3,4\}$$
$$S_o \cap (S-t) = \{5,6,7,9,10\}$$
$$S_u = \{8\}$$
$$S_v = \{2\}$$

$$
R = \begin{array}{c} \\ \end{array}
\begin{array}{cccccccccc}
5 & 6 & 7 & 9 & 10 & 1 & 3 & 4 & 8 & 2 \\
\end{array}
$$

$$
R = \left(\begin{array}{cccccccccc}
1 & & & & & -1 & & -1 & & \\
 & 1 & & & & & & +1 & & -1 \\
 & & 1 & & & & -1 & -1 & & \\
 & & & 1 & & 1 & -1 & & & \\
 & & & & 1 & -1 & & & & -1 \\
 & & & & & & 1 & & 1 & 1
\end{array}\right)
$$

Hence

$$
R_{12} = \begin{array}{c} \begin{array}{ccc} 1 & 3 & 4 \end{array} \\ \left(\begin{array}{ccc}
-1 & 0 & -1 \\
0 & 0 & +1 \\
0 & -1 & -1 \\
+1 & -1 & 0 \\
-1 & 0 & 0
\end{array}\right) \end{array}
\qquad
R_{22} = \begin{array}{c} \begin{array}{ccc} 1 & 3 & 4 \end{array} \\ \left(\begin{array}{ccc}
0 & +1 & 0
\end{array}\right) \end{array}
$$

$$
R_{14} = \begin{array}{c} 2 \\ \left(\begin{array}{c}
0 \\
-1 \\
0 \\
0 \\
-1
\end{array}\right) \end{array}
\qquad
R_{24} = \begin{array}{c} 2 \\ \left(\begin{array}{c} +1 \end{array}\right) \end{array} .
$$

(for the matrix R we have omitted the zero entries).

The 1-cycle space of the graph of N_1 is given by

$$V' = V \times (S_o \cup S_w) \cdot S_o \ , \quad M_{V'} = M_V \times (S_o \cup S_w) \cdot S_o \ .$$

The corresponding representative matrix for V' is given below :

$$R' = \begin{array}{c} \\ \end{array} \begin{array}{cccccccc} 5 & 6 & 7 & 9 & 10 & 1 & 3 & 4 \end{array}$$

$$R' = \begin{pmatrix} 1 & & & & & -1 & & -1 \\ & 1 & & & & & & 1 \\ & & 1 & & & & -1 & -1 \\ & & & 1 & & 1 & -1 & \\ & & & & 1 & -1 & & \end{pmatrix}$$

(R' being the fundamental circuit matrix of the graph of N_1 with respect to the cotree 567910). The oriented graph corresponding to N_1 is given in Figure 1.5. The original source vectors are

$$\begin{array}{cccccccccc} 5 & 6 & 7 & 9 & 10 & 1 & 3 & 4 & 8 & 2 \end{array}$$

$$\underline{\bar{i}}^t = \left(\begin{array}{ccccc|ccc|c|c} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{array} \right) = \left(\underline{\bar{I}}^t_{10} \quad \underline{\bar{I}}^t_{20} \quad \underline{\bar{I}}^t_{u} \quad \underline{\bar{I}}^t_{w} \right)$$

$$\underline{\bar{v}}^t = \left(\begin{array}{ccccc|ccc|c|c} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right) = \left(\underline{\bar{V}}^t_{10} \quad \underline{\bar{V}}^t_{20} \quad \underline{\bar{V}}^t_{u} \quad \underline{\bar{V}}^t_{w} \right)$$

The source vectors for N_1 are given in the proof of Theorem 1.3 as follows :

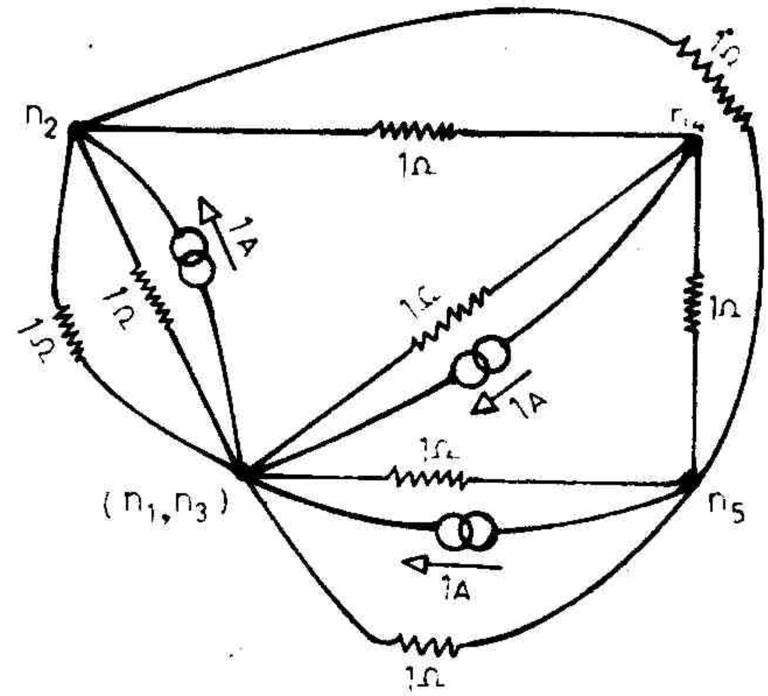


FIG. 1·4.

DIAGRAM OF N_1.

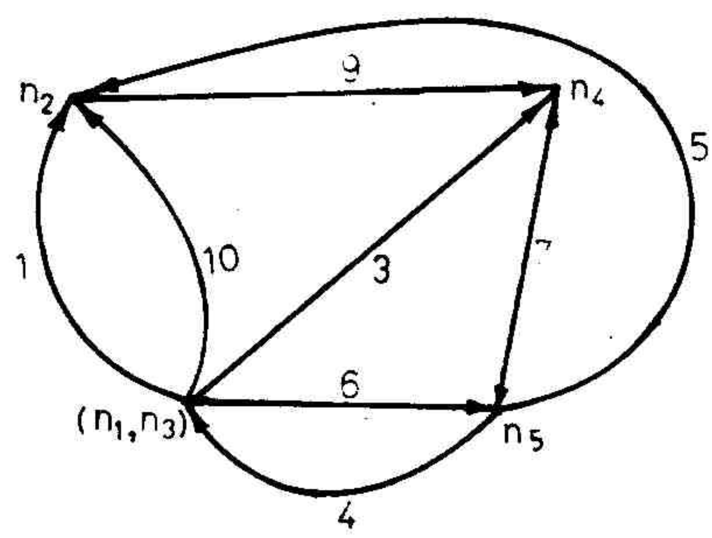


FIG. 1·5.

THE 'GRAPH' OF N_1.

$$\bar{\underline{u}}' = \begin{pmatrix} \bar{\underline{u}}_{10} \\ \text{-----------} \\ \bar{\underline{u}}_{20} - R_{22}^{t}\,\bar{\underline{u}}_{u} \end{pmatrix} \qquad \bar{\underline{v}}' = \begin{pmatrix} \bar{\underline{v}}_{10} + R_{14}\,\bar{\underline{v}}_{v} \\ \text{-----------} \\ \bar{\underline{v}}_{20} \end{pmatrix}$$

Hence the source vectors for N_1 are

$$\bar{\underline{i}}' = \begin{matrix} 5 \\ 6 \\ 7 \\ 9 \\ 10 \\ \\ 1 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \text{--} \\ 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \text{-------} \\ \begin{pmatrix} 0 \\ +1 \\ 0 \end{pmatrix} (1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \text{--} \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\bar{\underline{v}}' = \begin{matrix} 5 \\ 6 \\ 7 \\ 9 \\ 10 \\ \text{---} \\ 1 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ \text{---} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ \text{---} \\ 0 \\ 0 \\ 0 \end{pmatrix} (1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \text{---} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(See Figure 1.4)

Section 2 : Analysis of Canonical Networks by Tearing

In this section, we analyse canonical networks by 'tearing', i.e. we discuss Diakoptics and Codiakoptics and a useful extension. By the results of the previous section we lose no generality in limiting ourselves to canonical networks.

Let $N = (M_V, V, G, S)$ be a canonical network.

Let $S = S_1 \cup S_2$.

Let b be a base of M such that $b \cap S_1$ is a base of $M_V \times S_1$. Such a base exists by Axiom System 1. Let R be a standard representative matrix of M_V associated with $S - b$, and R^* be the corresponding standard representative matrix of M_V^* associated with b.

Let E be the set such that $e \in E$ iff

(a) $e \in S_2 \cap (S - b)$, and

(b) if L is the circuit such that $e \in L \subseteq e \cup b$, then $L \cap S_1$ is nonvoid.

Let C be the set such that $e \in C$ iff

(a) $e \in S_1 \cap b$, and

(b) There exists $d \in E$ such that if L is the circuit with $d \in L \subseteq d \cup b$, then $e \in L$.

It is clear from the definition that E and C are dual quantities, i.e. one could define C to be the set such that $e \in C$ iff

(a) $e \in S_1 \cap b$, and

(b) If L is the bond such that $e \in L \subseteq e \cup (S - b)$, then $L \cap S_2$ is nonvoid and E to be the set such that $e \in E$ iff

(a) $e \in S_2 \cap (S - b)$, and

(b) There exists $d \in C$ such that if L is the bond with $d \in L \subseteq d \cup (S - b)$, then $e \in L$.

We further define sets B and D such that

$$B \subseteq S_1 \cap (S - b)$$

$$F \subseteq S_2 \cap b \ .$$

We next define sets A, A_1, A_2, D, D_1, D_2 as

$$A = S_1 - (B \cup C)$$

$$A_1 = S_1 \cap (b - C)$$

$$A_2 = S_1 - b - B \ .$$

It is clear that $A = A_1 \cup A_2$.

$$D = S_2 - (E \cup F)$$

$$D_1 = S_2 - b - E$$

$$D_2 = S_2 \cap (b - F) \ .$$

It is clear that $D = D_1 \cup D_2$.

We make the following assumption on G for the preliminary part of the analysis.

G can be expressed as the following block diagonal positive definite matrix :

$$G = \begin{array}{c} C \\ A_1 \\ A_2 \\ F \\ D_1 \\ D_2 \\ B \\ E \end{array} \begin{pmatrix} G_C & & & & & & & \\ & G_{A_1} & G_{A_{12}} & & & & & \\ & G_{A_{21}} & G_{A_2} & & & & & \\ & & & G_F & & & & \\ & & & & G_{D_1} & G_{D_{12}} & & \\ & & & & G_{D_{21}} & G_{D_2} & & \\ & & & & & & G_B & \\ & & & & & & & G_E \end{pmatrix}$$

Hence $G^{-1} = H$ can be expressed as the following block diagonal positive definite matrix :

$$H = \begin{array}{c} C \\ A_1 \\ A_2 \\ F \\ D_1 \\ D_2 \\ B \\ E \end{array} \begin{pmatrix} H_C & & & & & & & \\ & H_{A_1} & H_{A_{12}} & & & & & \\ & H_{A_{21}} & H_{A_2} & & & & & \\ & & & H_F & & & & \\ & & & & H_{D_1} & H_{D_{12}} & & \\ & & & & H_{D_{21}} & H_{D_2} & & \\ & & & & & & H_B & \\ & & & & & & & H_E \end{pmatrix}$$

However for reasons that would be apparent later we prefer to write G and H in the following forms :

$$
G = \begin{array}{c} \\ C \\ A_1 \\ F \\ D_2 \\ B \\ A_2 \\ E \\ D_1 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} C & A_1 & F & D_2 & B & A_2 & E & D_1 \end{array} \\
\left(\begin{array}{cccccccc}
G_C & & & & & & & \\
 & G_{A_1} & & & & G_{A_{12}} & & \\
 & & G_F & & & & & \\
 & & & G_{D_2} & & & & G_{D_{21}} \\
 & & & & G_B & & & \\
 & G_{A_{21}} & & & & G_{A_2} & & \\
 & & & & & & G_E & \\
 & & & G_{D_{12}} & & & & G_{D_1}
\end{array} \right)
\end{array}
$$

$$
H = \begin{array}{c} C \\ A_1 \\ F \\ D_2 \\ B \\ A_2 \\ E \\ D_1 \end{array}
\left(\begin{array}{cccccccc}
H_C & & & & & & & \\
 & H_{A_1} & & & & H_{A_{12}} & & \\
 & & H_F & & & & & \\
 & & & H_{D_2} & & & & H_{D_{21}} \\
 & & & & H_B & & & \\
 & H_{A_{21}} & & & & H_{A_2} & & \\
 & & & & & & H_E & \\
 & & & H_{D_{12}} & & & & H_{D_1}
\end{array} \right)
$$

The solution and source vectors are correspondingly partitioned as follows :

$$\underline{u}^t = (\underline{u}_C{}^t \mid \underline{u}_{A_1}{}^t \mid \underline{u}_F{}^t \mid \underline{u}_{D_2}{}^t \mid \underline{u}_B{}^t \mid \underline{u}_{A_2}{}^t \mid \underline{u}_E{}^t \mid \underline{u}_{D_1}{}^t)$$

$$\underline{v}^t = (\underline{v}_C{}^t \mid \underline{v}_{A_1}{}^t \mid \underline{v}_F{}^t \mid \underline{v}_{D_2}{}^t \mid \underline{v}_B{}^t \mid \underline{v}_{A_2}{}^t \mid \underline{v}_E{}^t \mid \underline{v}_{D_1}{}^t)$$

$$\bar{\underline{u}}^t = (\bar{\underline{u}}_C{}^t \mid \bar{\underline{u}}_{A_1}{}^t \mid \bar{\underline{u}}_F{}^t \mid \bar{\underline{u}}_{D_2}{}^t \mid \bar{\underline{u}}_B{}^t \mid \bar{\underline{u}}_{A_2}{}^t \mid \bar{\underline{u}}_E{}^t \mid \bar{\underline{u}}_{D_1}{}^t)$$

$$\bar{\underline{v}}^t = (\bar{\underline{v}}_C{}^t \mid \bar{\underline{v}}_{A_1}{}^t \mid \bar{\underline{v}}_F{}^t \mid \bar{\underline{v}}_{D_2}{}^t \mid \bar{\underline{v}}_B{}^t \mid \bar{\underline{v}}_{A_2}{}^t \mid \bar{\underline{v}}_E{}^t \mid \bar{\underline{v}}_{D_1}{}^t)$$

We partition the matrix R^* as :

$$R^* = \begin{array}{c} \begin{array}{cccccccc} C & A_1 & F & D_2 & B & A_2 & E & D_1 \end{array} \\ \begin{pmatrix} U & 0 & 0 & 0 & R^*_{15} & R^*_{16} & R^*_{17} & R^*_{18} \\ 0 & U & 0 & 0 & R^*_{25} & R^*_{26} & R^*_{27} & R^*_{28} \\ 0 & 0 & U & 0 & R^*_{35} & R^*_{36} & R^*_{37} & R^*_{38} \\ 0 & 0 & 0 & U & R^*_{45} & R^*_{46} & R^*_{47} & R^*_{48} \end{pmatrix} \end{array} = \begin{pmatrix} R_1{}^* \\ R_2{}^* \\ R_3{}^* \\ R_4{}^* \end{pmatrix}$$

For this matrix we are able to effect a simplification proceeding as follows :

(a) Since $C \cup A_1$ is a base for $M_V \times S_1$ by definition, and $B \subseteq S_1$ and $A_2 \subseteq S_1$ we have by the use of Theorem T1 :

$$R^*_{35} = 0 \ , \quad R^*_{36} = 0 \ , \quad R^*_{45} = 0 \ , \quad R^*_{46} = 0 \ .$$

Again, by the definition of E and D_1 —

$$R^*_{18} = 0 \ , \quad R^*_{27} = 0 \ , \quad R^*_{28} = 0 \ .$$

We can, therefore, write R^* as follows :-

$$R^* = \begin{array}{c} \\ \end{array}\begin{matrix} C & A_1 & F & D_2 & B & A_2 & E & D_1 \end{matrix}$$

$$R^* = \begin{pmatrix} U & 0 & 0 & 0 & R^*_{15} & R^*_{16} & R^*_{17} & 0 \\ 0 & U & 0 & 0 & R^*_{25} & R^*_{26} & 0 & 0 \\ 0 & 0 & U & 0 & 0 & 0 & R^*_{37} & R^*_{38} \\ 0 & 0 & 0 & U & 0 & 0 & R^*_{47} & R^*_{48} \end{pmatrix} \quad (1)$$

(columns: C, A_1, F, D_2, B, A_2, E, D_1)

The standard representative matrix R associated with the cobase S - b can then be written by the use of Theorem T18, as :

(columns: C, A_1, F, D_2, B, A_2, E, D_1)

$$R = \begin{pmatrix} -R^{*t}_{15} & -R^{*t}_{25} & 0 & 0 & U & 0 & 0 & 0 \\ -R^{*t}_{16} & -R^{*t}_{26} & 0 & 0 & 0 & U & 0 & 0 \\ -R^{*t}_{17} & 0 & -R^{*t}_{37} & -R^{*t}_{47} & 0 & 0 & U & 0 \\ 0 & 0 & -R^{*t}_{38} & -R^{*t}_{48} & 0 & 0 & 0 & U \end{pmatrix} \equiv \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix}$$

... (2)

We now proceed to obtain the solution ($\underline{u}$, $\underline{v}$) by the use of the methods of Diakoptics when the source vectors $\bar{\underline{u}}$, $\bar{\underline{v}}$ are given.

Since $\underline{u} \in V$ and V and V^* are orthogonal complements, we have,

$$\begin{pmatrix} R_1^* \\ --- \\ R_2^* \end{pmatrix} \left[\underline{u} \right] = \begin{pmatrix} 0 \\ --- \\ 0 \end{pmatrix}$$

Hence

$$\begin{pmatrix} R_1^* \\ --- \\ R_2^* \end{pmatrix} \left[\underline{u} - \bar{\underline{u}} \right] = - \begin{pmatrix} R_1^* \\ --- \\ R_2^* \end{pmatrix} \left[\bar{\underline{u}} \right] \qquad (3)$$

Expanding (3) by the use of (1) we get

$$\begin{array}{ccccc} C & A_1 & B & A_2 & E \end{array}$$

$$\begin{pmatrix} U & 0 & R_{15}^* & R_{16}^* & R_{17}^* \\ 0 & U & R_{25}^* & R_{26}^* & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_C - \bar{\underline{u}}_C \\ \underline{u}_{A_1} - \bar{\underline{u}}_{A_1} \\ \underline{u}_B - \bar{\underline{u}}_B \\ \underline{u}_{A_2} - \bar{\underline{u}}_{A_2} \\ \underline{u}_E - \bar{\underline{u}}_E \end{pmatrix} = - \begin{pmatrix} R_1^* \\ R_2^* \end{pmatrix} \left[\bar{\underline{u}} \right]$$

Hence

$$\begin{pmatrix} U & 0 & R_{16}^* \\ 0 & U & R_{26}^* \end{pmatrix} \begin{pmatrix} u_C - \bar{\underline{u}}_C \\ u_{A_1} - \bar{\underline{u}}_{A_1} \\ u_{A_2} - \bar{\underline{u}}_{A_2} \end{pmatrix} = - \begin{pmatrix} R_1^* \\ R_2^* \end{pmatrix} \left[\bar{\underline{u}} \right] - \begin{pmatrix} R_{15}^* & R_{17}^* \\ R_{25}^* & 0 \end{pmatrix} \cdot \begin{pmatrix} u_B - \bar{\underline{u}}_B \\ u_E - \bar{\underline{u}}_E \end{pmatrix}$$

Hence by the use of the network equation (3) of Definition 1.2 of the last section

$$\begin{pmatrix} U & 0 & R_{16}^{*} \\ 0 & U & R_{26}^{*} \end{pmatrix} \begin{pmatrix} H_C & & \\ & H_{A_1} & H_{A_{12}} \\ & H_{A_{21}} & H_{A_2} \end{pmatrix} \begin{pmatrix} \underline{v}_C - \bar{\underline{v}}_C \\ \underline{v}_{A_1} - \bar{\underline{v}}_{A_1} \\ \underline{v}_{A_2} - \bar{\underline{v}}_{A_2} \end{pmatrix}$$

$$= - \begin{pmatrix} R_1^{*} \\ R_2^{*} \end{pmatrix} \left[\bar{\underline{u}} \right] - \begin{pmatrix} R_{15}^{*} & R_{17}^{*} \\ R_{25}^{*} & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_B - \bar{\underline{u}}_B \\ \underline{u}_E - \bar{\underline{u}}_E \end{pmatrix} \tag{4}$$

$$\begin{matrix} C \\ A_1 \\ F \\ D_2 \\ B \\ A_2 \\ E \\ D_1 \end{matrix} \begin{pmatrix} U & 0 & 0 & 0 \\ 0 & U & 0 & 0 \\ 0 & 0 & U & 0 \\ 0 & 0 & 0 & U \\ R_{15}^{*t} & R_{25}^{*t} & 0 & 0 \\ R_{16}^{*t} & R_{26}^{*t} & 0 & 0 \\ R_{17}^{*t} & 0 & R_{37}^{*t} & R_{47}^{*t} \\ 0 & 0 & R_{38}^{*t} & R_{48}^{*t} \end{pmatrix} \begin{pmatrix} \underline{v}_C \\ \underline{v}_{A_1} \\ \underline{v}_F \\ \underline{v}_{D_2} \end{pmatrix} = \begin{pmatrix} \underline{v}_C \\ \underline{v}_{A_1} \\ \underline{v}_F \\ \underline{v}_{D_2} \\ \underline{v}_B \\ \underline{v}_{A_2} \\ \underline{v}_E \\ \underline{v}_{D_1} \end{pmatrix} \tag{5}$$

Hence

$$\begin{pmatrix} \underline{v}_C \\ \hline \underline{v}_{A_1} \\ \hline \underline{v}_B \\ \hline \underline{v}_{A_2} \end{pmatrix} = \begin{pmatrix} U & 0 \\ \hline 0 & U \\ \hline R_{15}^{*t} & R_{25}^{*t} \\ \hline R_{16}^{*t} & R_{26}^{*t} \end{pmatrix} \begin{pmatrix} \underline{v}_C \\ \underline{v}_{A_1} \end{pmatrix} \tag{6}$$

Using (6) we modify (4) into

$$\begin{pmatrix} U & 0 & R_{16}^{*} \\ 0 & U & R_{26}^{*} \end{pmatrix} \begin{pmatrix} H_C & & \\ & H_{A_1} & H_{A_{12}} \\ & H_{A_{21}} & H_{A_2} \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \\ R_{16}^{*t} & R_{26}^{*t} \end{pmatrix} \begin{pmatrix} \underline{v}_C \\ \underline{v}_{A_1} \end{pmatrix}$$

$$= \begin{pmatrix} U & 0 & R_{16}^{*} \\ 0 & U & R_{26}^{*} \end{pmatrix} \begin{pmatrix} H_C & & \\ & H_{A_1} & H_{A_{12}} \\ & H_{A_{21}} & H_{A_2} \end{pmatrix} \begin{pmatrix} \bar{\underline{v}}_c \\ \bar{\underline{v}}_{A_1} \\ \bar{\underline{v}}_{A_2} \end{pmatrix} -$$

$$- \begin{pmatrix} U & 0 & R_{16}^{*} \\ 0 & U & R_{26}^{*} \end{pmatrix} \begin{pmatrix} \bar{\underline{u}}_C \\ \bar{\underline{u}}_{A_1} \\ \bar{\underline{u}}_{A_2} \end{pmatrix} - \begin{pmatrix} R_{15}^{*} & R_{17}^{*} \\ R_{25}^{*} & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_B \\ \underline{u}_E \end{pmatrix} \tag{7}$$

(Note that

$$- \begin{pmatrix} R_1^{*} \\ R_2^{*} \end{pmatrix} \bar{\underline{u}} + \begin{pmatrix} R_{15}^{*} & R_{17}^{*} \\ R_{25}^{*} & 0 \end{pmatrix} \begin{pmatrix} \bar{\underline{u}}_B \\ \bar{\underline{u}}_E \end{pmatrix} = - \begin{pmatrix} U & 0 & R_{16}^{*} \\ 0 & U & R_{26}^{*} \end{pmatrix} \begin{pmatrix} \bar{\underline{u}}_C \\ \bar{\underline{u}}_{A_1} \\ \bar{\underline{u}}_{A_2} \end{pmatrix} \quad)$$

Denoting

$$\begin{pmatrix} U & 0 & R_{16}^{*} \\ 0 & U & R_{26}^{*} \end{pmatrix} \begin{pmatrix} H_C & & \\ & H_{A_1} & H_{A_{12}} \\ & H_{A_{21}} & H_{A_2} \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \\ R_{16}^{*t} & R_{26}^{*t} \end{pmatrix} \quad \text{as} \quad \tilde{H}_{CA_1}$$

and

$$\begin{pmatrix} U & 0 & R_{16}^{*} \\ 0 & U & R_{26}^{*} \end{pmatrix} \begin{pmatrix} H_C & & \\ & H_{A_1} & H_{A_{12}} \\ & H_{A_{21}} & H_{A_2} \end{pmatrix} \begin{pmatrix} \bar{\underline{v}}_C \\ \bar{\underline{v}}_{A_1} \\ \bar{\underline{v}}_{A_2} \end{pmatrix} - \begin{pmatrix} U & 0 & R_{16}^{*} \\ 0 & U & R_{26}^{*} \end{pmatrix} \begin{pmatrix} \bar{\underline{u}}_C \\ \bar{\underline{u}}_{A_1} \\ \bar{\underline{u}}_{A_2} \end{pmatrix}$$

as $\tilde{\underline{u}}_1$, we can rewrite (7) as :

$$\tilde{H}_{CA_1} \begin{pmatrix} \underline{v}_C \\ --- \\ \underline{v}_{A_1} \end{pmatrix} = \tilde{\underline{u}}_1 - \begin{pmatrix} R_{15}^{*} & R_{17}^{*} \\ R_{25}^{*} & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_B \\ \underline{u}_E \end{pmatrix} \qquad (8)$$

(We take $\tilde{H}_{CA_1}$ to be suitably partitioned).

We now proceed to apply the dual process to $\underline{v}$.

$\underline{v} \in V^{*}$ and V and V^{*} are orthogonal complements.

Hence

$$\begin{pmatrix} R_3 \\ --- \\ R_4 \end{pmatrix} \underline{v} = \begin{pmatrix} 0 \\ --- \\ 0 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} R_3 \\ R_4 \end{pmatrix} \left[\underline{v} - \bar{\underline{v}} \right] = - \begin{pmatrix} R_3 \\ R_4 \end{pmatrix} \bar{\underline{v}} \qquad (9)$$

Since $\underline{u} \in V$ we have

$$
\begin{array}{c}
C \\ A_1 \\ F \\ D_2 \\ B \\ A_2 \\ E \\ D_1
\end{array}
\begin{pmatrix}
-R_{15}^{*} & -R_{16}^{*} & -R_{17}^{*} & 0 \\
-R_{25}^{*} & -R_{26}^{*} & 0 & 0 \\
0 & 0 & -R_{37}^{*} & -R_{38}^{*} \\
0 & 0 & -R_{47}^{*} & -R_{48}^{*} \\
U & 0 & 0 & 0 \\
0 & U & 0 & 0 \\
0 & 0 & U & 0 \\
0 & 0 & 0 & U
\end{pmatrix}
\begin{pmatrix}
\underline{u}_B \\ \underline{u}_{A_2} \\ \underline{u}_E \\ \underline{u}_{D_1}
\end{pmatrix}
=
\begin{pmatrix}
\underline{u}_C \\ \underline{u}_{A_1} \\ \underline{u}_F \\ \underline{u}_{D_2} \\ \underline{u}_B \\ \underline{u}_{A_2} \\ \underline{u}_E \\ \underline{u}_{D_1}
\end{pmatrix}
\qquad (10)
$$

Hence

$$
\begin{pmatrix}
\underline{u}_{D_2} \\ \underline{u}_E \\ \underline{u}_{D_1}
\end{pmatrix}
=
\begin{pmatrix}
-R_{47}^{*} & -R_{48}^{*} \\
U & 0 \\
0 & U
\end{pmatrix}
\begin{pmatrix}
\underline{u}_E \\ \underline{u}_{D_1}
\end{pmatrix}
\qquad (11)
$$

Now (9) can be written as :

$$
\begin{array}{ccccc}
C & F & D_2 & E & D_1
\end{array}
$$

$$
\begin{pmatrix}
R_{17}^{*} & -R_{37}^{*t} & -R_{47}^{*t} & U & 0 \\
0 & -R_{38}^{*t} & -R_{48}^{*t} & 0 & U
\end{pmatrix}
\begin{pmatrix}
\underline{v}_C - \bar{\underline{v}}_C \\
\underline{v}_F - \bar{\underline{v}}_F \\
\underline{v}_{D_2} - \bar{\underline{v}}_{D_2} \\
\underline{v}_E - \bar{\underline{v}}_E \\
\underline{v}_{D_1} - \bar{\underline{v}}_{D_1}
\end{pmatrix}
= -
\begin{pmatrix}
R_3 \\ R_4
\end{pmatrix}
\bar{\underline{v}}
$$

Hence

$$
\begin{matrix} \quad D_2 & \quad E & \quad D_1 \end{matrix}
$$

$$
\begin{pmatrix} -R_{47}^{*t} & U & 0 \\ -R_{48}^{*t} & 0 & U \end{pmatrix}
\begin{pmatrix} \underline{v}_{D_2} - \bar{\underline{v}}_{D_2} \\ \underline{v}_E - \bar{\underline{v}}_E \\ \underline{v}_{D_1} - \bar{\underline{v}}_{D_1} \end{pmatrix}
$$

$$
\begin{matrix} \qquad\qquad\qquad\qquad\qquad\quad C & \qquad F \end{matrix}
$$

$$
= - \begin{pmatrix} R_3 \\ R_4 \end{pmatrix} \bar{\underline{v}} - \begin{pmatrix} -R_{17}^{*t} & -R_{37}^{*t} \\ 0 & -R_{38}^{*t} \end{pmatrix} \begin{pmatrix} \underline{v}_C - \bar{\underline{v}}_C \\ \underline{v}_F - \bar{\underline{v}}_F \end{pmatrix}
$$

Using the network equation (3) of Definition 1.2 of the last section, we get,

$$
\begin{pmatrix} -R_{47}^{*t} & U & 0 \\ -R_{48}^{*t} & 0 & U \end{pmatrix}
\begin{pmatrix} G_{D_2} & & G_{D_{21}} \\ & G_E & \\ G_{D_{12}} & & G_{D_1} \end{pmatrix}
\begin{pmatrix} \underline{u}_{D_2} - \bar{\underline{u}}_{D_2} \\ \underline{u}_E - \bar{\underline{u}}_E \\ \underline{u}_{D_1} - \bar{\underline{u}}_{D_1} \end{pmatrix}
$$

$$
= - \begin{pmatrix} R_3 \\ R_4 \end{pmatrix} \bar{\underline{v}} - \begin{pmatrix} -R_{17}^{*t} & -R_{37}^{*t} \\ 0 & -R_{38}^{*t} \end{pmatrix} \begin{pmatrix} \underline{v}_C - \bar{\underline{v}}_C \\ \underline{v}_F - \bar{\underline{v}}_F \end{pmatrix}
$$

Using (11) and simplifying, we get,

$$\begin{bmatrix} -R_{47}^{*t} & U & 0 \\ -R_{48}^{*t} & 0 & U \end{bmatrix} \begin{bmatrix} G_{D_2} & & G_{D_{21}} \\ & G_E & \\ G_{D_{12}} & & G_{D_1} \end{bmatrix} \begin{bmatrix} -R_{47}^{*} & -R_{48}^{*} \\ U & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} \underline{u}_E \\ \underline{u}_{D_1} \end{bmatrix}$$

$$= \begin{bmatrix} -R_{47}^{*t} & U & 0 \\ -R_{48}^{*t} & 0 & U \end{bmatrix} \begin{bmatrix} G_{D_2} & & G_{D_{21}} \\ & G_E & \\ G_{D_{12}} & & G_{D_1} \end{bmatrix} \begin{bmatrix} \bar{\underline{u}}_{D_2} \\ \bar{\underline{u}}_E \\ \bar{\underline{u}}_{D_1} \end{bmatrix}$$

$$- \begin{bmatrix} -R_{47}^{*t} & U & 0 \\ -R_{48}^{*t} & 0 & U \end{bmatrix} \begin{bmatrix} \bar{\underline{v}}_{D_2} \\ \bar{\underline{v}}_E \\ \bar{\underline{v}}_{D_1} \end{bmatrix} - \begin{bmatrix} -R_{17}^{*t} & -R_{37}^{*t} \\ 0 & -R_{38}^{*t} \end{bmatrix} \begin{bmatrix} \underline{v}_C \\ \underline{v}_F \end{bmatrix}$$

.. (12)

Denoting

$$\begin{bmatrix} -R_{47}^{*t} & U & 0 \\ -R_{48}^{*t} & 0 & U \end{bmatrix} \begin{bmatrix} G_{D_2} & & G_{D_{21}} \\ & G_E & \\ G_{D_{12}} & & G_{D_1} \end{bmatrix} \begin{bmatrix} -R_{47}^{*} & -R_{48}^{*} \\ U & 0 \\ 0 & U \end{bmatrix}$$

as $\tilde{G}_{ED_1}$

and

$$\begin{bmatrix} -R_{47}^{*t} & U & 0 \\ -R_{48}^{*t} & 0 & U \end{bmatrix} \begin{bmatrix} G_{D_2} & & G_{D_{21}} \\ & G_E & \\ G_{D_{12}} & & G_{D_1} \end{bmatrix} \begin{bmatrix} \bar{\underline{u}}_{D_2} \\ \bar{\underline{u}}_E \\ \bar{\underline{u}}_{D_1} \end{bmatrix} - \begin{bmatrix} -R_{47}^{*t} & U & 0 \\ -R_{48}^{*t} & 0 & U \end{bmatrix} \begin{bmatrix} \bar{\underline{v}}_{D_2} \\ \bar{\underline{v}}_E \\ \bar{\underline{v}}_{D_1} \end{bmatrix}$$

as $\bar{\underline{w}}_2$

We can write (11) as

$$\tilde{G}_{ED_1} \begin{bmatrix} \underline{u}_E \\ \underline{u}_{D_1} \end{bmatrix} = \underline{\bar{\omega}}_2 - \begin{bmatrix} -R_{17}^{*t} & -R_{37}^{*t} \\ 0 & -R_{38}^{*t} \end{bmatrix} \begin{bmatrix} \underline{v}_C \\ \underline{v}_F \end{bmatrix} \qquad (13)$$

(We take $\tilde{G}_{ED_1}$ to be suitably partitioned).

We note here that $\tilde{R}_{CA_1}$ in equation (8) and $\tilde{G}_{ED_1}$ in equation (13) are positive definite by the use of Theorems T27 and T29.

Writing the network equations (3) of Definition 1.2 for B and F we have :

$$\begin{bmatrix} G_B & & & \\ & U & & \\ & & U & \\ & & & H_F \end{bmatrix} \begin{bmatrix} \underline{u}_B - \bar{\underline{u}}_B \\ \underline{u}_E - \bar{\underline{u}}_E \\ \underline{v}_C - \bar{\underline{v}}_C \\ \underline{v}_F - \bar{\underline{v}}_F \end{bmatrix} = \begin{bmatrix} \underline{v}_B - \bar{\underline{v}}_B \\ \underline{u}_E - \bar{\underline{u}}_E \\ \underline{v}_C - \bar{\underline{v}}_C \\ \underline{u}_F - \bar{\underline{u}}_F \end{bmatrix}$$

Using equations (5) and (10) the R.H.S. of the above equation reduces to

$$\text{R.H.S.} = - \begin{bmatrix} \bar{\underline{v}}_B \\ \bar{\underline{u}}_E \\ \bar{\underline{v}}_C \\ \bar{\underline{u}}_F \end{bmatrix} + \begin{bmatrix} R_{15}^{*t} & R_{25}^{*t} & 0 & 0 \\ 0 & 0 & U & 0 \\ U & 0 & 0 & 0 \\ 0 & 0 & -R_{37}^{*} & -R_{38}^{*} \end{bmatrix} \begin{bmatrix} \underline{v}_C \\ \underline{v}_{A_1} \\ \underline{u}_E \\ \underline{u}_{D_1} \end{bmatrix}.$$

Hence

$$\begin{pmatrix} G_B & & & \\ & U & & \\ & & U & \\ & & & H_F \end{pmatrix} \begin{pmatrix} \underline{u}_B \\ \underline{u}_E \\ \underline{v}_C \\ \underline{v}_F \end{pmatrix} = \begin{pmatrix} G_B & & & \\ & U & & \\ & & U & \\ & & & H_F \end{pmatrix} \begin{pmatrix} \bar{\underline{u}}_B \\ \bar{\underline{u}}_E \\ \bar{\underline{v}}_C \\ \bar{\underline{v}}_F \end{pmatrix} -$$

R.H.S. (contd.)

$$- \begin{pmatrix} \bar{\underline{v}}_B \\ \bar{\underline{u}}_E \\ \bar{\underline{v}}_C \\ \bar{\underline{v}}_F \end{pmatrix} + \begin{pmatrix} R_{15}^{*t} & R_{25}^{*t} & 0 & 0 \\ 0 & 0 & U & 0 \\ U & 0 & 0 & 0 \\ 0 & 0 & -R_{37}^{*} & -R_{38}^{*} \end{pmatrix} \begin{pmatrix} \underline{v}_C \\ \underline{v}_{A_1} \\ \underline{u}_E \\ \underline{u}_{D_1} \end{pmatrix} \qquad (14)$$

Representing the terms in the R.H.S. involving source vectors by $\bar{\underline{p}}'$ and the coefficient matrix in the L.H.S. by J and using equations (8) and (13), we obtain,

$$J \begin{pmatrix} \underline{u}_B \\ \underline{u}_E \\ \underline{v}_C \\ \underline{v}_F \end{pmatrix} = \bar{\underline{p}}' + \left(\begin{array}{cc|cc} R_{15}^{*t} & R_{25}^{*t} & 0 & 0 \\ 0 & 0 & U & 0 \\ U & 0 & 0 & 0 \\ 0 & 0 & -R_{37}^{*} & -R_{38}^{*} \end{array}\right) \left(\begin{array}{c|c} (\tilde{H}_{CA_1})^{-1} & \\ \hline & (\tilde{G}_{ED_1})^{-1} \end{array}\right) X$$

(X - represents matrix multiplication) R.H.S. (contd.)

$$X \left[\left(\begin{array}{c} \bar{\underline{u}}_1 \\ \hline \bar{\underline{w}}_2 \end{array}\right) - \left(\begin{array}{cccc} R_{15}^{*} & R_{17}^{*} & 0 & 0 \\ R_{25}^{*} & 0 & 0 & 0 \\ \hline 0 & 0 & -R_{17}^{*t} & -R_{37}^{*t} \\ 0 & 0 & 0 & -R_{38}^{*t} \end{array}\right) \begin{pmatrix} \underline{u}_B \\ \underline{u}_E \\ \underline{v}_C \\ \underline{v}_F \end{pmatrix} \right] \text{R.H.S. ends.}$$

Hence

$$\left\{ J + \begin{pmatrix} R_{15}^{*t} & R_{25}^{*t} & 0 & 0 \\ 0 & 0 & U & 0 \\ U & 0 & 0 & 0 \\ 0 & 0 & -R_{37}^{*} & -R_{38}^{*} \end{pmatrix} \begin{pmatrix} (\tilde{H}_{CA_1})^{-1} & \\ & (\tilde{G}_{ED_1})^{-1} \end{pmatrix} \times \right.$$

L.H.S. (contd.)

$$\left. \times \begin{pmatrix} R_{15}^{*} & R_{17}^{*} & 0 & 0 \\ R_{25}^{*} & 0 & 0 & 0 \\ 0 & 0 & -R_{17}^{*t} & -R_{37}^{*t} \\ 0 & 0 & 0 & -R_{38}^{*t} \end{pmatrix} \right\} \begin{pmatrix} \underline{u}_B \\ \underline{u}_E \\ \underline{v}_C \\ \underline{v}_F \end{pmatrix}$$

$$= \underline{\bar{p}}' + \begin{pmatrix} R_{15}^{*t} & R_{25}^{*t} & 0 & 0 \\ 0 & 0 & U & 0 \\ U & 0 & 0 & 0 \\ 0 & 0 & -R_{37}^{*} & -R_{38}^{*} \end{pmatrix} \begin{pmatrix} (\tilde{H}_{CA_1})^{-1} & \\ & (\tilde{G}_{ED_1})^{-1} \end{pmatrix} \begin{pmatrix} \underline{\bar{u}}_1 \\ \underline{\bar{v}}_2 \end{pmatrix}$$

$$= \underline{\bar{p}} \tag{16}$$

representing R.H.S. by $\underline{\bar{p}}$.

<u>Definition 2.1.</u> Let T be an n x n matrix such that if R is any n x b matrix TR results in a permutation of rows of R. Then T is called a permutation matrix.

We state the following well known results about permutation matrices in the form of a lemma.

Lemma 2.1. If T is an $n \times n$ permutation matrix

(i) T is nonsingular

(ii) $T^{-1} = T^t$

(iii) Let R be any $b \times n$ matrix.

Then RT results in a permutation of the columns of R.

Theorem 2.1. Let $N = (M_V, V, G, S)$ be a generalised network in the canonical form. Let $S = S_1 \cup S_2 \ldots \cup S_n$, where the S_i are elementary separators of the matroid M_V. Suppose there exists a permutation matrix T such that TGT^t has the following block diagonal form :

$$TGT^t = \begin{pmatrix} G_{S_1} & & & \\ & G_{S_2} & & \\ & & \ddots & \\ & & & G_{S_n} \end{pmatrix}$$

Then

(i) If $H = G^{-1}$,

$$THT^t = \begin{pmatrix} H_{S_1} & & & \\ & H_{S_2} & & \\ & & \ddots & \\ & & & H_{S_n} \end{pmatrix}$$

(ii) If R^* is the standard representative matrix of V^* with respect to the base b of M_V, and $b \cap S_i = b_i$ ($i \in \{1,2,\ldots n\}$), there exists a permutation matrix T_1 such that, (denoting $R^* H (R^*)^t$ by $\widetilde{H}$)

$$T_1 \widetilde{H} T_1^t = T_1 (R^*) H (R^*)^t T_1^t = \begin{bmatrix} \widetilde{H}_{b_1} & & & \\ & \widetilde{H}_{b_2} & & \\ & & \ddots & \\ & & & \widetilde{H}_{b_n} \end{bmatrix}$$

(i.e. $T_1 \widetilde{H} T_1^t$ has a block diagonal form corresponding to the partition of b into $b_1, b_2 \ldots b_n$).

(iii) If R is the standard representative matrix of V with respect to the cobase C of M_V and $C \cap S_i = C_i$ ($i \in \{1,2,\ldots n\}$), there exists a permutation matrix T_2 such that, (denoting RGR^t by $\widetilde{G}$)

$$T_2 \widetilde{G} T_2^t = T_2 R G R^t T_2^t = \begin{bmatrix} \widetilde{G}_{C_1} & & \\ & \ddots & \\ & & \widetilde{G}_{C_n} \end{bmatrix}$$

(i.e. $T_2 \widetilde{G} T_2^t$ has a block diagonal form corresponding to the partition of C into $C_1, C_2 \ldots C_n$).

<u>Proof.</u> $T G^{-1} T^t = (T G T^t)^{-1}$, since $T^{-1} = T^t$.

Hence $THT^t = TG^{-1}T^t =$

$$= \begin{bmatrix} (G_{S_1})^{-1} & & \\ & \ddots & \\ & & (G_{S_n})^{-1} \end{bmatrix} = \begin{bmatrix} H_{S_1} & & \\ & \ddots & \\ & & H_{S_n} \end{bmatrix}$$

(each G_{S_i} is a positive definite matrix by Theorem T87).

(ii) Let $R^{*} = \begin{bmatrix} U & F \end{bmatrix}$
$\quad\quad\quad\quad\quad\quad b \quad S-b$

and let T_1 be the permutation matrix that permutes the rows of R^{*} as follows :

$$T_1 R^{*} = \begin{bmatrix} I_{|b_1|} & & & F_1 \\ & I_{|b_2|} & & F_2 \\ & & \ddots & \vdots \\ & & I_{|b_n|} & F_n \end{bmatrix}$$

$I_{|b_i|}$ being the unit submatrix corresponding to b_i and F_i corresponding to $(S-b) \cap (S_i)$. Now post multiplication of $T_1 R^{*}$ by the matrix T^t clearly results in a rearrangement of the columns of $T_1 R^{*}$ as follows :

$$T_1 R^{*} T^t = \begin{bmatrix} I_{|b_1|} & K_{(S_1-b_1)} & & & & \\ & & I_{|b_2|} & K_{(S_2-b_2)} & & \\ & & & & \ddots & \\ & & & & I_{|b_n|} & K_{(S_n-b_n} \end{bmatrix}$$

where $K_{(S_i-b_i)}$ is obtained by the rearrangement of columns of F_i according to the permutation defined by T^t. Hence

$$T_1 \widetilde{H} T_1^{\,t} = T_1 (R^{*}) H (R^{*})^t T_1^{\,t}$$

$$= T_1 (R^{*}) T^t T H T^t T (R^{*})^t T_1^{\,t}$$

$$= \begin{bmatrix} I_{|b_1|} & K_{(S_1-b_1)} & \cdots & & \\ & & & I_{|b_n|} & K_{(S_n-b_n)} \end{bmatrix} \begin{bmatrix} H_{S_1} & & & \\ & H_{S_2} & & \\ & & \ddots & \\ & & & H_{S_n} \end{bmatrix} \quad X$$

(X - represents matrix multiplication). R.H.S. (contd.)

$$X \begin{bmatrix} I_{|b_1|} & & \\ K^t_{(S_1-b_1)} & \ddots & \\ & & I_{|b_n|} \\ & & K^t_{(S_n-b_n)} \end{bmatrix} = \begin{bmatrix} \tilde{H}_{b_1} & & & \\ & \tilde{H}_{b_2} & & \\ & & \ddots & \\ & & & \tilde{H}_{b_n} \end{bmatrix}$$

The proof of (iii) is similar to the above.

Q.E.D.

We will now consider a number of cases.

Case 1. $S_2 = \Phi$ (hence) $C = \Phi$.

The matrix R^* in (1) would then have the following form. [We have not changed the indices of the submatrices for ease in identification].

$$R^* = \begin{array}{c} \begin{matrix} A_1 & & B & & A_2 \end{matrix} \\ \begin{bmatrix} U & & R_{25}^* & & R_{26}^* \end{bmatrix} \end{array} \qquad (16)$$

The matrix R in (2) would have the following form

$$R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} -(R_{25}^{*})^t & U & 0 \\ -(R_{26}^{*})^t & 0 & U \end{bmatrix} \qquad (17)$$

(columns: A_1, B, A_2)

Suppose now B is so chosen that $M_V \times A$ has k elementary separators say $A^1, A^2 \ldots A^k$. We know from Theorems T19, T20 (see Defn. 0.35) that $(V \times A)^*$ has $R^*(A)_L$ as a standard representative matrix. We will assume that there exists a permutation matrix T such that

$$T\,G\,T^t = \begin{matrix} A^1 \\ A^2 \\ \vdots \\ A^k \\ B \end{matrix} \begin{bmatrix} G_1 & & & & \\ & G_2 & & & \\ & & \ddots & & \\ & & & G_k & \\ & & & & G_B \end{bmatrix}$$

(When G has off-diagonal terms B has to be so chosen that the above assumption is satisfied. It may be noted that this assumption is effectively the same as saying that there should be no magnetic link between the separated pieces or the set B. If G is a positive diagonal matrix the assumption is automatically satisfied).

Let us denote by G_A the matrix obtained from $G(A)$ by deleting the zero rows. Since G_A is a principal submatrix of G and G is positive definite it follows that G_A is positive definite and therefore nonsingular by Theorems T27 and T28.

Hence $(G_A)^{-1}$ exists. Let $H_A = (G_A)^{-1}$.

It is easy to see that H_A is a principal submatrix of H, and that its rows and columns correspond to the set A. Now by considering the network ($M_{V \times A}$, V X A, G_A , A) and using Theorem 2.1 we find that there exists permutation matrix T_1 such that

$$T_1(R^*(A))H_A(R^*(A))^t\, T_1^t = \tilde{H}_{A_1} = \begin{bmatrix} \tilde{H}_1 & & & \\ & \tilde{H}_2 & & \\ & & \ddots & \\ & & & \tilde{H}_k \end{bmatrix}$$

We, therefore, lose no generality in assuming that $R^*(A)$ has its rows so arranged that

$$R^*(A)\, H_A\, (R^*(A))^t = \tilde{H}_{A_1} \quad .$$

Henceforth we will make this assumption.

Equation (7) will now have the following form :

$$\left[R^*(A)\, (H_A)\, (R^*(A))^t \right] (\underline{W}_{A_1}) = \bar{\mathcal{U}}_1 - \left[R_{25}^* \right] \underline{n}_B \qquad (18)$$

i.e.

$$(\tilde{H}_{A_1})\,(\underline{W}_{A_1}) = \bar{\underline{\mathcal{U}}}_1 - \left[R_{25}^* \right] \underline{n}_B \quad . \qquad (19)$$

(We note that the source vectors have to be calculated in accordance with the basic assumptions $S_2 = \Phi$, $C = \Phi$).

Equation (15) can be written as

$$\left[J_B + (R_{25}^*)^t \left(\tilde{H}_{A_1} \right)^{-1} (R_{25}^*) \right] \underline{u}_B = \bar{\underline{p}} \qquad (20)$$

Denoting the coefficient matrix on the L.H.S. by $\tilde{J}_B$ we get,

$$\tilde{J}_B \, \underline{u}_B = \bar{\underline{p}} \quad . \qquad (21)$$

Now $J_B = G_B$ and hence by Theorem T27,

J_B is positive definite.

$(\tilde{H}_{A_1})$ is positive definite by Theorem T 29

$(\tilde{H}_{A_1})^{-1}$ is positive definite by Theorem T 28

$(R_{25}^*)^t (\tilde{H}_{A_1})^{-1} R_{25}^*$ is positive semidefinite by Theorem T29. Hence by Theorem T30, $\tilde{J}_B$ is positive definite and therefore nonsingular.

Hence $\underline{u}_B = (\tilde{J}_B)^{-1} \bar{\underline{p}}$. (21')

Substituting in equation (19) we obtain

$$\tilde{H}_{A_1} \underline{w}_{A_1} = \bar{\underline{y}}_1 - R_{25}^* (\tilde{J}_B)^{-1} \bar{\underline{p}} \quad .$$

Hence

$$\underline{v}_{A_1} = (\breve{H}_{A_1})^{-1} \left[\sigma\bar{\underline{u}}_1 - R_{25}^{*} (\breve{J}_B)^{-1} \; \bar{\underline{p}} \right] \tag{22}$$

i.e.

$$\underline{v}_{A_1} = \begin{pmatrix} (\breve{H}_1)^{-1} & & \\ & \ddots & \\ & & (\breve{H}_k)^{-1} \end{pmatrix} \left[\sigma\bar{\underline{u}}_1 - R_{25}^{*} (\breve{J}_B)^{-1} \bar{\underline{p}} \right] .$$

Hence $\underline{v}_{A_1}$ is known.

Since A_1 is a base of M_V, by Theorem 1.5, this is equivalent to the complete determination of the solution ($\underline{u}$, $\underline{v}$) for the generalized network N.

Let us now compare this procedure with that in the proof of Theorem 1.2.

The equation (A) of Theorem 1.2 would have the following form

$$\overset{\begin{matrix} A_1 \quad & B \quad & A_2 \end{matrix}}{\begin{bmatrix} U & R_{25}^{*} & R_{26}^{*} \end{bmatrix}} \begin{pmatrix} H_{A_1} & & \\ & H_B & \\ & & H_{A_2} \end{pmatrix} \begin{pmatrix} U \\ (R_{25}^{*})^t \\ (R_{26}^{*})^t \end{pmatrix} \begin{bmatrix} \underline{v}_{A_1} \end{bmatrix}$$

$$= \begin{bmatrix} U & R_{25}^{*} & R_{26}^{*} \end{bmatrix} \begin{pmatrix} H_{A_1} & & \\ & H_B & \\ & & H_{A_2} \end{pmatrix} \begin{pmatrix} \bar{\underline{v}}_{A_1} \\ \bar{\underline{v}}_B \\ \bar{\underline{v}}_{A_2} \end{pmatrix} - \begin{bmatrix} U & R_{25}^{*} & R_{26}^{*} \end{bmatrix} \begin{pmatrix} \bar{\underline{u}}_{A_1} \\ \bar{\underline{u}}_B \\ \bar{\underline{u}}_{A_2} \end{pmatrix}$$

(23)

We will denote the coefficient matrix on the L.H.S. by $\widetilde{H}$.

Let us now suppose that $\bar{\underline{u}}_{A_2} = 0$, $\bar{\underline{u}}_B = 0$ and $\bar{\underline{v}} = 0$.

Then from equation (7) we have

$$\sigma\bar{\underline{u}}_1 = \begin{bmatrix} U & R_{26}^* \end{bmatrix} \begin{pmatrix} \bar{\underline{u}}_{A_1} \\ \bar{\underline{u}}_{A_2} \end{pmatrix} = \bar{\underline{u}}_{A_1} .$$

From equation (15) we have

$$\bar{\underline{p}} = [(R_{25}^*)^t] (\widetilde{H}_{A_1})^{-1} \sigma\bar{\underline{u}}_1 = [(R_{25}^*)^t][\widetilde{H}_{A_1}]^{-1} \bar{\underline{u}}_{A_1} .$$

Hence equation (22) can be written as

$$\underline{v}_{A_1} = (\widetilde{H}_{A_1})^{-1} [(I_{|A_1|}) - R_{25}^* [\widetilde{J}_B]^{-1} (R_{25}^*)^t (\widetilde{H}_{A_1})^{-1}] \bar{\underline{u}}_{A_1} .$$

whereas equation (23) can be written as

$$\underline{v}_{A_1} = (\widehat{H})^{-1} \bar{\underline{u}}_{A_1} \qquad (24)$$

Since $(\bar{\underline{u}}_{A_1})^t$ can take any of the values $[1, 0, \ldots 0]$, $[0, 1, \ldots 0] \ldots [0, 0 \ldots 0, 1]$ it follows that

$$(\widehat{H})^{-1} = (\widetilde{H}_{A_1})^{-1} - (\widetilde{H}_{A_1})^{-1} R_{25}^* (\widetilde{J}_B)^{-1} (R_{25}^*)^t (\widetilde{H}_{A_1})^{-1}$$

$$= (\widetilde{H}_{A_1})^{-1} - Q \quad \text{say} . \qquad (25)$$

Then

$$\underline{w}_{A_1} = \left[(\widetilde{H}_{A_1})^{-1} - Q \right] \left\{ \left[U \quad R_{25}^{*} \quad R_{26}^{*} \right] \begin{pmatrix} H_{A_1} & & \\ & H_B & \\ & & H_{A_2} \end{pmatrix} \begin{pmatrix} \bar{\underline{v}}_{A_1} \\ \bar{\underline{v}}_B \\ \bar{\underline{v}}_{A_2} \end{pmatrix} - \left[U \quad R_{25}^{*} \quad R_{26}^{*} \right] \begin{pmatrix} \bar{\underline{u}}_{A_1} \\ \bar{\underline{u}}_B \\ \bar{\underline{u}}_{A_2} \end{pmatrix} \right\} \qquad (25(a))$$

We know that

$$(\widetilde{H}_{A_1})^{-1} = \begin{bmatrix} (\widetilde{H}_1)^{-1} & & \\ & \ddots & \\ & & (\widetilde{H}_k)^{-1} \end{bmatrix}$$

Let the order of $\widetilde{H}_{A_1}$ be $m \times m$ and the order of $\widetilde{H}_i$ be $n_i \times n_i$, (with $\sum_{i=1}^{k} n_i = m$), that of $\widetilde{J}_B$ be $n_{k+1} \times n_{k+1}$.

Now if we take the time required to invert a symmetric matrix of order $n \times n$ to be $\alpha \cdot n^3$ (where α is a constant) the time required to invert $\widetilde{H}$ directly would be $\alpha \cdot m^3 = t_o$, whereas the time required to invert indirectly as in R.H.S. of (25)

$t_d \simeq \sum_{i=1}^{k} \alpha \cdot n_i^3 + \alpha \cdot n_{k+1}^3$ (neglecting the time required to form products and sums of matrices).

In general $t_d \ll t_o$ since $F \propto . n_1^3 \ll \propto . m^3$ with n_{k+1} relatively small for reasonable values of k. Very large values of k can increase n_{k+1} to a value that nullifies the aim of our procedure.

Our purpose should be to minimise t_d for a given k. Clearly this means that B should be so chosen that the n_i's are approximately equal while n_{k+1} is as low as possible.

Diakoptics for electrical networks is, essentially, inverting $\widetilde{H}$ by the above indirect procedure by the use of equation (26). (See Note at the end of Case III).

Case I'. $S_1 = \Phi$ (and hence), $E = \Phi$.

This case is the dual of Case 1 i.e. we work with M_V^*, V^*, $\underline{u}$, $\underline{v}$, R^*, R, $\bar{\underline{u}}$, $\bar{\underline{v}}$ instead of M_V, $\underline{V}$, $\underline{v}$, $\underline{u}$, R, R^*, $\bar{\underline{v}}$, $\bar{\underline{u}}$.

We note that we need corresponding block diagonal conditions on G. We do not trace out the whole procedure all over again but merely write out the analogue of equation (26).

$$(\widetilde{G})^{-1} = (\widetilde{G}_{D_1})^{-1} - (\widetilde{G}_{D_1})^{-1} (-R_{38}^{*})^{t}(\widetilde{J}_F)^{-1}(-R_{38}^{*})(\widetilde{G}_{D_1})^{-1} \qquad .. \quad (26)'$$

Note :- As far as electrical networks are concerned Case I suffices both for Diakoptics and codiakoptics.

For Diakoptics we take

$\underline{u} = \underline{i}$, $\underline{v} = \underline{v}$, $\bar{\underline{u}} = \bar{\underline{I}}$, $\bar{\underline{v}} = \bar{\underline{V}}$

R = fundamental circuit matrix

R^* = fundamental cutset matrix.

For Codiakoptics we take

$\underline{u} = \underline{v}$, $\underline{w} = \underline{i}$, $\bar{\underline{u}} = \bar{\underline{V}}$, $\bar{\underline{w}} = \bar{\underline{I}}$

R = fundamental cutset matrix

R^* = fundamental circuit matrix.

Here we state Case I' separately only because it is a convenient reference when we consider Case III.

The next two cases should rightly be called 'Mixed Analysis'.

<u>Case II.</u> (Mixed Analysis (1))

$$B = \Phi , \quad F = \Phi \ .$$

We take as our starting point equation (8) and equation (13)

$$\left(\tilde{H}_{CA_1}\right) \begin{bmatrix} \underline{w}_C \\ \\ \underline{w}_{A_1} \end{bmatrix} = \bar{\underline{\mathcal{U}}}_1 - \begin{bmatrix} R_{17}^* \\ \\ 0 \end{bmatrix} \begin{bmatrix} \underline{u}_E \end{bmatrix} \qquad (8')$$

$$\left(\tilde{G}_{ED_1}\right) \begin{bmatrix} \underline{u}_E \\ \\ \underline{u}_{D_1} \end{bmatrix} = \bar{\underline{\mathcal{W}}}_1 - \begin{bmatrix} -R_{17}^{*t} \\ \\ 0 \end{bmatrix} \begin{bmatrix} \underline{w}_C \end{bmatrix} \qquad (13')$$

We will rewrite these equations as

$$\left(\tilde{H}_{CA_1}\right)\begin{bmatrix}\underline{v}_C\\ \\ \underline{v}_{A_1}\end{bmatrix} = \bar{\underline{u}}_1 - \begin{bmatrix}R_{17}^{*} & 0\\ \\ 0 & 0\end{bmatrix}\begin{bmatrix}\underline{u}_E\\ \\ \underline{u}_{D_1}\end{bmatrix} \tag{26}$$

$$\left(\tilde{G}_{ED_1}\right)\begin{bmatrix}\underline{u}_E\\ \\ \underline{u}_{D_1}\end{bmatrix} = \bar{\underline{w}}_2 - \begin{bmatrix}-R_{17}^{*t} & 0\\ \\ 0 & 0\end{bmatrix}\begin{bmatrix}\underline{v}_C\\ \\ \underline{v}_{A_1}\end{bmatrix} \tag{27}$$

Therefore,

$$\left(\tilde{G}_{ED_1}\right)\begin{bmatrix}\underline{u}_E\\ \\ \underline{u}_{D_1}\end{bmatrix} = \bar{\underline{w}}_2 - \left[\begin{bmatrix}(-R_{17}^{*})^t & 0\\ \\ 0 & 0\end{bmatrix}\left(\tilde{H}_{CA_1}\right)^{-1} \times \left[\left[\bar{\underline{u}}_1\right] - \begin{bmatrix}R_{17}^{*} & 0\\ \\ 0 & 0\end{bmatrix}\begin{bmatrix}\underline{u}_E\\ \\ \underline{u}_{D_1}\end{bmatrix}\right]\right]$$

$$= \bar{\underline{w}}_2 - \begin{bmatrix}(-R_{17}^{*})^t & 0\\ 0 & 0\end{bmatrix}\left(\tilde{H}_{CA_1}\right)^{-1}\left[\bar{\underline{u}}_1\right]$$

$$+ \begin{bmatrix}(-R_{17}^{*})^t & 0\\ \\ 0 & 0\end{bmatrix}\left(\tilde{H}_{CA_1}\right)^{-1}\begin{bmatrix}R_{17}^{*} & 0\\ \\ 0 & 0\end{bmatrix}\begin{bmatrix}\underline{u}_E\\ \\ \underline{u}_{D_1}\end{bmatrix}$$

Hence

$$\left[\left(\widetilde{G}_{ED_1}\right)+\begin{pmatrix}(R_{17}^{*})^t & 0\\ 0 & 0\end{pmatrix}(\widetilde{H}_{CA_1})^{-1}\begin{pmatrix}R_{17}^{*} & 0\\ 0 & 0\end{pmatrix}\right]\begin{pmatrix}\underline{u}_E\\ \underline{u}_{D_1}\end{pmatrix}$$

$$= \bar{\underline{\omega}}_2 - \begin{pmatrix}(-R_{17}^{*})^t & 0\\ 0 & 0\end{pmatrix}(\widetilde{H}_{CA_1})^{-1}\left[\bar{\underline{u}}_1\right] \qquad (28)$$

$= \bar{\underline{\omega}}_3$ say .

Calling the coefficient matrix on the L.H.S. as $\tilde{\widetilde{G}}_{ED_1}$ we have,

$$\left(\tilde{\widetilde{G}}_{ED_1}\right)\begin{pmatrix}\underline{u}_E\\ \underline{u}_{D_1}\end{pmatrix} = \bar{\underline{\omega}}_3 .$$

Now $\widetilde{G}_{ED_1}$ is a positive definite matrix by the use of Theorem T29 since G_{ED_1} is positive definite.

Similarly $\widetilde{H}_{CA_1}$ is positive definite and therefore $(\widetilde{H}_{CA_1})^{-1}$ is positive definite by Theorem T28.

Hence $$\begin{pmatrix}(R_{17}^{*})^t & 0\\ 0 & 0\end{pmatrix}\left(\widetilde{H}_{CA_1}\right)^{-1}\begin{pmatrix}R_{17}^{*} & 0\\ 0 & 0\end{pmatrix}$$

is positive semidefinite by Theorem T29.

Hence, by the use of Theorem T 30, $\tilde{\tilde{G}}_{ED_1}$ is positive definite and therefore nonsingular.

Therefore

$$\begin{pmatrix} \underline{u}_E \\ \\ \underline{u}_{D_1} \end{pmatrix} = \left(\tilde{\tilde{G}}_{ED_1}\right)^{-1} \left[\overline{\underline{w}}_3\right]$$

By using equation (26) we have

$$\begin{pmatrix} \underline{w}_C \\ \\ \underline{w}_{A_1} \end{pmatrix} = (\tilde{H}_{CA_1})^{-1} \left(\left[\overline{\underline{U}}_1\right] - \begin{pmatrix} K_{17}^{*} & 0 \\ 0 & 0 \end{pmatrix} (\tilde{\tilde{G}}_{ED_1})^{-1} \left[\overline{\underline{w}}_3\right] \right)$$

Hence we have determined the vector

$$\underline{q} = \begin{pmatrix} \underline{u}_E \\ \underline{u}_{D_1} \\ \underline{w}_C \\ \underline{w}_{A_1} \end{pmatrix}$$

By Theorem 1.6 this is equivalent to the determination of the solution ($\underline{u}$, $\underline{w}$) .

Now suppose $M_V \times S_1$ has a number of elementary separators. Then, as in Case I provided a corresponding block diagonal condition holds for G, $\tilde{H}_{CA_1}$ would be of the block diagonal form (perhaps after pre- and post- multiplication by a permutation matrix and its transpose), and therefore easy to invert. Also, note that the vector $\underline{q}$ may well have less number of rows than the rank or nullity of M_V. The number of rows of $\underline{q}$ is equal to $[\, r(M_V \times S_1) + \mu(M_V \cdot S_2)\,]$. (In Example 1.7 we consider a case where this number is less than $r(M_V)$ or $\mu(M_V)$). In this connection a very important problem is to partition the set S in such a fashion as to minimise

$$[\, r(M_V \times S_1) + \mu(M_V \cdot S_2)\,] .$$

This problem was completely solved by Kishi and Kajitani [Ki 2] for graphs. Essentially their procedure is to divide the edges of the graph into three sets S_o, S_1, S_2 such that (M_V being the polygon matroid of the graph).

$$r[\, M_V \times S_1 \,] + \mu[\, M_V \cdot (S_o \cup S_2)\,]$$

$$= r[\, M_V \times (S_1 \cup S_o)\,] + \mu[\, M_V \cdot S_2 \,]$$

$$= \min_{\substack{S_i,\, S_j \\ S_i \cup S_j = S}} [\, r(M_V \times S_i) + \mu(M_V \cdot S_j)\,] = r_H(M_V) .$$

The results of Kishi and Kajitani have been extended to Matroids by Bruno and Weinberg [Br 2].

Case II is far superior to solving a network by direct inversion of the matrices $\widetilde{H}$ or $\widetilde{G}$. For in this case we are inverting two matrices $\widetilde{H}_{CA_1}$ and $\widetilde{G}_{ED_1}$, the sum of whose orders may well be less than the order of $\widetilde{H}$ or $\widetilde{G}$. Further $\widetilde{H}_{CA_1}$ is of the block diagonal form. Thus there is considerable saving in computational labour. Even then, however, Case II is not sufficient for the full utilization of Kishi and Kajitani's powerful result. For, a partition of S corresponding to $r_H (M_V)$:

(1) need not leave $M_V \times S_1$ with a suitable number of elementary separators.

(2) we may have $r (M_V \times S_1) \approx \mu (M_V . S_2)$;
since only $\widetilde{H}_{CA_1}$ and not $\widetilde{\widetilde{G}}_{BD_1}$ is of the block diagonal form the procedure is not as efficient as one could wish.

Case III gets around these difficulties.

Cases I and II were originally formulated by G. Kron in 'Diakoptics' [Kr 1]. A treatment in terms of matroids of vector spaces and generalised networks is however believed to be not available prior to this thesis. Case III is believed to be new.

(If one were to treat only Cases II and I the best method would be to develop Case II first and derive Case I as a special case. We have proceeded differently, primarily in order to give a unified treatment for Cases I, II and III).

<u>Computational time for Case II</u> :

As before let us assume the time required for inverting a matrix of order n x n to be $\alpha \cdot n^3$. Let N be a network (M_V , V, G, S).

Let us decompose S into S_1 and S_2 . Suppose $M_V \times S_1$ has elementary separators $S^1, S^2 \ldots S^k$ and let,

$$r (M_V \times S^i) = r_i \quad (i \in \{1,2 \ldots k\})$$

and let $\mu (M_V \cdot S_2) = \mu_2$.

Then the time t_1 required to solve the network by Case II is (neglecting time for multiplication and addition of matrices)

$$t_1 = \alpha \cdot \sum_{i=1}^{k} (r_i)^3 + \alpha \cdot (\mu_2)^3 .$$

<u>Case III.</u> (Mixed Analysis(2)) .

In this case also we take B = Φ and F = Φ but proceed differently.

R^* and R will have the following forms (from equations (1) and (2)).

$$R^* = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc} C & A_1 & D_2 & A_2 & E & D_1 \\ \end{array}$$

$$R^* = \begin{pmatrix} U & 0 & 0 & R_{16}^* & R_{17}^* & 0 \\ 0 & U & 0 & R_{26}^* & 0 & 0 \\ 0 & 0 & U & 0 & R_{47}^* & R_{48}^* \end{pmatrix}$$

(columns: C, A_1, D_2, A_2, E, D_1)

$$R = \begin{pmatrix} (-R_{16}^*)^t & (-R_{26}^*)^t & 0 & U & 0 & 0 \\ (-R_{17}^*)^t & 0 & (-R_{47}^*)^t & 0 & U & 0 \\ 0 & 0 & (-R_{48}^*)^t & 0 & 0 & U \end{pmatrix}$$

(columns: C, A_1, D_2, A_2, E, D_1)

Equation (16) would have the following form :

$$\left[\begin{pmatrix} I_{|E|} & \\ & I_{|C|} \end{pmatrix} + \begin{pmatrix} 0 & 0 & U & 0 \\ U & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (\tilde{H}_{CA_1})^{-1} & \\ & (\tilde{G}_{ED_1})^{-1} \end{pmatrix} \begin{pmatrix} R_{17}^* & 0 \\ 0 & 0 \\ 0 & (-R_{17}^*)^t \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \underline{u}_E \\ \underline{v}_C \end{pmatrix}$$

(columns of the second matrix: C, A_1, E, D_1)

$$= \bar{\underline{p}}$$ ($\bar{\underline{p}}$ being evaluated in conformity with the assumptions of this case).

We can rewrite the above equation as :

$$\left[\begin{pmatrix} U & \\ & U \end{pmatrix} + \left(\begin{array}{cc|cc} 0 & 0 & U & 0 \\ U & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c|c} (\breve{H}_{CA_1})^{-1} & \\ \hline & (\breve{G}_{ED_1})^{-1} \end{array}\right) \left(\begin{array}{cc} 0 & U \\ 0 & 0 \\ \hline U & 0 \\ 0 & 0 \end{array}\right) \; X \right.$$

(X - represents matrix multiplication)

$$\left. X \begin{pmatrix} 0 & (-R_{17}^{*})^{t} \\ R_{17}^{*} & 0 \end{pmatrix} \right] \begin{pmatrix} \underline{u}_E \\ \underline{v}_C \end{pmatrix} = \bar{\underline{p}} \qquad (29)$$

Let us write the coefficient matrix in the L.H.S. of (29) as $\breve{J}$.

Then

$$\left(\breve{J} \right) \begin{pmatrix} \underline{u}_E \\ \underline{v}_C \end{pmatrix} = \bar{\underline{p}} \; .$$

Now $\breve{J}$ has the following form

$$\breve{J} = I + K\,L$$

where I is the identity matrix of the same order as $\breve{J}$,

$$L = \begin{pmatrix} 0 & (-R_{17}^{*})^{t} \\ R_{17}^{*} & 0 \end{pmatrix}$$

L is a skew symmetric matrix and hence positive semidefinite.

$$K = \left(\begin{array}{cc|cc} 0 & 0 & U & 0 \\ U & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c|c} (\breve{H}_{CA_1})^{-1} & \\ \hline & (\breve{G}_{ED_1})^{-1} \end{array}\right) \left(\begin{array}{cc} 0 & U \\ 0 & 0 \\ \hline U & 0 \\ 0 & 0 \end{array}\right)$$

K can be seen to be symmetric positive definite by the same arguments as in Case I.

<u>Lemma 3.2.</u> $\tilde{J}$ is nonsingular.

<u>Proof.</u> Let the order of K be m. Since K is a symmetric positive definite matrix there exists an orthogonal matrix T such that

$$TKT^t = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix} \quad \text{by Theorem T31.}$$

(λ_i positive for $i \in \{1, 2, \ldots m\}$)

$$= [\lambda] \text{ say .}$$

Hence $T \tilde{J} T^t = [\, T I T^t + T K T^t T B T^t \,]$

$$= I_m + \{\lambda\} (C)$$

where $(C) = T B T^t$.

By Theorem T29 (C) is positive semidefinite.

$$T \tilde{J} T^t = \{\lambda\} [\, \{\lambda\}^{-1} + (C) \,] .$$

Now $\{\lambda\}^{-1}$ is a positive diagonal matrix and therefore positive definite while (C) is positive semidefinite.

Hence $(\{\lambda\}^{-1} + (C))$ is positive definite by Theorem T30 and hence nonsingular. Hence $T \tilde{J} T^t$ is nonsingular and therefore $\tilde{J}$ is nonsingular.

Thus we are justified in writing

$$\begin{bmatrix} \underline{u}_E \\ \underline{v}_C \end{bmatrix} = \left[\tilde{J}\right]^{-1} \cdot \bar{\underline{p}} \quad .$$

For this case equations (8) and (13) would have the following forms.

$$\tilde{H}_{CA_1} \begin{bmatrix} \underline{v}_C \\ \underline{v}_{A_1} \end{bmatrix} = \bar{\underline{\mathcal{U}}}_1 - \begin{bmatrix} R_{17}^{*} \\ 0 \end{bmatrix} \left[\underline{u}_E\right] \tag{8'}$$

$$\tilde{G}_{ED_1} \begin{bmatrix} \underline{u}_E \\ \underline{u}_{D_1} \end{bmatrix} = \bar{\underline{\omega}}_2 - \begin{bmatrix} (-R_{17}^{*})^t \\ 0 \end{bmatrix} \left[\underline{v}_c\right] \tag{13'}$$

These equations can be rewritten as

$$\tilde{H}_{CA_1} \begin{bmatrix} \underline{v}_C \\ \underline{v}_{A_1} \end{bmatrix} = \bar{\underline{\mathcal{U}}}_1 - \begin{bmatrix} R_{17}^{*} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{u}_E \\ \underline{v}_C \end{bmatrix} \tag{30}$$

$$\tilde{G}_{ED_1} \begin{bmatrix} \underline{u}_E \\ \underline{u}_{D_1} \end{bmatrix} = \bar{\underline{\omega}}_2 - \begin{bmatrix} 0 & (-R_{17}^{*})^t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{u}_E \\ \underline{v}_C \end{bmatrix} \tag{31}$$

Hence

$$\begin{bmatrix} \underline{v}_C \\ \underline{v}_{A_1} \end{bmatrix} = (\breve{H}_{CA_1})^{-1} \left(\bar{\underline{u}}_1 - \begin{bmatrix} R_{17}^* & 0 \\ 0 & 0 \end{bmatrix} \left[\breve{J} \right]^{-1} \left[\bar{\underline{p}} \right] \right) \qquad (32)$$

$$\begin{bmatrix} \underline{u}_E \\ \underline{u}_{D_1} \end{bmatrix} = (\breve{G}_{ED_1})^{-1} \left(\bar{\underline{\omega}}_1 - \begin{bmatrix} 0 & (-R_{17}^*)^t \\ 0 & 0 \end{bmatrix} \left[\breve{J} \right]^{-1} \bar{\underline{p}} \right) \qquad (33)$$

Equations (32) and (33) involve the inversion of $\breve{H}_{CA_1}$ and $\breve{G}_{ED_1}$. Note that $\breve{J}$ also contains the inverses of these matrices. Now inverting these matrices essentially means application of Case I and Case I' to the generalized networks

$$N_1 = (M_V \times S_1 , \; V \times S_1 , \; G_{S_1} , \; S_1)$$

and

$$N_2 = (M_V \cdot S_2 , V \cdot S_2 , G_{S_2} , S_2) .$$

Hence these matrices can again be inverted indirectly by using equations (25) and (25'), after choosing suitable sets corresponding to B and F.

Finally we note that as in Case II determination of the vector

$$\begin{bmatrix} \underline{v}_C \\ \underline{v}_{A_1} \\ \underline{u}_E \\ \underline{u}_{D_1} \end{bmatrix}$$ is equivalent to the determination of the solution ($\underline{u}$, $\underline{v}$).

The advantage of this method is that full use of Kishi and Kajitani's principal partition can be made. We however pay the penalty of having to invert an extra unsymmetrical matrix $[\tilde{J}]$. However cases can easily be conceived where the advantages of this procedure far outweigh the disadvantages.

Example : $r_H(M_V) \ll r(M_V)$, $\mu(M_V)$ and if S_1, S_2 is a partition of S corresponding to

$$r_H(M_V),\ r(M_V \times S_1) \approx \mu(M_V \cdot S_1),$$

but $|E \cup C|$ reasonably small. See Example 1.7.
In this example both Case I and Case II will be relatively inefficient while Case III would be very efficient.

(We may note that when G has non-diagonal terms a partition corresponding to $r_H(M_V)$ may not be effective, for, our analysis will work only when the block diagonal condition holds for G i.e. we must be able express G as

$$G = \begin{bmatrix} G_{S_1} & \\ & G_{S_2} \end{bmatrix} .$$

In a network there might be a number of partitions of S into S_i and S_j such that

$$r(M_V \times S_i) + r(M_V \cdot S_j) = r_H(M_V) .$$

For such partitions the number $|E \cup C|$ might vary to some extent. We would like to choose one of such partitions which has $|E \cup C|$ minimum. It would, therefore, be useful to have a method for generating the set of all partitions corresponding to $r_H(M_V)$. Such a method has been obtained by T. Ohtsuki et al [Oht. 1] for graphs. In the next chapter we show that this method is a special case of our general method for partitioning molecular matroids.

Computational time for Case III :

Let $N = (M_V, V, G, S)$ be the network to be solved by Case III.

Let S be decomposed into S_1 and S_2. We assume G to have the suitable block diagonal form. We now analyse $N_1 = (M_V \times S_1, V \times S_1, G_{S_1}, S_1)$ by Case I. Let $B_1 \subseteq S_1$ such that B_1 is a subset of some cobase of $M_V \times S_1$. Let $M_V \times (S_1 - B_1)$ have the elementary separators $S^1, S^2 \ldots S^p$, and let $r(M_V \times S^i) = r_i \quad (i \in \{1,2 \ldots p\})$.

Let E and C be defined as in the theory and $|E \cup C| = k$.

Next we analyse $N_2 = [(M_V \cdot S_2)^*, (V \cdot S_2)^*, H_{S_2}, S_2)]$ by Case I (i.e. $N_2' = [(M_V \cdot S_2), V \cdot S_2, G_{S_2} \cdot S_2)]$ by Case I').

Let $B_2 \subseteq S$ such that B_2 is a subset of some cobase of $(M_V \cdot S_2)^* = M_V^* \times S_2$. Let $M_V^* \times (S_2 - B_2)$ have elementary separator $R^1, R^2 \ldots R^j$ and let

$$r(M_V^* \times R^i) = \mu_i \quad (i \in \{1,2 \ldots j\})$$

Then the time required for solution of N is :

$$t = \sum_{i=1}^{p} (r_i)^3 + \sum_{i=1}^{j} (\mu_i)^3 + k^3 .$$

[Note :- (For Case I) : In equation (25) we have expressed $(\breve{H})^{-1}$ in terms of $(\breve{H}_{A_1})^{-1}$ and $(\breve{J}_B)^{-1}$. If the order of $\breve{J}_B$ is large (when the network considered is very large) it might be worthwhile to carry this procedure out in a series of steps instead of in a single step as in equation (25). Instead of removing all the elements of B at once, we might choose to remove a sequence of sets of elements $B_1, B_2 \ldots B_{k-1}$ such that

(1) $\bigcup_{i=1}^{k-1} B_i = B$.

(2) $M_V \times (S - (\bigcup_{i=1}^{j} B_i)]$ has $j + 1$ elementary separators.

We may then express

$(\breve{H})^{-1}$ in terms of $(\breve{H}^{1}_{A_1})^{-1}$ and a certain matrix $(\breve{J}_{B_1})^{-1}$

$(\breve{H}^{1}_{A_1})^{-1}$ in terms of $(\breve{H}^{2}_{A_1})^{-1}$ and a certain matrix $(\breve{J}_{B_2})^{-1}$

$(\breve{H}_{A_1}^{k-2})^{-1}$ in terms of $(\breve{H}_{A_1}^{k-1})^{-1} = (\breve{H}_{A_1})^{-1}$ and $(\breve{J}_{B_{k-1}})^{-1}$

(order of $\breve{J}_{B_1} = |B_1|$). This of course would increase the 'clerical' labour].

Example 1.2. For the illustration of Case I we use the network N_o in Fig. 1.6. Let S be the set of edges of the graph. (of N_o).

We put $\underline{u} = \underline{i}$, $\bar{\underline{u}} = \bar{\underline{i}}$

$\underline{v} = \underline{v}$, $\bar{\underline{v}} = \bar{\underline{v}}$

V = 1-cycle space of the graph of N_o

M_V = Polygon matroid of the graph

R^* is the fundamental cutset matrix

R is the fundamental circuit matrix.

G is the branch impedance matrix

H is the branch admittance matrix.

$$G = \begin{array}{c|ccccccccc|} & 1 & 2 & 3 & 4 & 10 & 11 & 12 & 13 & 14 \\ \hline 1 & 1 & & & & & & & & \\ 2 & & 1s & & & & & & 1/2s & \\ 3 & & & 1 & & & & & & \\ 4 & & & & 1 & & & & & \\ 10 & & & & & 1 & & & & \\ 11 & & & & & & 1 & & & \\ 12 & & & & & & & 1 & & \\ 13 & & 1/2s & & & & & & 1s & \\ 14 & & & & & & & & & 1 \end{array}$$

We will obtain our results in the frequency domain. We notice that deletion of the edge 10 leaves the network in two nonseparable parts. Since this is a convenient division we put $B = \{10\}$. Note that in the matroid M_V we are performing the contraction $M_V \times (S - \{10\})$.

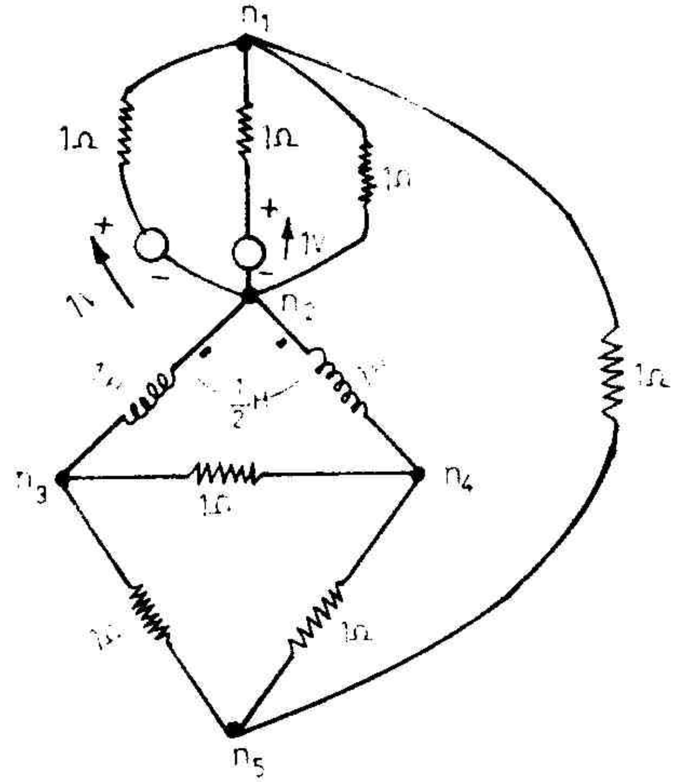


FIG. 1·6.

'DIAGRAM' OF No

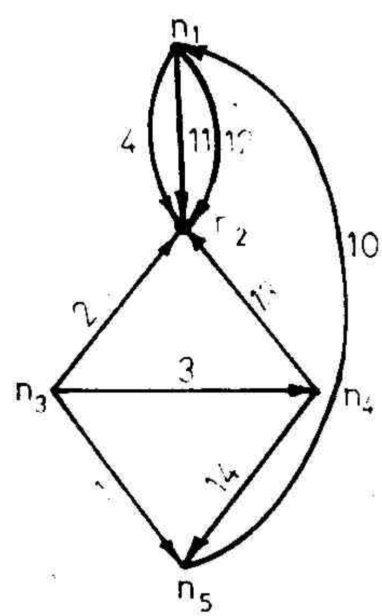


FIG. 1·7.

'GRAPH' OF No

$$R^{*} = \left(\begin{array}{cccc|c|cccc} 1 & 2 & 3 & 4 & 10 & 11 & 12 & 13 & 14 \\ \hline 1 & & & & -1 & & & & +1 \\ & 1 & & & +1 & & & +1 & \\ & & 1 & & & & & -1 & -1 \\ & & & 1 & -1 & +1 & +1 & & \end{array}\right)$$

(column 10 is marked B)

We will use the notation used in Section 2 of the theory. The above matrix would then have the following form :

$$\begin{array}{ccc} A_1 & B & A_2 \\ \end{array}$$

$$\left[U \quad R_{25}^{*} \quad R_{26}^{*} \right] ; \quad R^{*}(A_1 \cup A_2) = \left[U \quad R_{26}^{*} \right]$$

with

$$R_{25}^{*} = \begin{array}{c} 10 \\ \begin{pmatrix} -1 \\ +1 \\ 0 \\ -1 \end{pmatrix} \end{array}$$

$$R_{26}^{*} = \begin{array}{cccc} 11 & 12 & 13 & 14 \\ \end{array}$$

$$R_{26}^{*} = \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & -1 & -1 \\ +1 & +1 & 0 & 0 \end{pmatrix}$$

The branch admittance matrix $H = G^{-1}$

$$\therefore H = \begin{array}{c|ccccccccc} & 1 & 2 & 3 & 4 & 10 & 11 & 12 & 13 & 14 \\ \hline 1 & 1 & & & & & & & & \\ 2 & & 4/3s & & & & & & -4/6s & \\ 3 & & & 1 & & & & & & \\ 4 & & & & 1 & & & & & \\ 10 & & & & & 1 & & & & \\ 11 & & & & & & 1 & & & \\ 12 & & & & & & & 1 & & \\ 13 & & -4/6s & & & & & & 4/3s & \\ 14 & & & & & & & & & 1 \end{array}$$

$$\tilde{H}_{CA_1} = \begin{bmatrix} U & R_{26}^{x} \end{bmatrix} \begin{bmatrix} H(A_1 \cup A_2) \end{bmatrix} \begin{pmatrix} U \\ R_{26}^{x\,t} \end{pmatrix}$$

$$= \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 4/3s & -2/3s & 0 \\ -1 & -2/3s & 2 + 4/3s & 0 \\ \hline 0 & 0 & 0 & 3 \end{array}\right)$$

$$(\tilde{H}_{CA_1})^{-1} = \left(\begin{array}{ccc|c} \frac{1}{2} + \frac{s}{2(3s+2)} & \frac{s}{2(3s+2)} & \frac{s}{3s+2} & 0 \\ \frac{s}{2(3s+2)} & \frac{3s}{4} + \frac{s}{2(3s+2)} & \frac{s}{3s+2} & 0 \\ \frac{s}{3s+2} & \frac{s}{3s+2} & \frac{2s}{3s+2} & 0 \\ \hline 0 & 0 & 0 & 1/3 \end{array}\right)$$

The source vectors are

$$
\begin{array}{cccccccccccc}
 & & & 1 & 2 & 3 & 4 & 10 & 11 & 12 & 13 & 14 \\
\bar{\underline{u}}^t & = & \bar{\underline{i}}^t = & [\,0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\,] \\
\bar{\underline{w}}^t & = & \bar{\underline{v}}^t = & [\,0 & 0 & 0 & +1/s & 0 & +1/s & 0 & 0 & 0\,]
\end{array}
$$

(We have taken $\bar{\underline{v}}_{10} = \bar{\underline{v}}_4 = 0, t < 0$; $\bar{\underline{v}}_{10} = \bar{\underline{v}}_4 = 1, t \geq 0$).

We now evaluate the source terms $\bar{\mathcal{U}}_1$ and $\bar{\underline{p}}$ in equation (22). From equation (7) and (8) we have when $\bar{\underline{u}} = \underline{0}$,

$$
\bar{\mathcal{U}}_1 = \begin{bmatrix} U & R_{26}^{x} \end{bmatrix} \begin{pmatrix} H_{A_1} & H_{A_{12}} \\ H_{A_{21}} & H_{A_2} \end{pmatrix} \begin{pmatrix} \bar{\underline{v}}_{A_1} \\ \bar{\underline{v}}_{A_2} \end{pmatrix} =
$$

$$
= \begin{bmatrix} U & R_{26}^{x} \end{bmatrix} \begin{pmatrix} 1 & & & & & & & \\ & 4/3s & & & & & -2/3s & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & -2/3s & & & & & 4/3s & \\ & & & & & & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ +1 \\ +1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

Therefore

$$
\bar{\mathcal{U}}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ +2/s \end{pmatrix}
$$

Since the edge 10 has no voltage source or current source associated with it, $\underline{\bar{p}}' = 0$.

Hence from equation (15),

$$\underline{\bar{p}} = R_{35}^{\pi t} \left(\tilde{H}_{CA_1} \right)^{-1} \left[\underline{\bar{u}}_1 \right]$$

$$= \begin{bmatrix} -1 & +1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} + \frac{s}{2(3s+2)} & \frac{s}{2(3s+2)} & \frac{s}{3s+2} & 0 \\ \frac{s}{2(3s+2)} & \frac{3s}{4} + \frac{s}{2(3s+2)} & \frac{s}{3s+2} & 0 \\ \frac{s}{3s+2} & \frac{s}{3s+2} & \frac{2s}{3s+2} & 0 \\ 0 & 0 & 0 & 1/3 \end{bmatrix} \times$$

$$\times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2/s \end{bmatrix}$$

$$= \begin{bmatrix} -1 & +1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2/3s \end{bmatrix} = -2/3s \quad .$$

We will next evaluate $\tilde{J}_B$.

$$\tilde{J}_B = J_B + R_{26}^{x^t} \tilde{H}_{A_1}^{-1} R_{26}^{x}$$

$$= 1 + \begin{bmatrix} -1 & +1 & 0 & -1 \end{bmatrix} \begin{pmatrix} \frac{1}{2} + \frac{s}{2(3s+2)} & \frac{s}{2(3s+2)} & \frac{s}{3s+2} & 0 \\ \frac{s}{2(3s+2)} & \frac{3s}{4} + \frac{s}{2(3s+2)} & \frac{s}{3s+2} & 0 \\ \frac{s}{3s+2} & \frac{s}{3s+2} & \frac{2s}{3s+2} & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix} \times$$

$$\times \begin{pmatrix} -1 \\ +1 \\ 0 \\ -1 \end{pmatrix}$$

$$= 1 + \begin{bmatrix} -1 & +1 & 0 & -1 \end{bmatrix} \begin{pmatrix} -1/2 \\ \frac{3s}{4} \\ 0 \\ -1/3 \end{pmatrix}$$

$$= 1 + \frac{1}{2} + \frac{3s}{4} + \frac{1}{3}$$

$$= \frac{11}{6} + \frac{3s}{4} = \frac{22 + 9s}{12}$$

Hence $(\tilde{J}_B)^{-1} = \dfrac{12}{22 + 9s}$.

Hence $\underline{u}_B = (\tilde{J}_B)^{-1} \; \bar{\underline{p}}$

$$= \frac{12}{22 + 9s} \times \left(\frac{-2}{3s} \right) = \frac{-24}{3s(22 + 9s)} = \frac{-8}{s(22+9B)}$$

From equation (22) we have

$$\underline{v}_{A_1} = (\tilde{H}_{CA_1})^{-1} \left[\bar{u}_1 - \left[R_{25}^{x} \; [\tilde{J}_B]^{-1} \; \bar{\underline{p}} \right] \right]$$

$$= (\tilde{H}_{CA_1})^{-1} \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 2/s \end{pmatrix} - \begin{pmatrix} -1 \\ +1 \\ 0 \\ -1 \end{pmatrix} \frac{12}{22 + 9s} \times \frac{-2}{3s} \right]$$

$$= (\tilde{H}_{CA_1})^{-1} \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 2/s \end{pmatrix} - \begin{pmatrix} +8/s(22+9s) \\ -8/s(22+9s) \\ 0 \\ +8/s(22+9s) \end{pmatrix} \right]$$

$$= (\tilde{H}_{CA_1})^{-1} \left[\begin{pmatrix} -8/s(22+9s) \\ +8/s(22+9s) \\ 0 \\ \dfrac{18s + 36}{s(22+9s)} \end{pmatrix} \right]$$

$$\underline{V}_{A_1} = \begin{pmatrix} \frac{1}{2} + \frac{s}{2(3s+2)} & \frac{s}{2(3s+2)} & \frac{s}{3s+2} & 0 \\ \frac{3}{2(3s+2)} & \frac{3s}{4} + \frac{s}{2(3s+2)} & \frac{s}{3s+2} & 0 \\ \frac{s}{3s+2} & \frac{s}{3s+2} & \frac{2s}{3s+2} & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} \frac{-8}{s(22+9s)} \\ \frac{+8}{s(22+9s)} \\ 0 \\ \frac{18s+36}{s(22+9s)} \end{pmatrix}$$

$$= \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} -4/s(22+9s) \\ 6/(22+9s) \\ 0 \\ \frac{6s+12}{s(22+9s)} \end{pmatrix}$$

$$\underline{V} = (R^{x})^{t} \; \underline{V}_{A_1}$$

$$= \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{matrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ -1 & +1 & 0 & -1 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 \\ 0 & +1 & -1 & 0 \\ +1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -4/s(22+9s) \\ 6/22+9s \\ 0 \\ \frac{6s+12}{s(22+9s)} \end{pmatrix}$$

Hence the branch voltage vector $\underline{v}$ =

$$\begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{array} \left[\begin{array}{l} -4/s(22+9s) \\ 6/22+9s \\ 0 \\ (6s+12)/s(22+9s) \\ -8/s(22+9s) \\ (12+6s)/s(22+9s) \\ (6s+12)/s(22+9s) \\ 4/s(22+9s) \\ -4/s(22+9s) \end{array} \right]$$

$$\underline{v} - \bar{\underline{v}} = \underline{v} - \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{array} \left[\begin{array}{c} \\ \\ \\ +1/s \\ \\ +1/s \\ \\ \\ \\ \end{array} \right] = \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{array} \left[\begin{array}{l} -4/s(22+9s) \\ 6/22+9s \\ 0 \\ -(10+3s)/s(22+9s) \\ -8/s(22+9s) \\ -(10+3s)/s(22+9s) \\ (6s+12)/s(22+9s) \\ 4/s(22+9s) \\ -4/s(22+9s) \end{array} \right]$$

$\underline{u} - \bar{\underline{u}} = H(\underline{v} - \bar{\underline{v}})$. This can be seen to be

$$\begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{array} \left[\begin{array}{l} -4/s(22+9s) \\ 4/(22+9s)s \\ 0 \\ -(10+3s)/s(22+9s) \\ -8/s(22+9s) \\ -(10+3s)/s(22+9s) \\ (6s+12)/s(22+9s) \\ 4/(22+9s)s \\ -4/s(22+9s) \end{array} \right] = \underline{u} = \underline{i}$$

since the source vector $\bar{i}$ is null.

<u>Example 1.3.</u> For the illustration of Case II we use the network N_4 in Figure 1.8. We consider N_4 under the following substitution

$$\underline{u} = \underline{i}, \quad \bar{\underline{u}} = \bar{\underline{i}}, \quad \underline{w} = \underline{v}, \quad \bar{\underline{w}} = \bar{\underline{v}} \quad ;$$

$\underline{i}$ being the current vector,
$\underline{v}$ being the voltage vector and $\bar{\underline{i}}$, $\bar{\underline{v}}$ the corresponding source vectors.

It follows that R^* is a fundamental cutset matrix ;

R is the fundamental circuit matrix

V = vector space generated by the rows of R

V^*= vector space generated by the rows of R^*

M_V = Polygon matroid of the graph of N_4

G = Branch impedance matrix

H = Branch admittance matrix

Let S be the set of edges of the graph of N_4 .

We take G to be I_9 . Since $G^{-1} = H$, it follows that $H = I_9$.
Let $S_1 = \{1,2,3,4,6,7,8\}$, $S_2 = \{5,9\}$.

We will analyse $M_V \times S_1$ by 'nodal analysis' and $M_V \cdot S_1$ by 'mesh analysis'. Also, we shall make use of the disconnectedness of $M_V \times S_1$. We have

$C = \{1,2,3\}$. $B = \Phi$. $E = \{9\}$. $A_1 = \{4\}$. $A_2 = \{6,7,8\}$
$F = \Phi$. $D_2 = \{5\}$. $D_1 = \Phi$.

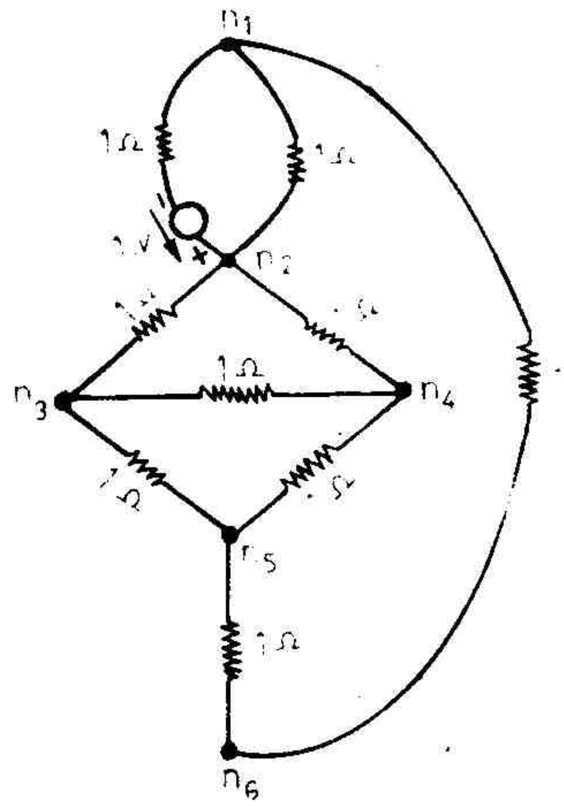


FIG. 1.8

'DIAGRAM' OF N_4

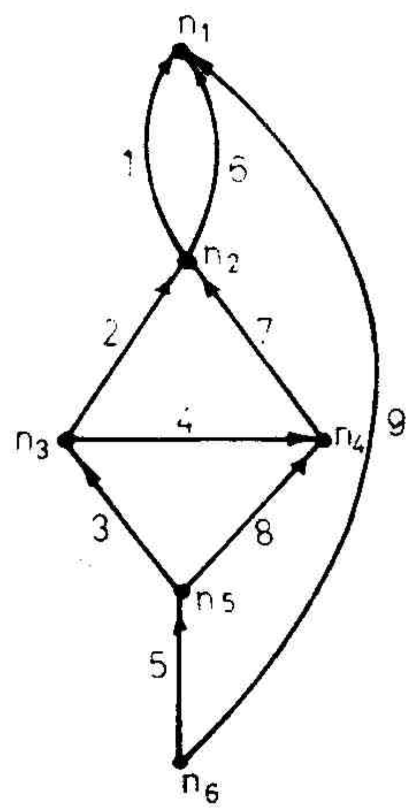


FIG. 1.9.

'GRAPH' OF N_4.

$$R^* = \left(\begin{array}{ccc|c|c|ccc|c}
\multicolumn{3}{c|}{C} & A_1 & D_2 & \multicolumn{3}{c|}{A_2} & E \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
+1 & & & & & +1 & & & +1 \\
 & +1 & & & & & +1 & & +1 \\
 & & +1 & & & & & +1 & +1 \\
\hline
 & & & +1 & & & -1 & +1 & \\
\hline
 & & & & +1 & & & & +1
\end{array}\right)$$

In the notation of the theory of Section 2 of this chapter we then have

$$R_{15}^* = \Phi . \quad R_{16}^* = \begin{array}{c} \begin{matrix} 6 & 7 & 8 \end{matrix} \\ \begin{bmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{bmatrix} \end{array} \qquad R_{17}^* = \begin{array}{c} 9 \\ \begin{pmatrix} +1 \\ +1 \\ +1 \end{pmatrix} \end{array}$$

$$R_{25}^* = \Phi . \quad R_{26}^* = \begin{bmatrix} 0 & -1 & +1 \end{bmatrix}$$

$$R_{37}^* = R_{38}^* = R_{48}^* = \Phi \quad . \qquad R_{47}^* = \begin{bmatrix} +1 \end{bmatrix} .$$

The source vectors are

$$\begin{array}{lcccccccccc}
 & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\bar{\underline{u}}^t = \underline{I}^t & = & [0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0] \\
\bar{\underline{v}}^t = \bar{\underline{V}}^t & = & [1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0] .
\end{array}$$

We now evaluate the source terms $\bar{\underline{\mu}}_1$ and $\bar{\underline{\phi}}_2$ in equations (26) and (27), using the defining equations (7) and (12).

$\tilde{\underline{u}}_1$ when $\bar{\underline{u}} = \underline{0}$ and G is positive diagonal is given by

$$\tilde{\underline{u}}_1 = \begin{pmatrix} U & 0 & R_{16}^{*} \\ 0 & U & R_{26}^{*} \end{pmatrix} \begin{pmatrix} H_C & & \\ & H_{A_1} & \\ & & H_{A_2} \end{pmatrix} \begin{pmatrix} \bar{\underline{w}}_C \\ \bar{\underline{w}}_{A_1} \\ \bar{\underline{w}}_{A_2} \end{pmatrix}$$

We then find

$$\tilde{\underline{u}}_1 = \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ \hline 0 \end{array}\right) \begin{array}{c} C \\ \\ A_1 \end{array}$$

$\bar{\underline{w}}_2$ can be seen to be $\underline{0}$ for this example. We now evaluate the matrices $\tilde{H}_{CA_1}$ and $\tilde{G}_{ED_1}$.

$$\tilde{H}_{CA_1} = \begin{pmatrix} U & 0 & R_{16}^{*} \\ 0 & U & R_{26}^{*} \end{pmatrix} \begin{pmatrix} H_C & & \\ & H_{A_1} & \\ & & H_{A_2} \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \\ (R_{16}^{*})^t & (R_{26}^{*})^t \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & +1 \\ 0 & -1 & +1 & 2 \end{pmatrix} \quad \text{(Notice the block diagonal form)}$$

Hence

$$(\tilde{H}_{CA_1})^{-1} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 5/8 & -1/8 & 1/4 \\ 0 & -1/8 & 5/8 & -1/4 \\ 0 & 1/4 & -1/4 & 1/2 \end{bmatrix}$$

Since $R_{48}^{*} = \phi$,

$$\tilde{G}_{ED_1} = \begin{bmatrix} -(R_{47}^{*})^t & U \end{bmatrix} \begin{bmatrix} G_{D_2} & \\ & G_E \end{bmatrix} \begin{bmatrix} -R_{47}^{*} \\ U \end{bmatrix} = 2$$

$$\therefore (\tilde{G}_{ED_1})^{-1} = 1/2 .$$

For the present case equation (13) reads as follows :

$$(\tilde{G}_{ED_1}) \begin{bmatrix} \underline{u}_E \end{bmatrix} = \bar{\underline{w}}_2 - \begin{bmatrix} -R_{17}^{*t} & 0 \end{bmatrix} \begin{bmatrix} w_C \\ w_{A_1} \end{bmatrix}$$

We have from equation (28)

$$(\tilde{\tilde{G}}_{ED_1}) \ \underline{u}_E = \bar{\underline{w}}_2 - \begin{bmatrix} -(R_{17}^{*})^t & 0 \end{bmatrix} (\tilde{H}_{CA_1})^{-1} \begin{bmatrix} \sigma \bar{\underline{v}}_1 \end{bmatrix}$$

Now $\tilde{\tilde{G}}_{ED_1} = \tilde{G}_{ED_1} + \begin{bmatrix} (R_{17}^{*})^t & 0 \end{bmatrix} (\tilde{H}_{CA_1})^{-1} \begin{bmatrix} R_{17}^{*} \\ 0 \end{bmatrix}$.

$$= 2 + \frac{3}{2} = \frac{7}{2} .$$

Now

$$\bar{\underline{v}}_2 - \left[-(R_{17}^{*})^t \quad 0 \right] (\breve{H}_{CA_1})^{-1} \left[\bar{\underline{u}}_1 \right] = 0 - (-1/2)$$

$$= 1/2$$

Hence $\underline{u}_E = \frac{2}{7} \times \frac{1}{2} = \frac{1}{7}$.

$$\begin{pmatrix} \underline{v}_C \\ \underline{v}_{A_1} \end{pmatrix} = (\breve{H}_{CA_1})^{-1} \left[\bar{\underline{u}}_1 \right] - \begin{pmatrix} R_{17}^{*} \\ 0 \end{pmatrix} \underline{u}_E = \begin{pmatrix} 3/7 \\ -1/14 \\ -1/14 \\ 0 \end{pmatrix}$$

From equation (6) we have

$$\begin{pmatrix} \underline{v}_C \\ \underline{v}_{A_1} \\ \underline{v}_B \\ \underline{v}_{A_2} \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U \\ (R_{15}^{*})^t & (R_{25}^{*})^t \\ (R_{16}^{*})^t & (R_{26}^{*})^t \end{pmatrix} \begin{pmatrix} \underline{v}_C \\ \underline{v}_{A_1} \end{pmatrix}$$

We can therefore evaluate $\underline{v}_{S-}\ \{5,9\}$ and find this to be

$$(\underline{v}_{S-}\ \{5,9\})^t = \begin{bmatrix} \overset{1}{3/7} & \overset{2}{-1/14} & \overset{3}{-1/14} & \overset{4}{0} & \overset{6}{3/7} & \overset{7}{-1/14} & \overset{8}{-1/14} \end{bmatrix}$$

From equation (10) we have (for this problem).

$$\begin{pmatrix} -R_{47}^{*} \\ U \end{pmatrix} \underline{u}_E = \begin{pmatrix} \underline{v}_{D_2} \\ \underline{u}_E \end{pmatrix}$$

Hence $\begin{pmatrix} \underline{u}_5 \\ \underline{u}_9 \end{pmatrix} = \begin{pmatrix} -1/7 \\ +1/7 \end{pmatrix} \begin{matrix} 5 \\ 9 \end{matrix}$

We can therefore evaluate $\begin{pmatrix} \underline{v}_5 \\ \underline{v}_9 \end{pmatrix}$ and find this to be

$$\begin{pmatrix} \underline{v}_5 \\ \underline{v}_9 \end{pmatrix} = \begin{pmatrix} -1/7 \\ +1/7 \end{pmatrix}$$

Hence

$$\begin{matrix} & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \underline{v}^t & = & [\,3/7 & -1/14 & -1/14 & 0 & -1/7 & 3/7 & -1/14 & -1/14 & 1/7\,] \end{matrix}$$

$$\underline{v}^t - \bar{\underline{v}}^t = \begin{bmatrix} -4/7 & -1/14 & -1/14 & 0 & -1/7 & 3/7 & -1/14 & -1/14 & 1/7 \end{bmatrix}$$

$$= \underline{u}^t - \bar{\underline{u}}^t, \text{ since } G = [U].$$

$$= \underline{u}^t$$

since $\bar{\underline{u}}^t = \underline{0}$.

We now give a series of three examples which together serve to illustrate Case III.

Example 1.4. For the illustration of Case I we use the network N_1 in Fig. 1.10. For convenience we retain the original labels for the edges (in conformity with Fig. 1.15).

Let S be the set of edges of the network.

We put $\underline{u} = \underline{i}$, $\bar{\underline{u}} = \bar{\underline{I}}$

$\underline{v} = \underline{v}$, $\bar{\underline{v}} = \bar{\underline{V}}$

$\underline{i}$ being the current vector

$\underline{v}$ the voltage vector

$\bar{\underline{I}}, \bar{\underline{v}}$ the corresponding source vectors.

It follows that

R^* is a fundamental cutset matrix of the oriented graph of N_1

R is the fundamental circuit matrix

V is the vector space defined by the rows of R

V^* = space defined by the rows of R^*

M_V =Polygon matroid of graph

G = Branch impedance matrix

H = Branch admittance matrix

$$G = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{array} \begin{bmatrix} 1 &&&&&&&& \\ & 1 &&&&&&& \\ && 1 &&&&&& \\ &&& 1 &&&&& \\ &&&& 1 &&&& \\ &&&&& 1 &&& \\ &&&&&& 1 && \\ &&&&&&& 1 & \\ &&&&&&&& 1 \end{bmatrix}$$

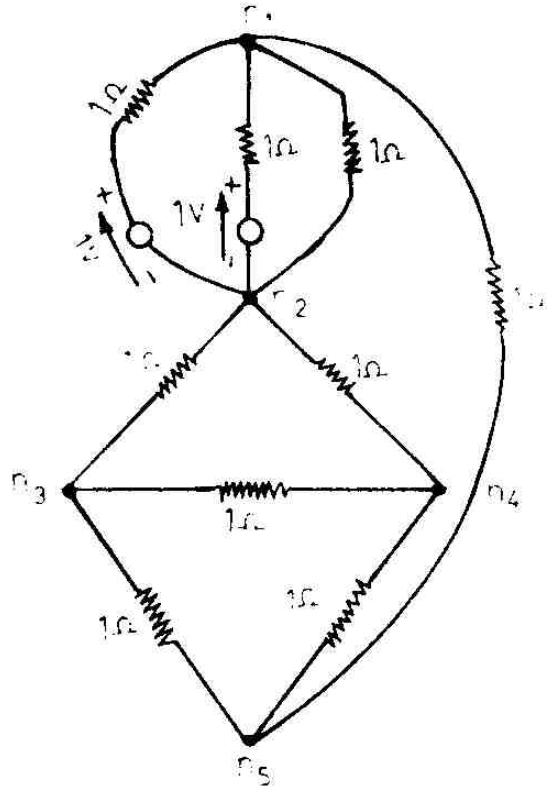


FIG. 1.10.

'DIAGRAM' OF N1

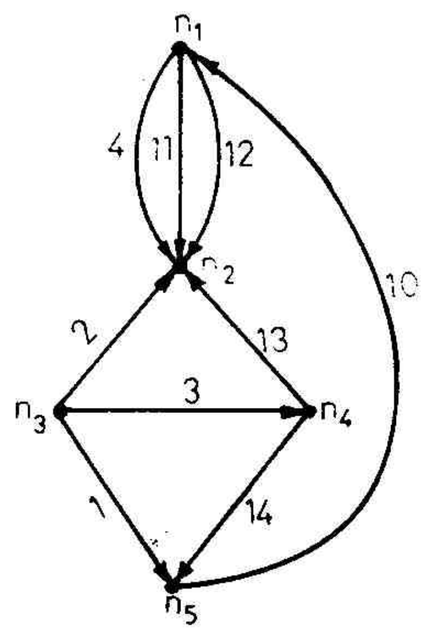


FIG. 1.11.

'GRAPH' OF N1.

We notice that deletion of the edge 10 leaves the network in two nonseparable parts. Since this is a convenient division we put $B = \{10\}$. Note that in the matroid M_V we are performing the contraction $M_V \times (S - \{10\})$.

$$R^{*} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 10 & 11 & 12 & 13 & 14 \end{matrix} \\ \begin{pmatrix} +1 & & & & -1 & & & & +1 \\ & +1 & & & +1 & & & +1 & \\ & & +1 & & & & & -1 & -1 \\ & & & +1 & -1 & +1 & +1 & & \end{pmatrix} \end{array}$$

We will use the notation used in Section 2 of the theory. The above matrix would then have the following form :

$$\begin{array}{c|c|c} A_1 & B & A_2 \\ \left[\; U \right. & R_{25}^{*} & \left. R_{26}^{*} \;\right] \end{array} ; \quad R^{*}_{(A_1 \cup A_2)} = \left[\, U \quad R_{26}^{*} \,\right]$$

with

$$R_{25}^{*} = \begin{pmatrix} -1 \\ +1 \\ 0 \\ -1 \end{pmatrix} \qquad R_{26}^{*} = \begin{array}{c} \begin{matrix} 11 & 12 & 13 & 14 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & -1 & -1 \\ +1 & +1 & 0 & 0 \end{pmatrix} \end{array}$$

The branch admittance matrix $H = G^{-1}$.

$$\therefore \quad H = I_{|14|} \; .$$

$$\tilde{H}_{A_1} = \begin{bmatrix} U & R_{26}^{*} \end{bmatrix} \begin{bmatrix} H(A_1 \cup A_2) \end{bmatrix} \begin{bmatrix} U \\ R_{26}^{*t} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$(\tilde{H}_{A_1})^{-1} = \begin{bmatrix} 5/8 & 1/8 & 1/4 & 0 \\ 1/8 & 5/8 & 1/4 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 0 & 0 & 0 & 1/3 \end{bmatrix}$$

The source vectors are

$$\begin{array}{cccccccccccc} & & & 1 & 2 & 3 & 4 & 10 & 11 & 12 & 13 & 14 \\ \bar{u}^t = \bar{i}^t = & \Big[& & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ \Big] \\ \bar{v}^t = \bar{v}^t = & \Big[& & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \ \Big] \end{array}$$

We now evaluate the source terms $\bar{u}_1$, and $\bar{p}$ in equation (32).
From equation (7), (8) we have (when $\bar{u} = 0$)

$$\tilde{u}_1 = \begin{bmatrix} U & R_{26}^{*} \end{bmatrix} \begin{bmatrix} H_{A_1} & \\ & H_{A_2} \end{bmatrix} \begin{bmatrix} \bar{v}_{A_1} \\ \bar{v}_{A_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

Since the edge 10 has no voltage or current source associated with it (see equation (15)).

$$\bar{\underline{p}}' = 0$$

Hence from equation (15)

$$\bar{\underline{p}} = R_{25}^{*t} (\tilde{H}_{A_1})^{-1} [\bar{\underline{u}}_1]$$

$$= -\frac{2}{3} .$$

We will next evaluate $\tilde{J}_B$.

$$\tilde{J}_B = [J_B + R_{25}^{*t} (\tilde{H}_{A_1})^{-1} R_{25}^{*}]$$

$$= [1 + \frac{4}{3}] = \frac{7}{3} .$$

Hence $(\tilde{J}_B)^{-1} = \frac{3}{7}$.

$$\underline{u}_B = (\tilde{J}_B)^{-1} \bar{\underline{p}} \qquad \text{(equation 21')}$$

Hence, $\underline{u}_B = \frac{3}{7} \times \frac{-2}{3} = -\frac{2}{7}$.

From equation (22), we have

$$\underline{y}_{A_1} = (\tilde{H}_{A_1})^{-1} [\bar{\underline{u}}_1 - R_{25}^{*} (\tilde{J}_B)^{-1} \bar{\underline{p}}] .$$

$$Y_{A_1} = \underline{y}_{A_1} = \begin{pmatrix} +1/7 \\ 1/7 \\ 0 \\ 4/7 \end{pmatrix}$$

$$\underline{v} = (R^{*})^{t}\, \underline{v}_{A_1}$$

$$= \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{array} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ -1 & +1 & 0 & -1 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & +1 \\ 0 & +1 & -1 & 0 \\ +1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1/7 \\ 1/7 \\ 0 \\ 4/7 \end{pmatrix} = \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{array} \begin{pmatrix} -1/7 \\ 1/7 \\ 0 \\ 4/7 \\ -2/7 \\ 4/7 \\ 4/7 \\ 1/7 \\ -1/7 \end{pmatrix}$$

$$\underline{v} - \bar{\underline{v}} = \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{array} \begin{pmatrix} -1/7 \\ 1/7 \\ 0 \\ 4/7 \\ -2/7 \\ 4/7 \\ 4/7 \\ 1/7 \\ -1/7 \end{pmatrix} - \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{array} \begin{pmatrix} -1/7 \\ 1/7 \\ 0 \\ -3/7 \\ -2/7 \\ -3/7 \\ 4/7 \\ 1/7 \\ -1/7 \end{pmatrix}$$

$$\underline{u} - \bar{\underline{u}} = H\,(\underline{v} - \bar{\underline{v}})$$

However since for this example $\bar{\underline{u}} = 0$.

$$\underline{u} = H \left[\, \underline{v} - \bar{\underline{v}} \,\right]$$

Therefore,

$$\underline{u}^{t} = \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 10 & 11 & 12 & 13 & 14 \\ \end{array}$$
$$\left[-\tfrac{1}{7} \quad \tfrac{1}{7} \quad 0 \quad -\tfrac{3}{7} \quad -\tfrac{2}{7} \quad -\tfrac{3}{7} \quad \tfrac{4}{7} \quad \tfrac{1}{7} \quad -\tfrac{1}{7} \right]$$

Let us now evaluate the inverse of the node pair admittance matrix as in (25). We have

$$(\widetilde{H}_A)^{-1} = (\widetilde{H}_{A_1})^{-1}\left[[U] - R_{25}^{*}(\widetilde{J}_B)^{-1} R_{25}^{*t}(\widetilde{H}_{A_1})^{-1}\right]$$

$$= \begin{pmatrix} 29/56 & 13/56 & 1/4 & -1/14 \\ 13/56 & 29/56 & 1/4 & 1/14 \\ 1/4 & 1/4 & 1/2 & 0 \\ -1/14 & 1/14 & 0 & 2/7 \end{pmatrix}$$

From equation (25 a) after deleting the current source terms

$$V_{A_1} = (\widetilde{H}_A)^{-1} \begin{pmatrix} U & R_{25}^{*} & R_{26}^{*} \end{pmatrix} \begin{pmatrix} H_{A_1} & & \\ & H_B & \\ & & H_{A_2} \end{pmatrix} \begin{pmatrix} \bar{V}_{A_1} \\ \bar{V}_B \\ \bar{V}_{A_2} \end{pmatrix}$$

$$= (\widetilde{H}_A)^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1/7 \\ 1/7 \\ 0 \\ 4/7 \end{pmatrix}$$

<u>**Example 1.5.**</u> We now give an example to illustrate Case I under dual substitutions.

(We consider the network N_2 in Figure 1.12), i.e. we put

$$\underline{u} = \underline{v} \ ; \ \bar{\underline{u}} = \bar{\underline{v}} \ ; \ \underline{v} = \underline{i} \ ; \ \bar{\underline{v}} = \bar{\underline{i}} \ .$$

where $\underline{i}$ is the current vector

$\bar{\underline{i}}$ is the current source vector

$\underline{v}$ is voltage vector

$\bar{\underline{v}}$ is the voltage source vector.

Hence R^* is the fundamental circuit matrix of the oriented graph of N_2

R is the fundamental cutset matrix

V is the space defined by the rows of R

V^* is the space defined by the rows of R^*

G is the branch admittance matrix

H is the branch impedance matrix

M_V is the Polygon matroid of the graph

Let S be the set of edges of the graph. We have

$$G = I_9$$

We notice that the contraction (short circuiting) of edge 5 leaves the network in two nonseparable parts. Since this is a convenient division we put $B = \{5\}$. (Note that in the matroid M_V we are performing the contraction $M_V \times (S - \{5\}$

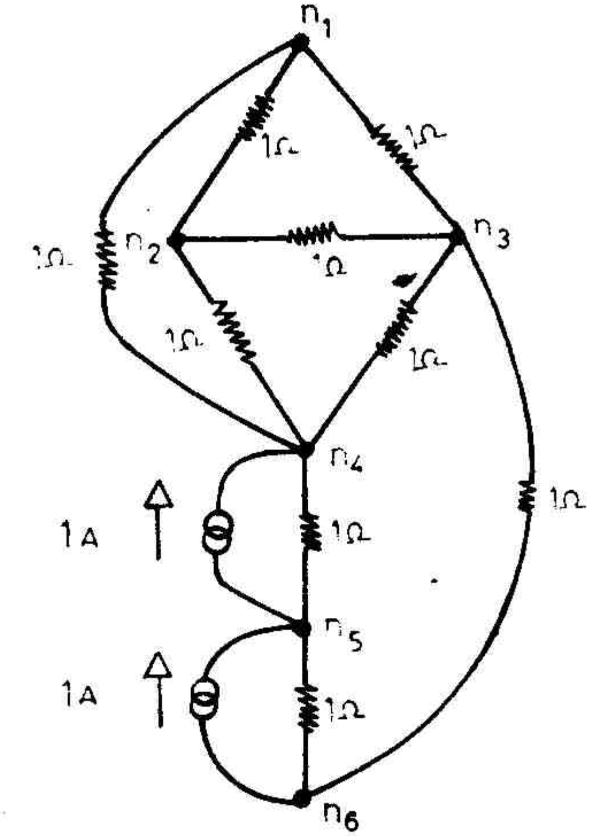


FIG.1.12.

'DIAGRAM' OF N_2.

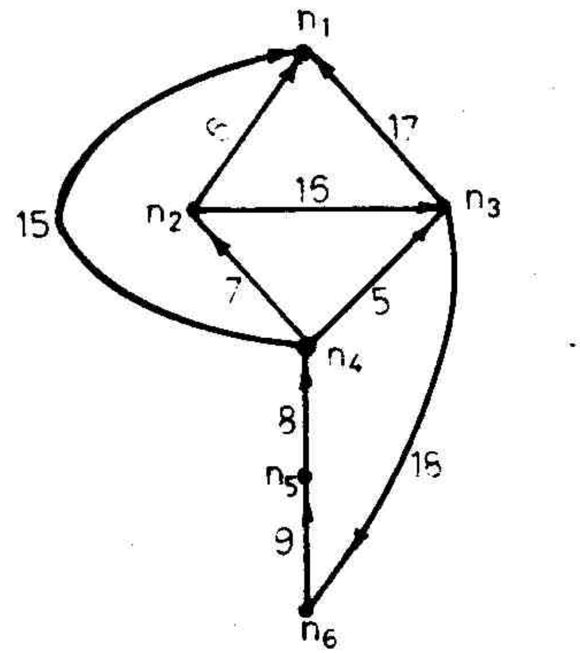


FIG.1.13.

'GRAPH' OF N_2.

$$
R^{*} = \begin{array}{c} \\ \end{array}
\begin{pmatrix}
\overset{15}{+1} & \overset{16}{} & \overset{17}{} & \overset{18}{} & \overset{5}{} & \overset{6}{-1} & \overset{7}{-1} & \overset{8}{} & \overset{9}{} \\
 & +1 & & & -1 & & +1 & & \\
 & & +1 & & +1 & -1 & -1 & & \\
 & & & +1 & +1 & & & +1 & +1
\end{pmatrix}
$$

In the notation of the theory the above matrix would have the following form.

$$
\begin{array}{ccc} A_1 & B & A_2 \\ \end{array}
$$

$$
\begin{bmatrix} U & R_{25}^{*} & R_{26}^{*} \end{bmatrix}
$$

$$
R^{*} (A_1 \; U \; A_2) = \begin{bmatrix} U & R_{26}^{*} \end{bmatrix}
$$

We have

$$
R_{25}^{*} = \begin{pmatrix} 0 \\ -1 \\ +1 \\ +1 \end{pmatrix} \qquad
R_{26}^{*} = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & +1 & +1 \end{pmatrix}
$$

The branch impedance matrix $H = G^{-1}$

$$
H = \begin{bmatrix} U \end{bmatrix}
$$

$$
\widehat{H}_{A_1} = \begin{bmatrix} U & R_{26}^{*} \end{bmatrix} \begin{bmatrix} H(A_1 \; U \; A_2) \end{bmatrix} \begin{pmatrix} U \\ R_{26}^{*t} \end{pmatrix}
$$

$$\widetilde{H}_{A_1} = \begin{pmatrix} 3 & -1 & 2 & 0 \\ -1 & 2 & -1 & 0 \\ 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Hence

$$(\widetilde{H}_{A_1})^{-1} = \begin{bmatrix} 5/8 & 1/8 & -3/8 & \\ 1/8 & 5/8 & 1/8 & \\ -3/8 & 1/8 & 5/8 & \\ & & & 1/3 \end{bmatrix}$$

The source vectors are

$$\begin{matrix} & 15 & 16 & 17 & 18 & 5 & 6 & 7 & 8 & 9 \\ \bar{\underline{u}}^t = \bar{\underline{v}}^t = & [\,0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\,] \\ \bar{\underline{w}}^t = \bar{\underline{i}}^t = & [\,0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & +1\,] \end{matrix}$$

We now evaluate the source terms $\bar{\underline{u}}_1$ and $\bar{\underline{p}}$ in the equation (22). From equations (7), (8) we have when $\bar{\underline{u}} = 0$.

$$\bar{\underline{u}}_1 = \begin{bmatrix} U & R_{26}^{*} \end{bmatrix} \begin{pmatrix} H_{A_1} & \\ & H_{A_2} \end{pmatrix} \begin{pmatrix} \bar{w}_{A_1} \\ \bar{w}_{A_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

Since edge 5 has no voltage or current source associated with it

$$\bar{\underline{p}}' = 0$$

Hence from equation 15

$$\bar{\underline{p}} = R_{25}^{*t} (\breve{H}_{A_1})^{-1} \bar{\underline{u}}_1 = + \frac{2}{3}$$

We will next evaluate $\breve{J}_B$.

$$\breve{J}_B = J_B + R_{25}^{*t} (\breve{H}_{A_1})^{-1} R_{25}^{*}$$

$$= 1 + (0 \quad -1 \quad +1 \quad +1) (\breve{H}_{A_1})^{-1} \begin{pmatrix} 0 \\ -1 \\ +1 \\ +1 \end{pmatrix}$$

$$= 7/3 .$$

Hence $(\breve{J}_B)^{-1} = 3/7$

Hence $\underline{u}_B = \left(\breve{J}_B \right)^{-1} \bar{\underline{p}}$

$$= \frac{3}{7} \times \frac{2}{3} = \frac{2}{7} .$$

From equation (22) we have

$$\underline{v}_{A_1} = (\breve{H}_{A_1})^{-1} \left[\bar{\underline{u}}_1 - R_{25}^{*} \left(\breve{J}_B \right)^{-1} \bar{\underline{p}} \right] .$$

$$
\underline{w}_{A_1} = \begin{matrix} 15 \\ 16 \\ 17 \\ 18 \end{matrix} \begin{pmatrix} 1/7 \\ 1/7 \\ -1/7 \\ 4/7 \end{pmatrix}
$$

$$
\underline{v} = (R^{*})^{t}\, \underline{w}_{A_1} = \begin{matrix} 15 \\ 16 \\ 17 \\ 18 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ & -1 & +1 & +1 \\ -1 & & -1 & \\ -1 & +1 & -1 & \\ & & & +1 \\ & & & +1 \end{pmatrix} \begin{pmatrix} 1/7 \\ 1/7 \\ -1/7 \\ 4/7 \end{pmatrix}
$$

$$
\begin{array}{lccccccccc}
 & 15 & 16 & 17 & 18 & 5 & 6 & 7 & 8 & 9 \\
\underline{v}^{t} = & [\,1/7 & 1/7 & -1/7 & 4/7 & 2/7 & 0 & 1/7 & 4/7 & 4/7\,] \\
\underline{v}^{t} - \underline{\bar{v}}^{t} = & [\,1/7 & 1/7 & -1/7 & 4/7 & 2/7 & 0 & 1/7 & 4/7 & 4/7\,] - \\
 - & [\,0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & +1\,] \\
 = & [\,1/7 & 1/7 & -1/7 & 4/7 & 2/7 & 0 & 1/7 & -3/7 & -3/7\,]
\end{array}
$$

$\underline{u} - \underline{\bar{u}} = H(\underline{v} - \underline{\bar{v}})$; $\underline{\bar{u}} = 0$; $H = [U]$

Hence $\underline{u} = \underline{v} - \underline{\bar{v}}$.

Let us now evaluate the inverse of the node-pair admittance matrix as in (25). We have

$$(\widetilde{H}_A)^{-1} = (\widetilde{H}_{A_1})^{-1}\left[[U] - R_{25}^{*}(\widetilde{J}_B)^{-1} R_{25}^{*t}(\widetilde{H}_{A_1})^{-1}\right]$$

$$= \begin{matrix} 15 \\ 16 \\ 17 \\ 18 \end{matrix} \begin{pmatrix} 29/56 & 1/56 & -15/56 & 1/4 \\ 1/56 & 29/56 & 13/56 & 1/14 \\ -15/56 & 13/56 & 29/56 & -1/14 \\ 1/14 & 1/14 & -1/14 & 2/7 \end{pmatrix}$$

Example 1.6.

For the illustration of Case III we use the network N in Fig. 1.14.

Let S be the set of edges of the graph.

We put $\underline{u} = \underline{i}$; $\bar{\underline{u}} = \bar{\underline{i}}$; $\underline{v} = \underline{v}$; $\bar{\underline{v}} = \bar{\underline{v}}$.

Hence R^{*} is the fundamental cutset matrix

R is the fundamental circuit matrix

V is the space defined by the rows of R

V^{*} is the space defined by the rows of R^{*}

M_V is the bond matroid of the graph

G is the branch impedance matrix

H is the branch admittance matrix.

We have from the Figure 1.14,

$$G = I_{18}$$

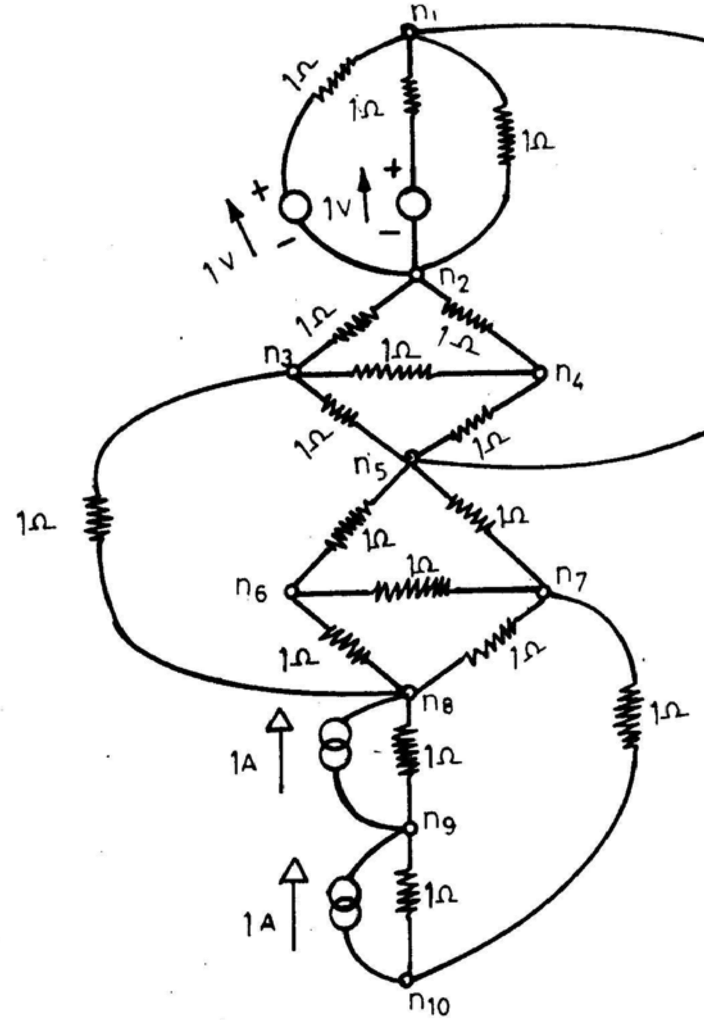


FIG. 1.14.

DIAGRAM OF N_3

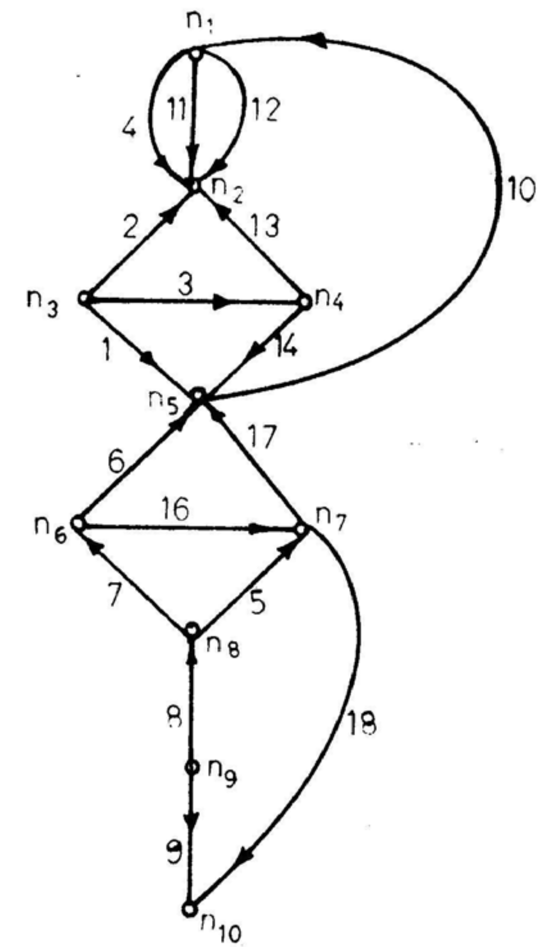


FIG. 1.15.

GRAPH OF N_3

We notice if we choose $\{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\}$ as a tree of the network, and

$$S_1 = \{1,2,3,4,\ 10,\ 11,\ 12,\ 13,\ 14\}$$

$$S_2 = \{5,6,7,8,9,\ 15,\ 16,\ 17,\ 18\}$$

$C = \{1\}$ and $E = \{15\}$. The sets B and F are absent. We now form the fundamental cutset matrix R^* with respect to the tree $\{1,2,3,4,5,6,7,8,9.\}$ (See next page).

$$A_1 = \{2,3,4\} \qquad D_1 = \{16,\ 17,\ 18\}$$

$$A_2 = \{10,11,12,13,14\} \qquad D_2 = \{5,6,7,8,9\}$$

The source terms are

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\bar{\underline{j}}^t = \bar{\underline{i}}^t =$	0	0	0	0	0	0	0	+1	+1	0	0	0	0	0	0	0	0	0
$\bar{\underline{v}}^t = \bar{\underline{v}}^t =$	0	0	0	+1	0	0	0	0	0	0	+1	0	0	0	0	0	0	0

Denote

$$\begin{pmatrix} U & 0 & R_{16}^* \\ 0 & U & R_{26}^* \end{pmatrix} \begin{pmatrix} H_C & & \\ & H_{A_1} & \\ & & H_{A_2} \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \\ R_{16}^{*t} & R_{26}^{*t} \end{pmatrix} \text{ as } \breve{H}_{CA_1}$$

G	A_1			D_2					A_2					E	D_1		
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
+1									-1				+1	-1			
	+1								+1			+1					
		+1										-1	-1				
			+1						-1	+1	+1						
				+1											+1	-1	-1
					+1									+1		+1	
						+1								+1	-1	+1	
							+1										-1
								+1									-1

$= R^*$

Notice that the inverse of the above matrix has been evaluated in example 1.4 but is denoted there by $(\breve{H}_A)^{-1}$.

Hence

$$(\breve{H}_{CA_1})^{-1} = \begin{pmatrix} 29/56 & 13/56 & 1/4 & -1/14 \\ 13/56 & 29/56 & 1/4 & 1/14 \\ 1/4 & 1/4 & 1/2 & 0 \\ -1/14 & 1/14 & 0 & 2/7 \end{pmatrix}$$

From equation (7), when $\begin{pmatrix} \bar{\underline{u}}_C \\ \bar{\underline{u}}_{A_1} \\ \bar{\underline{u}}_{A_2} \end{pmatrix} = \underline{0}$,

$$\begin{pmatrix} U & 0 & R_{16}^{*} \\ 0 & U & R_{26}^{*} \end{pmatrix} \begin{pmatrix} H_C & & \\ & H_{A_1} & \\ & & H_{A_2} \end{pmatrix} \begin{pmatrix} \bar{\underline{V}}_C \\ \bar{\underline{V}}_{A_1} \\ \bar{\underline{V}}_{A_2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

Denote

$$\begin{pmatrix} -R_{47}^{*t} & U & 0 \\ -R_{48}^{*t} & 0 & U \end{pmatrix} \begin{pmatrix} G_{D_2} & & \\ & G_E & \\ & & G_{D_1} \end{pmatrix} \begin{pmatrix} -R_{47}^{*} & -R_{48}^{*} \\ U & 0 \\ 0 & U \end{pmatrix}$$

by $\breve{G}_{ED_1}$.

Note that the inverse of this matrix has already been evaluated in example 1.6 but is denoted there by $(\tilde{F}_A)^{-1}$.

Hence

$$(\tilde{G}_{ED_1})^{-1} = \begin{pmatrix} 29/56 & 1/56 & -15/56 & 1/14 \\ 1/56 & 29/56 & 13/56 & 1/14 \\ -15/56 & 13/56 & 29/56 & -1/14 \\ 1/14 & 1/14 & -1/14 & 2/7 \end{pmatrix}$$

From equation (12) when $\begin{pmatrix} \bar{w}_{D_2} \\ \bar{w}_E \\ \bar{w}_{B_1} \end{pmatrix} = \underline{0}$

we have

$$\bar{\underline{w}}_2 = \begin{pmatrix} -R_{47}^{*} & U & 0 \\ -R_{48}^{*} & 0 & U \end{pmatrix} \begin{pmatrix} G_{D_2} & & \\ & G_E & \\ & & G_{D_1} \end{pmatrix} \begin{pmatrix} \bar{u}_{D_2} \\ \bar{u}_E \\ \bar{u}_{D_1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

Clearly we have $\underline{\bar{p}}' = 0$

$$\underline{\bar{p}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (\tilde{H}_{CA_1})^{-1} & \\ & (\tilde{G}_{ED_1})^{-1} \end{pmatrix} \begin{pmatrix} \bar{\underline{u}}_1 \\ \bar{\underline{w}}_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1/7 \\ -1/7 \end{pmatrix}$$

We next evaluate $\tilde{J}$.

By equation (29)

$$\tilde{J} = [U] + \begin{pmatrix} \overset{C}{0} & \overset{A_1}{0} & 0 & 0 & \overset{E}{1} & \overset{D_1}{0} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} (\tilde{H}_{CA_1})^{-1} & \\ & (\tilde{G}_{ED_1})^{-1} \end{bmatrix} \begin{pmatrix} R_{17}^{*} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -(R_{17}^{*})^t \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\tilde{J} = \begin{pmatrix} 1 & 29/56 \\ 29/56 & 1 \end{pmatrix}$$

Hence $(\tilde{J})^{-1} = \dfrac{56 \times 56}{3877} \begin{pmatrix} 1 & -29/56 \\ 29/56 & 1 \end{pmatrix}$

From equations (32) and (33)

$$\begin{pmatrix} \underline{V}_C \\ \\ \underline{V}_{A_1} \end{pmatrix} = (\tilde{H}_{CA_1})^{-1} \left(\bar{\underline{V}}_1 - \begin{matrix} C \\ \\ A_1 \end{matrix} \begin{pmatrix} R_{17}^{*} & 0 \\ \\ 0 & 0 \end{pmatrix} (\tilde{J})^{-1} \; \bar{\underline{p}} \right)$$

.. (32)

$$\begin{pmatrix} u_E \\ u_{D_1} \end{pmatrix} = (\tilde{G}_{ED_1})^{-1} \left(\bar{\underline{w}}_2 - \begin{matrix} E \\ D_1 \end{matrix} \begin{pmatrix} 0 & -{R_{17}^{*}}^{t} \\ 0 & 0 \end{pmatrix} (\tilde{J})^{-1} \; \bar{p} \right) \quad .. \quad (39)$$

$$\begin{pmatrix} w_C \\ \hline w_{A_1} \end{pmatrix} = \begin{pmatrix} -.0544 \\ \hline .1828 \\ .0428 \\ .5600 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$\begin{pmatrix} u_E \\ \hline \\ \\ u_{D_1} \end{pmatrix} = \begin{pmatrix} .171 \\ \hline .142 \\ .158 \\ .576 \end{pmatrix} \begin{matrix} 15 \\ 16 \\ 17 \\ 18 \end{matrix}$$

We can then obtain

$$\underline{w}_{A_2} = \left[(R_{16}^{*})^{t} \quad (R_{26}^{*})^{t} \right] \begin{pmatrix} \underline{w}_C \\ \underline{w}_{A_1} \end{pmatrix}$$

$$\underline{u}_{A_2} = (G_{A_2})^{-1} (\underline{w}_{A_2} - \bar{\underline{w}}_{A_2})$$

$$\begin{pmatrix} \underline{u}_C \\ \underline{u}_{A_1} \end{pmatrix} = \begin{pmatrix} (G_C)^{-1} & \\ & (G_{A_1})^{-1} \end{pmatrix} \begin{pmatrix} \underline{w}_C - \bar{\underline{w}}_C \\ \bar{\underline{w}}_{A_1} - \bar{\underline{w}}_{A_1} \end{pmatrix}$$

$$\underline{u}_{D_2} = \begin{bmatrix} -R_{47}^{*} & -R_{48}^{*} \end{bmatrix} \begin{pmatrix} \underline{u}_E \\ \underline{u}_{D_1} \end{pmatrix}$$

$$\underline{v}_{D_2} = G_{D_2} \left(\underline{u}_{D_2} - \bar{\underline{u}}_{D_2} \right)$$

$$\begin{pmatrix} \underline{v}_E \\ \underline{v}_{D_1} \end{pmatrix} = \begin{pmatrix} G_E & \\ & G_{D_1} \end{pmatrix} \begin{pmatrix} \underline{u}_E - \bar{\underline{u}}_E \\ \underline{u}_{D_1} - \bar{\underline{u}}_{D_1} \end{pmatrix}$$

We can thus evaluate the current and voltage vectors and obtain the following result :

Currents : (in Amperes)

$u_1 = -.0544$	$u_7 = .130$	$u_{13} = .140$
$u_2 = .1828$	$u_8 = .576$	$u_{14} = -.0972$
$u_3 = .0428$	$u_9 = .576$	$u_{15} = .171$
$u_4 = -.440$	$u_{10} = -.323$	$u_{16} = .143$
$u_5 = .275$	$u_{11} = -.440$	$u_{17} = -.158$
$u_6 = -.013$	$u_{12} = .560$	$u_{18} = .576$

Voltages : (in Volts)

$v_1 = -.0544$	$v_7 = .130$	$v_{13} = .140$
$v_2 = .1828$	$v_8 = -.424$	$v_{14} = -.0972$
$v_3 = .0428$	$v_9 = -.424$	$v_{15} = .171$
$v_4 = .560$	$v_{10} = -.323$	$v_{16} = .143$
$v_5 = .275$	$v_{11} = .560$	$v_{17} = -.158$
$v_6 = -.013$	$v_{12} = .560$	$v_{18} = .576$

<u>Example 1.7.</u> In Figure 1.16 we have the graph of a network that is best analysed by Case III. Let M_V be the polygon matroid of the graph and V the 1-cycle space of the graph. Let N = (M_V, V, G, S). (Let us assume G to be diagonal). If one solves the network by Case I or Case II the time t required for solution can be seen to be (at best)

$$t = \alpha \,.\, [\, 5^3 + 4^3 + 5^3 + 5^3 + 3^3 \,] = \alpha \,.\, 466.$$

This corresponds to the decomposition shown in Figure 1.17 with $S_2 = \Phi$, $S_1 = S$ and $B = \{25,\ 31,\ 32\}$. (The reader may note that this corresponds to nodal analysis by tearing).

Now let us use Case III and the decomposition shown in Fig. 1.18 we obtain

$$S_1 = \{1, 2 \ldots 25\}, \quad S_2 = \{26, 27, \ldots 37\}$$
$$E = \{31\}, \quad C = \{28\}$$

(E and C can of course be chosen in a number of ways)

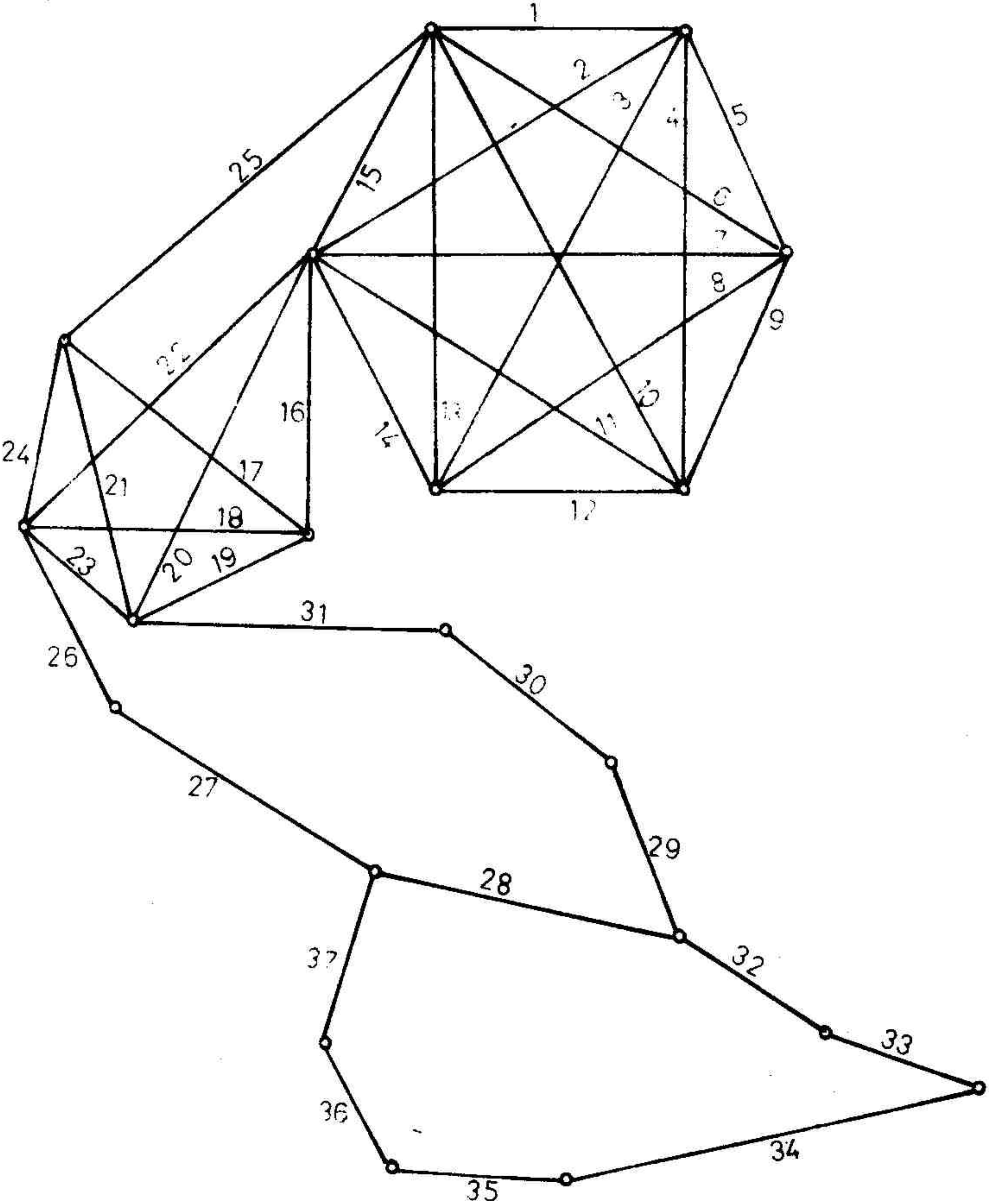
We obtain

$$r\,[\, M_V \times S_1 \,] = 9$$

$$\mu\,[\, M_V \,.\, S_2 \,] = 2$$

$$|E \cup C| = 2$$

Using Case I on N_1 = ($M_V \times S_1$, $V \times S_1$, G_{S_1} , S_1) and taking $B = \{25\}$ we obtain the decomposition in Figure 1.19. (This corresponds to nodal analysis by tearing).


1
2
3
4
5
6
7
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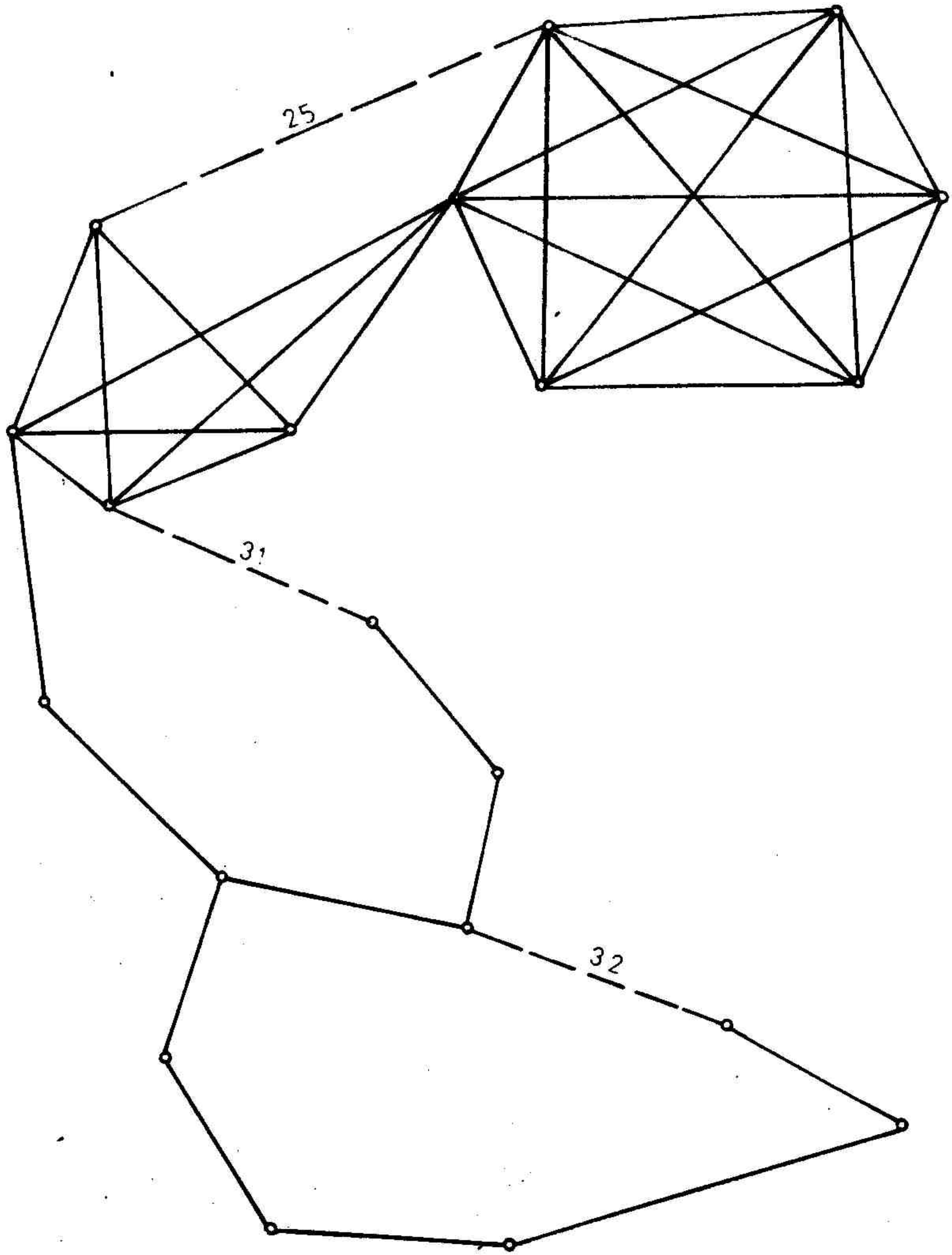
37

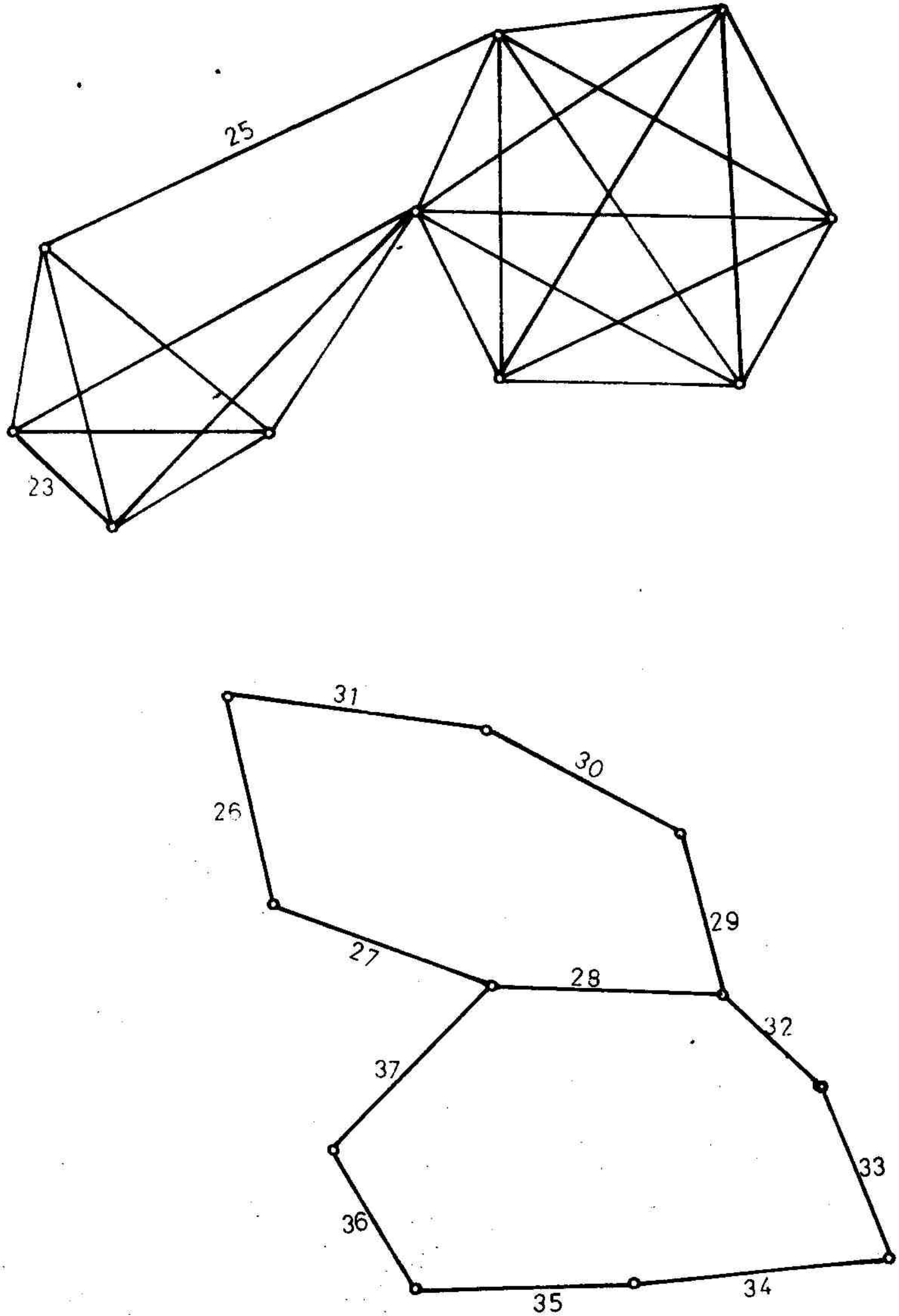
FIG · 1·16·



B = {25, 31, 32}

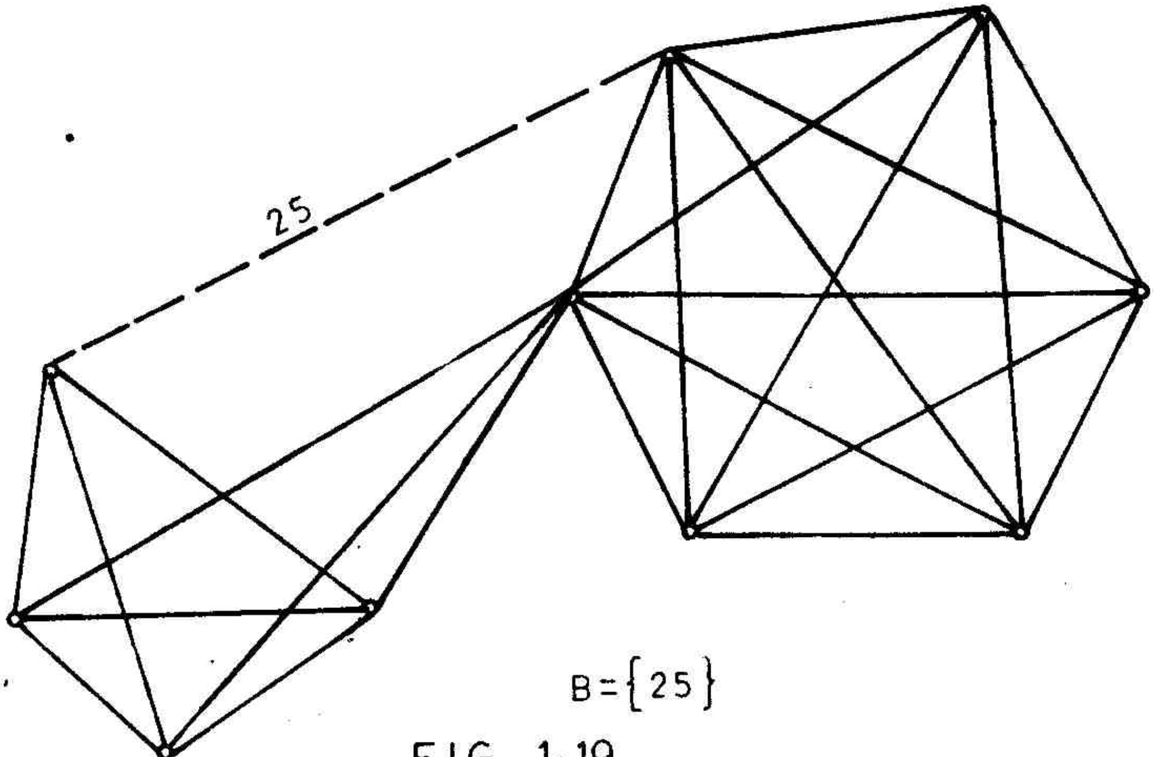
FIG. 1.17.

'BEST DECOMPOSITION' FOR CASE I OR CASE II.



E = {31}   C = {23}

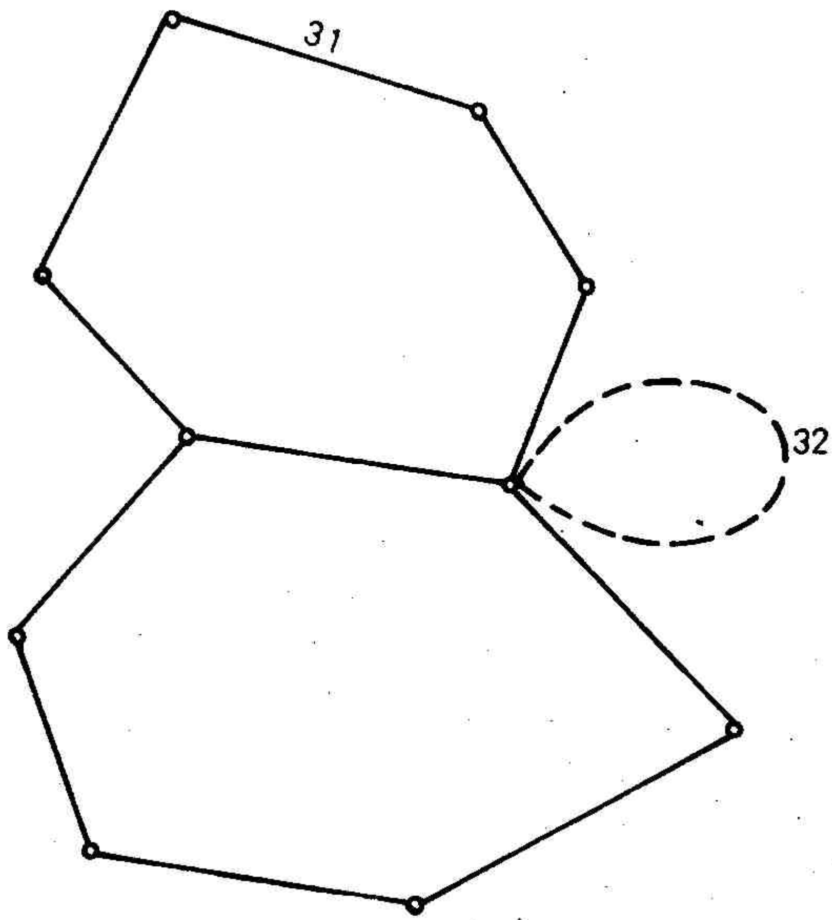
FIG·1·18. DEÇOMPOSITION FOR CASE III.



B = {25}

FIG. 1·19.

DECOMPOSITION FOR CASE I ON MvXS₁.



B = {32}

FIG. 1·20.

DECOMPOSITION FOR CASE I′ ON Mv S₂.

Using Case I on $N_2 = [(M_V . S_2)^*, (V.S_2)^*, H_{S_2} , S_2]$

(i.e. Case I' on $N_2' = [(M_V . S_2), V . S_2 , G_{S_2} , S_2]$)

and taking B = $\{32\}$ we obtain the decomposition in Figure 1.20. (This corresponds to loop analysis by 'shorting').

We then obtain t =

$$\alpha [[5^3 + 4^3 + (1^3)] + (2)^3 + [1^3 + 1^3 + (1^3)]$$

$$= \alpha . 201.$$

It is thus seen that Case III is superior to Case II or Case I for the solution of this network.

CHAPTER 2

A PARTITION FOR MATROIDS

In this Chapter we consider a method of partitioning the underlying set of a matroid into subsets invariant under the automorphisms of the matroid. This partition is an extension of the 'Principal partition of a graph' due to Kishi and Kajitani, and is, in a certain sense, the finest possible. Such studies seem to have originated with the papers of Tutte and Nash Williams [Tu 7][Na 1]. In the context of Matroid theory Edmonds [Ed 1], [Ed 6] and Nashwilliams [Na 3] have dealt with some of the basic problems. Independently of these workers (though a bit later) and approaching from a different point of view Kishi and Kajitani [Ki 2] arrived at essentially the same concepts while tackling a fundamental problem in electrical engineering viz. the determination of the topological degree of freedom of a network. Bruno and Weinberg in [Br 2] have extended Kishi-Kajitani's ideas in the context of Matroid Theory. They have however not related their results to the Matroid union theorem. Our contributions here are as follows :

(1) We have refined the partition of Bruno and Weinberg. Our refinement is in a certain sense the finest possible. We have also given efficient algorithms for this partition.

(2) We have made a detailed study of this partition and investigated the changes that occur in this partition when we consider certain functions (termed admissible functions) of the original matroid. We have also studied the partition of a series or parallel connection of the original matroids (See Ch.3) and single element extension of the original matroid.

(3) We have, in addition, obtained our preliminary partition (which we term a P-sequence) in terms of coloops of admissible functions of the original matroid. Admissible functions are obtained by the use of the unary dualization operator (*) and the binary matroid union operator (V).

Matroids which cannot be partitioned by our methods we have termed as 'atomic'. These matroids are shown to have a certain interesting property with respect to the matroid union operation. A study of such matroids should be of help in determining 'heuristic' algorithms for the isomorphism problem of graphs. Also we note that, all the 'forbidden minors' that occur 'naturally' in matroid theory seem to be atomic. Finally, one of the algorithms that are described here is of use in a suitable decomposition of electrical networks for mixed analysis.

Section 1 : Matroid Unions

Unless otherwise stated all the matroids considered in the subsequent pages are finite matroids (i.e. the set of definition is finite). The proof of Theorem 1.1 is however valid even when the matroids are infinite. The theorem is due originally to Edmonds and Nashwilliams [Ed 6], [Na 3]. Our proof here, though longer than the other known proofs, is based on an efficient algorithm for the construction of a base of the union of matroids. Also, the methods described in this proof are used to a considerable extent in this chapter. We make a reasonably detailed study of the matroid union operation in this section.

Definition 1.1. Let M_1 and M_2 be two given matroids on a set S. We define $M_3 = M_1 \vee M_2$ as follows :

M_3 is the pair (S,I), where I is the class of all subsets P such that

(1) $P \subseteq b_1 \cup b_2$ where b_1 is a base of M_1, b_2 is a base of M_2.

Theorem 1.1. Let M_1 and M_2 be matroids on S. Then $M_3 = M_1 \vee M_2$ is a matroid on S.

Proof. We need to show that M_3 satisfies conditions (a) and (b) of Axiom system 1 for a matroid.

That condition (a) is satisfied, is obvious.

To show that condition (b) is satisfied, we proceed as follows :

Let $A \subseteq S$. Let $P = b_1 \cup b_2$ where b_1 is a base of $M_1 \times A$, and b_2 is a base of $M_2 \times A$. Let F be the class of all such subsets P. We will show that maximal subsets of A satisfying (1) in Definition 1.1 above have the same number of elements. If $R \subseteq A$ and R satisfies (1) above, $R \subseteq b_{1,i} \cup b_{2,j}$ where $b_{1,i}$ is a base of $M_1 \times A$ and $b_{2,j}$ is a base of $M_2 \times A$. Hence we need only show that maximal members of F have the same number of elements.

From $P = b_1 \cup b_2$, b_1 a base of $M_1 \times A$, b_2 a base of $M_2 \times A$, we attempt to construct a larger set

$$P_1 = b_{1,k} \cup b_{2,p}, \quad b_{1,k} \text{ a base of } M_1 \times A$$
$$b_{2,p} \text{ a base of } M_2 \times A,$$

as follows. Let $Q = A - (b_1 \cup b_2)$. Let $e_1 \in Q$. Then $e_1 \cup b_1$ contains a circuit say C_1 of M_1 by Theorem T1. Let $e_2 \in C_1 \cap b_1$. If $e_2 \notin b_2$, $e_2 \cup b_2$ contains a circuit C_2 of M_2. We pick any e_3 from $C_2 \cap b_2$ and if $e_3 \notin b_1$ consider the C_3 in $e_3 \cup b_1$. We proceed thus until we encounter an element e_n such that $e_n \in b_1 \cap b_2$. In the sequence $e_1, \ldots e_r, e_{r+1}, \ldots e_n$, it is clear that

(a) $e_1 \notin b_1 \cup b_2$

(b) if $e_r \in b_1$ (b_2), then $e_{r+1} \in C_r \cap b_2$ $(C_r \cap b_1)$ and $C_r \subseteq e_r \cup b_2$ $(e_r \cup b_1)$, C_r being a circuit of M_2 (M_1).

From this sequence it is always possible to obtain (as explained later) a sequence with the same first and last elements which satisfies (a), (b) above, and further the conditions (c) there are no repetitions in the sequence (d) $e_r \notin C_j$ when $j < r-1$.

We shall call a sequence satisfying conditions (a), (b), (c) and (d), an alternating sequence with respect to (b_1, b_2). If one can construct an alternating sequence $e_1, e_2 \ldots e_r$ with respect to (b_1, b_2) starting from e_1, we say that e_r is accessible from e_1 with respect to (b_1, b_2).

Let the alternating sequence obtained from the sequence $e_1, e_2 \ldots e_n$ be $e_1, e_2 \ldots e_k$. We now use the alternating sequence $e_1, e_2 \ldots e_k$ to change (b_1, b_2) to $(b_{1,k}, b_{2,p})$ as follows :

$$b_{1,k} = \left(b_1 - \left(\bigcup_{r=1}^{\langle n/2 \rangle} \left\{ e_{2r} \right\} \right) \right) \cup \left(\bigcup_{r=0}^{\langle \frac{n-1}{2} \rangle} \left\{ e_{2r+1} \right\} \right) .$$

$$b_{2,p} = \left(b_2 - \left(\bigcup_{r=0}^{\langle \frac{n-1}{2} \rangle} \left\{ e_{2r+1} \right\} \right) \right) \cup \left(\bigcup_{r=1}^{\langle n/2 \rangle} \left\{ e_{2r} \right\} \right) ,$$

where $\langle n/2 \rangle$ is the highest integer $\leq n/2$.

It follows from Theorem T1 that $b_{1,k}$ and $b_{2,p}$ are bases of $M_1 \times A$ and $M_2 \times A$ respectively. Also since

$b_{1,k} \cup b_{2,p} = (b_1 \cup b_2) \cup e_1$, it follows that

$b_{1,k} \cup b_{2,p} = \left| b_1 \cup b_2 \right| + 1$.

Let $Q_1 = Q - \{e_1\}$.

Now we take any element of Q_1 and proceed as before. Ultimately we construct a Q_f such that if $(b_{1,f}, b_{2,f})$ be the bases of $M_1 \times A$ and $M_2 \times A$ respectively at that stage of construction, no element of $b_{1,f} \cap b_{2,f}$ is accessible from an element of Q_f with respect to $(b_{1,f}, b_{2,f})$. If, however, no element of $b_1 \cap b_2$ is accessible from an element of Q with respect to (b_1, b_2), we take $(b_{1,f}, b_{2,f})$ to be (b_1, b_2).

Let us take each element of Q_f to be accessible from itself. Let S_1 be the set of all elements which are accessible from the elements of Q_f with respect to $(b_{1,f}, b_{2,f})$. Then

$$S_1 \supseteq Q_f$$

$$S_1 \cap b_{1,f} \cap b_{2,f} = \Phi$$

Let us call $b_{1,f} \cap S_1$ as b^1 and $b_{2,f} \cap S_1$ as b^2.

If e_1 be an element of Q_f, $e_1 \cup b_{1,f}$ contains a circuit of M_1. Since each element of this circuit is accessible from e_1, this circuit is a subset of $e_1 \cup b^1$. Similarly $e_1 \cup b^2$ contains a circuit of M_2. Let $e_2 \in b^1$. Then $e_2 \cup b_{2,f}$ contains a circuit of M_2 However e_2 is accessible from some element e_1 of Q_f. Hence every element of this circuit is accessible from e_1 with respect to $(b_{1,f}, b_{2,f})$. Therefore, this circuit is contained in $e_2 \cup b^2$. Similarly, if e_3 be any element of b^2, $e_3 \cup b^1$ contains a circuit of M_1. Thus b^1 is a base of $M_1 \times A \times S_1 = M_1 \times S_1$.

and b^2 is a base of $M_2 \times A \times S_1 = M_2 \times S_1$.

From Theorem T3 we have

(2) $M_1 \times A . (A - S_1)$ has $b_{1,f} \cap (A - S_1)$ as a base.

$M_2 \times A . (A - S_1)$ has $b_{2,f} \cap (A - S_1)$ as a base.

Further since $Q_f \subseteq S_1$

$$b_{1,f} \cup b_{2,f} \supseteq A - S_1 .$$

We will now show that $b_{1,f} \cup b_{2,f}$ is a maximal element of F.

Let $b_{1,i}$, $b_{2,j}$ be bases of $M_1 \times A$, $M_2 \times A$ respectively. From Theorem T2 it is clear that $b_{1,i} \cap S_1$, $b_{2,j} \cap S_1$ are independent sets of $M_1 \times A \times S_1 = M_1 \times S_1$ and $M_2 \times S_1$ respectively. Hence $|b_{1,i} \cup b_{2,j}| \leq |A - S_1| + r(M_1 \times S_1) + r(M_2 \times S_1)$. Therefore $|b_{1,i} \cup b_{2,j}| \leq |b_{1,f} \cup b_{2,f}|$.

Thus $b_{1,f} \cup b_{2,f}$ is a maximal element of F. Now let $b_{1,i}$, $b_{2,j}$ be bases of $M_1 \times A$, $M_2 \times A$ respectively such that $(b_{1,i} \cup b_{2,j})$ is a maximal element of F. By the proof of the maximality of $(b_{1,f} \cup b_{2,f})$ it is easy to see that

$$|b_{1,i} \cup b_{2,j}| \leq |b_{1,f} \cup b_{2,f}| .$$

Let S_1' be the set of all elements accessible from A - $(b_{1,i} \cup b_{2,j})$ with respect to $(b_{1,i}, b_{2,j})$. Then it is clear that

(1) $A - S_1' \subseteq b_{1,i} \cup b_{2,j}$

(2) $b_{1,i} \cap S_1'$ is a base of $M_1 \times (S_1')$, $b_{2,j} \cap S_1'$ is a base of $M_2 \times (S_1')$

(3) $b_{1,i} \cap b_{2,j} \cap S_1' = \phi$.

Hence we can show that

$$| b_{1,f} \cup b_{2,f} | \leq | b_{1,i} \cup b_{2,j} |$$

Thus it follows that

$$| b_{1,i} \cup b_{2,j} | = | b_{1,f} \cup b_{2,f} |$$

[It may be noted that the only way the above equation can hold is to have

$$A - S_1' = A - S_1 \quad .$$

Hence since $S_1', S_1 \subseteq A$

$$S_1' = S_1 \text{]}$$

Thus condition (b) of axiom system 1 for a matroid is satisfied by M_3. Hence M_3 is a matroid.

Q.E.D.

From the method of proof of Theorem 1.1, it is clear that there exists a set $K \subseteq S$ such that $(M_1 \vee M_2) \times K$ contains disjoint bases b_1, b_2 of $M_1 \times K$ and $M_2 \times K$ such that K contains all the elements in the complement of any base of $M_1 \vee M_2$ and every element of b_1, b_2 is accessible from some element in

$K - (b_1 \cup b_2)$ through an alternating sequence with respect to (b_1, b_2). This sequence can be used to find bases b_3, b_4 of $M_1 \times K$, $M_2 \times K$ respectively such that for a particular element e, $e \notin b_3 \cup b_4$. Also $M_1 \vee M_2 \cdot (S - K)$ is the matroid M_u on $S - K$. Hence

$$K = A(M_1 \vee M_2) \cup C(M_1 \vee M_2)$$

$$S - K = B(M_1 \vee M_2).$$

The following is an obvious Corollary of Theorem 1.1.

<u>Corollary 1.</u> $M_o \vee M = M$ and $M_u \vee M = M_u$, all the matroids being on S.

<u>Corollary 2.</u> Let M_1 and M_2 be two matroids on S. Define $M_3 = M_1 \wedge M_2$ as the pair (S, I), where I is the class of all P, $P \subseteq b_i \cap b_j$ where b_i is a base of M_1 and b_j is a base of M_2 such that if b_m, b_n are any bases of M_1, M_2 respectively, $|b_i \cap b_j| \leq |b_m \cap b_n|$. Then M_3 is a matroid.

<u>Proof.</u> It is easy to see that

$$M_3 = (M_1^* \vee M_2^*)^* .$$

Q.E.D.

<u>Corollary 3.</u> $M_u \wedge M = M$ and

$M_o \wedge M = M_o$, all the matroids being on S.

<u>Corollary 4.</u> Let M_1 and M_2 be matroids defined on S. Let $A \subseteq S$. Then

(a) $(M_1 \times A) \vee (M_2 \times A) = (M_1 \vee M_2) \times A$.

(b) Any set independent in $(M_1 \cdot A \vee M_2 \cdot A)$ is also independent in $(M_1 \vee M_2) \cdot A$.

Hence, $r[(M_1 \vee M_2) \cdot A] \geq r[M_1 \cdot A \vee M_2 \cdot A]$.

Proof. (a) is obvious from the method of proof of Theorem 1.1.

(b) Let C be a base in $M_1 \cdot A \vee M_2 \cdot A$. Then $C = C_1 \cup C_2$ where C_1 is a base of $M_1 \cdot A$, C_2 is a base of $M_2 \cdot A$.

Then by Theorem T3 there exist bases B_1 of M_1 and B_2 of M_2 such that

$$B_1 = C_1 \cup D_1$$

$$B_2 = C_2 \cup D_2 \quad .$$

where D_1 is a base of $M_1 \times (S - A)$.

D_2 is a base of $M_2 \times (S - A)$.

Let $D_1 \cup D_2 = D$, then D is independent in $(M_1 \vee M_2) \times (S-A)$. Let E be a base of $(M_1 \vee M_2) \times (S - A)$ such that $D \subseteq E$. Then $(E \cup B_1 \cup B_2)$ is an independent set in $M_1 \vee M_2$ by Theorem T3. Also $(E \cup B_1 \cup B_2) \cap (S - A)$ is a base of $(M_1 \vee M_2) \times (S - A)$. Hence $(E \cup B_1 \cup B_2) \cap A$ is independent in $(M_1 \vee M_2) \cdot A$ by Theorem T2. But $(E \cup B_1 \cup B_2) \cap A = C$. Hence C is independent in $(M_1 \vee M_2) \cdot A$.

Q.E.D.

Let $M_1, M_2 \ldots M_n$ be matroids on S. Then we can define a matroid $M_1 \vee M_2 \vee \ldots \vee M_n = M_{n+1}$ as a pair (S,I) where I is a family of subsets of S, each of which is a union of independent subsets of $M_1, M_2 \ldots M_n$. It is clear that this notion is well defined. In constructing larger independent sets from a given independent set of M_{n+1}, the alternating sequence is defined similar to the case of $M_1 \vee M_2$, the condition being that successive elements in the sequence should belong to bases of different matroids.

<u>Definition 1.2.</u> Let $b_1, \ldots b_n$ be bases of $M_1, \ldots M_n$ respectively. Let $a \in S - (b_1 \cup \ldots \cup b_n)$.

Then an alternating sequence starting from a is defined as follows :

Let $a_o, a_1 \ldots a_k$ be a sequence of elements of S such that

$$a = a_o, \; C_1 \subseteq a_o \cup b_{i,1}, \; a_1 \in C_1, \; C_2 \subseteq a_1 \cup b_{i,2}, \; a_2 \in C_2, \ldots$$

$$\ldots\ldots C_k \subseteq a_{k-1} \cup b_{i,k}, \; a_k \in C_k, \; a_k \in b_{i,k} \cap b_{i,r}$$

$$b_{i,r} \neq b_{i,k} .$$

where (a) $b_{i,j}$ is any one of $b_1, \ldots b_n$.

(b) $b_{i,j}, b_{i,p}$ are not necessarily distinct.

(c) C_j is a circuit in the matroid M_p whose base is $b_{i,j}$, $p \in \{1,2 \ldots n\}$.

(d) There are no repetitions in the sequence.

(e) $a_r \notin C_j$ when $j < r - 1$.

(Note :- From a sequence which does not satisfy conditions (d) and (e) it is possible to construct an alternating sequence with the same first and last element as follows.

(1) Let the element d occur for the first time as a_j and the last time as a_p ; then we have a new sequence with less repetitions —

$$a_o \ldots a_{j-1} , d = a_j = a_p , a_{p+1} \ldots a_k \text{ .}$$

This process is continued until there are no repetitions.

(2) Let the sequence satisfy (d). Let C_j be the first circuit containing more than one element of the sequence $a_1, \ldots a_k$, and let a_i, a_p be the first and the last elements respectively, of the sequence, belonging to C_j. We then have a new sequence, fewer elements of which do not satisfy condition (e), namely,

$$a = a_o, \ldots a_{i-1}, a_p , a_{p+1} \ldots a_k \text{ .}$$

The process is applied to each circuit containing more than one elements of the sequence so that all the elements of the reduced sequence satisfy condition (e)].

Let $b_{i,k}$ be a base of M_p, (for some $p \in \{1,2 \ldots n\}$). Now we start from the last term of the alternating sequence, and change $b_{i,k}$ to $b'_{i,k}$ where $b'_{i,k} = (a_{k-1} \cup b_{i,k}) - a_k$.

From Theorem T1 we know that $b'_{i,k}$ is a base of M_p. Suppose $b_{i,k}$ occurs earlier in the sequence as say $b_{i,j}$, with $a_j \in C_j$, $C_j \subseteq a_{j-1} \cup b_{i,j}$.

Since $a_k \notin C_j$, $C_j \subseteq a_{j-1} \cup b'_{i,k}$.

We now replace $b_{i,j}$ by $b'_{i,k}$ and call this new base $b_{i,j}$. This is done for each occurrence of $b_{i,k}$. It is clear that the sequence which results is again an alternating sequence which has a_{k-1} as its last term. The process is repeated now with $b_{i,\,k-1}$ and so on backwards until the number of terms in the sequence reduces to zero. Let the bases at this stage be $b_{2,1} \ldots b_{2,n}$. Then $b_{2,1} \cup \ldots \cup b_{2n} = a \cup (b_1 \cup \ldots b_n)$ and this is a larger independent set of M_{n+1}. The process is repeated until we have a set of bases $b_{i,1} \ldots b_{i,n}$ such that $\bigcup_{j=1}^{n} b_{i,j}$ cannot be increased in size by our algorithm.

We now restate the above procedure in the form of an efficient algorithm.

Algorithm 1.1. [Determination of a base of $M_1 \vee M_2 \vee \ldots \vee M_n$]. Let M be a matroid on S, and let $b = \bigcup_{i=1}^{n} b_i$ [where b_i is a base of M_i]. Let $D = S - \bigcup_{i=1}^{n} b_i$, and

$B = \{ e \mid e \in S$ and e belongs to at least two of the bases $b_1, b_2 \ldots b_n \}$.

__Step 1.__ If $D = \Phi$ or $B = \Phi$, b is a base of $M_1 \vee M_2 \ldots \vee M_n$.

__Step 2.__ Pick some $e \in D$. $e \cup b_i$ contains a circuit C_i of M_i for $i = 1, \ldots n$.

Let A_1 denote the union of all these circuits. Each member of A_1 is not a member of some of the bases $b_1, b_2 \ldots b_n$, and thus forms a unique circuit in the corresponding matroids with these bases. Let A_2 denote the union of all such circuits formed by each of the members of A_1 with the bases $b_1, b_2 \ldots b_n$. In a similar manner A_{n+1} is obtained from A_n for $n = 2, 3 \ldots$. There are two possible cases.

__Case 1.__ There is a least positive integer s such that $A_s = A_{s+1}$ and $A_s \cap B = \Phi$. Update D by $D - \{e\}$ and return to step 1.

__Case 2.__ There is a least positive integer p such that $A_p \cap B \neq \Phi$. Hence there exists an alternating sequence starting from $e \in D$ and ending with some $d \in A_p \cap B$:

$$e = e_0,\ C_1 \subseteq e_0 \cup b_{i,1},\ e_1 \in C_1 \ldots e_k \in C_k,\ e_k = d$$

$$C_k \subseteq C_{k-1} \cup b_{i,k}$$

$$e_k \in b_{i,k} \cap b_{i,r}$$

$$b_{i,r} \neq b_{i,k}.$$

(Alternating sequence being as defined earlier).

We update $b_{i_{k-p+1}}$ to $(e_{k-p} \cup b_{i_{k-p+1}} - e_{k-p+1})$ starting with p = 1 and proceeding upto p = k to obtain the updated version of $b = \bigcup_{i=1}^{n} b_i$. Set $D = [\, S - b \,]$ and

$$B = \{ e / e \in S \text{ and } e \text{ belongs to at least two of the bases in } b_1, b_2 \ldots b_n \}$$

Go to step 1.

Algorithm ends.

<u>Example 2.1.1.</u> Let M_1, M_2, M_3 be matroids on $\{1,2,3,4,5,6,7,8,9,10,11,12,13,14\}$

The required circuits and bases for the matroid are described as we proceed.

We start with bases b_1, b_2, b_3 of matroids M_1, M_2, M_3 respectively. (For convenience we have taken $r(M_1) = r(M_2) = r(M_3)$. The procedure is equally valid even when this is not so).

$b = b_1 \cup b_2 \cup b_3$ is independent in $M_1 \vee M_2 \vee M_3$. We will now construct an independent set of this matroid that properly contains b.

$b_1 = \{1,2,3,4,5\}$, $b_2 = \{5,6,7,8,9\}$, $b_3 = \{9,10,11,12,13\}$.

$\{14\} \cup \{1,2,3,4,5\} \supseteq C_1 = \{14,1,2\}$ C_1 is a circuit of M_1

$b_{i_1} = b_1$

$\{1\} \cup \{5,6,7,8,9\} \supseteq C_2 = \{1,6,7\}$, C_2 is a circuit of M_2

$b_{i_2} = b_2$

$\{7\} \cup \{9,10,11,12,13\} \supseteq C_3 = \{7,10,11\}$, C_3 is a circuit of M_3

$b_{i_3} = b_3$.

$\{10\} \cup \{5,6,7,8,9\} \supseteq C_4 = \{10,7,8\}$, C_4 is a circuit of M_2

$b_{i_4} = b_2$

$\{8\} \cup \{9,10,11,12,13\} \supseteq C_5 = \{8,10,13\}$, C_5 is a circuit of M_3

$b_{i_5} = b_3$

$\{13\} \cup \{1,2,3,4,5\} \supseteq C_6 = \{13,5,2\}$, C_6 is a circuit of M_1

$b_{i_6} = b_1$

$5 \notin b_1 \cap b_2$

The alternating sequence is (14, 1,7, 10, 8, 13) .

We convert b_{i_6} to $\{13\} \cup b_{i_6} - \{5\} = \{1,2,3,4,13\}$

= the updated b_1 .

The updated sequence of bases at this stage therefore is

$b_{i_1} = \{1,2,3,4,13\}$, $b_{i_2} = \{5,6,7,8,9\}$, $b_{i_3} = \{9,10,11,12,13\}$

$b_{i_4} = \{5,6,7,8,9\}$ $b_{i_5} = \{9,10,11,12,13\}$.

The updated b_1,b_2,b_3 are $b_1 = \{1,2,3,4,13\}$

$b_2 = \{5,6,7,8,9\}$, $b_3 = \{9,10,11,12,13\}$.

Next we convert b_{i_5} to $\{8\} \cup b_{i_5} - \{13\} = \{8,9,10,11,12\}$

= the updated b_3 .

$\{7\} \cup \{9,10,11,12,13\} \supseteq C_3 = \{7,10,11\}$, C_3 is a circuit of M_3

$b_{i_3} = b_3$.

$\{10\} \cup \{5,6,7,8,9\} \supseteq C_4 = \{10,7,8\}$, C_4 is a circuit of M_2

$b_{i_4} = b_2$

$\{8\} \cup \{9,10,11,12,13\} \supseteq C_5 = \{8,10,13\}$, C_5 is a circuit of M_3

$b_{i_5} = b_3$

$\{13\} \cup \{1,2,3,4,5\} \supseteq C_6 = \{13,5,2\}$, C_6 is a circuit of M_1

$b_{i_6} = b_1$

$$5 \in b_1 \cap b_2$$

The alternating sequence is (14, 1,7, 10, 8, 13) .

We convert b_{i_6} to $\{13\} \cup b_{i_6} - \{5\} = \{1,2,3,4,13\}$

= the updated b_1 .

The updated sequence of bases at this stage therefore is

$b_{i_1} = \{1,2,3,4,13\}$, $b_{i_2} = \{5,6,7,8,9\}$, $b_{i_3} = \{9,10,11,12,13\}$

$b_{i_4} = \{5,6,7,8,9\}$ $b_{i_5} = \{9,10,11,12,13\}$.

The updated b_1, b_2, b_3 are $b_1 = \{1,2,3,4,13\}$

$b_2 = \{5,6,7,8,9\}$, $b_3 = \{9,10,11,12,13\}$.

Next we convert b_{i_5} to $\{8\} \cup b_{i_5} - \{13\} = \{8,9,10,11,12\}$

= the updated b_3 .

The updated sequence of bases at this stage therefore is

$b_{i_1} = \{1,2,3,4,13\}$, $b_{i_2} = \{5,6,7,8,9\}$, $b_{i_3} = \{8,9,10,11,12\}$,

$b_{i_4} = \{5,6,7,8,9\}$.

The updated b_1, b_2, b_3 are $b_1 = \{1,2,3,4,13\}$,

$b_2 = \{5,6,7,8,9\}$, $b_3 = \{8,9,10,11,12\}$.

Next we convert b_{i_4} to $\{10\} \cup b_{i_4} - \{8\} = \{5,6,7,9,10\}$

= the updated b_2.

The updated sequence of bases at this stage is

$b_{i_1} = \{1,2,3,4,13\}$, $b_{i_2} = \{5,6,7,9,10\}$, $b_{i_3} = \{8,9,10,11,12\}$

The updated b_1, b_2, b_3 are

$b_1 = \{1,2,3,4,13\}$, $b_2 = \{5,6,7,9,10\}$, $b_3 = \{8,9,10,11,12\}$.

Next we convert b_{i_3} to $\{7\} \cup b_{i_3} - \{10\} = \{7,8,9,11,12\}$

= the updated b_3

The updated sequence of bases at this stage is

$b_{i_1} = \{1,2,3,4,13\}$, $b_{i_2} = \{5,6,7,9,10\}$

The updated b_1, b_2, b_3 are

$b_1 = \{1,2,3,4,13\}$, $b_2 = \{5,6,7,9,10\}$, $b_3 = \{7,8,9,11,12\}$.

Next we convert b_{i_2} to $\{1\} \cup b_{i_2} - \{7\} = \{1,5,6,9,10\}$

= the updated b_2.

The updated sequence of bases at this stage is

$b_{i_1} = \{1,2,3,4,13\}$.

The updated b_1, b_2, b_3 are

$b_1 = \{1,2,3,4,13\}$, $b_2 = \{1,5,6,9,10\}$, $b_3 = \{7,8,9,11,12\}$.

Next we convert b_{i_1} to $\{14\} \cup b_{i_1} - \{1\}$

$= \{2,3,4,13,14\}$ = the updated b_1.

The updated b_1, b_2, b_3 are

$b_1 = \{2,3,4,13,14\}$, $b_2 = \{1,5,6,9,10\}$, $b_3 = \{7,8,9,11,12\}$.

$b_1 \cup b_2 \cup b_3 = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14\}$.

Thus we have an independent set of $M_1 \vee M_2 \vee M_3$ whose cardinality is one more than the independent set we started with.

Let $b_1, b_2 \ldots b_n$ be the bases of $M_1, \ldots M_n$ respectively such that $b = b_1 \cup \ldots \cup b_n$ is a base of $M_1 \vee \ldots \vee M_n = M_{n+1}$. Let K be the set of all elements accessible from elements of $D = S - b$ through alternating sequences with respect to $(b_1, \ldots b_n)$. If $e \in b_i$, e is accessible from some element say $d \in S - b$ with respect to $(b_1, \ldots b_n)$ through an alternating sequence. This can be used to obtain a new base b' of M_{n+1} such that $d \in b'$ but $e \notin b'$. Also, proceeding as in Theorem 1.1 one can show that $S - K \subseteq b$ for any base b of M_{n+1}. We can now summarise the above discussion in the form of the following Theorem.

Theorem 1.2. Let $M_1, M_2 \ldots M_n$ be matroids on S. Let $M_{n+1} = M_1 \vee M_2 \vee \ldots \vee M_n$. Let $b_1, b_2 \ldots b_n$ be bases of $M_1, M_2 \ldots M_n$ respectively such that $b_1 \cup b_2 \cup \ldots \cup b_n = b$ is a base of M_{n+1}. Then if $K = A(M_{n+1}) \cup C(M_{n+1})$, it will have the following properties :

(1) $b_i \cap K$ is a base of $M_i \times K$ $(i = 1, 2, \ldots n)$

(2) $b_1 \cap K$, $b_2 \cap K \ldots ; b_n \cap K$ are pairwise disjoint

(3) No element of $S - K$ is accessible from any element of $K - b$ with respect to $(b_1, b_2 \ldots b_n)$.

(4) Every element of $b \cap K$ is accessible from some element of $K - b$ with respect to $(b_1, b_2 \ldots b_n)$.

(5) $S - K \subseteq b$.

Corollary 1. Let $M_1, M_2 \ldots M_n$ be matroids on S. If $A \subseteq S$ such that $(M_1 \vee \ldots \vee M_n) \times A$ has no coloops, and b is a base of $(M_1 \vee \ldots \vee M_n) \times A$, then $b = b_1 \cup b_2 \ldots \cup b_n$ where $b_1, b_2 \ldots b_n$ are some bases of $M_1 \times A$, $M_2 \times A_1 \ldots M_n \times A$ respectively.

Theorem 1.3. Let M_1, M_2 be matroids on S. Then if

$$r[(M_1 \vee M_2) \times (S-A)] = r[M_1 \times (S-A)] + r[M_2 \times (S-A)]$$

then $(M_1 \vee M_2) \cdot A = (M_1 \cdot A) \vee (M_2 \cdot A)$.

Proof. Let b be a base of $M_1 \vee M_2$ such that $b \cap A$ is a base of $(M_1 \vee M_2) \cdot A$. Hence by Theorem T3, $b \cap (S-A)$ is a base of $(M_1 \vee M_2) \times (S-A)$. Let $b = b_1 \cup b_2$, b_1 a base of M_1, b_2 a base of M_2.

Since $r[(M_1 \vee M_2) \times (S - A)] = r[M_1 \times (S-A)] + r[M_2 \times (S-A)]$, $b_1 \cap (S-A)$ and $b_2 \cap (S-A)$ are disjoint bases of $M_1 \times (S-A)$, $M_2 \times (S-A)$ respectively. Hence $b_1 \cap A$, $b_2 \cap A$ are bases of $M_1 . A$, $M_2 . A$ respectively. Hence $(b_1 \cap A) \cup (b_2 \cap A)$ is independent in $(M_1 . A) \vee (M_2 . A)$. Hence every base of $(M_1 \vee M_2) . A$ is independent in $(M_1 . A) \vee (M_2 . A)$. But by corollary 4 of Theorem 1.1, every base of $(M_1 . A) \vee (M_2 . A)$ is independent in $(M_1 \vee M_2) . A$.

$$\text{Hence } (M_1 \vee M_2) . A = (M_1 . A) \vee (M_2 . A).$$

Q.E.D.

<u>Corollary 1.</u> Let M_1, M_2 be matroids on S. Let $A \subseteq S$. If $(M_1 \vee M_2) \times (S - A)$ is coloop-free, then $(M_1 \vee M_2) . A = (M_1 . A) \vee (M_2 . A)$.

<u>Proof.</u> By Corollary 1 of Theorem 1.2 $r[(M_1 \vee M_2) \times (S - A)] = r[M_1 \times (S - A)] + r[M_2 \times (S - A)]$. The result now follows from Theorem 1.3 above.

Q.E.D.

<u>Corollary 2.</u> (Of Theorem 1.3). Let $M_1, M_2 \ldots M_n$ be matroids on S. Let $A \subseteq S$. Then if,

$$r\left([M_1 \vee M_2 \ldots \vee M_n] \times (S - A)\right)$$

$$= r[M_1 \times (S - A)] + \ldots + r[M_n \times (S - A)], \text{ then}$$

$$[M_1 \vee M_2 \vee \ldots \vee M_n] . A = (M_1 . A) \vee \ldots \vee (M_n . A).$$

We next describe a circuit in a union of matroids.

Theorem 1.4. Let $M_1 \ldots M_n$ be matroids on S, and let $M_{n+1} = M_1 \vee \ldots \vee M_n$.

Let $b_1, \ldots, b_n$ be bases of $M_1, \ldots M_n$ respectively such that $b = b_1 \cup \ldots \cup b_n$ is a base of M_{n+1}. Let C be the set of all elements accessible from a ($a \in S - b$) with respect to $(b_1, \ldots b_n)$. (We take a to be accessible from itself). Then C is the fundamental circuit of the base $b_1 \cup \ldots \cup b_n$ with respect to a in the matroid M_{n+1}.

Proof. Let $K = A(M_{n+1}) \cup C(M_{n+1})$.

Then, by Theorem 1.2, $b_1 \cap K, \ldots b_n \cap K$ are disjoint bases of $M_1 \times K \ldots M_n \times K$ respectively, and $a \in K$, and $C \subseteq K$. It is clear that $b_1 \cap C, \ldots b_n \cap C$ are pairwise disjoint bases of $M_1 \times C \ldots M_n \times C$ respectively since C does not contain coloops of $M_{n+1} \times C$. Therefore C cannot be contained in any base of M_{n+1}. Again if $d \in C$ and $d \neq a$, d is accessible from a with respect to $(b_1, \ldots b_n)$. Hence $[(b_1 \cup \ldots \cup b_n) - d] \cup a$ is a base in M_{n+1}, i.e. $C-d$ is independent in M_{n+1}. Hence C is minimally dependent in M_{n+1}. Also $C - (b_1 \cup \ldots \cup b_n) = \{a\}$. Thus C is the fundamental circuit of a with respect to $b_1 \cup \ldots \cup b_n$ in the matroid M_{n+1}.

Q.E.D.

From the above theorem it follows that when $b_1, \ldots b_n$ are bases of matroids $M_1, \ldots M_n$ respectively such that $b_1 \cup \ldots \cup b_n$ is a base of M_{n+1}, then d is accessible from a with respect to $(b_1, \ldots b_n)$ iff d is accessible from a with respect to $(b_{2,1} \ldots b_{2,n})$ where $b_{2,1} \cup \ldots \cup b_{2,n} = b_1 \cup .. \cup b_n$. Hence in this case we can say that d is accessible from a with respect to $(b_1 \cup \ldots \cup b_n)$ without ambiguity.

We next prove an important theorem originally due to Nashwilliams [Na 3].

<u>Theorem 1.5.</u> (Nashwilliams Rank Formula). Let $M_1, M_2 \ldots M_n$ be matroids on S. Let $M = M_1 \vee M_2 \vee \ldots \vee M_n$ and let $\rho_1, \rho_2 \ldots \rho_n, \rho$ denote the rank functions of $M_1, M_2 \ldots M_n, M$ respectively. Then for each $F \subseteq S$ we have

$$\rho(F) = \min_{X \subseteq F} \left[\rho_1(X) + \rho_2(X) + \ldots + \rho_n(X) + |F - X| \right].$$

<u>Proof.</u> From Corollary 4 of Theorem 1.1 we have,

$$M \times F = (M_1 \times F) \vee (M_2 \times F).$$

Let $K = A [M \times F] \cup C [M \times F]$. Then by Theorem 1.2, if b is any base of $M \times F$,

(i) $b \supseteq F - K$

(ii) $b = b_1 \cup b_2 \cup \ldots \cup b_n$ where $b_1, b_2 \ldots b_n$ are bases of $M_1 \times F \ldots M_n \times F$ respectively, and $b_1 \cap K, b_2 \cap K \ldots b_n \cap K$ are pairwise disjoint.

(iii) $b_1 \cap K$, $b_2 \cap K \dots b_n \cap K$ are bases of $M_1 \times K \dots M_n \times K$ respectively.

Hence $\rho(F) = \rho_1(K) + \rho_2(K) + \dots \rho_n(K) + |F - K|$.

Let $X \subseteq F$. Then

$\rho(F) = r[M \times X] + r[M \times F.(F - X)]$ by Theorems T3,T7.

But $\rho(X) = r[M \times X] \leq \rho_1(X) + \rho_2(X) + \dots + \rho_n(X)$.

Hence $\rho(F) \leq \rho_1(X) + \rho_2(X) + \dots + \rho_n(X) + |F - X|$.

Hence $\rho(F) = \rho_1(K) + \rho_2(K) + \dots + \rho_n(K) + |F - K|$

$$= \min_{X \subseteq F} (\rho_1(X) + \rho_2(X) + \dots + \rho_n(X) + |F - X|).$$

Q.E.D.

<u>Theorem 1.6.</u> Let $M_1, M_2 \dots M_n$ be matroids on S. Let $M = M_1 \vee M_2 \vee \dots \vee M_n$ have no coloops. Then S_1 is a separator for M iff S_1 is a separator for each of the matroids $M_1, M_2 \dots M_n$.

<u>Proof.</u> Suppose S_1 is a separator for M. By Theorem T8, we have, $M \times S_1 = M \cdot S_1$, $M \times (S - S_1) = M \cdot (S - S_1)$ and hence $r[M \times S_1] = r[M \cdot S_1]$ and

$$r[M \times (S - S_1)] = r[M \cdot (S - S_1)] .$$

Clearly $M \cdot (S - S_1)$ has no coloops since M has no coloops. Hence $M \times (S - S_1)$ has no coloops. By Corollary 1 of Theorem 1.2, we then have,

$$r[M \; X \; (S-S_1)] = r[M_1 \; X \; (S-S_1)] + \dots r[M_n \; X \; (S-S_1)]$$

Using Corollary 2 of Theorem 1.3, we have,

$$[M_1 \vee M_2 \vee \dots \vee M_n] \cdot S_1 = [M_1 \cdot S_1] \vee \dots \vee [M_n \cdot S_1].$$

However, since $M \cdot S_1$ has no coloops (M has no coloops)

$$r[M \cdot S_1] = r[M_1 \cdot S_1] + \dots + r[M_n \cdot S_1]$$

$M \; X \; S_1 = M \cdot S_1$ has no coloops. Hence

$$r[M \; X \; S_1] = r[M_1 \; X \; S_1] + \dots + r[M_n \; X \; S_1].$$

But by Theorem T2,

$$r[M_i \; X \; S_1] \geq r[M_i \cdot S_1] \quad \text{for } i = 1,2 \dots n$$

Hence $r[M_i \; X \; S_1] = r[M_i \cdot S_1]$ for $i = 1,2 \dots n$.

Hence by Theorem T8, S_1 is a separator for M_i for all $i \in \{1,2 \dots n\}$.

Suppose each M_i ($i \in \{1,2 \dots n\}$) has S_1 as one of its separators. Then clearly $(S - S_1)$ is a separator for each M_i , $i \in \{1,2 \dots n\}$. Since M has no coloops

$$r[M] = r[M_1] + r[M_2] + \dots r[M_n]$$

$$= \sum_{i=1}^{n} r[M_i \cdot S_1] + \sum_{i=1}^{n} r[M_i \; X \; (S - S_1)]$$

by Theorem T3.

But $r[M_i \cdot S_1] = r[M_i \; X \; S_1]$ for all $i \in \{1,2 \dots n\}$.

Hence

$$r\,[\,M\,] = \sum_{i=1}^{n} r\,[\,M_i \times S_1\,] + \sum_{i=1}^{n} r\,[\,M_i \times (S - S_1)\,].$$

Hence

$$r\,[\,M\,] = r\,[\,M \times S_1\,] + r\,[\,M \times (S-S_1)\,]$$

But

$$r\,[\,M\,] = r\,[\,M \cdot S_1\,] + r\,[\,M \times (S - S_1)\,]$$

by Theorem T_3 .

Hence $r\,[\,M \cdot S_1\,] = r\,[\,M \times S_1\,]$.

Hence S_1 is a separator for M.

Q.E.D.

Section 2 : P-sequences

In this section we partition the set of definition of a matroid through a sequence of subsets which we term a P-sequence. The resulting partition is a refinement of the partition resulting from Bruno and Weinberg's Principal-r-minors and augmented principal-r-minors. It is shown that a P-sequence is unique. Certain properties of the P-sequence are also studied.

Definition 2.1. Let M be a matroid on S. We denote

$\frac{|S|}{r(M)}$ by d(M). We call d(M) the density of M in S.

Theorem 2.1. Let M be a matroid on S. Let T_1, T_2 be subsets of S such that

$$d(M \times T_1) = d(M \times T_2) = \max_{R \subseteq S} d(M \times R)$$

Then $d(M \times T_1) = d(M \times (T_1 \cup T_2)) = d[M \times (T_1 \cap T_2)]$

Proof. Let ρ be the rank function of M.

$$d(M \times R) = \frac{|R|}{\rho(R)}$$

Hence

$$\frac{\rho(R)}{|R|} \geq \frac{\rho(T_1)}{|T_1|} \quad \text{for any } R \subseteq S \qquad .. \quad (1)$$

and

$$\frac{\rho(T_1)}{|T_1|} = \frac{\rho(T_2)}{|T_2|} \qquad .. \quad (2)$$

Hence $\rho(T_2) = |T_2| \cdot \dfrac{\rho(T_1)}{|T_1|}$ and from (1)

$$\frac{\rho(T_1 \cup T_2)}{|T_1 \cup T_2|} \geq \frac{\rho(T_1)}{|T_1|} \qquad .. \quad (3)$$

$$\frac{\rho(T_1 \cap T_2)}{|T_1 \cap T_2|} \geq \frac{\rho(T_1)}{|T_1|} \qquad .. \quad (4)$$

Therefore $\rho(T_1 \cup T_2) \geq \rho(T_1) \cdot \dfrac{|T_1 \cup T_2|}{|T_1|}$.

Suppose

$$\frac{\rho(T_1 \cup T_2)}{|T_1 \cup T_2|} > \frac{\rho(T_1)}{|T_1|}$$

then

$$\rho(T_1 \cup T_2) > \rho(T_1) \cdot \frac{|T_1 \cup T_2|}{|T_1|}$$

$$> \rho(T_1) \cdot \frac{|T_1| + |T_2| - |T_1 \cap T_2|}{|T_1|}$$

$$> \rho(T_1) + \frac{\rho(T_1) \cdot |T_2|}{|T_1|} - \frac{\rho(T_1) \cdot |T_1 \cap T_2|}{|T_1|} \qquad .. \quad (5)$$

Hence from (2), (4) and (5)

$$\rho(T_1 \cup T_2) > \rho(T_1) + \rho(T_2) - \rho(T_1 \cap T_2)$$

By Theorem T9 this is impossible.

Hence $\dfrac{\rho(T_1 \cup T_2)}{|T_1 \cup T_2|} = \dfrac{\rho(T_1)}{|T_1|}$

Suppose $\dfrac{\varphi(T_1 \cap T_2)}{|T_1 \cap T_2|} > \dfrac{\varphi(T_1)}{|T_1|}$ .. (6)

We have

$$\varphi(T_1 \cup T_2) \geq \varphi(T_1) \cdot \frac{|T_1| + |T_2| - |T_1 \cap T_2|}{|T_1|}$$

$$\geq \varphi(T_1) + \frac{\varphi(T_1) \,.\, |T_2|}{|T_1|} - \frac{\varphi(T_1) \,.\, |T_1 \cap T_2|}{|T_2|} \quad .. \quad (7)$$

Then by (2), (6) and (7)

$$\varphi(T_1 \cup T_2) > \varphi(T_1) + \varphi(T_2) - \varphi(T_1 \cap T_2).$$

This is impossible by Theorem T9.

Hence

$$\frac{\varphi(T_1 \cap T_2)}{|T_1 \cap T_2|} = \frac{\varphi(T_1)}{|T_1|}$$

Thus

$$d(M \times T_1) = d\,[\,M \times (T_1 \cup T_2)\,] = d\,[\,M \times (T_1 \cap T_2)\,].$$

Q.E.D.

Corollary 1. The maximal set T such that

$$d\,(M \times T) = \max_{R \subseteq S} d(M \times R) \text{ is unique.}$$

Proof. Let R_1 and R_2 be two such maximal sets. Then by Theorem 2.1.

$$d(M \times R_1) = d[M \times (R_1 \cup R_2)]$$

Hence $R_1 = R_1 \cup R_2$ i.e. $R_1 = R_2$.

Q.E.D.

[Note :- If T_1, T_2 are disjoint, then $T_1 \cap T_2 = \phi$. Hence the minimal set T such that

$$d[M \times T] = \max_{R \subseteq S} d[M \times R]$$

is void. Clearly in this case the minimal non-void sets with this property will not be unique.] We now give a partial list of definitions of concepts used in the rest of the section.

Definition 2.2. A matroid M on S is said to be molecular iff for every $R \subset S$

$$d(M \times R) \leq d(M).$$

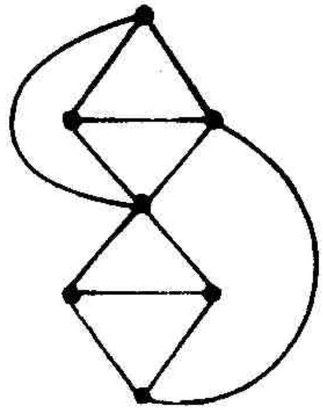
Definition 2.3. A matroid M on S is said to be atomic iff for every $R \subset S$, $d(M \times R) < d(M)$.

Example 2.2.1. Molecular matroid. Let M_1, M_2 be the polygon matroids of the graphs shown in Fig. 2.2.1(a), (b) respectively. M_1 can be shown to be molecular but not atomic. M_2 can be shown to be atomic.

Definition 2.4. Let M be a matroid on S. Let $P_1, P_2 \ldots P_n$ be a sequence of pairwise disjoint subsets of S such that

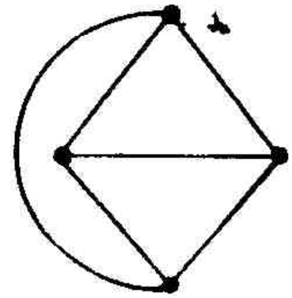
$$(i)\quad d[M \times (\bigcup_{i=1}^{r} P_i) \cdot P_r] > d[M \times (\bigcup_{i=1}^{m} P_i) \cdot P_m]$$

whenever $m > r$.



GRAPH OF A MOLECULAR MATROID

FIG. 2.2.1 (a).



GRAPH OF AN ATOMIC

FIG. 2.2.1.(b).

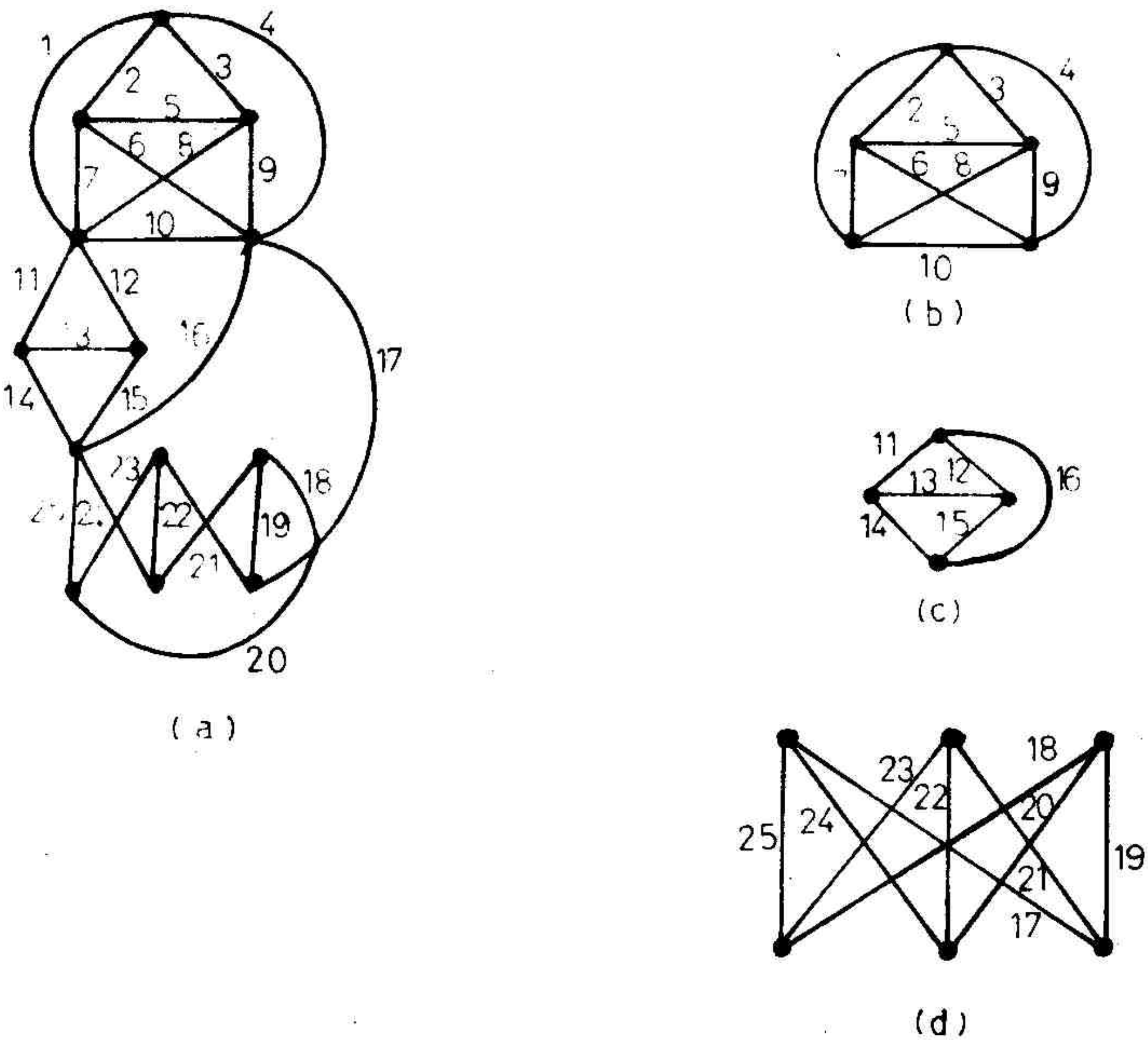


FIG. 2.2.2.

(ii) $M \times (\bigcup_{i=1}^{r} P_i) \cdot P_r$ is molecular for $(i = 1, \ldots n)$.

(Note :- If P_1 is a set of loops we take $d(M \times P_1) = \infty$).

Example of P-sequence :

Example 2.2.2. Let M be the polygon matroid of the graph shown in Fig. 2.2.2(a). Then M can be shown to have the following P-sequence :

$P_1 = \{1,2,3,4,5,6,7,8,9,10\}$

$P_2 = \{11,12,13,14,15,16\}$

$P_3 = \{17,18,19,20,21,22,23,24,25\}$

$M \times P_1$, $M \times (P_1 \cup P_2) \cdot P_2$ and $M \cdot P_3$ are polygon matroids of graphs shown in Fig. 2.2.2 (b),(c) and (d) respectively. Further it can be shown that these matroids are molecular and

$d(M \times P_1) = 2\ 1/2$

$d(M \times P_1 \cup P_2 \cdot P_2) = 2$

$d(M \cdot P_3) = 9/5$

Definition 2.5. Let M be a matroid on S. Let $a,b \in S$. We say that a,b are in parallel iff either

(i) $\{a,b\}$ is a circuit of M

or

(ii) $\{a\}$, $\{b\}$ are circuits of M

We say that $a_1, a_2 \ldots a_n$ are in parallel iff every pair of them are in parallel.

Definition 2.6. Let M be a matroid on S. Let M_1 be a matroid on $S \cup \{a\}$, $a \notin S$, such that

(i) a is parallel in M_1 to some element $d \in S$.

(ii) $M = M_1 \times S$.

Then we say that M_1 is a parallel extension of M or that M_1 is obtained by adding a parallel to d in M. Similarly if M_2 is a matroid on $S \cup \{a_1, \ldots a_n\}$, $S \cap \{a_1, \ldots a_n\} = \emptyset$ with

(i) $d, a_1 \ldots, a_n$ parallel in M_1, $d \in S$.

(ii) $M_1 \times S = M$,

then we say that M_1 is obtained by adding $a_1, \ldots a_n$ parallel to d in M.

The following theorem on parallel extensions is easy to see.

Theorem 2.2. Let M be a matroid on S and let M_1 be a parallel extension of M on $S \cup \{a\}$, $a \notin S$, such that a is parallel to some $d \in S$. Then

(i) Any base of M is a base of M_1.

(ii) If $a \in K \subseteq S \cup \{a\}$ and $d \in K$ then $M_1 \times K$ is a parallel extension of $M \times (K - \{a\})$ and

$M_1 \cdot K$ is a parallel extension of $M \cdot (K - \{a\})$.

Definition 2.7. Let M be a matroid defined on S and α be a natural number.

Let $S = \{a_1^1, a_1^2, \ldots a_1^n\}$.

Let $S_\alpha = \{a_1^1, a_2^1 \ldots a_\alpha^1, a_1^2, a_2^2 \ldots a_\alpha^2, \ldots a_1^n, a_2^n \ldots a_\alpha^n\}$

with $a_j^i = a_m^k$ iff $i = k$ and $j = m$.

Let M_α be the matroid obtained from M by adding

$a_2^1, \ldots a_\alpha^1$ parallel to a_1^1 .

$a_2^2, \ldots a_\alpha^2$ parallel to a_1^2 .

$a_2^n, \ldots a_\alpha^n$ parallel to a_1^n .

Then M_α is said to be a parallel α-copy of M.

<u>Definition 2.8.</u> Let Υ be the map from S to P (S_α) [P (S_α) being the class of all subsets of S_α] such that if $a \in S$

$$\Upsilon(a) = \{e \mid e = a \text{ or } e \in S_\alpha - S \text{ and } e \text{ is an element added parallel to } a \text{ in } M_\alpha\}.$$

This map induces the following map from P(S) to P(S_α) which we still denote by Υ .

If $R \subseteq S$

$$\Upsilon(R) = \bigcup_{a \in R} \Upsilon(a) .$$

Clearly this map from P(S) to P(S_α) is one to one into. Hence $\Upsilon(S-T) = \Upsilon(S) - \Upsilon(T) = S_\alpha - \Upsilon(T)$. Hence the map Υ^{-1} from Υ(P(S)) onto P(S) is one to one onto. Since M_α can be obtained from M by a sequence of parallel extensions, we obtain from Theorem 2.2 the following simple result :

<u>Theorem 2.3.</u> Let M be a matroid on S. Let M_α be a parallel-α-copy of M on S_α . Then

(i) Any base of M is a base of M_α .

(ii) $M_\alpha \times \gamma(T)$ is a parallel α-copy of $M \times T$ for any $T \subseteq S$.

(iii) $M_\alpha \cdot \gamma(T)$ is a parallel extension of $M \cdot T$ for any $T \subseteq S$.

(iv) $M_\alpha \times \gamma(T_1) \cdot (\gamma(T_2))$ is a parallel extension of $M \times T_1 \cdot T_2$ for any T_1 and T_2 such that $T_2 \subseteq T_1 \subseteq S$.

Corollary 1. Let M be a matroid on S and M_α be its parallel α-copy on S_α .

Then $d(M_\alpha) = \alpha \cdot d(M)$.

Proof. $r(M_\alpha) = r(M)$

$$S_\alpha = \alpha \cdot |S| .$$

Q.E.D.

Example 2.2.3. Let M_1, M_2 be the polygon matroids of the graphs shown in Fig. 2.2.3(a), (b) respectively.

M_2 can be seen to be the parallel-2-copy of M_1.

$$\gamma\{a_1^1, a_1^2, a_1^3, a_1^4, a_1^5, a_1^6\}$$

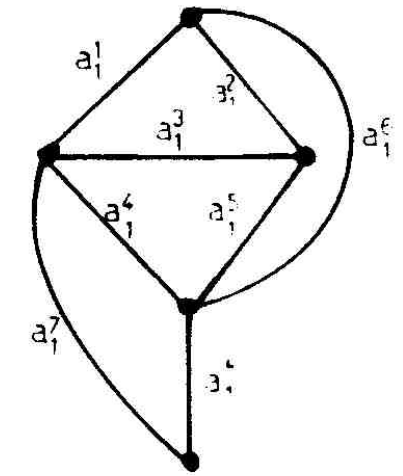
$$= \{a_1^1, a_2^1, a_1^2, a_2^2, a_1^3, a_2^3, a_1^4, a_2^4, a_1^5, a_2^5, a_1^6, a_2^6\} .$$

The following theorem relates the P-sequences of M and M_α.

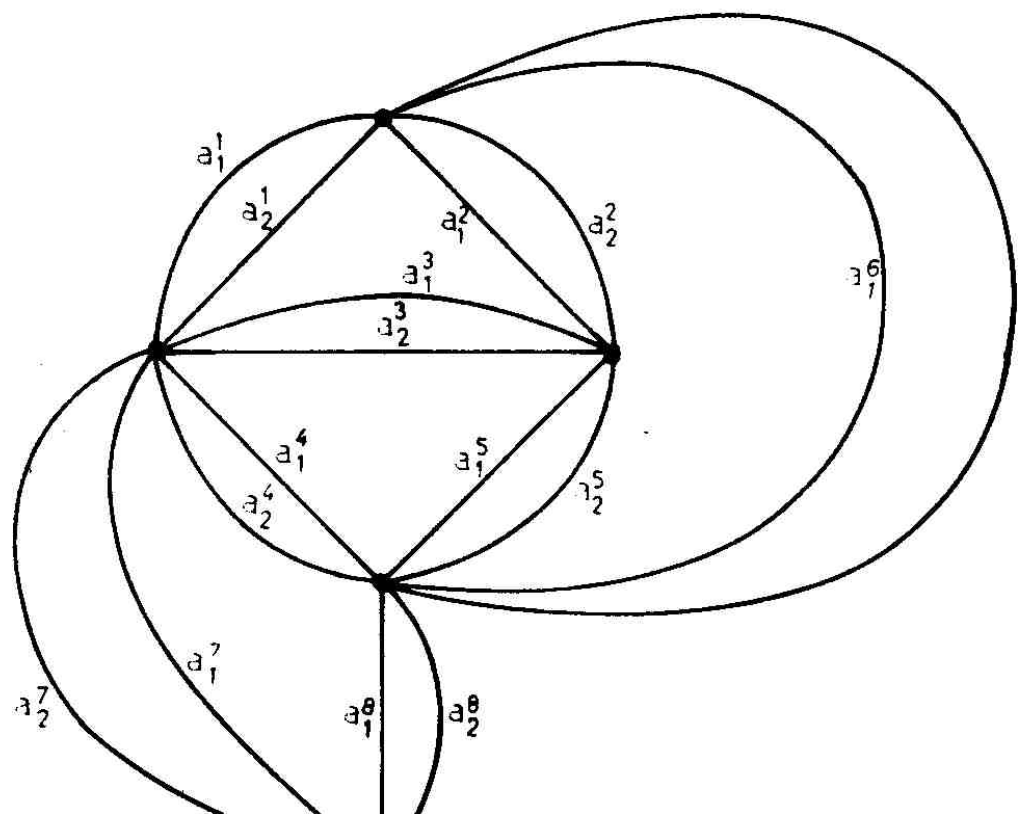
Theorem 2.4. Let M be a matroid on S. Let M_α be its parallel α-copy on S_α .

Let $P_1, P_2 \ldots P_n$ be a P-sequence in M.

Then $\gamma(P_1), \gamma(P_2) \ldots \gamma(P_n)$ is a P-sequence in M_α .



(a)



(b)

FIG.2.2.3

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Proof. Suppose $M_\alpha \times \mathcal{T}(P_1)$ is not molecular. Then there exists a set $T_1 \subseteq \mathcal{T}(P_1)$ such that $d(M_\alpha \times T_1) > d(M_\alpha \times P_1)$. Let $T_2 = \mathcal{T}(T_1 \cap S)$. Clearly $T_1 \cap S \subseteq P_1$, and $T_2 \supseteq T_1$. By Theorem 2.3, $M_\alpha \times T_2$ is a parallel-α-copy of $M \times (T_1 \cap S)$. Also $M_\alpha \times T_1$ is obtained by adding parallel elements to elements of $M \times (T_1 \cap S)$. Hence by repeated use of Theorem 2.2

$$r\,(M_\alpha \times T_1) = r\,(M \times (T_1 \cap S))$$

$$= r\,(M_\alpha \times T_2) \text{ by Theorem 2.3.}$$

Hence,

$$d(M_\alpha \times T_2) \geq d\,(M_\alpha \times T_1) > d\,(M_\alpha \times \mathcal{T}(P_1)) \qquad .. \quad (1)$$

Now by Corollary 1 of Theorem 2.3,

$$d\,(M_\alpha \times T_2) = \alpha \,.\, d\,[\,M \times (T_1 \cap S)\,]$$

and $\quad d\,[\,M_\alpha \times \mathcal{T}(P_1)\,] = \alpha \,.\, d\,(M \times P_1)$.

By (1) this means that

$$d\,[\,M \times (T_1 \cap S)\,] > d\,(M \times P_1) \text{ which is a contradiction.}$$

Hence, $M_\alpha \times \mathcal{T}(P_1)$ is molecular.

Now by Theorem 2.3, $M_\alpha \,.\, (S_\alpha - \mathcal{T}(P_1))$ is a parallel-α-copy of $M \,.\, (S-P_1)$. Hence one can repeat the previous argument to show that

$$M_\alpha \,.\, (S_\alpha - \mathcal{T}(P_1)) \times \mathcal{T}(P_2)$$

$$= M_\alpha \times (\mathcal{T}(P_1) \cup \mathcal{T}(P_2)) \,.\, \mathcal{T}(P_2) \;\; [\text{By Theorem T7}]\,.$$

Also since by Theorem 2.3, $M_\alpha \times [\tau(P_1) \cup \tau(P_2)] \cdot \tau(P_2)$ is a parallel-α-copy of $M \times (P_1 \cup P_2) \cdot P_2$, by Corollary 1 of Theorem 2.3 we have

$$d[M_\alpha \times [\tau(P_1) \cup \tau(P_2)] \cdot \tau(P_2)]$$

$$= \alpha . d[M \times (P_1 \cup P_2) \cdot P_2] .$$

Hence,

$$d[M_\alpha \times \tau(P_1 \cup P_2) . \tau(P_2)] < d[M_\alpha \times \tau(P_1)] .$$

Repeating the above argument one sees that

$[M_\alpha \times \bigcup_{i=1}^{r} \tau(P_i) \cdot \tau(P_r)]$ is molecular for all $r \in \{1, \ldots n\}$

$$d[M_\alpha \times \bigcup_{i=1}^{r} \tau(P_i) \cdot \tau(P_r)]$$

$$> d[M_\alpha \times \bigcup_{i=1}^{m} \tau(P_i) \cdot \tau(P_m)]$$

whenever $m > r$.

Hence $\tau(P_1), \tau(P_2) \ldots \tau(P_n)$ is a P-sequence of M_α .

<u>Corollary 1.</u> Let M be a matroid on S and let M_α be its parallel-α-copy on S_α.

Then (i) M is molecular iff M_α is molecular

(ii) M is atomic iff M_α is atomic.

Proof. (i) is obvious from Theorem 2.4.

(ii) We proceed as in the first part of the proof of Theorem 2.4.

Let M be atomic. Suppose M_α is not atomic. Then there exists a set $T_1 \subset S_\alpha$ such that $d(M_\alpha \times T_1) \geq d(M_\alpha)$.

Let $T_2 = \mathcal{T}(T_1 \cap S)$. Clearly $T_2 \supseteq T_1$.

By Theorem 2.3, $M_\alpha \times T_2$ is a parallel-α-copy of $M \times (T_1 \cap S)$.

Also $M_\alpha \times T_1$ is obtained by adding parallel elements to elements of $M \times (T_1 \cap S)$. Hence by repeated use of Theorem 2.2,

$$r(M_\alpha \times T_1) = r(M \times (T_1 \cap S))$$

$$= r(M_\alpha \times T_2) \text{ by Theorem 2.3.}$$

Hence $d(M_\alpha \times T_2) \geq d(M_\alpha \times T_1) \geq d(M_\alpha)$ .. (1)

Now by Corollary 1 of Theorem 2.3

$$d(M_\alpha \times T_2) = \alpha \,.\, d(M \times (T_1 \cap S))$$

and

$$d(M_\alpha) = \alpha \,.\, d(M) \,.$$

By (1) this means that

$$d(M \times (T_1 \cap S)) \geq d(M) \,.$$

Hence

$$T_1 \cap S = S \,.$$

But then if $T_1 \neq S_\alpha$.

$$|S_\alpha| > |T_1|$$

while $r(M_\alpha) = r(M_\alpha \times T_1)$

Hence $d(M_\alpha) > d(M_\alpha \times T_1)$ which is a contradiction.

Hence M_α is atomic.

Suppose M is not atomic. Then there exists $T \subset S$ such that

$$d(M \times T) \geq d(M)$$

But then $d(M_\alpha \times \mathcal{T}(T)) = \alpha \cdot d(M \times T)$

and $d(M_\alpha) = \alpha \cdot d(M)$ by Theorem 2.3 and its Corollary.

Hence $d[M_\alpha \times \mathcal{T}(T)] \geq d(M_\alpha)$.

Also $\mathcal{T}(T) \subset S_\alpha$. Hence M_α is not atomic.

Q.E.D.

We need the following Lemma for the proof of Theorem 2.5 (the uniqueness theorem). We denote the matroid

$$M \vee M \vee \ldots \vee M \text{ (k times) by } M^k.$$

<u>Lemma 2.1.</u> Let M be a molecular matroid on S. Then if $d(M) = k$, where k is an integer, there exists a set of k pairwise disjoint bases $b_1, b_2 \ldots b_k$ of M such that

$$\bigcup_{i=1}^{k} b_i = S.$$

<u>Proof.</u> Suppose S is not the base of M^k. Then,

$$R = A(M^k) \cup C(M^k)$$ is non-void. Hence by Theorem 1.2 there exist bases $b_1, b_2 \ldots b_k$ of M such that

$b_1 \cap R$, ... $b_k \cap R$ are pairwise disjoint bases of $M \times R$.

Also $R \notin \bigcup_{i=1}^{k} b_i$. Hence $d(M \times R) > d(M)$ which is a contradiction. Hence, $S = B(M^k)$.

But $|S| = k \cdot r(M)$.

Hence there exists a set of k pairwise disjoint bases $b_1, b_2 \ldots b_k$ of M such that

$$\dot{\bigcup_{i=1}^{k}} b_i = S .$$

Q.E.D.

<u>Notation</u> : We will henceforth use p.d. to denote 'pairwise disjoint'.

<u>Theorem 2.5.</u> Let M be a matroid on S. Let $P_1^1, P_2^1 \ldots P_n^1$ and $P_1^2, P_2^2 \ldots P_m^2$ be two P-sequences of M. Then,

(i) $n = m$

(ii) $P_i^1 = P_i^2$ for $(i = 1, \ldots n)$.

<u>Proof.</u> Let $r(M) = k$. Construct a parallel-α-copy of M on S_α with $\alpha = k!$.

Then by Theorem 2.4,

$\mathcal{T}(P_1^1), \mathcal{T}(P_2^1) \ldots \mathcal{T}(P_n^1)$

$\mathcal{T}(P_1^2), \mathcal{T}(P_2^2) \ldots \mathcal{T}(P_n^2)$ are P-sequences of M_α .

Also by Theorem 2.3, $M_\alpha \times (\bigcup_{i=1}^{t} \mathcal{T}(P_i^1))$. $\mathcal{T}(P_t^1)$ is a parallel-α-copy of $M \times \bigcup_{i=1}^{t} P_i^1 \cdot P_t^1$ for all $t \in \{1, 2 \ldots n\}$.

Now since $r\ (M \times \bigcup_{i=1}^{t} P_i^1 \cdot P_t^1)$ is a factor of $\alpha = k!$,

by Corollary 1 of Theorem 2.3,

$$d \left[M_\alpha \times (\bigcup_{i=1}^{t} \tau(P_i^1)) \cdot \tau(P_t^1) \right]$$

has integral values for all $t \in \{1, 2 \ldots n\}$.

Suppose

$$d \left[M_\alpha \times \tau(P_1^1) \right] = p_1$$

$$\vdots$$

$$d \left[M_\alpha \times (\bigcup_{i=1}^{t} \tau(P_i^1)) \cdot \tau(P_t^1) \right] = p_t$$

$$\vdots$$

$$d \left[M_\alpha \cdot \tau(P_n^1) \right] = p_n$$

the p_i being integral.

By Corollary 1 of Theorem 2.4

$M_\alpha \times (\bigcup_{i=1}^{t} \tau(P_i)) \cdot \tau(P_t)$ is molecular and hence by Lemma 1 has p_t p.d. bases $b_1^t, b_2^t \ldots b_{p_t}^t$ such that

$$\bigcup_{i=1}^{p_t} b_i^t = \tau(P_t).$$

Using Theorem T3 repeatedly we have $b_{i_1}^1 \cup b_{i_2}^2 \ldots \cup b_{i_n}^n$ is independent in M_α. Hence it is possible to find p.d.

independent sets $L_1, L_2 \ldots L_{p_1}$ such that

$$\bigcup_{i=1}^{p_1} L_i = S_\alpha$$

Also, $\Upsilon(P_1)$ cannot be covered with less than p_1 independent sets. Hence p_1 is the minimum number of independent sets required to cover S_α.

Hence $d[M_\alpha \times \Upsilon(P_1^1)] = \max_{R \subseteq S_\alpha} d(M_\alpha \times R)$.

By the same argument $d(M_\alpha \times \Upsilon(P_1^2))$ is the minimum member of independent sets required to cover S_α.

Hence $d[M_\alpha \times \Upsilon(P_1^2)] = d[M_\alpha \times \Upsilon(P_1^1)]$.

Next suppose $\Upsilon(P_1^2) \neq \Upsilon(P_1^1)$.

Then by Theorem 2.1 and Lemma 2.1

$M_\alpha \times (\Upsilon(P_1^2) \cup \Upsilon(P_1^1))$ has p_1 p.d. bases $b_1, b_2 \ldots b_{p_1}$

such that

$$\bigcup_{i=1}^{p_1} b_i = \Upsilon(P_1^2) \cup \Upsilon(P_1^1),$$

and

$$M_\alpha \times [\Upsilon(P_1^2) \cup \Upsilon(P_1^1)] \cdot [\Upsilon(P_1^2) - \Upsilon(P_1^1)]$$

has p_1 p.d. bases by the use of Theorem T3.

But by Theorem T7

$$M_\alpha \times [\ \tau(P_1^{\,2}) \cup \tau(P_1^{\,1})\ ] \cdot [\ \tau(P_1^{\,2}) - \tau(P_1^{\,1})\ ]$$

$$= M_\alpha \cdot [\ S_\alpha - \tau(P_1^{\,1})\ ] \times [\ \tau(P_1^{\,2}) - \tau(P_1^{\,1})\ ] .$$

Hence $M_\alpha \cdot [\ S_\alpha - \tau(P_1^{\,1})\ ] \times [\ \tau(P_1^{\,2}) - \tau(P_1^{\,1})\ ]$

has p_1 p.d. bases whose union is $\tau(P_1^{\,2}) - \tau(P_1^{\,1})$.

Hence $M_\alpha \cdot [\ S_\alpha - \tau(P_1^{\,1})\ ]$ cannot be covered by less than p_1 p.d. independent sets of $M_\alpha \cdot [\ S_\alpha - \tau(P_1^{\,1})\ ]$.

But using the argument in the first part of the proof $S_\alpha - \tau(P_1^{\,1})$ can be covered by

$$d\ [\ M_\alpha \times (\ \tau(P_1^{\,1}) \cup \tau(P_2^{\,1})) \cdot \tau(P_2^{\,1})\ ] = p_2 < p_1$$

independent sets. This is a contradiction.

Hence $\tau(P_1^{\,1}) = \tau(P_1^{\,2})$.

We can next consider $M_\alpha \cdot [\ S_\alpha - \tau(P_1^{\,1})\ ]$ and repeat the same argument.

Hence $\tau(\ P_2^{\,2}) = \tau(P_2^{\,1})$.

Repeated application of this argument then yields

$$\tau(P_1^{\,1}) = \tau(P_2^{\,1}) \ldots \tau(P_n^{\,1}) = \tau(P_n^{\,2})\ ; \quad m \text{ being equal to } n.$$

But the map $\tau : P(S) \to P(S_\alpha)$ is one to one into.

Hence $P_1^{\,1} = P_1^{\,2} \ldots P_n^{\,1} = P_n^{\,2}$.

Q.E.D.

<u>Corollary 1.</u> Let M be a matroid on S. Let $P_1, P_2 \dots P_n$ be its P-sequence. Then P_i for all $i \in \{1,2 \dots n\}$ is invariant under the automorphisms of M.

<u>Proof.</u> Let σ be any automorphism of M. Then, $\sigma(P_1), \sigma(P_2) \dots \sigma(P_n)$ is a P-sequence of M. But then by Theorem 2.5,

$$\sigma(P_i) = P_i \, , i \in \{1,2 \dots n\} .$$

Hence P_i are invariant under the automorphisms of M.

Q.E.D.

The next theorem gives a characteristic of P-sequences.

<u>Theorem 2.6.</u> Let M be a matroid on S. Let $P_1^1, P_2^1 \dots P_n^1$ be a sequence of p.d. subsets such that P_q^1 is the maximal set satisfying,

$$d \left[M . \left(\bigcup_{i=q}^{n} P_i^1 \right) \times P_q^1 \right] = \max_{Q \subseteq \bigcup_{i=q}^{n} P_i^1} d \left(M . \bigcup_{i=q}^{n} P_i^1 \times Q \right) .$$

Then $P_1^1, P_2^1 \dots P_n^1$ is the P-sequence of M.

Theorem 2.6 follows easily from the following Lemma :-

<u>Lemma 2.2.</u> Let M be a matroid on S. Let P be the maximal subset of S such that

$$d (M \times P) = \max_{R \subseteq S} d (M \times R) .$$

If Q is the maximal set such that :

(i) $Q \subseteq S - P$

(ii) $d [M . (S-P) X Q] = \max_{R \subseteq S-P} d [M . (S-P) X R]$,

then

$$d (M X P) > d (M . (S-P) X Q).$$

<u>Proof.</u> Suppose $d [M . (S-P) X Q] \geq d (M X P)$, then,

$$\frac{|Q|}{r [M . (S-P) X Q]} \geq \frac{|P|}{r (M X P)}$$

Hence

$$\frac{|Q| + |P|}{r [M . (S-P) X Q] + r(M X P)} \geq \frac{|P|}{r (M X P)}.$$

But by Theorem T3

$$r [M X (P \cup Q)] = r [M X P] + r [M X (P \cup Q) . Q].$$

But $M X (P \cup Q) . Q = M . (S-P) X Q$ by Theorem T7.

Hence

$$r [M X (P \cup Q)] = r (M X P) + r [M . (S-P) X Q].$$

Hence

$$\frac{|P \cup Q|}{r [M X (P \cup Q)]} \geq \frac{|P|}{r (M X P)}$$

which contradicts the definition of P.

Q.E.D.

Using Theorem 2.5 and 2.6 we obtain the following :

Corollary 1 :- Let M be a matroid on S. Let $P_1, P_2 \dots P_n$ be the P-sequence of M. Let $Q = \bigcup_{i=r}^{k} P_i$ ($1 \leq r \leq k \leq n$).

Then $P_r, P_{r+1} \dots P_k$ is the P-sequence of

$$M \times (\bigcup_{i=1}^{k} P_i) . Q \ .$$

Theorem 2.7. Let M be a matroid on S. Let $P_1^{\,1}, P_2^{\,1} \dots P_n^{\,1}$; $P_1^{\,2}, P_2^{\,2} \dots P_n^{\,2}$ be the P-sequences of M, M^* respectively. Then,

(i) n = m

(ii) $P_n^{\,1} = P_1^{\,2} \dots P_r^{\,1} = P^{2}_{n-r+1} \dots P_1^{\,1} = P_n^{\,2}$.

Theorem 2.7 follows easily from the following Lemmas :

Lemma 2.3. Let M be a matroid on S. Then

$$d (M \times P) > d (M) \text{ iff } d (M_* . (S-P)) < d (M) .$$

Proof. We have,

$$\frac{|P|}{r(M \times P)} > \frac{|S|}{r(M)} = \frac{|P| + |S - P|}{r (M \times P) + r[M . (S-P)]}$$

by Theorem T3.

Now it is easy to see that if $\frac{x}{a} > \frac{y}{b}$ x,y,a,b positive real numbers $x \leq y$, $a \leq b$,

Then, $\frac{y - x}{b - a} < \frac{y}{b}$. Hence $\frac{|S - P|}{r[M . (S-P)]} < \frac{|S|}{r(M)}$.

The converse is proved similarly. Q.E.D.

Lemma 2.4. Let M be a molecular matroid on S. Then M^* is molecular.

Proof. If possible, let $d(M^* \times P) > d(M^*)$, for some $P \subset S$

i.e. $d[(M \cdot P)^*] > d(M^*)$ since by Theorem T7

$$(M^* \times P) = (M \cdot P)^* .$$

Now it is easy to see that for any two matroids M_1 and M_2,

$$\text{if } d(M_1) > d(M_2), \quad d(M_1^*) < d(M_2^*)$$

Hence $d(M \cdot P) < d(M)$. Hence $d[M \times (S-P)] > d(M)$ by Lemma 2.3, which is a contradiction.

Q.E.D.

Proof of Theorem 2.7. Let $P_1^1 \ldots P_n^1$ be the P-sequence of M. Then by the use of Lemma 2.3,

$$d(M \cdot P_n^1) < d[M \cdot (P_n^1 \cup P_{n-1}^1) \times P_{n-1}^1] \ldots$$

$$< d(M \cdot \bigcup_{i=1}^{n} P_i^1 \times P_1^1) = d(M \times P_1^1) .$$

Hence

$$d(M^* \times P_n^1) > d[M^* \times (P_n^1 \cup P_{n-1}^1) \times P_{n-1}^1] \ldots$$

$$> d(M^* \times \bigcup_{i=1}^{n} P_i^1 \cdot P_1^1) .$$

Also, $M^* \times \bigcup_{i=r}^{n} P_i^1 \cdot P_r^1 = (M \cdot \bigcup_{i=r}^{n} P_i^1 \times P_r^1)^* =$

$$= (M \times \bigcup_{i=1}^{r} P_i^1 \cdot P_r^1)^* \text{ by the use of Theorem T7.}$$

Since $\left(M \times \bigcup_{i=1}^{r} P_i^{\,1} \,.\, P_r^{\,1} \right)$ is molecular,

by Lemma 2.4,

$$\left(M^{*} \times \bigcup_{i=r}^{n} P_i^{\,1} \,.\, P_r^{\,1} \right) \text{ is molecular.}$$

Thus, $P_n^{\,1}, P_{n-1}^{\,1} \ldots P_1^{\,1}$ is the P-sequence of M^{*}.

Q.E.D.

Section 3 : P-sequences and Matroid Unions

In this section we study the P-sequence of the union of two matroids relative to the P-sequences of these matroids when they are related in a certain fashion. We are then able to determine the P-sequence of the images, under certain functions (admissible functions), of a matroid wholly in terms of its P-sequence in a very simple manner. In addition we show that the P-sequence of any matroid is determined completely by the sets of coloops of the images of M under certain admissible functions.

The following is the central theorem of this section.

Theorem 3.1. Let M_1 and M_2 be two matroids on a set S. If $\max_{Q \subseteq S} d(M_i \times Q) = d(M_i \times P)$ for $i = 1,2$, then

$$\max_{Q \subseteq S} d[(M_1 \vee M_2) \times Q] = d[(M_1 \vee M_2) \times P].$$

Proof. Suppose $(M_1 \vee M_2) \times P$ has $(P-T)$ as its set of coloops. Let b be a base of $(M_1 \vee M_2) \times T$. Let $b = b_1 \cup b_2$ where b_1, b_2 are bases of $M_1 \times P$, $M_2 \times P$ respectively.

Then $b_1 \cap T$ is a base of $M_1 \times T$,

$b_2 \cap T$ is a base of $M_2 \times T$,

and $b_1 \cap b_2 \cap T = \phi$ by Corollary 1 of Theorem 1.2.

Let $a = |b_1 \cap T|$, $f = |b_2 \cap T|$

$c = |T - (b_1 \cup b_2) \cap T|$

$d = |b_1 \cap (P-T)|$, $e = |b_2 \cap (P-T)|$

Let ρ, ρ_1, ρ_2 be the rank functions of $M_1 \vee M_2$, M_1, M_2 respectively.

Since $\dfrac{\rho_1(P)}{|P|} \leq \dfrac{\rho_1(T)}{|T|}$ we have

$$\frac{\rho_1(P)}{|P|} \leq \frac{a + d}{a + f + c + d + e} \leq \frac{a}{a + f + c}$$

Hence $d(a + f + c) \leq a(d + e)$

i.e. $d(f + c) \leq ae$. (1)

Now since

$$\frac{\rho_2(P)}{|P|} \leq \frac{\rho_2(T)}{|T|}$$

we get similarly

$$\frac{\rho_2(P)}{|P|} \leq \frac{f + e}{a + f + c + d + e} \leq \frac{f}{a + f + c}$$

i.e. $e(a + f + c) \leq f(d + e)$

i.e. $e(a+c) \leq fd$ (2)

But by (1) above,

$$d(f + c) \leq ae$$

Hence $ec = 0$ and $de = 0$

i.e. $c = 0$ or $e = d = 0$.

<u>Case I.</u> $d = e = 0, \quad c \neq 0$.

Let $K \subseteq S$ such that

$$d \lfloor (M_1 \vee M_2) \times K \rfloor = \max_{Q \subseteq S} d \lfloor (M_1 \vee M_2) \times Q \rfloor$$

If $(M_1 \vee M_2) \times K$ has $(K-L)$ as its set of coloops, then

$$d \lfloor (M_1 \vee M_2) \times L \rfloor > d \lfloor (M_1 \vee M_2) \times K \rfloor$$

if L, $K - L$ are not void.

This is a contradiction.

Hence, $L = \phi$ or $K - L = \phi$. If now $L = \phi$ since $c \neq 0$ and

$$1 < \frac{|b_1 \cup b_2| + c}{|b_1 \cup b_2|} ,$$

$$d \lfloor (M_1 \vee M_2) \times K \rfloor < d \lfloor (M_1 \vee M_2) \times P \rfloor ,$$

which is a contradiction.

Hence $L \neq \phi$ and $K - L = \phi$.

Hence, by Corollary 1 of Theorem 1.2,

$$\frac{\rho(K)}{|K|} = \frac{\rho_1(K) + \rho_2(K)}{|K|}$$

$$\geq \frac{\rho_1(P) + \rho_2(P)}{|P|}$$

since $$\frac{\rho_1(K)}{|K|} \geq \frac{\rho_1(P)}{|P|}$$

and

$$\frac{\rho_2(K)}{|K|} \geq \frac{\rho_2(P)}{|P|}$$

Hence $d[(M_1 \vee M_2) \times P] \geq d[(M_1 \vee M_2) \times K]$.

<u>Case II.</u> $c = 0$

$$\rho_1(P) + \rho_2(P) \geq |P|$$

i.e. $d[(M_1 \vee M_2) \times P] = 1$.

Let $K \subseteq S$ be such that

$(M_1 \vee M_2) \times K$ has no coloops.

Then, $\frac{\rho(K)}{|K|} = \frac{\rho_1(K)}{|K|} + \frac{\rho_2(K)}{|K|}$ by the use of Corollary 1 of Theorem 1.2.

Hence

$$\frac{\rho(K)}{|K|} \geq \frac{\rho_1(P)}{|P|} + \frac{\rho_2(P)}{|P|}$$

Hence $K = \phi$ and

$d[(M_1 \vee M_2) \times P] \geq d[(M_1 \vee M_2) \times K]$ $(K \subseteq S.)$.

Q.E.D.

<u>Corollary 1.</u> Let M_1 and M_2 be two matroids on a set S. If $\max_{Q \subseteq S} d[(M_i \times Q)] = d[(M_i \times P)]$ for i = 1,2 and

$[(M_1 \vee M_2) \times P] = M_u$ on P.

Then $M_1 \vee M_2 = M_u$ on S.

<u>Corollary 2.</u> If M_1, M_2 are molecular matroids on S then $M_1 \vee M_2$ is molecular.

<u>Corollary 3.</u> If M_1, M_2 are atomic matroids on S, then $M_1 \vee M_2$ is atomic if $M_1 \vee M_2 \neq M_u$ on S.

<u>Proof.</u> We need consider only Case I of the proof of Theorem 3.1.,

since $M_1 \vee M_2 \neq M_u$ on S.

Firstly by the first part of the proof of Theorem 3.1, $M_1 \vee M_2$ has no coloops.

Hence for $K \subseteq S$, we have

$$\frac{r((M_1) \vee (M_2))}{|S|} = \frac{r(M_1) + r(M_2)}{|S|} < \frac{r(M_1 \times K) + r(M_2 \times K)}{|K|}$$

Hence,

$$d[M_1 \vee M_2] > d[(M_1 \vee M_2) \times K] \quad (K \subseteq S).$$

Q.E.D.

<u>Corollary 4.</u> Let M be a molecular matroid on S. Let $r(M) = \alpha$, and let M^k represent the matroid

$$M \vee M \vee \ldots \vee M \quad (k \text{ times})$$

Then $r[M^k] = \min[k\alpha, |S|]$.

Proof. Let M^k have no coloops. Then by Corollary 1 of Theorem 1.2,

$$r[M^k] = k.\alpha .$$

Suppose M^k has coloops. Since M^k is molecular, by Corollary 2 above it follows that M^k is the matroid M_u on S.

Hence $k . \alpha \geq r[M^k] = |S|$.

Thus $r[M^k] = \min[k\alpha, |S|]$.

Q.E.D.

Definition 2.1. Let M_1, M_2 be two matroids on S. Let M_1 have the P-sequence

$$P_1^{\ 1}, P_2^{\ 1} \ldots P_n^{\ 1}$$

and M_2 have the P-sequence

$P_1^{\ 2}, P_2^{\ 2}, \ldots P_m^{\ 2}$. Then the ordered pair (M_1, M_2) has overlapping P-structure iff one of the following condition holds :

(1) $P_1^{\ 1} = P_1^{\ 2}$, $P_2^{\ 1} = P_2^{\ 2}$, ... $P_m^{\ 2} = P_m^{\ 1}$,

n being equal to m.

(2) $P_1^{\ 1} = P_1^{\ 2}$, $P_2^{\ 1} = P_2^{\ 2}$, ... $P_{m-1}^{1} = P_{m-1}^{2}$,

$P_m^{\ 2} = \bigcup_{i=m}^{n} P_i^{\ 1}$ and $M_2 \cdot P_m^{\ 2} = M_u$ on $P_m^{\ 2}$.

(3) $\bigcup_{i=1}^{k} P_i^{1} = P_1^{2}$, $P_{k+1}^{1} = P_2^{2}$, ... $P_{k+m-1}^{1} = P_m^{2}$,

and $M_2 \times P_1^{2} = M_o$ on P_1^{2} .

(4) $\bigcup_{i=1}^{k} P_i^{1} = P_1^{2}$, $P_{k+1}^{1} = P_2^{2}$, ... $\bigcup_{i=k+m-1}^{n} P_i^{1} = P_m^{2}$

and $M_2 \times P_1^{2} = M_o$ on P_1^{2} , $M_2 \cdot P_m^{2} = M_u$ on P_m^{2} .

(5) $\bigcup_{i=1}^{k} P_i^{1} = P_1^{2}$, $P_{k+1}^{1} = P_2^{2}$ ... $P_n^{1} = \bigcup_{i=n-k+1}^{m} P_i^{2}$

and $M_2 \times P_1^{2} = M_o$ on P_1^{2} , $M_1 \cdot P_n^{1} = M_u$ on P_n^{1} .

We say the matroids M_1 and M_2 are 'aligned' iff (M_1, M_2) or (M_2, M_1) have overlapping P-structure. [See Figure 2.3.1]

Example 2.3.1. In the Figures 2.3.1 (1) to 2.3.1 (5) examples of various cases in Definition 3.1 are represented schematically. The strips into which the rectangles are divided represent the sets of the P-sequences, strips in the 'same line' in the two rectangles denote the same set.

Definition 3.2. A P-sequence defines a partition of the set of definition of the matroid. Let M_1, M_2 be aligned matroids on S with the P-sequences

$$\left\{P_i^{1}\right\}_{i=1}^{n} = P_1^{1} , P_2^{1} \ldots P_n^{1}.$$

$$\left\{P_i^{2}\right\}_{i=1}^{m} = P_1^{2}, P_2^{2} \ldots P_m^{2} .$$

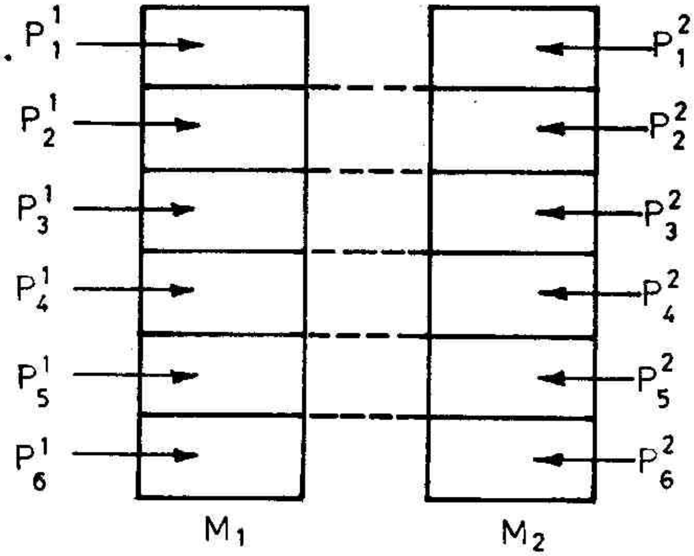


FIG. 2.3.1. (1).

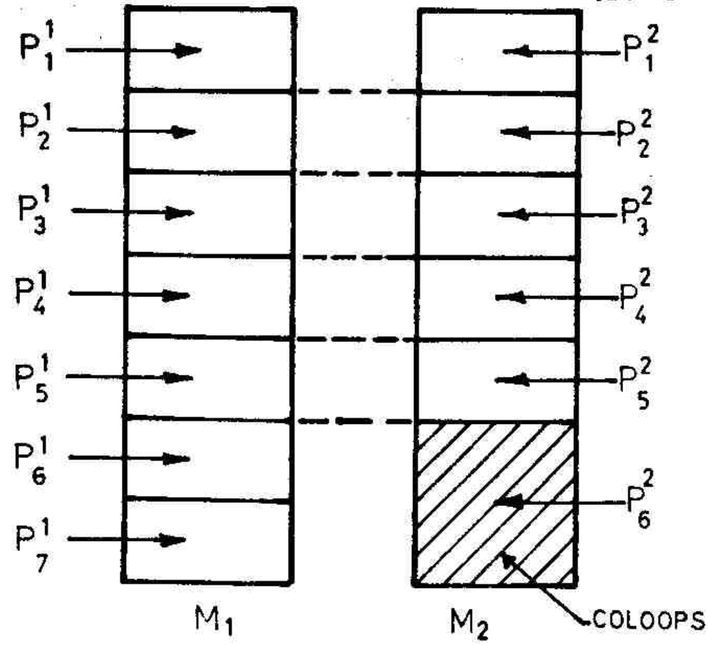


FIG. 2.3.1. (2).

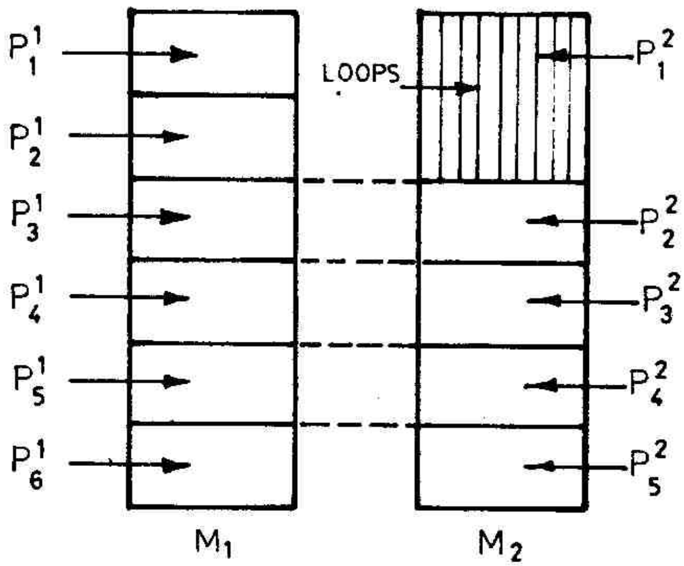


FIG. 2.3.1, (3).

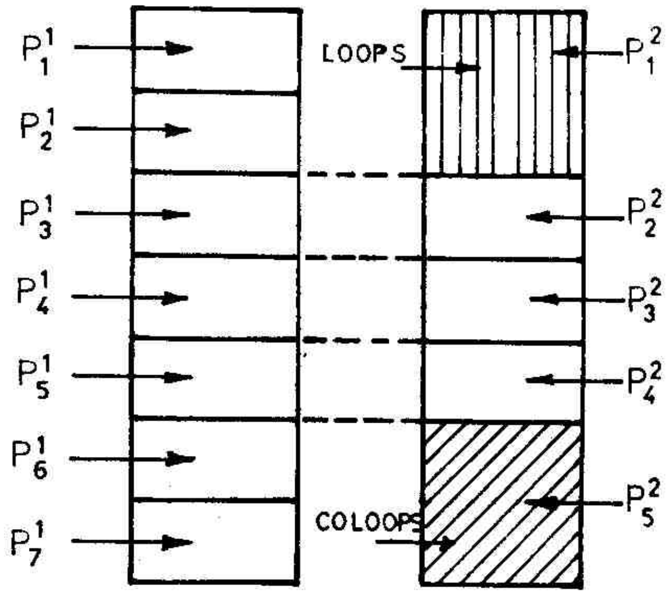


FIG. 2.3.1. (4).

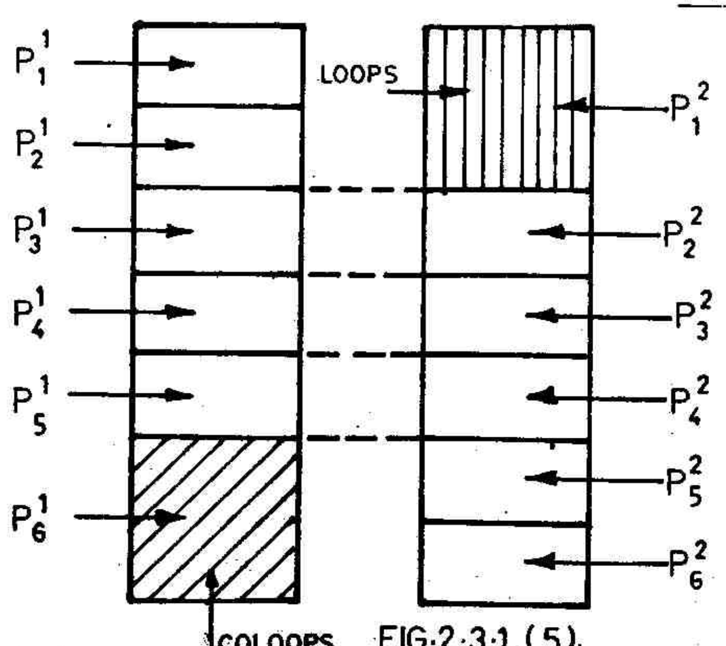


FIG. 2.3.1, (5).

FIG. 2.3.1.

We then say $\{P_i^1\}_{i=1}^n$ is coarser (finer) than $\{P_i^2\}_{i=1}^m$ iff the partition defined by $\{P_i^1\}_{i=1}^n$ is coarser (finer) than the partition defined by $\{P_i^2\}_{i=1}^m$. For aligned matroids M_1, M_2 on S with P- sequences $\{P_i^1\}_{i=1}^n$, $\{P_i^2\}_{i=1}^m$ we would say M_1 is coarser (finer) than M_2 iff $\{P_i^1\}_{i=1}^n$ is coarser (finer) than $\{P_i^2\}_{i=1}^m$. Again, we can talk of the infimum and supremum of P-sequences $\{P_i^1\}_{i=1}^n$, $\{P_i^2\}_{i=1}^m$ of aligned matroids, by considering the corresponding infimum and supremum of the partitions defined by them.

The following theorem is now easy to verify.

Theorem 3.2. Let M_1, M_2, M_3 be matroids on S.

(1) If M_1, M_2 and M_1, M_3 are aligned with M_1 coarser than M_2 and M_3 coarser than M_1, then M_2 and M_3 are aligned with M_3 coarser than M_2.

(2) If M_2 is coarser than M_1 and M_3 is coarser than M_1, then M_3, M_2 are aligned.

We next show that certain special kinds of reductions of matroids can be carried over to their union.

Theorem 3.3. Let M_1, M_2 be two matroids on S. Let M_1 have the P-sequence $P_1^1, P_2^1 \ldots P_n^1$ and let M_2 have the

P-sequence $P_1^2, P_2^2 \ldots P_m^2$. Then, if

(a) $\bigcup_{i=1}^{k} P_i^1 = P_1^2 = T$ with $M_2 \times P_1^2 = M_0$ on P_1^2

or

(b) $P_1^1 = P_1^2 = T$, then

$$(M_1 \vee M_2) \cdot (S-T) = M_1 \cdot (S-T) \vee M_2 \cdot (S-T) .$$

<u>Proof.</u> (a) Clearly $M_1 \times T$ and $M_2 \times T$ have disjoint bases since $M_2 \times T$ has ϕ as the base. The conditions of Theorem 1.3 are therefore satisfied. Hence,

$$(M_1 \vee M_2) \cdot (S-T) = M_1 \cdot (S-T) \vee M_2 \cdot (S-T) .$$

(b) <u>Case 1.</u> Let $(M_1 \vee M_2) \times T$ have no coloops. Then using Corollary 1 of Theorem 1.3, we have

$$(M_1 \vee M_2) \cdot (S-T) = (M_1 \cdot (S-T)) \vee (M_2 \cdot (S-T)).$$

<u>Case 2.</u> Let $(M_1 \vee M_2) \times T$ have coloops. Then by Corollary 2 of Theorem 3.1,

$(M_1 \vee M_2) \times T$ is the matroid M_u on T. Hence,

$$r(M_1 \times T) + r(M_2 \times T) \geq |T| .$$

Let us suppose

$(M_1 \cdot (S-T)) \vee (M_2 \cdot (S-T))$ is not M_u on $(S-T)$. Hence there exists $Q \subseteq (S-T)$ such that

$((M_1 \cdot (S-T)) \vee (M_2 \cdot (S-T)) \times Q$ has no coloops. Hence,

$$r[(M_1 \cdot (S-T)) \times Q] + r[(M_2 \cdot (S-T) \times Q] < |Q| .$$

But this implies either

$$\frac{r[(M_1 \cdot (S-T)) \times Q]}{|Q|} < \frac{r[M_1 \times T]}{|T|} \qquad \text{or}$$

$$\frac{r[(M_2 \cdot (S-T)) \times Q]}{|Q|} < \frac{r[M_2 \times T]}{|T|} .$$

By Theorem 3.6, this is impossible. Hence,

$(M_1 \cdot (S-T)) \vee (M_2 \cdot (S-T))$ is the matroid M_u on $(S-T)$.

Since $(M_1 \vee M_2) \times T$ is M_u on T, by Corollary 1 of Theorem 3.1, $(M_1 \vee M_2)$ is M_u on S. Hence $(M_1 \vee M_2) \cdot (S-T)$ is M_u on $(S-T)$. Hence,

$$(M_1 \vee M_2) \cdot (S-T) = M_1 \cdot (S-T) \vee M_2 \cdot (S-T) .$$

Q.E.D.

By repeated application of Theorem 3.3 we have the following Corollary.

<u>Corollary 1.</u> Let M_1, M_2 be two matroids on S. Let M_1 have the P-sequence $P_1^1, P_2^1 \dots P_n^1$ and M_2 have the P-sequence $P_1^2, P_2^2 \dots P_m^2$. Then, if $\bigcup_{i=1}^{k} P_i^1 = \bigcup_{i=1}^{j} P_i^2 = T$,

$$(M_1 \cdot (S-T)) \vee (M_2 \cdot (S-T)) = (M_1 \vee M_2) \cdot (S-T) ,$$

when $M_1 \times T$ and $M_2 \times T$ are aligned.

The following simple Lemma is needed for the proof of Theorem 3.4.

<u>Lemma 3.1.</u> Let M_1, M_2 be two matroids on S. Let $T_i \subseteq S$ (i=1,2) be the maximal set such that

$$d[M_i \times T_i] = \max_{Q \subseteq S} d[M_i \times Q], \quad (i = 1,2).$$

Then if

(i) $T_1 = T_2 = T$ and

(ii) $r[M_1 \times T] + r[M_2 \times T] < |T|$.

Then

$$d[(M_1 \vee M_2) \times T] > d[(M_1 \vee M_2) \times P]$$

$$\text{if } P - T \neq \phi .$$

<u>Proof.</u> Since $P - T \neq \phi$, we have

$$\frac{1}{d[M_i \times T]} = \frac{r[M_i \times T]}{|T|} < \frac{r[M_i \times P]}{|P|} \quad \text{for } (i=1,2).$$

By Corollary 2 of Theorem 3.1

$(M_1 \vee M_2) \times T$ is molecular.

Also $r[M_1 \times T] + r[M_2 \times T] < |T|$.

Hence $(M_1 \vee M_2) \times T$ has no coloops.

Therefore, by Corollary 1 of Theorem 1.2,

$$r[M_1 \times T] + r[M_2 \times T] = r[(M_1 \vee M_2) \times T]$$

and

$$\frac{1}{d \left[(M_1 \vee M_2) \times T \right]} = \frac{1}{d \left[M_1 \times T \right]} + \frac{1}{d \left[M_2 \times T \right]} .$$

It is now easy to see that

$$d \left[(M_1 \vee M_2) \times T \right] > d \left[(M_1 \vee M_2) \times P \right]$$

Q.E.D.

<u>Theorem 3.4.</u> Let M_1, M_2 be two matroid on S. Then if M_1, M_2 are aligned then $M_1 \vee M_2$, M_1 and $M_1 \vee M_2$, M_2 are aligned. Also the P-sequence of $M_1 \vee M_2$ is coarser than the infimum of the P-sequences of M_1 and M_2.

Proof. Let $P_1^1, P_2^1 \ldots P_n^1$

$$P_1^2, P_2^2 \ldots P_m^2$$

be the P-sequences of M_1, M_2 respectively.

Suppose there exists a least number k such that

(i) $P_k^1 = P_j^2$ for some $j \in \{1, 2, \ldots, m\}$

(ii) $\left[M_1 \vee M_2 \right] \times (\bigcup_{i=1}^{k} P_i^1) . P_k^1 = M_u$ on P_k^1 .

Then using Theorem (of Tutte) T7, Corollary 1 of Theorem 3.1 and Corollary 1 of Theorem 3.2 it is clear that

$$(M_1 \vee M_2) . (\bigcup_{i=k}^{n} P_i^1) = M_u \text{ on } \bigcup_{i=k}^{n} P_i^1 .$$

Hence $B(M_1 \vee M_2) = \bigcup_{i=k}^{n} P_i^{1}$.

If there exists no such k

$$B(M_1 \vee M_2) = B(M_1) \text{ or } B[M_2]$$

depending on $B[M_1] \supseteq B[M_2]$

or $B[M_2] \supseteq B[M_1]$.

<u>Case 1</u>

If $P_1^{1} = P_1^{2}$, using Lemma 3.1, Corollary 1 of Theorem 3.3 and Theorem 2.6, it is easy to see that

if $B[M_1 \vee M_2] \neq \phi$

$P_1^{1}, P_2^{1} \ldots , B[M_1 \vee M_2]$ is the P-sequence of $M_1 \vee M_2$,

and if $B[M_1 \vee M_2] = \phi$,

$P_1^{1}, P_2^{1} \ldots P_n^{1}$ is the P-sequence of $M_1 \vee M_2$.

<u>Case 2.</u>

If $\bigcup_{i=1}^{p} P_i^{2} = P_1^{1}$

with $M_1 \times P_1^{1} = M_o$ on P_1^{1} ,

since $M_o \vee M = M$ for any matroid M, again using Theorem 2.6, Corollary 1 of Theorem 3.3 and Lemma 1 we have

Hence $B (M_1 \vee M_2) = \bigcup_{i=k}^{n} P_i^1$.

If there exists no such k

$$B (M_1 \vee M_2) = B (M_1) \text{ or } B [M_2]$$

depending on $B [M_1] \supseteq B [M_2]$

or $B [M_2] \supseteq B [M_1]$.

Case 1

If $P_1^1 = P_1^2$, using Lemma 3.1, Corollary 1 of Theorem 3.3 and Theorem 2.6, it is easy to see that

if $B [M_1 \vee M_2] \neq \phi$

$P_1^1, P_2^1 \ldots , B [M_1 \vee M_2]$ is the P-sequence of $M_1 \vee M_2$,

and if $B [M_1 \vee M_2] = \phi$,

$P_1^1, P_2^1 \ldots P_n^1$ is the P-sequence of $M_1 \vee M_2$.

Case 2.

If $\bigcup_{i=1}^{p} P_i^2 = P_1^1$

with $M_1 \times P_1^1 = M_0$ on P_1^1 ,

since $M_0 \vee M = M$ for any matroid M, again using Theorem 2.6, Corollary 1 of Theorem 3.3 and Lemma 1 we have

$$P_1^2, P_2^2 \ldots P_p^2, P_{p+1}^2 \ldots B[M_1 \vee M_2]$$

if $\quad B[M_1 \vee M_2] \neq \phi$

and

$$P_1^2, P_2^2 \ldots P_p^2, P_{p+1}^2 \ldots P_m^2$$

if $B[M_1 \vee M_2] = \phi$

as the P-sequence of $M_1 \vee M_2$.

In either case we see that M_1 and M_2 are aligned and the P-sequence of $M_1 \vee M_2$ is coarser than the infimum of the P-sequences of M_1 and M_2 .

Q.E.D.

Let $\mathcal{M}$ be the class of all matroids. We would like to consider certain special kinds of functions from $\mathcal{M}$ into $\mathcal{M}$. To overcome certain minor technical difficulties we first define 'expressions'.

Expressions are defined as follows :-

(1) f_o is an expression

(2) If f is an expression $(f)^*$ is an expression.

(3) If f_1 and f_2 are expressions $f_1 \vee f_2$ is an expression

(4) The only expressions are those given by the above.

We define positive expressions as :

(1) f_o is a positive expression

(2) If f is a negative expression $(f)^*$ is a positive expression

(3) If f_1 and f_2 are positive expressions $f_1 \vee f_2$ is a positive expression

(4) The only positive expressions are those given by the above.

The order of a positive expression is the number of (V) operation needed to define it in terms of f_o.

Example

f_o, $f \equiv ((f_o \vee f_o \vee f_o)^* \vee (f_o \vee f_o)^*)^*$ are positive expressions. f_o has order 0. f has order 4.

We define negative expressions as :

(1) If f is a positive expression $(f)^*$ is a negative expression

(2) If f_1 and f_2 are negative expressions $f_1 \vee f_2$ is a negative expression

(3) The only negative expressions are those given by the above.

The order of a negative expression is the number of (V) operations needed to define it in terms of f_o.

<u>Example</u> : $(f_o)^*$, $f' \equiv (f_o \vee f_o \vee f_o)^* \vee (f_o \vee f_o)^*$ are negative expressions. Order of $(f_o)^*$ is 0. Order of f' is 4.

We will denote the set of all expressions by $\mathcal{E}$.

Now expressions can be made to represent functions from $\mathcal{M}$ into $\mathcal{M}$ as follows :

(1) Let f_o represent the function such that $f_o(\alpha) = \alpha$ for all $\alpha \in \mathcal{M}$.

(2) If $f = f_i \vee f_j$, f represents the function defined by

$$f(\alpha) = f_i(\alpha) \vee f_j(\alpha) \text{ for all } \alpha \in \mathcal{M} .$$

(3) If $f = [f_i]^*$ f represents the function defined by

$$f(\alpha) = [f_i(\alpha)]^* \text{ for all } \alpha \in \mathcal{M} .$$

An expression representing a function will be said to be a representation of the function. The same function can have several representations. (f_o and $((f_o)^* \vee f_o)^* \vee f_o$ represent the same function).

<u>Definition 3.3.</u> A function f mapping $\mathcal{M}$ into $\mathcal{M}$ is said to be positive iff there is at least one positive expression that is a representation of f. Order of a positive function f is equal to $\min_{g \in \mathcal{E}} \{ \text{order of } g \mid g$ is a representation of f and g is a positive expression $\}$.

<u>Definition 3.4.</u> A function f mapping $\mathcal{M}$ into $\mathcal{M}$ is said to be negative iff there is at least one negative expression that is a representation of f. Order of a negative function f is equal to $\min_{g \in \mathcal{E}} \{ \text{order of } g \mid g$ is a representation of f and g is a negative expression $\}$.

<u>Note</u> :- There is some room for confusion in the Definitions 3.3 and 3.4. We do not at this stage know whether a function can be represented both by a positive and a negative expression. However the next theorem indicates that this is impossible. This can also be seen directly by considering the effect of positive and negative functions on a matroid $M \equiv M_u$ on some set S.

We will henceforth not attempt to distinguish between a function and its representation. For instance, instead of saying 'let f be a function with a representation $f_i \vee f_j$ ' we will say 'let $f = f_i \vee f_j$ '.

<u>Definition 3.5.</u> Let F be the class of functions mapping $\mathcal{M}$ into $\mathcal{M}$ such that $F = \{ f \mid f$ is a psotive or negative function $\}$. Then we call F as the class of admissible functions.

<u>Theorem 3.5.</u> Let M be a matroid on S. If f is a positive (negative) function f(M) is aligned with M (M^*) and is coarser than M (M^*).

<u>Proof.</u> Let f be a positive function.

(1) If $f = f_o$, the theorem is trivially true.

(2) Let $f = f_i \vee f_j$, where f_i, f_j are positive functions whose orders are less than that of f. By induction on the order of a positive function we can assume the theorem to hold for f_i and f_j i.e. $f_i(M)$, $f_j(M)$ are aligned to M and are coarser than M. Hence the infimum of the P-sequences of $f_i(M)$, $f_j(M)$ is coarser than the P-sequence of M. Now by Theorem 3.4, f(M) is aligned to $f_i(M)$, $f_j(M)$ and the P-sequence of f(M) is coarser than the infimum of the P-sequences of $f_i(M)$ and $f_j(M)$. Hence f(M) is aligned to M and is coarser than M.

The proof when f is a negative function and $f = f_i \vee f_j$, where f_i, f_j are negative functions of orders less than the order of f, is identical.

Now suppose f is a positive function such that $f = (f_i)^*$, where f_i is a negative function and $f_i = f_k \vee f_j$, where f_k, f_j are negative functions whose orders are less than that of f_i. $f_i(M)$ is then aligned to M^* and is coarser than M^*. By Theorem 3.7, f(M) is aligned to M and is coarser than M. The proof of the case where f is a negative function such that $f = (f_i)^*$ where f_i is a positive function and $f_i = f_k \vee f_j$, where f_k, f_j are positive functions, is similar.

Q.E.D.

<u>Corollary 1.</u> Let M be a matroid on S. Let $P_1, P_2 \dots P_n$ be the P-sequence of M. Let f be an admissible function. Then, if $P_k \subseteq B[f(M)]$, $\bigcup_{i=k}^{n} P_i \subseteq B[f(M)]$ or $\bigcup_{i=1}^{k} P_i \subseteq B[f(M)]$ according as f is positive or negative.

The following Corollary of Theorem 3.5 is also quite easy to verify.

<u>Corollary 2.</u> If M_1, M_2 are aligned matroids on S, and f is an admissible function, $f(M_1)$, $f(M_2)$ are aligned matroids.

<u>Theorem 3.6.</u> Let f be a positive function. Then if a matroid M has the P-sequence $P_1, P_2 \dots P_n$ with $P_1 = T$ then

$$f[M \cdot (S-T)] = f(M) \cdot (S-T)$$

$$f[M \times T] = f(M) \times T$$

If f is a negative function

$$f[M \cdot (S-T)] = f(M) \times (S-T)$$

$$f[M \times T] = f(M) \cdot T .$$

Proof. Let f be a positive function. If $f = f_0$, the theorem is trivially true. Let $f = f_i \vee f_j$, where f_i, f_j are positive functions whose orders are less than that of f. By induction on the order of a positive function we may assume the theorem to be true for f_i and f_j. Hence,

$$f_i [M \cdot (S-T)] = (f_i(M)) \cdot (S-T)$$

$$f_j [M \cdot (S-T)] = (f_j(M)) \cdot (S-T) .$$

Since $f_i(M)$, $f_j(M)$, M are aligned, by the use of Theorem 3.3 we have

$$\begin{aligned}(f(M)) \cdot (S-T) &= ((f_i(M)) \cdot (S-T)) \vee ((f_j(M)) \cdot (S-T)) . \\ &= (f_i(M \cdot (S-T))) \vee (f_j(M \cdot (S-T))) \\ &= f(M \cdot (S-T)).\end{aligned}$$

Again by the induction assumption

$$f_i(M \times T) = f_i(M) \times T$$

$$f_j(M \times T) = f_j(M) \times T$$

By Corollary 4 of Theorem 1.1,

$$\begin{aligned}f(M) \times T &= ((f_i(M)) \times T) \vee ((f_j(M)) \times T) \\ &= (f_i(M \times T)) \vee (f_j(M \times T)) = f(M \times T) .\end{aligned}$$

Now let f be a negative function. If $f = (f_o)^*$ the theorem is true, by Theorem 2.7 and Theorem T7. Now let $f = f_i \vee f_j$, where f_i, f_j are negative functions whose orders are less than that of f. By induction, we can take the theorem to hold for f_i, f_j. We have, by Corollary 4 of Theorem 1.1,

$$(f(M)) \times (S-T) = [(f_i(M)) \times (S-T)] \vee [f_j(M) \times (S-T)]$$
$$= [f_i(M \cdot (S-T))] \vee [f_j(M \cdot (S-T))]$$
$$= f[M \cdot (S-T)] .$$

We know, by Theorem 3.5 that $f_i(M)$, $f_j(M)$ are aligned with M^*. If T contains any coloops of say $f_i(M)$, $T \subseteq B[f_i(M)]$ since $f_i(M)$ is aligned with M^*. In this case it is trivially true that

$$(f(M)) \cdot T = [(f_i(M)) \cdot T] \vee [(f_j(M)) \cdot T]$$
$$= [f_i(M \times T)] \vee [f_j(M \times T)]$$
$$= f(M \times T) .$$

If T contains no coloops of $f_i(M)$ or $f_j(M)$, by Corollary 1 of Theorem 3.3

$$(f(M)) \cdot T = [(f_i(M)) \cdot T] \vee [(f_j(M)) \cdot T]$$
$$= [f_i(M \times T)] \vee [f_j(M \times T)]$$
$$= f(M \times T) .$$

Next let f be a positive function with $f = (f_i)^*$ where f_i is a negative function which can be expressed as $f_i = f_j \vee f_k$, where f_j, f_k are negative functions of order less than the order of f_i. By the above proof the theorem holds for $f_i(M)$.

$$f(M) \,.\, (S-T) = [(f_i(M)) \times (S-T)]^* \quad \text{by Theorem T7.}$$

$$= [f_i(M \,.\, (S-T))]^* = f(M \,.\, (S-T))$$

$$f(M) \times T = [f_i(M) \,.\, T]^* = [f_i(M \times T)]^*$$

$$= f(M \times T).$$

The proof for the case where f is a negative function such that $f = (f_i)^*$ where f_i is a positive function is similar to the above.

Q.E.D.

The following Corollary is obtained by repeated application of Theorem 3.6.

<u>Corollary 1.</u> Let a matroid M on S have the P-sequence $P_1, P_2 \ldots P_n$ with $\bigcup_{i=1}^{k} P_i = T$.

If f is a positive function

$$f[M \,.\, (S-T)] = (f(M)) \,.\, (S-T)$$

$$f[M \times T] = (f(M)) \times T$$

If f is a negative function

$$f[M \,.\, (S-T)] = (f(M)) \times (S-T)$$

$$f[M \times T] = (f(M)) \,.\, T.$$

Theorems 3.5, 3.6 and their corollaries are sufficient for the complete determination of the P-sequence of $f(M)$ along with the ranks of the various molecular matroids (f being admissible) in terms of the P-sequence of M, in a very simple manner. (See example 2.3.3 at the end of this section).

For convenience henceforth we will denote $M \vee M \vee \ldots \vee M$ (k times) by M^k and $f \vee f \vee \ldots \vee f$ (k times) by f^k.

We now show that it is possible to obtain the P-sequence of a matroid by the use of its admissible functions.

<u>Definition 3.6.</u> Let $P_1, P_2 \ldots P_n$ be the P-sequence of a matroid M on S. Let $S = S_1 \cup S_2$ such that $S_1 = \bigcup_{i=1}^{k} P_i$, $S_2 = \bigcup_{i=k+1}^{n} P_i$.

We say an admissible function f distinguishes between S_1 and S_2 iff

(a) $S_2 = B[f(M)]$ or

(b) $S_1 = B[f(M)]$.

By Theorem 3.5, Case (a) corresponds to a positive function while Case (b) corresponds to a negative function. The next theorem describes functions which distinguish between any S_1 and S_2 when S_1 and S_2 are defined as above.

<u>Theorem 3.7.</u> Let M be a matroid on S. Let $P_1, P_2 \ldots P_n$ be the P-sequence of M.

Let $S_1 = \bigcup_{i=1}^{k} P_i$, $S_2 = \bigcup_{i=k+1}^{n} P_i$, $k < n$.

Then there exists an admissible function f which distinguishes between S_1 and S_2 .

<u>Proof.</u> Let us denote

$$M \times \left(\bigcup_{i=1}^{k} P_i \right) \cdot P_k \text{ by } M_1$$

$$M \times \left(\bigcup_{i=1}^{k+1} P_i \right) \cdot P_{k+1} \text{ by } M_2 .$$

By the definition of P-sequences we know M_1 and M_2 to be molecular, and $d(M_1) > d(M_2)$.

We will now consider three cases.

<u>Case 1.</u> $d(M_1) = m$

$d(M_2) < m$ where m is a natural number.

Consider the admissible function f such that

$$f = [[(f_o)^{m-1}]^*]^m$$

$$f [M] = [(M^{m-1})^*]^m$$

Let $r(M_1) = a$. Since M_1 is molecular, M_1^{m-1} is molecular by Corollary 2 of Theorem 3.1. Also since $d(M_1) = m$ clearly M_1^{m-1} is not M_u on P_k . Hence by Corollary 4 of Theorem 3.1,

$$r(M_1^{m-1}) = (m-1) a$$

$$r[[M_1^{m-1}]^*] = ma - (m-1)a = a .$$

Again by the use of an argument similar to the above,

$$r[(M_1^{m-1})^*]^m = ma .$$

Hence $d[f(M_1)] = 1$.

Let $r(M_2) = d$, and let $e = |P_{k+1}| - (m-1)d$, if this value is positive, and $e = 0$ if this value is negative or zero. Then using an argument similar to that used for M_1 we have,

$$r(M_2^{m-1}) = \min[(m-1)d, |P_{k+1}|] ,$$

$$r(M_2^{m-1})^* = e ; \quad r[[(M_2^{m-1})^*]^m] = me < |P_{k+1}|$$

since $e < d$ because $d(M_2) < m$. Hence $f(M_2)$ is not the matroid M_u on P_{k+1}. Since $f(M)$ is clearly negative and therefore aligned with M^*,

$$B[f(M)] = S_1 .$$

We may note here that another function that distinguishes between S_1 and S_2 in the above case is the function g such that

$$g(M) = (M^{m-1})^* \vee M^* .$$

Since M_1 is molecular $g(M_1)$ is molecular. If $g(M_1) \neq M_u$ on P_k,

$$r[g(M_1)] = r[(M_1)^{m-1}]^* + r(M_1^*) .$$

But it is easyto see that $r[(M_1^{m-1})^*] = r(M_1)$ since M_1 is molecular and $d(M) = m$.

Hence $r [g(M_1)] = |P_k|$.

We thus arrive at a contradiction. Hence we conclude that

$$g(M_1) = M_u \text{ on } P_k .$$

Again, since M_2 is molecular,

$g(M_2)$ is molecular.

Suppose $g(M_2) = M_u$ on P_{k+1}.

Then $|P_{k+1}| = r(g(M_2)) \leq r((M_2^{m-1})^*) + r(M_2^*)$

But since $d(M_2) < m$, $r((M_2^{m-1})^*) < r(M)$

Hence $r(g(M_2)) < r(M) + r(M^*)$.

Hence $g(M_2) \neq M_u$ on P_{k+1}.

We thus arrive at a contradiction. Hence we conclude that

$$B [g(M)] = S_1 .$$

Case 2. $d(M_1) > m$

$d(M_2) \leq m$.

Consider the admissible function f such that $f = (f_o)^m$ that is, $f(M) = M^m$. Let $r(M_1) = a$. By Corollary 4 of Theorem 3.1, we have $r(M_1^m) = m\,a < |P_k|$.

This means that M_1^m has no coloops in P_k.

However for M_2 if $r(M_2) = d$, clearly $md \geq |P_{k+1}|$.

Hence $r(M_2^m) = |P_{k+1}|$ by Corollary 4 of Theorem 3.1.

Hence $M_2^m = M_u$ on P_{k+1}.

Now f is positive and therefore f(M) is aligned with M.

Hence $B(M^m) = S_2$.

<u>Case 2.</u> $d(M_1) = m + (\alpha_1 / a)$, $d(M_2) = m + (\beta_1 / d)$

Let $|P_k| = e_1$, $|P_{k+1}| = e_2$, $\frac{\alpha_1}{a}$, $\frac{\beta_1}{d}$ being proper fractions, m being an integer.

$r(M_1) = a$, $r(M_2) = d$.

Clearly $\frac{\alpha_1}{a} > \frac{\beta_1}{d}$.

We now proceed to write e_1 and e_2 as follows :

(1) $e_1 = ma + \alpha_1$	(1) $e_2 = md + \beta_1$
(2) $e_1 = k_1\alpha_1 + \alpha_2$	(2) $e_2 = p_1\beta_1 + \beta_2$
⋮	⋮
(i) $e_1 = k_{i-1}\alpha_{i-1} + \alpha_i$	(i) $e_2 = p_{i-1}\beta_{i-1} + \beta_i$
⋮	⋮
(q) $e_1 = k_{q-1}\alpha_{q-1}$	(s) $e_2 = p_{s-1}\beta_{s-1}$.

We note that $\alpha_1, \alpha_2 \ldots \alpha_{q-1}$ forms a decreasing sequence of positive integers and so does $\beta_1, \beta_2 \ldots \beta_{s-1}$.

Let j be the least number such that $k_{j-1} \neq p_{j-1}$.

We now define a sequence of admissible functions $f_1, f_2 \ldots f_j$ recursively as follows :

$$f_1 = [(f_o)^m]^*$$
$$f_2 = [(f_1)^{k_1}]^*$$
$$\vdots$$
$$f_j = [(f_{j-1})^{\delta}]^*$$

where $\delta = \max [k_{j-1} , p_{j-1}]$.

We note that f_i is a positive function if i is even,

f_i is a negative function if i is odd.

Since M_1, M_2 are molecular, it is easy to see that, $f_i(M_1)$, $f_i(M_2)$ are molecular and

$$r [f_i(M_1)] = \alpha_i , \quad r [f_i (M_2)] = \beta_i .$$

Suppose $k_{j-1} > p_{j-1}$.

We have $(k_{j-1}) (\alpha_{j-1}) < e_1$

$$(k_{j-1}) (\beta_{j-1}) > e_2 .$$

Since $f_{j-1}(M_1)$, $f_{j-1}(M_2)$ are molecular, it follows from Corollary 4 of Theorem 3.1, that $f_j(M_2) = M_u$ on P_{k+1}, while $f_j(M_1)$ has no coloops.

Since f_j is an admissible function, $f_j(M)$ is aligned with M or M^*.

Hence it follows that $B [f_j(M)] = S_2$.

(We note that this means that f_j is positive and hence j is even).

Similarly if $p_{j-1} > k_{j-1}$, one can show that $B [f_j(M)] = S_1$.

(In this case f_j will be a negative function and hence j will be odd).

Thus in all three cases we have admissible functions which distinguish between S_1 and S_2.

Q.E.D.

It is worth noting that when $d\left(M \times \bigcup_{i=1}^{k} P_i \cdot P_k \right) < 2$.

Since by Theorem T7,

$$\left[M \times \bigcup_{i=1}^{k} P_i \cdot P_k \right]^* = \left[M^* \cdot \bigcup_{i=1}^{k} P_i \times P_k \right]$$

$$= \left[M^* \times \bigcup_{i=k}^{n} P_i \cdot P_k \right],$$

$$d\left[M^* \times \bigcup_{i=k}^{n} P_i \cdot P_k \right] > 2 .$$

In some of the above cases we might be able to use Case 1 and Case 2 of the above theorem using admissible functions of M^* in place of admissible functions of M. We can now construct a sequence coarser than the P-sequence using only Case 1 and Case 2 as follows :

TABLE 1

Let k be the minimum number such that

$$M^k = M_u \text{ on } S - C(M),$$

m be the minimum number such that

$$(M^*)^m = M_u \text{ on } S - C(M^*) .$$

Then we define

$$G_{k,1} = B\,[((M^{k-1})^{*})^{k}] \qquad k > 2$$

$$G_{k,2} = A\,[M^{k-1}] - B\,[((M^{k-1})^{*})^{k}]$$

$$\vdots$$

$$G_{i,1} = B\,[((M^{i-1})^{*})^{i}] - [A\,[M^{i}] \cup C\,[M^{i}]]$$

$$G_{i,2} = A\,[M^{i-1}] - B\,[((M^{i-1})^{*})^{i}]. \qquad (i > 2).$$

$$G_{3,2} = A\,[M^{2}] - B\,[((M^{2})^{*})^{3}]$$

$$G_{2,1} = B\,[(M^{*})^{2}] - [A\,[M^{2}] \cup C\,[M^{2}]]$$

$$= B\,[M^{2}] - [A\,[M^{*}]^{2} \cup C\,[M^{*}]^{2}] = g_{2,1}$$

$$g_{3,2} = A\,[(M^{*})^{2}] - B\,[(((M^{*})^{2})^{*})^{3}]$$

$$\vdots$$

$$g_{j,2} = A\,[(M^{*})^{j-1}] - B\,[(((M^{*})^{j-1})^{*})^{j}] \qquad j > 2$$

$$g_{j,1} = B\,[(((M^{*})^{j-1})^{*})^{j}] - [A\,[(M^{*})^{j}] \cup C\,[(M^{*})^{j}]] \qquad j > 2$$

$$\vdots$$

$$g_{m,2} = A\,[(M^{*})^{m-1}] - B\,[(((M^{*})^{m-1})^{*})^{m}]$$

$$g_{m,1} = B\,[(((M^{*})^{m-1})^{*})^{m}].$$

Deleting the void sets from the sequence

$C(M)$, $G_{k,1}$, $G_{k,2}$ ...... $g_{m,1}$, $C(M^{*})$, we obtain the sequence

$Q_1, Q_2 \dots Q_p$.

<u>Definition 3.7.</u> The above sequence we shall call the Q-sequence of M. (We may note here that the partition of S into

$$A(M^2) \ , \ B(M^*)^2 - A\ (M^2) \ , \ A \ [\ (M^*)^2 \ ]$$

corresponds to Kishi-Kajitani's Principal partition of a graph).

We state some of the obvious properties of Q-sequences in the form of a theorem which follows from Theorem 3.7.

<u>Theorem 3.8.</u> Let M be a matroid on S. Let $P_1, P_2, \dots P_n$ be the P-sequence and $Q_1, Q_2 \dots Q_p$ be the Q sequence of M.

(1) If $P_s \cap Q_i \neq \phi$, then $P_s \subseteq Q_i$ $s \leq n$, $i \leq p$.

(2) If $P_s \subseteq Q_i$, $P_r \subseteq Q_j$ and $j > i$, then $r > s$.

(3) If $d\ (\ M \times \bigcup_{i=1}^{r} Q_i \cdot Q_r\ ) = k$

or

if $d\ (\ M^* \times \bigcup_{i=r}^{p} Q_i \cdot Q_r\ ) = k$

where k is any positive integer, then there exists a set. P_s in the P-sequence of M such that $Q_r = P_s$.

Assuming one has obtained the Q-sequence of M , condition 3 of the above theorem enables us to identify some of those members of the Q-sequence which are also members of the P-sequence. It is possible to obtain the entire P-sequence by partitioning the remaining members of the Q-sequence.

We now state the important properties of $A(M^r) \cup C(M^r)$ and $B \lfloor ((M^{r-1})^*)^r \rfloor$ in the form of a theorem.

<u>Theorem 3.9.</u> Let M be a matroid on S. Let $b_1, b_2 \ldots b_r$ be r bases of M such that $\bigcup_{i=1}^{r} b_i$ is a base of M^r.

Let $T \subseteq S$ be such that

(1) $b_i \cap T$ is a base of $M \times T$

(2) $b_i \cap b_j \cap T = \phi$ if $i \neq j$ ($i,j \in \{1,2 \ldots r\}$.)

(3) $T \supseteq S - \bigcup_{i=1}^{r} b_i$.

Then $T \supseteq A(M^r) \cup C(M^r)$

and $T \subseteq B \lfloor ((M^{r-1})^*)^r \rfloor$.

<u>Proof.</u> $T \supseteq S - \bigcup_{i=1}^{r} b_i$.

Also T contains the set of all elements accessible from any element of $S - \bigcup_{i=1}^{r} b_i$ with respect to $b_1, b_2 \ldots b_r$. Hence by Theorem 1.2, $T \supseteq A(M^r) \cup C(M^r)$.

We know that $B \lfloor ((M^{r-1})^*)^r \rfloor = \bigcup_{i=1}^{k} P_i$,

where $P_1, P_2 \ldots P_n$ is the P-sequence of M, and k is the largest number such that

$$d (M \times \bigcup_{i=1}^{k} P_i \cdot P_k) \geq r .$$

Let us suppose $T - \bigcup_{i=1}^{k} P_i \neq \phi$.

Consider the matroid $M \times [T \cup (\bigcup_{i=1}^{k} P_i)]$. $b_i \cap T$ is independent in this matroid by Theorem T2. Build each $b_i \cap T$ into a base of $b_{i,1}$ of $M \times [T \cup (\bigcup_{i=1}^{k} P_i)]$. Then by Theorem T3, $(b_{i,1} \cap [T - \bigcup_{i=1}^{k} P_i])$ contains a base of $M \times [T \cup (\bigcup_{i=1}^{k} P_i)] \cdot (T - \bigcup_{i=1}^{k} P_i)$.

Hence it follows that

$$d[M \times (T \cup [\bigcup_{i=1}^{k} P_i]) \cdot (T - \bigcup_{i=1}^{k} P_i)] \geq r .$$

This clearly contradicts Theorem 2.6, since

$$M \times [T \cup (\bigcup_{i=1}^{k} P_i)] \cdot (T - \bigcup_{i=1}^{k} P_i)$$

$$= M \cdot (\bigcup_{i=k+1}^{n} P_i) \times (T - \bigcup_{i=1}^{k} P_i)$$ by Theorem T 7-3.

Q.E.D.

We now give algorithms for determination of $A(M^r) \cup C(M^r)$ and $B[((M^{r-1})^*)^r]$.

Algorithm for $A(M^r) \cup C(M^r)$.

<u>Algorithm 3.1.</u> Let M be a matroid on S. By algorithm 1.1 we can construct a set of bases $b_1, b_2 \dots b_r$ of M such that $\bigcup_{i=1}^{r} b_i$ is a base of M^r. Let $D = S - \bigcup_{i=1}^{r} b_i$. Let Q_a be the set of all elements accessible from $a \in D$ with respect to $b_1, b_2 \dots b_r$. (If $a \in D$, we take a to be accessible from itself).

Form $\bigcup_{a \in D} Q_a$. Then $A(M^r) \cup C(M^r) = \bigcup_{a \in D} Q_a$.

The above algorithm follows clearly from Theorem 1.2.

Algorithm for $B[((M^{r-1})^*)^r]$.

<u>Algorithm 3.2.</u> Let M be a matroid on S. Let $b_1, b_2 \ldots b_r$ be a set of r bases of M such that

$$\bigcup_{i=1}^{r} b_i = b \text{ is a base of } M^r .$$

Let $S_1 = A(M^r) \cup C(M^r)$. Let $Q = \{e \mid e \in S$ and e belongs to at least two of the bases $b_1, b_2 \ldots b_r\}$.

If $b_1 - (S_1 \cup Q) = \Phi$, $S_1 = B[((M^{r-1})^*)^r]$.

Otherwise pick some $e \in b_1 - (S_1 \cup Q)$. The element e forms a unique circuit in all but one of the bases $b_1, b_2 \ldots b_r$. Let A_1^e be the union of all such circuits formed by e with the bases $b_1, b_2 \ldots b_r$. Each member of A_1^e is not a member of certain of the bases $b_1, b_2 \ldots b_r$ and accordingly forms a unique circuit in each of these bases. Let A_2^e denote the union of all such circuits formed by each of the members of A_1^e with the bases $b_1, b_2 \ldots b_r$. In a similar manner A_{n+1}^e is obtained from A_n^e for $n = 2, 3 \ldots$.

There are two cases.

<u>Case 1.</u> There is a least positive integer p such that

$$A_p^e \cap Q \neq \Phi, \quad \text{or}$$

Case 2. There is a least positive integer s_e such that

$A^e_{s_e} = A^e_{s_e+1}$ and $A^e_{s_e} \cap Q = \Phi$.

Let $F = \bigcup_{e \in D} A^e_{s_e}$ where

(1) $D \subseteq b_1 - (S_1 \cup Q)$

(2) $A^e_{s_e} = A^e_{s_e+1}$ and $A^e_{s_e} \cap Q = \Phi$ iff $e \in D$

Then $F \cup S_1 = B \angle^- ((M^{r-1})^*)^r \rfloor$.

In order to justify the above algorithm we note firstly that $F \cup S_1 \supseteq A (M^r) \cup C (M^r)$. Secondly by the method of construction of $F \cup S_1$ it is clear that $M \times (F \cup S_1)$ has $b_i \cap (F \cup S_1)$ $(i = 1,2, \ldots r)$ as a base.

Also $b_i \cap b_j \cap (F \cup S_1) = \Phi$, $i \neq j$, $i,j \in \{1,2 \ldots r\}$. Hence by Theorem 3.9, $F \cup S_1 \subseteq B \angle^- ((M^{r-1})^*)^r \rfloor$.

To show that $B \angle^- ((M^{r-1})^*)^r \rfloor \subseteq F \cup S_1$ we need merely prove the following lemma.

Lemma 3.2. Let M be a matroid on S. Let $b_1, b_r \ldots b_r$ be a set of bases of M such that $\bigcup_{i=1}^{r} b_i = b$ is a base of M^r. Let $S_2 = B \angle^-((M^{r-1})^*)^r \rfloor$.

Then $b_i \cap S_2$ is a base of $M \times S_2$ and

$b_i \cap b_j \cap S_2 = \Phi$, $i \neq j$, $i,j \in \{1,2 \ldots r\}$.

Proof. Let $P_1, P_2 \ldots P_n$ be the P-sequence of M. Then we know from Theorem 3.7 that $A(M^r) \cup C(M^r) = \bigcup_{i=1}^{k} P_i$ where k is the largest number such that $d\,(\,M \times \bigcup_{i=1}^{k} P_i \cdot P_k\,) > r$. If $d\,(\,M \times \bigcup_{i=1}^{k+1} P_i \cdot P_{k+1}\,) < r$, we can see from the proof of Theorem 3.7 that $A(M^r) \cup C\,(M^r) = B\,[\,((M^{r-1})^{*})^{r}\,]$. Hence the Lemma holds for this case.

Suppose $d\,(\,M \times \bigcup_{i=1}^{k+1} P_i \cdot P_{k+1}\,) = r$. Firstly $P_{k+1} \subseteq B(M^r)$. Hence $P_{k+1} \subseteq \bigcup_{i=1}^{r} b_i$. Now $b_i \cap S_2$ is independent in $M \times S_2$ ($i = 1,2 \ldots r$). But by Theorem 1.2 $b_i \cap (\bigcup_{i=1}^{k} P_i)$ ($i \in \{1,2 \ldots r\}$) is a base for $M \times (\bigcup_{i=1}^{k} b_i)$. Hence by application of Theorem T2 we find that $b_i \cap P_{k+1}$ is independent in $(\,M \times \bigcup_{i=1}^{k+1} P_i \cdot P_{k+1}\,)$. But $d(\,M \times \bigcup_{i=1}^{k+1} P_i \cdot P_{k+1}\,) = r$. Hence $b_i \cap P_{k+1}$ for ($i \in \{1,2 \ldots r\}$) is a base for $M \times \bigcup_{i=1}^{k+1} P_i \cdot P_{k+1}$ and $b_1 \cap P_{k+1}, b_2 \cap P_{k+1}, \ldots b_r \cap P_{k+1}$ are pairwise disjoint.

Now by using Theorem T3 we find that

$b_i \cap S_2$ (for $i \in \{1,2 \ldots r\}$) is a base for $M \times S_2$. Also we know that $b_i \cap b_j \cap [\,A(M^r) \cup C(M^r)\,] = \Phi$, $i \neq j$, $i, j \in \{1,2 \ldots r\}$. Hence $b_i \cap b_j \cap S_2 = \Phi$, for $i \neq j$, $i,j \in \{1,2 \ldots r\}$.

Q.E.D.

The sets $A(M^r) \cup C(M^r)$ and $B\,[((M^{r-1})^*)^r]$ are called Principal - r - minor and Augmented - principal - r - minor respectively by Bruno and Weinberg [Br 2]. Algorithms 3.1 and 3.2 are due essentially to them. However, their description of these sets is not in terms of the matroid union theorem.

We now need to partition the Q-sequence further to obtain the P-sequence.

<u>Definition 3.8.</u> Let M be a matroid on S, with

$Q_1, Q_2 \ldots Q_k$ as its Q-sequence.

$$\left.\begin{array}{l} \text{Let } M_r = (M \times \bigcup_{i=1}^{r} Q_i \cdot Q_r) \\ \quad \text{if } d (M \times \bigcup_{i=1}^{r} Q_i \cdot Q_r) > 2 \end{array}\right\} \text{(condition 1)}$$

and

$$\left.\begin{array}{l} \quad = (M^* \times \bigcup_{i=r}^{p} Q_i \cdot Q_r) \\ \quad \text{if } d (M^* \times \bigcup_{i=r}^{p} Q_i \cdot Q_r) > 2 \end{array}\right\} \text{(condition 2)}$$

Let $P_1^r, P_2^r \ldots P_{k_r}^r$ be the P-sequence of M_r if M_r satisfies condition 1 and let $P_{k_r}^r, \ldots P_2^r, P_1^r$ be the sequence of M_r if M_r satisfies condition 2.

Then it follows from Corollary 1 of Theorem 2.6 that

$P_1^1, \ldots P_{k_1}^1, P_1^2, \ldots P_{k_2}^2 \ldots P_1^p, \ldots P_{k_p}^p$ is the P-sequence of M.

We also note that if d (M_r) is an integer, then by Theorem 3.8, $k_r = 1$.

The next theorem helps us in obtaining the P-sequence of the matroid M_r when $d(M_r)$ is not integral.

<u>Theorem 3.10.</u> Let M be a matroid on S such that d(M) is not an integer. Let $r(M) = \alpha$. Let M_α be the parallel-α-copy of M on S_α. If M is not molecular, $B(M_\alpha^{|S|}) \neq S_\alpha$.

<u>Proof.</u> $d(M_\alpha) = \alpha \cdot d(M) = \dfrac{\alpha \cdot |S|}{\alpha} = |S|$.

If $B(M_\alpha^{|S|}) = S_\alpha$, it follows that there exist $|S|$ p.d. bases $b_1, b_2 \ldots, b_{|S|}$ for M_α such that their union is S_α. Let $T \subseteq S_\alpha$. Then each $b_i \cap T$ ($i = 1,2 \ldots |S|$) is either void or independent in $M_\alpha \times T$ by Theorem T2.

Hence we require atmost $|S|$ p.d. independent sets of $M_\alpha \times T$ to cover T.

Hence $d(M_\alpha \times T) \leq |S|$. Hence M_α is molecular. Hence M is molecular by Corollary 1 of Theorem 2.4. The result follows by contraposition.

Q.E.D.

Let $P_1, P_2 \ldots P_n$ be the P-sequence of M. Then $\Upsilon(P_1), \Upsilon(P_2) \ldots$ $\ldots \Upsilon(P_n)$ is the P-sequence of M_α (Υ as defined in Definition 2.8). Since $M_\alpha^{|S|}$ is aligned with M_α, it is clear that for some k, $B(M_\alpha^{|S|}) = \bigcup_{i=k}^{n} \Upsilon(P_i)$.

Let $S_1 = \bigcup_{i=1}^{k-1} P_i$ and $S_2 = S - S_1$. Then by Corollary 1 of Theorem 2.6, $M \times S_1$ has the P-sequence $P_1, P_2 \ldots P_{k-1}$ and $M \cdot S_2$ has the P-sequence $P_k , \ldots P_n$. By repeated applications of Theorem T3 it is easy to see that $d (M \times S_1) > d (M \cdot S_2)$. We note that S_1 and S_2 can clearly be obtained by using Algorithm 3.1 on M_α . We now construct parallel - α - copies of $M \times S_1$ and $M \cdot S_2$ with α respectively equal to $r(M \times S_1)$ and $r (M \cdot S_2)$. Clearly this procedure when continued in this manner will ultimately lead to the molecular matroids

$$M \times R_1, \ldots M \times \bigcup_{i=1}^{j} R_i \cdot R_j \ldots M \cdot R_m \text{ with}$$

$$d(M \times \bigcup_{i=1}^{j} R_i \cdot R_j) > d (M \times \bigcup_{i=1}^{r} R_i \cdot R_r) \text{ for } (r > j) .$$

Hence by Theorem 2.5 it follows that $m = n$ and

$$P_1 = R_1 , \; P_2 = R_2 \ldots \; P_n = R_n \; .$$

Thus by repeated use of Algorithm 3.1 we are able to obtain the P-sequence of M.

We could use the above procedure to obtain the P-sequence of M_r in definition 3.8. As noted before this completes the procedure for obtaining the P-sequence of the matroid M.

We outline this procedure briefly below.

Procedure for obtaining the P-sequence of a matroid M :

(1) Use algorithms 3.1 and 3.2 to obtain the Q-sequence of M.

(2) Let M_r be defined as in Definition 3.8. Proceed to find the P-sequence of M_r as described by Theorem 3.10 and the discussion following it.

(3) Build up the P-sequence of M through the P-sequences of M_r as described immediately after Definition 3.8.

We may note here that while the algorithms used for determining the Q-sequence of M are very efficient, the procedure described by Theorem 3.10 and subsequent to it are not quite so efficient. The loss in efficiency is mainly due to the fact that we are forced to work with parallel-α-copies instead of the original matroid. The procedure should, however, be still classified as a 'good' algorithm by Edmond's definition. [Let S be the set of definition of the matroid. Then, we say an algorithm is 'good' iff the time required to compute the algorithm has a finite polynomial on $|S|$ as an upper bound]. Algorithms 3.1 and 3.2 as well as our procedure assume that there is a simple method of detecting the fundamental circuit formed when an element from the cobase is added to the base. For graphs this is indeed very simple.

To illustrate our procedure for construction of the P-sequence of a matroid we give the following example :

<u>Example 2.3.2.</u> Let M be a matroid on S. Let $P_1, P_2, \ldots P_9$ be the P-sequence of M. Further let the following hold :

$d(M \times P_1) = 4$ $\qquad$ $r(M \times P_1) = 1,\ |P_1| = 4$

$d(M \times \bigcup_{i=1}^{2} P_i \cdot P_2) = 3\frac{3}{4}$ $\qquad$ $r(M \times (P_1 \cup P_2) \cdot P_2) = 4$

$|P_2| = 15$

$d(M \times \bigcup_{i=1}^{3} P_i \cdot P_3) = 3\frac{2}{3}$ $\qquad$ $r(M \times \bigcup_{i=1}^{3} P_i \cdot P_3) = 3$

$|P_3| = 11$

$d(M \times \bigcup_{i=1}^{4} P_i \cdot P_4) = 2\frac{3}{5}$ $\qquad$ $r(M \times \bigcup_{i=1}^{4} P_i \cdot P_4) = 5$

$|P_4| = 13$

$d(M \times \bigcup_{i=1}^{5} P_i \cdot P_5) = 2$ $\qquad$ $r(M \times \bigcup_{i=1}^{5} P_i \cdot P_5) = 3$

$|P_5| = 6$

$d(M \times \bigcup_{i=1}^{6} P_i \cdot P_6) = 1\frac{3}{4}$

$d(M^* \times \bigcup_{i=6}^{9} P_i \cdot P_6) = 2\frac{1}{3}$ $\qquad$ $r(M^* \times \bigcup_{i=6}^{9} P_i \cdot P_6) = 3$

$|P_6| = 7$

$d(M^* \times \bigcup_{i=7}^{9} P_i \cdot P_7) = 2\frac{2}{3}$ $\qquad$ $r(M^* \times \bigcup_{i=7}^{9} P_i \cdot P_7) = 3$

$|P_7| = 8$

$d(M^* \times \bigcup_{i=8}^{9} P_i \cdot P_8) = 2\frac{3}{4}$ $\qquad$ $r(M^* \times \bigcup_{i=8}^{9} P_i \cdot P_8) = 4$

$|P_8| = 11$

$d(M^* \times P_9) = 3$ $\qquad$ $r(M^* \times P_9) = 3,\ |P_9| = 9.$

We will describe the procedure for obtaining the above P-sequence.

(1) For this matroid $C(M) = C(M^*) = \Phi$.

We start by finding $A(M^2)$, $A(M^3)$ using Algorithm 3.1.

We find that $A(M^4) = \Phi$ so we do not proceed anymore.

We now have
$$A(M^3) = \bigcup_{i=1}^{3} P_i$$
$$A(M^2) = \bigcup_{i=1}^{4} P_i$$

Next we use Algorithm 3.1 on M^* and find $A[(M^*)^2]$. We find $A[(M^*)^3] = \Phi$. We thus have $A(M^*)^2 = \bigcup_{i=6}^{9} P_i$.

Next we use Algorithm 3.2 and find

$B[(M^*)^2]$, $B[((M^2)^*)^3]$, $B[((M^3)^*)^4]$

We know that $B[((M^4)^*)^5] \subseteq A(M^4) = \Phi$.

Using the same Algorithm on M^* we find

$B(M^2)$, $B[(((M^*)^2)^*)^3]$; we know that

$B[(((M^*)^3)^*)^4] \subseteq A[(M^*)^3] = \Phi$.

We thus have

$B[(M^*)^2] = \bigcup_{i=1}^{5} P_i$; $B[((M^2)^*)^3] = \bigcup_{i=1}^{3} P_i$;

$B[((M^3)^*)^4] = P_1$; $B(M^2) = \bigcup_{i=6}^{9} P_i$;

$B \angle^{-} (((M^{*})^{2})^{*})^{3} \rfloor = P_9$.

We thus have the Q-sequence

$Q_1 = P_1$, $Q_2 = P_2 \cup P_3$, $Q_3 = P_4$, $Q_4 = P_5$, $Q_5 = P_6 \cup P_7 \cup P_8$,
$Q_6 = P_9$.

We start by constructing parallel-α-copies of the matroids

$M_1 = M \times Q_1$, $M \times Q_1 \cup Q_2 \cdot Q_2 = M_2$,

$M \times \bigcup_{i=1}^{3} Q_i \cdot Q_3 = M_3$, $M \times \bigcup_{i=1}^{4} Q_i \cdot Q_4 = M_4$,

$M^{*} \times \bigcup_{i=5}^{6} Q_i \cdot Q_5 = M_5$, $M^{*} \times Q_6 = M_6$, taking α equal to their respective ranks i.e.

$\alpha = 1, 7, 5, 3, 10, 3.$

We find that

$((M_1))^{4} = M_u$ on Q_1 , $((M_3)_{\alpha=5})^{13} = M_u$ on $(Q_3)_5$,

$((M_4)_{\alpha=3})^{6} = M_u$ on $(Q_4)_3$, $((M_6)_{\alpha=3})^{9} = M_u$ on $(Q_6)_3$.

We, therefore, conclude that M_1, M_3, M_4, M_6 are molecular and turn our attention to M_2 and M_5.

Since $d \angle^{-} (M_2)_{\alpha=7} \times \mathcal{T}_2 (P_2) \rfloor = \frac{15 \times 7}{4} = 26\frac{1}{4}$ and

$$d \angle^{-} (M_2)_{\alpha=7} \times \mathcal{T}_2 (P_2 \cup P_3) \cdot \mathcal{T}_2(P_3) \rfloor = \frac{11 \times 7}{3} = 25\frac{2}{3}$$

(Υ_2 having the usual meaning Υ has for a parallel-α-copy).

Hence $B \angle^{-} (\angle^{-} M_2 \rfloor_{\alpha=7})^{26} \rfloor = \Upsilon_2 (P_3) \subset Q_2$

($|Q_2|$ being 26).

We next similarly check $M_2 \times P_2$ and $M_2 \cdot P_3$ for molecularity and find them to be so. Hence the P-sequence of M_2 can be seen to be P_2, P_3.

Now we go through the same procedure with M_5.

$r(M_5) = 10 \quad , \quad |Q_5| = 26$

(We will use Υ_5 for the Υ function in considering parallel-10-copy of M_5).

We have

$$d \angle^{-} (M_5)_{\alpha=10} \times \Upsilon_5 (P_8) \rfloor = \frac{11}{4} \times 10 = 27\frac{1}{2}$$

$$d \angle^{-} (M_5)_{\alpha=10} \times \Upsilon_5 (P_7 \cup P_8) \cdot \Upsilon_5 (P_7) \rfloor = \frac{8}{3} \times 10 = 26\frac{2}{3}$$

$$d \angle^{-} (M_5)_{\alpha=10} \cdot \Upsilon_5 (P_6) \rfloor = \frac{7}{3} \times 10 = 23\frac{1}{3} \ .$$

We obtain

$$B \angle^{-} ((M_5)_{\alpha=10})^{26} \rfloor = P_6 \ .$$

We check for molecularity of $M_5 \cdot P_6$ and find it to be so. We next turn our attention to

$$M_{10} = M_5 \times P_7 \cup P_8 \cdot r(M_{10}) = 7,$$

$$|P_7 \cup P_8| = 19.$$

We form $(M_{10})_{\alpha=7}$

$$d \angle^{-} (M_{10})_{\alpha=7} \times \mathcal{T}_{10}(P_8) \rfloor = \frac{11}{4} \times 7 = \frac{77}{4} = 19\frac{1}{4}$$

$$d \angle^{-} (M_{10})_{\alpha=7} \cdot \mathcal{T}_{10}(P_7) \rfloor = \frac{8}{3} \times 7 = \frac{56}{3} = 18\tfrac{2}{3} .$$

Hence

$$B \angle^{-} ((M_{10})_{\alpha=7})^{19} \rfloor = \mathcal{T}_{10}(P_7)$$

The usual molecularity checks for $M_{10} \times P_8$ and $M_{10} \cdot P_7$ indicates that they are molecular.

P-sequence of M_{10} therefore is P_8, P_7, P_6 .

Hence we get the P-sequence of M as

$$P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9 .$$

Example 2.3.3 Let M be a matroid on S.

Let the P-sequence of M be P_1, P_2, P_3, P_4, P_5.

Let $M \times P_1 = M_1$, $M \times P_1 \cup P_2 \cdot P_2 = M_2$,

$M \times \bigcup_{i=1}^{3} P_i \cdot P_3 = M_3$, $M \times \bigcup_{i=1}^{4} P_i \cdot P_4 = M_4$,

$M \cdot P_5 = M_5$.

Let $r(M_1) = 2$,	$\|P_1\| = 8$	$\therefore d(M_1) = 4$
$r(M_2) = 3$,	$\|P_2\| = 9$	$\therefore d(M_2) = 3$
$r(M_3) = 5$,	$\|P_3\| = 13$	$\therefore d(M_3) = 13/5$
$r(M_4) = 6$,	$\|P_4\| = 12$	$\therefore d(M_4) = 2$
$r(M_5) = 5$,	$\|P_5\| = 9$	$\therefore d(M_5) = 9/5$

We would like to study the effect of the negative function $f = (f_o \vee f_o)^* \vee f_o^*$ on M. Let $M_6 = f(M) = (M \vee M)^* \vee M^*$. We note firstly that $f(M_1)$, $f(M_2)$, $f(M_3)$, $f(M_4)$, $f(M_5)$ are molecular.

If $f[M_1] \neq M_u$ on P_1

$$r[f(M_1)] = r(M_1 \vee M_1)^* + r(M_1^*)$$

$$= [\,|P_1| - 2 \cdot r(M_1)\,] + r(M_1^*)$$

$$= [\,8 - 4\,] + 6 = 10$$ which is impossible.

Hence $f(M_1) = M_u$. Similarly $f(M_2) = M_u$.

Since $f(M_3)$ is molecular, if $M_3 = M_u$ on P_3 our calculation of $r[f(M_3)]$ must yield a value greater than or equal to $|P_3|$.

But $r[f(M_3)] = r[M_3 \vee M_3]^* + r([M_3]^*)$

$$= [13 - 2 \times 5] + 8 = 11.$$

Since this value is less than $|P_3|$,

$$r[f(M_3)] = 11. \quad d[f(M_3)] = \frac{13}{11}.$$

Since f is a negative function $f(M_4)$ and $f(M_5)$ cannot have coloops.

Hence $r[f(M_4)] = [12 - 2 \times 6] + 6 = 6$

$$d[f(M_4)] = 2$$

$$r[f(M_5)] = [0] + (9-5)$$

$$= 4$$

$$d[f(M_5)] = 9/4.$$

(Note that $(f(M_5)) \vee (f(M_5)) = M_u$ on P_5 since

$$r[f(M_5)] + r[f(M_5)] = 5 + 5 > |P_5| = 9 \ ).$$

$f(M_5) = M_6 \times P_5$, $f(M_4) = M_6 \times P_5 \cup P_4 \cdot P_4 = M_6 \cdot \bigcup_{i=1}^{4} P_i \times P_4$

$f(M_3) = M_6 \times P_5 \cup P_4 \cup P_3 \cdot P_3$, $f(M_2) = M_6 \times (\bigcup_{i=2}^{5} P_i) \cdot P_2$,

$f(M_1) = M_6 \cdot P_1$ using Theorem 3.6 or its Corollary 1.

Since these matroids are molecular and

$$d[f(M_5)] > d[f(M_4)] > d[f(M_3)]$$

we have

$$P_5, P_4, P_3, P_1 \cup P_2 \text{ as the P-sequence of } f(M)$$

where $P_1 \cup P_2 = B[f(M)]$.

$$r[f(M)] = |P_1 \cup P_2| + r[f(M_3)] + r[f(M_4)] +$$

$$+ r[f(M_5)]$$

$$= 17 + 11 + 6 + 4 = 38.$$

It is clear from the above example that given any admissible function we can carry out the calculations similar to the above routinely. Also note that if $r[f(M)]$ were not required we do not need the values of $r(M_1)$ etc. The values of $d(M_i)$ $i \in \{1, 2 \ldots 5\}$ would suffice.

Section 4 : Set of Atoms of a Molecular Matroid

In this section we describe a procedure for 'breaking up' a molecular matroid into atomic matroids. Further, we discuss the effect of admissible functions on this partition.

Definition 4.1. A matroid on S is said to be a matroid of kind (n) iff there exists a set of n pairwise disjoint bases $b_1, b_2 \ldots b_n$ of M such that $\bigcup_{i=1}^{n} b_i = S$.

Definition 4.2. Let M be a matroid on S which has a set of n pairwise disjoint (n.p.d.) bases $b_1, b_2 \ldots b_n$.

Let $a_o, a_1, a_2 \ldots a_k$ be a sequence with $a_i \in \bigcup_{j=1}^{n} b_j$ for all $i \in \{0,1,2 \ldots k\}$, where

(1) $a_o \in b_{i,o}$, $C_1 \subseteq a_o \cup b_{i,1}, \ldots, C_k \subseteq a_{k-1} \cup b_{i,k}$
$a_1 \in C_1$ $\quad a_k \in C_k$;

C_r being a circuit of M for $r = 1,2 \ldots k$.

(2) $b_{i,r}$ is any one of the bases $b_1, b_2 \ldots b_n$ for each $r \in \{0,1,2 \ldots k\}$.

(3) $b_{i,r}$, $b_{i,p}$ ($r \neq p$) are not necessarily distinct.

(4) There are no repetitions of elements in the sequence $a_o, a_1, a_2 \ldots a_k$.

(5) $a_r \notin C_j$ ($j < r-1$) .

Then the sequence $a_o, a_1, \dots a_k$ is said to be an a - sequence from a_o to a_k with respect to $(b_1, b_2 \dots b_n)$.

<u>Definition 4.3.</u> Let M be a matroid on S, which has a set of n.p.d. bases $b_1, b_2 \dots b_n$. Then $d \in b_i$ ($i \in \{1, 2 \dots n\}$) is approachable from $a \in b_j$ ($j \in \{1, 2, \dots n\}$) iff there exists an a-sequence from a to d with respect to $b_1, b_2 \dots b_n$.

It is easy to see that from any sequence satisfying conditions (1), (2) and (3) of Definition 4.2 we can construct an a- sequence with the same first and last elements. We note here the following properties of approachability.

(a) Approachability with respect to $(b_1, b_2 \dots b_n)$ is transitive.

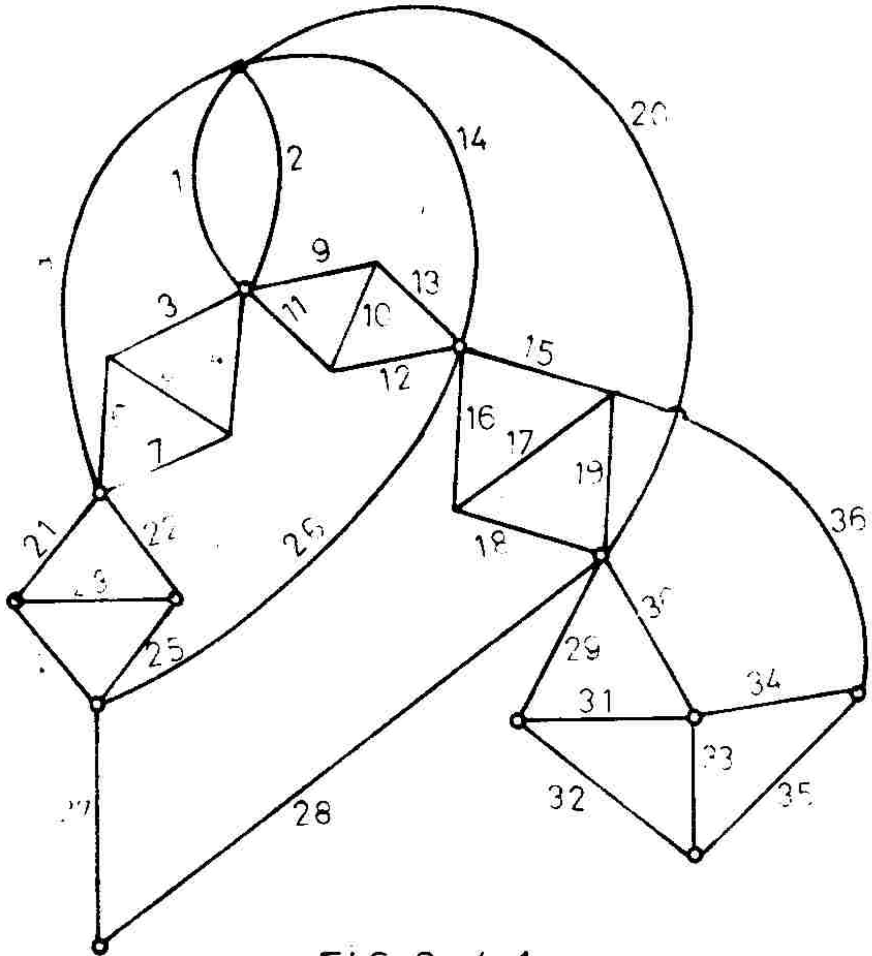
(b) Approachability with respect to $(b_1, b_2 \dots b_n)$ is not necessarily symmetric (See example 2.4.D below).

(c) We take approachability to be reflexive by definition i.e. d is approachable from itself.

To see (a), suppose we have an a-sequence from a to d with respect to $(b_1, b_2 \dots b_n)$ and another from d to e again with respect to $(b_1, b_2 \dots b_n)$. Then we can construct a sequence satisfying conditions (1), (2) and (3) of Definition 4.2 from a to e. From this sequence we can construct an a- sequence from a to e.

<u>Example 2.4.0.</u> Let M be the polygon matroid of the graph shown in Fig. 2.4.1.

We may choose our 2 p.d. bases as :



FIG·2·4·1.

THE GRAPH G.

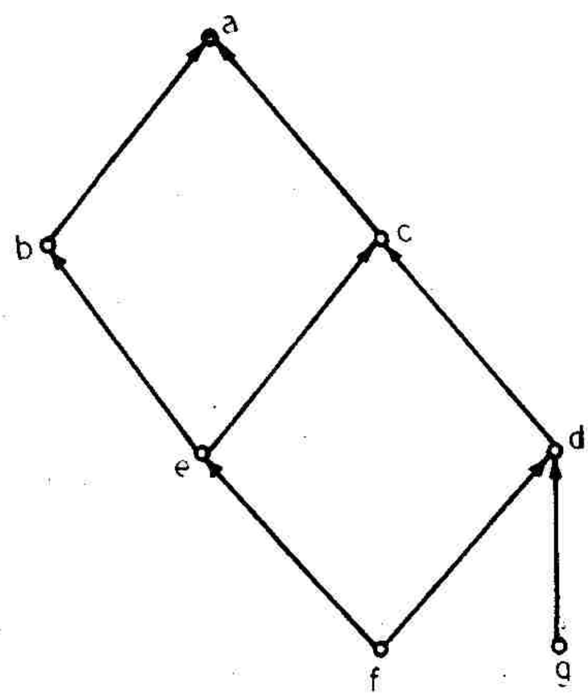


FIG. 2·4·2.

'DIAGRAM' OF L

$b_1 = \{1,3,5,7,9,10,12,21,23,25,16,17,19,27,30,31,32,35\}$

$b_2 = \{1,2 \ldots 36\} - b_1$

The following is an a-sequence from 28 to 1 with respect to (b_1, b_2) : 28, 9, 2, 1 .

But there exists no a-sequence from 1 to 28.

We now prove that a matroid of kind (n) is molecular.

<u>Theorem 4.1.</u> Let M be a matroid of kind (n) on S. Then M is molecular.

<u>Proof</u> : It is easy to see that d (M) = n. Let $T \subseteq S$. Now let $b_1, b_2 \ldots b_n$ be a set of n p.d. bases of M. Then $\bigcup_{i=1}^{n} b_i = S$. By Theorem T3, $b_i \cap T$ for each $i \in \{1,2, \ldots n\}$ is independent in M X T. Also $\bigcup_{i=1}^{n} (b_i \cap T) = T$. Hence $d(M X T) \leq n$. Hence for any $T \subseteq S$, $d(M X T) \leq d(M)$.

Q.E.D.

The following lemmas are needed in the proof of Theorem 4.2.

<u>Lemma 4.1.</u> Let M be a matroid of kind (n) on S. If M X P , ($P \subseteq S$), is a matroid of kind (n) and $a \in P$, then the set A of all elements approachable from a with respect to any set of n p.d. bases $(b_1, b_2 \ldots b_n)$ is contained in P.

<u>Proof.</u> Let $(b_1, b_2 \ldots b_n)$ be any set of n p.d. bases of M. Then $b_1 \cap P, \ldots b_n \cap P$ must obviously be n p.d. bases of M X P. If an a-sequence w.r.t. $(b_1, b_2 \ldots b_n)$ from a to d

has just two elements a and d, and if $d \in b_1$, then since $a \cup (b_1 \cap P)$ contains a circuit, we must have $d \in b_1 \cap P$. This shows that no element of S-P is approachable from a w.r.t. $(b_1, b_2 \ldots b_n)$ through an a- sequence of two elements.

Next let us suppose that the lemma is true for an a- sequence w.r.t. $(b_1, \ldots b_n)$ of (k-1) elements. Consider any a- sequence w.r.t. $(b_1, \ldots b_n)$ of k elements. Let e be the $(k-1)^{th}$ element in this sequence and d be the k^{th} element. Then $e \in P$ by the induction assumption. Suppose $d \in b_1$. But $b_1 \cap P$ is a base for M X P. Hence $e \cup (b_1 \cap P)$ contains a circuit. Hence $d \in (b_1 \cap P)$ and therefore $d \in P$. Thus by the induction hypothesis the lemma is true for all elements d approachable from a with respect to $(b_1, b_2 \ldots b_n)$.

Q.E.D.

<u>Lemma 4.2.</u> Let M be a matroid of kind (n) on S. Let $b_1, b_2 \ldots b_n$ be n p.d. bases of M. Let P be the set of all elements approachable from the elements of a given set $Q \subseteq S$. Then M X P is of kind (n).

<u>Proof.</u> Let $a \in P$. Then a is approachable from some element d of Q. All elements approachable from a are also approachable from d. Hence if $a \notin b_1$ and C_1 is the circuit contained in $a \cup b_1$, all the elements of C_1 are approachable from d. It follows that $b_i \cap P$ for all $i \in \{1, 2 \ldots, n\}$ is a base for M X P. Hence M X P is of kind (n).

Q.E.D.

Theorem 4.2. Let M be a matroid of kind (n) on S. Then approachability is independent of the n p.d. bases chosen.

Proof. We will show that the set of all elements approachable from any element $a \in S$ is independent of the n p.d. bases chosen. Let P be the set of all elements approachable from a with respect to $(b_1, b_2 \ldots b_n)$.

Then from Lemma 4.2, M X P is of kind (n). Let $(b_{2,1}, b_{2,2} \ldots b_{2,n})$ be any other set of n p.d. bases of M. From Lemma 4.1 it is clear that the set of all elements Q approachable from a with respect to $(b_{2,1}, \ldots b_{2,n})$ is contained in P. By the same argument interchanging roles of $(b_1, b_2 \ldots b_n)$ and $(b_{2,1}, \ldots b_{2,n})$ it follows that $P \subseteq Q$. Hence $P = Q$.

Q.E.D.

Lemma 4.3. Let M be a matroid of kind (n) on S. Let $P \subseteq S$. Then M X P is of kind (n) iff M . (S-P) is of kind (n).

Proof. Let $b_1, b_2 \ldots b_n$ be a set of n.p.d. bases of M. Then $\bigcup_{i=1}^{n} b_i = S$. Now if M X P (M . (S-P)) is of kind (n), $(b_1 \cap P, \ldots b_n \cap P)$ $((b_1 \cap (S-P), \ldots b_n \cap (S-P)))$ is a set of n p.d. bases of M X P (M.(S-P)). From Theorem T3, it follows that $(b_1 \cap (S-P), \ldots b_n \cap (S-P))$ $((b_1 \cap P, \ldots b_n \cap P))$ is a set of n p.d. bases of M. (S-P) (M X P). Hence M.(S-P)(M X P) is of kind (n).

Q.E.D.

Theorem 4.3. Let M be a matroid of kind (n) on S. Let $g \in S$

and let $Q = \{p \mid p$ approachable from g, but g not approachable from $p\}$. Then

(a) The matroid M . (S - Q) is of kind (n).

(b) $d \in (S - Q)$ is approachable from $a \in (S - Q)$ in this matroid iff d is approachable from a in M.

Proof. From Lemma 4.2 we conclude that M X Q is of kind (n). From Lemma 4.3 it follows that M . (S - Q) is of kind (n).

(b) Let $b_1, b_2, \ldots b_n$ be a set of n p.d. bases for M. If d is approachable from a, there exists an a- sequence with respect to $(b_1, b_2 \ldots b_n)$ from a to d. Every element of this sequence then belongs to S - Q, by the use of Lemma 4.1. Let e be any element in this sequence and f be the element next to it. Let $e \in b_i$ and $f \in b_j$. Then there exists a circuit C such that $C \subseteq e \cup b_j$ and $f \in C$. Since $b_j \cap Q$ is a base of M X Q, it is clear that we will now have a circuit C_1 of M . (S - Q) such that

$$\begin{aligned} C_1 &= C \cap (b_j \cup e) \cap (S - Q) \\ &= C \cap (S - Q) . \end{aligned}$$

Hence $e, f \in C_1$. This proves that the original a- sequence can still be used in M . (S - Q) for going from a to d.

Conversely, let d be approachable from a in M . (S - Q). Then there exists an a- sequence with respect to $(b_1 \cap (S - Q), \ldots b_n \cap (S - Q))$ from a to d in M . (S - Q). Let $e \in b_i \cap (S - Q)$ be any element in this sequence and $f \in b_j \cap (S - Q)$ be the element next to it. Then there exists a circuit C_1 in M . (S - Q) such that $C_1 \subseteq e \cup (b_j \cap (S - Q))$ and $f \in C_1$. Now there exists a circuit C in M such that $C \supseteq C_1$ and $C \subseteq e \cup b_j$. Hence there exists an a- sequence with respect to $(b_1, \ldots b_n)$ from a to d in M.

Q.E.D.

<u>Theorem 4.4.</u> Let M be a matroid of kind (n) on S. Let P, Q be disjoint subsets of S such that

(a) $P \cup Q = S$ and

(b) No element of P(Q) is approachable from any element of Q(P). Then P and Q are separators of M.

<u>Proof.</u> Since P(Q) is the union of elements approachable from P(Q), we have from Lemma 4.2, that M X P (M X Q) is of kind (n). Let $b_1, b_2 \ldots b_n$ be a set of n p.d. bases of M. Then $b_1 \cap P, \ldots b_n \cap P$ and $b_1 \cap Q, \ldots b_n \cap Q$ are p.d. bases for M X P and M X Q respectively. But then by Theorem T3 $b_1 \cap Q$ is a base for M . Q. Hence r (M X Q) = r (M . Q). Therefore by Theorem T8 Q is a separator of M and hence P is a separator of M.

Q.E.D.

<u>Definition 4.4.</u> Let M be a matroid of kind (n) on S. If $d \in S$, define $S_d = \{ a \mid a \in S$ and d and a are mutually approachable $\}$ and $S' = \{S_d \mid d \in S\}$. We will refer to the set S' as the set of atoms of M.

We can define a partial order L on $S^{\prime}$ as follows :

$S_a > S_d$ in L iff an element of S_d is approachable from an element of S_a but no element of S_a is approachable from an element of S_d. We will refer to this partial order as the usual partial order on $S^{\prime}$.

We may note that Lemma 4.1 implies that when M is a molecular matroid of kind (n) on S, $d(M \times T) = d(M)$ $(T \subseteq S)$ only if T is a union of some of the elements of $S^{\prime}$.

<u>Definition 4.4'.</u> From Lemma 4.1 one can see that an equivalent way of defining the usual partial order L on $S^{\prime}$ is to take $S_d > S_a$ $(S_d, S_a \in S')$ iff there exists no $T \subseteq S - S_a$ such that $S_d \subseteq T$ and $M \times T$ is of kind (n).

We sometimes use " $S_a < S_d$" (S_a 'less than' S_d) instead of " $S_d > S_a$" (S_d 'greater than' S_a), " $S_a \geq S_d$" in place of " $S_a > S_d$ or $S_a = S_d$" , " $S_a \not> S_d$" for the negation of " $S_a > S_d$" and " $S_a \not< S_d$" for the negation of " $S_a < S_d$".

The following theorem gives the structure of a suitably defined matroid on an element of $S^{\prime}$.

<u>Theorem 4.5.</u> Let M be a matroid of kind (n) on S. Let $S^{\prime}$ and the partial order L on $S^{\prime}$ be defined as above. Let $S_a \in S^{\prime}$.

Let $P_1 = \{S_p \mid S_p < S_a\}$

$Q_1 = \{S_p \mid S_p > S_a\}$

$R_1 = \{S_p \mid S_p \not> S_a$ and $S_p \not< S_a\}$

and let $T_1 \subseteq R_1$.

Let $A = S_a$

$$P = \{ e \mid e \in S_d ,\ S_d \in P_1 \}$$

$$Q = \{ e \mid e \in S_d ,\ S_d \in Q_1 \}$$

$$R = \{ e \mid e \in S_d ,\ S_d \in R_1 \}$$

$$T = \{ e \mid e \in S_d ,\ S_d \in T_1 \}$$

Let $M_1 = M \times (P \cup R \cup A) . A$.

Then

(1) M_1 is of kind (n)

(2) $M_1 = M \times (P \cup T \cup A) . A = M \times (P \cup A) . A$

(3) M_1 has no proper contractions or reductions of kind (n)

i.e. M_1 is atomic and is of kind (n).

<u>Proof.</u> The subset $P \cup R \cup A$ is the set of all elements approachable from elements of $P \cup R \cup A$. Hence, $M \times (P \cup R \cup A)$ is a matroid of kind (n), by Lemma 4.2. Similarly $M \times (P \cup R \cup A) \times P = M \times P$ is a matroid of kind (n). Hence $M \times (P \cup R \cup A) . (R \cup A)$ is a matroid of kind (n), from Lemma 4.3. Also from Theorem 4.3, it follows that in this matroid every element approachable from any $d \in A$ belongs to A. Hence

$(M \times (P \cup R \cup A) . (R \cup A)) \times A$ is a matroid of kind (n).

Now, in $M \times (P \cup R \cup A) . (R \cup A)$, by Theorem 4.3, no element of $R(A)$ is approachable from any element of $A(R)$. Also R and A are disjoint subsets of $R \cup A$. Hence R and A are separators of $M \times (P \cup R \cup A) . (R \cup A)$ by Theorem 4.4. Therefore, we have, using Theorem T8,

$$(M \times (P \cup R \cup A) . (R \cup A)) \times A = (M \times (P \cup R \cup A) . (R \cup A)) . A$$
$$= M \times (P \cup R \cup A) . A = M_1 .$$

Hence M_1 is of kind (n). Let $R - T = T'$.

Now by Theorem T7 -(2)

$$M_1 = (M \times (P \cup R \cup A) . (R \cup A)) . (T' \cup A) . A$$
$$= (M \times (P \cup R \cup A) . (R \cup A)) . (T' \cup A) \times A \quad \text{by Theorem T8.}$$
$$= (M \times (P \cup R \cup A) . (T' \cup A)) \times A$$
$$= (M . (Q \cup T' \cup A) \times (T' \cup A)) \times A \quad \text{by Theorem T7-(3).}$$
$$= M . (Q \cup T' \cup A) \times A$$
$$= M \times (P \cup T \cup A) . A \quad \text{by Theorem T7-(4)}$$

Thus $M_1 = M \times (P \cup T \cup A) . A$. If we put $T = \phi$, we get,

$$M_1 = M \times (P \cup A) . A .$$

From Lemma 4.1 if M_1 has a proper contraction of kind (n), there exist elements d, e $\in$ A such that d is not approachable from e in the matroid $M \times (P \cup A) . A$. Hence by Theorem 4.3 d is not approachable from e in $M \times P \cup A$ and hence in M.

This is a contradiction of the definition of A. Hence M_1 is of kind (n) and atomic.

Q.E.D.

<u>Definition 4.5.</u> A one to one mapping α from S' onto S' is an automorphism with respect to the partial order L on S' iff it preserves the partial order i.e. for S_a, $S_d \in S'$, $\alpha(S_a) > \alpha(S_d)$ iff $a > d$.

Theorem 4.6. Let M be a matroid of kind (n) on S. Let σ be an automorphism of M. Then $a \in S$ is approachable from $d \in S$, iff $\sigma(a)$ is approachable from $\sigma(d)$.

Proof. Let $b_1, b_2, \ldots b_n$ be n p.d. bases of M. We have an a-sequence from d to a,

$$d = a_o \text{ , } C_1 \subseteq a_o \cup b_{i,1} \text{ , } \ldots\ldots C_k \subseteq a_{k-1} \cup b_{i,k} \text{ ,}$$
$$a_1 \in C_1 \text{ , } \qquad a_k \in C_k$$
$$a_k = a$$

where the C_i are circuits of M and $b_{i,j}$ ($j \in \{1,2 \ldots k\}$) is one of the bases $b_1, b_2 \ldots b_k$. Now from Theorem T32, $\sigma(b_1), \ldots \sigma(b_n)$ are n p.d. bases of M and $\sigma(C_i)$ are circuits of M. We, therefore, have the a-sequence from $\sigma(d)$ to $\sigma(a)$

$$\sigma(d) = \sigma(a_o) \text{ ; } \sigma(C_1) \subseteq \sigma(a_o) \cup \sigma(b_{i,1}) \ldots$$
$$\sigma(a_1) \in \sigma(C_1)$$
$$\ldots\ldots\ldots\ldots\ldots C_k \subseteq \sigma(a_{k-1}) \cup \sigma(b_{i,k})$$
$$\sigma(a_k) \in \sigma(C_k)$$
$$\sigma(a_k) = \sigma(a)$$

Hence $\sigma(a)$ is approachable from $\sigma(d)$.

Now by Theorem T32, σ^{-1} is an automorphism of M. Hence if $\sigma(a)$ is approachable from $\sigma(d)$, $\sigma^{-1}(\sigma(a))$ is approachable from $\sigma^{-1}(\sigma(d))$ i.e. a is approachable from d.

Q.E.D.

__Corollary 1.__ Let M be a matroid of kind (n) on S. Let $a \in S$. Let S' be the set of atoms of M. If $a \in S_f$, $S_f \in S'$, then for some automorphism σ, $\sigma(a) \in S_d$, $S_d \in S'$ only if there exists an automorphism α of the partial order L on S' such that $\alpha(S_f) = S_d$.

__Corollary 2.__ Let P_1 be a subset of S' invariant under the automorphisms of the partial order L on S'. Let $P = \{a \mid a \in S_f, S_f \in P_1\}$. Then P is invariant under the automorphisms of the matroid M.

__Theorem 4.7.__ Let M be a matroid of kind (n) on S. Let S' be the set of atoms of M. Let σ be an automorphism of M. If $a \in S_f$, $S_f \in S'$ and $\sigma(a) \in S_d$, $S_d \in S'$ then $\sigma(g) \in S_d$ for every $g \in S_d$, i.e. $\sigma(S_f) = S_d$.

__Proof.__ Consider the set of all elements $S_e \in S'$ such that $\alpha(S_f) = S_e$ for some automorphism α of the partial order L on S'. Let this set be B_1. From Corollary 1 of Theorem 4.6 $S_d \in B_1$. If S_e, $S_h \in B_1$, it is clear that $S_e \not> S_h$ and $S_h \not> S_e$. Let $B = \{a \mid a \in S_e, S_e \in B_1\}$. Then B is invariant under the automorphisms of M by Corollary 2 of Theorem 4.6. Now let $a, g \in S_f$, $S_f \in S'$. We know that $a(g)$ is approachable from $g(a)$. Hence by Theorem 4.6, $\sigma(a)$ $(\sigma(g))$ is approachable from $\sigma(g)$ $(\sigma(a))$. But $\sigma(a), \sigma(g) \in B$. Hence if $\sigma(a) \in S_d$, $S_d \in B_1$ then $\sigma(g) \in S_d$.

Q.E.D.

We next show that one obtains closed sets of M or M^* by considering suitable unions of elements of S'.

<u>Theorem 4.8.</u> Let M be a matroid of kind (n) ($n \geq 2$) on S. Let S' be the set of atoms of M. Let $B_1 \subseteq S'$ and $D_1 = \{S_f \mid S_f \in S', S_f \geq S_d, S_d \in B_1\}$. If $D = \{a \mid a \in S_f, S_f \in D_1\}$ then D is closed in M and S-D is closed in M^*.

<u>Proof.</u> Let $b_1, b_2 \ldots b_n$ be n p.d. bases of M. Then $b_1 \cap D, \ldots b_n \cap D$ are n p.d. bases of $M \times D$. Let $a \in (S-D)$ and let if possible $a \in \bar{D}$ in M. Let $a \in b_j$. But then a depends on $b_j \cap D$, i.e. $a \cup (b_j \cap D)$ contains a circuit of M, which is a contradiction of the definition of a base.

Now $d(M \times D) = \mathcal{M}$ ($n \geq 2$). We find then that $M \times D$ has no coloops. Hence by Theorem T10, S-D is closed in M^*.

Q.E.D.

For convenience we describe our procedure for obtaining S' and the partial order L on S' in the form of two algorithms.

<u>Algorithm 4.1.</u> (Approachability algorithm).

Let M be a matroid of kind (n) on S. Let $b_1, b_2 \ldots b_n$ be a set of n p.d. bases of M obtained by using Algorithm 1.1. Let $a \in b_1$. Then a forms a circuit with each of the bases $b_2 \ldots b_n$. Let A_1 represent the union of all these circuits. Each member of A_1 forms a circuit with (n-1) bases among $b_1, b_2 \ldots b_n$. Let A_2 represent the union of all these circuits. In a similar manner A_{n+1} is obtained from A_n. Let s be the least number such that $A_s = A_{s+1}$. Then $Q_a = A_s$ is the set of all elements approachable from a.

<u>Algorithm 4.2.</u> (Algorithm for S^1 and L)

Let M be a matroid of kind (n) on S. Let $b_1, b_2 \ldots b_n$ be a set of n p.d. bases of M, obtained by using Algorithm 1.1.

For each $a \in b_1$ form the set Q_a of all elements approachable from a. Form the family of sets

$$S^1 = \left\{ P \;\middle|\; P = Q_d - \bigcup_{a \in D} Q_a \;,\; d \in b_1 \;,\; D \subseteq b_1 \text{ such that } Q_d - \bigcup_{a \in D} Q_a \neq \phi \text{ but } Q_d - \bigcup_{a \in T} Q_a = \phi \text{ if } T \supset D . \right\}$$

To form the partial order L proceed as follows. Let $S_d, S_g \in S^1$. Choose $e \in b_1 \cap S_d$ and $f \in b_1 \cap S_g$. Check if $Q_f \subseteq Q_e$ or $Q_e \subseteq Q_f$. If $Q_f \supseteq Q_e$ put $S_g > S_d$.

If $Q_e \supseteq Q_f$ put $S_d > S_g$.

If $Q_f \nsubseteq Q_e$ and $Q_e \nsubseteq Q_f$, then S_d, S_g are not comparable in L. (EN

We next consider the case where M is a molecular matroid on S but d(M) is not an integer. Our procedure here is simple. We first consider a parallel - α - copy M_α of M such that $d(M_\alpha)$ is an integer, i.e. M_α is a matroid of kind (n) for some positive integer n. We can then use the theory developed so far in this section to construct the set $(S_\alpha)^1$ of atoms of M_α and the partial order L_α on $(S_\alpha)^1$. We then use the map Υ to construct the corresponding set S' and the partial order L for M.

We now proceed to justify this procedure.

Theorem 4.9. Let M be a molecular matroid on S. Let M_α be its parallel - α - copy on S_α. Let $T_2 \subseteq S_\alpha$. Then $d(M_\alpha \times T_2) = d(M_\alpha)$ iff $T_2 = \Upsilon(T_1)$ such that $T_1 \subseteq S$ and $d(M \times T_1) = d(M)$.

Proof. Let $T_1 \subseteq S$ such that $d(M \times T_1) = d(M)$. Then $d[M_\alpha \times \Upsilon(T_1)] = \alpha \cdot d(M \times T_1) = \alpha \cdot d(M)$, using Theorem 2.3 and its Corollary 1. But $d(M_\alpha) = \alpha \cdot d(M)$. Therefore, $d[M_\alpha \times \Upsilon(T_1)] = d(M_\alpha)$. Now conversely let $T_2 \subseteq S_\alpha$ be such that

$$d(M_\alpha \times T_2) = d(M_\alpha). \text{ Let } T_2 \cap S = T_1.$$

Suppose $\Upsilon(T_1) \supset T_2$. Hence $|\Upsilon(T_1)| > |T_2|$. Now we have $r(M_\alpha \times T_2) = r(M \times T_1) = r[M_\alpha \times \Upsilon(T_1)]$. Hence it follows that

$$d[M_\alpha \times \Upsilon(T_1)] > d[M_\alpha \times T_2] = d(M_\alpha).$$

But this is impossible since M_α is molecular. Hence $\Upsilon(T_1) = T_2$. Hence by Theorem 2.3, $d(M \times T_1) = d(M)$.

Q.E.D.

Definition 4.6. Let M be a matroid on S. Let M_α be its parallel - α - copy on S_α. If σ is a one to one mapping from S_α onto S_α, we denote by σ' the restriction of σ to S.

Theorem 4.10. Let M be a matroid on S. Let M_α be its parallel - α - copy on S_α. If β is an automorphism of M, then there exists an automorphism σ of M_α such that $\sigma' = \beta$.

Proof : Let $S = \left\{ a_1^1 , a_1^2 \dots a_1^k \right\}$.

Then M_α on S_α is obtained by adding $a_2^1, \dots a_\alpha^1$ parallel to $a_1^1 \dots$ $a_2^k, \dots a_\alpha^k$ parallel to a_1^k, as in Definition 2.7. Let β be an automorphism of M. We may define an extension of β to an automorphism σ of M_α as follows :

Let $a_1^i, a_1^j \in S$ such that $\beta(a_1^i) = a_1^j$. Let ξ be a permutation of $(1,2, \dots \alpha)$ such that $\xi(1) = 1$, i.e.

$$\xi = \begin{pmatrix} 1, 2, \dots\dots\dots \alpha \\ 1, \xi(2), \dots \xi(\alpha) \end{pmatrix}$$

Then $\sigma(a_t^i) = a^j_{\xi(t)}$.

We then have $\sigma' = \beta$ and it is not difficult to see that σ is an automorphism of M_α.

Q.E.D.

Let σ be any automorphism of M_α. We can then construct a corresponding automorphism θ of M as follows :

If $\sigma(a_1^i) = a_n^m$ $\qquad \theta(a_1^i) = a_1^m$.

We can therefore easily see the following corollary using Theorem 4.10 and the above remark.

<u>Corollary 1.</u> Let M be a matroid on S and let M_α be its parallel - α - copy on S_α. Let $T \subseteq S$. Then $\Upsilon(T)$ is invariant under the automorphisms of M_α iff T is invariant under the automorphisms of M.

<u>Theorem 4.11.</u> Let M be a matroid on S, and let M_α be its parallel - α - copy on S_α. Let $T \subseteq S$. Then $\Upsilon(T)$ is a closed set in M_α iff T is closed in M.

<u>Proof.</u> If $\Upsilon(T)$ is closed in M_α it is easy to see that T is closed in M. Suppose $\Upsilon(T)$ is not closed in M_α. $M_\alpha \times \Upsilon(T)$ is the parallel-α-copy of M X T by Theorem 2.3. Let b be a base of M X T. Then b is also a base of $M_\alpha \times \Upsilon(T)$. Since $\Upsilon(T)$ is not closed in M_α there exists an $a_j^1 \in S_\alpha - \Upsilon(T)$ such that $a_j^1 \in \overline{\Upsilon(T)}$. Hence there exists a circuit C such that $C \cap (S_\alpha - \Upsilon(T) = \{a_j^1\}$ and $(C - a_j^1) \cup a_1^1$ is a circuit of M. But $a_1^1 \in S - \Upsilon(T) = S - T$. Hence T is not closed in M.

Q.E.D.

<u>Definition 4.7.</u> Let M be a molecular matroid on S, and let M_α be its parallel-α-copy on S_α such that $\alpha = r\,(M)$. Then M_α is a matroid of kind (n) where $n = \alpha \cdot d\,(M) = |S|$.

Let $(S_\alpha)^1$ be the set of atoms of M_α and let L_α be the partial order on $(S_\alpha)^1$ (as in Definition 4.6). We now construct the set S^1 from elements of S by taking

$$S^1 = \left\{ S_d \mid S_d = (\Upsilon)^{-1}(S_e^{\alpha}),\ \ S_e^{\alpha} \in (S_\alpha)^1 \right\}$$

(This set is well defined by Theorem 4.9). We will call the set S^1 the set of atoms of M. Next we define the partial order L on S^1, by taking $S_a > S_d$ (S_a, $S_d \in S^1$) in L

$$\text{iff} \quad \mathcal{T}(S_a) \underset{\alpha}{>} \mathcal{T}(S_d) \text{ in } L_\alpha .$$

L will be called the usual partial order on S^1 for M. It is easy to see that L_α is isomorphic to L.

We may note that Lemma 4.1 and Theorem 4.9 imply that if M is a molecular matroid on S, $d(M \times T) = d(M)$ ($T \subseteq S$) only if T is a union of some of the elements of S^1.

<u>Definition 4.7'.</u> From Theorem 4.9 and Definition 4.4' one can see that an equivalent way of defining the partial order L on S^1 is to take $S_d > S_a$ (S_d , $S_a \in S^1$) iff there exists no $T \subseteq S - S_a$ such that $S_d \subseteq T$ and $d(M \times T) = d(M)$.

Now by the use of Corollary 2 of Theorem 4.6 and Corollary 1 of Theorem 4.10 the following Theorem is immediate.

<u>Theorem 4.12.</u> Let M be a molecular matroid on S. Let S^1 be the set of atoms of M. Let $P_1 \subseteq S^1$ be a set invariant under the automorphisms of the partial order L on S^1. Let $P = \{a \mid a \in S_d, S_d \in P_1\}$. Then P is invariant under the automorphisms of M.

We now rewrite Theorem 4.5 for any general molecular matroid M. The result follows readily from Theorems 2.3, 4.5 and 4.9.

<u>Theorem 4.13.</u> Let M be a molecular matroid on S. Let S^1 be its set of atoms and let L be the partial order on S^1 defined as usual.

Let $S_a \in S^1$.

Let $P_1 = \{ S_p \mid S_p < S_a \}$

$Q_1 = \{ S_p \mid S_p > S_a \}$

$R_1 = \{ S_p \mid S_p \ngtr S_a \text{ and } S_p \nless S_a \}$

and let $T_1 \subseteq R_1$.

Let $A = S_a$

$P = \{ d \mid d \in S_e ,\ S_e \in P_1 \}$

$Q = \{ d \mid d \in S_e ,\ S_e \in Q_1 \}$

$R = \{ d \mid d \in S_e ,\ S_e \in R_1 \}$

$T = \{ d \mid d \in S_e ,\ S_e \in T_1 \}$.

Let $M_1 = M \times (P \cup R \cup A) . A$.

Then

(a) $M_1 = M \times (P \cup T \cup A) . A = M \times (P \cup A) . A$

(b) M_1 is atomic and $d(M_1) = d(M)$.

Now by using Theorem 4.8, Theorem 4.11 and Theorem T10 we have,

<u>Theorem 4.14.</u> Let M be a molecular matroid on S with $d(M) > 1$. Let S^1 be its set of atoms. Let $B_1 \subseteq S^1$ and $D_1 = \{ S_d \mid S_d \in S^1, S_d \leq S_e , S_e \in B_1 \}$ (the partial order L on S' being as in Definition 4.7).

If $D = \left\{ a \mid a \in S_d , S_d \in D_1 \right\}$ then

(a) D is closed in M, and

(b) S - D is closed in M^*.

<u>Proof of (b).</u> We need merely prove that M X D has no coloops. Now $d [M_\alpha X \gamma(D)] = d [M_\alpha]$ by Lemma 4.2. Hence $d (M X D) = d (M)$ by Theorem 4.9. Hence $d(M X D) > 1$. Also M X D is clearly molecular and therefore M X D has no coloops. Hence S - D is closed in M^*.

Q.E.D.

We now outline the procedure for obtaining the set S^1 of atoms and the partial order L on S^1 for a molecular matroid M, when d(M) is not an integer.

(1) Let $r(M) = \alpha$. Construct the parallel-α-copy of M on S_α. Then $d(M_\alpha) = \alpha . d(M)$. Hence M_α is a matroid of kind (n) where $n = |S|$.

(2) Construct the set $(S_\alpha)^1$ of atoms and the partial order L_α on $(S_\alpha)^1$ by using Algorithms 4.1 and 4.2.

(3) Use the function γ from P(S) into $P(S_\alpha)$ to construct the set S^1 of atoms and the partial order L on S^1 for M, using Definition 4.7.

So far in this section we have considered methods of partitioning the set of definition of a molecular matroid into subsets invariant under its automorphisms. We would now like to link these results with our results on P-sequences. In this manner we would obtain a partition finer than P-sequences.

We first prove a simple Lemma.

Lemma 4.5. Let M be any matroid on S. Let Q be a subset of S invariant under the automorphisms of M. Then if σ is an automorphism of M, the mapping σ restricted to Q is an automorphism of M X Q and M . Q.

Proof. Let b be any base of M X Q. Then b is independent in M. $\sigma(b) \subseteq Q$, since Q is invariant under the automorphisms of M. Also $\sigma(b)$ is independent in M and $|\sigma(b)| = |b|$. Hence $\sigma(b)$ is a base of M X Q. Hence from Theorem T32, the mapping σ restricted to Q is an automorphism of M X Q. By Theorem T32, σ is an automorphism of M^*. Hence σ restricted to Q is an automorphism of M^* X Q, that is of $(M . Q)^*$ and therefore of M . Q.

Q.E.D.

Corollary 1. Let M be a matroid on S. Let P, Q be subsets of S invariant under the automorphisms of M, such that $P \subseteq Q$. If $R \subseteq P$ is invariant under the automorphisms of M X Q . P or M . Q X P then R is invariant under the automorphisms of M.

The next theorem follows immediately from Corollary 1 of Theorem 2.5, Lemma 4.5 and its Corollary and Theorem 4.12.

Theorem 4.15. Let M be a matroid on S. Let $P_1, P_2 \dots P_n$ be its P-sequence. Let

$$M_1 = M \times \bigcup_{i=1}^{k} P_i \cdot P_k$$

and let R^1 be its set of atoms, and L be the partial order on R^1 as in Definition 4.7. Let $Q_1 \subseteq R^1$ be invariant under the automorphisms of L. Then $Q = \{ a \mid a \in S_d , \quad , S_d \in Q_1 \}$ is invariant under the automorphisms of M.

<u>Theorem 4.16.</u> Let M be a matroid on S. Let $P_1, P_2 \ldots P_n$ be its P-sequence. Then $(\bigcup_{i=1}^{k} P_i)$ for all $k \in \{1,2 \ldots n\}$ is closed in M and $(\bigcup_{i=j}^{n} P_i)$ for all $j \in \{1,2, \ldots n\}$ is closed in M^*.

<u>Proof.</u> If M is the matroid M_u on S, the theorem is trivial. Otherwise $M \times \bigcup_{i=1}^{j-1} P_i$ for all $j \in \{2, \ldots n\}$ has no coloops. Hence by Theorem T10 $\bigcup_{i=j}^{n} P_i$ is closed in M^*. By Theorem 2.7, M^* has the P- sequence $P_n , P_{n-1}, \ldots P_1$. If $M^* = M_u$ on S the theorem is trivial. Otherwise $M^* \times \bigcup_{j+1}^{n} P_i$ (for all $j \in \{1,2, \ldots n\}$) has no coloops. Hence,

$$\bigcup_{i=1}^{j} P_i \text{ is closed in M for all } j \in \{1,2, \ldots n\} .$$

Q.E.D.

<u>Lemma 4.6.</u> Let M be a matroid on S. Let $P \subseteq S$.

(a) If Q is closed in $M.(S-P)$, then $P \cup Q$ is closed in M.

(b) If P is closed in M and $(S - Q) \subseteq P$, then $P \cap Q$ is closed in $M . Q$.

<u>Proof.</u> Assume $P \cup Q$ is not closed in M. Then there exists a circuit C in M such that $C - (P \cup Q) = \{a\}$ say. Further, if

possible let C_2 be a circuit in M such that $C_2 - P = \{a\}$; then clearly $\{a\}$ is a circuit of M . (S-P) and hence $a \in Q$ by the definition of closed sets. This is a contradiction and hence there exists no circuit C_2 in M such that $C_2 - P = \{a\}$. Hence $C \cap Q \neq \Phi$ for all such C. Hence there exists a circuit C_3 in M . (S - P) such that $C_3 \subseteq C$, $C_3 \cap Q \neq \Phi$ and $a \in C_3$. This contradicts the fact that Q is closed in M . (S - P). Hence P U Q is closed in M.

(b) Assume $P \cap Q$ is not closed in M . Q. Then there exists a circuit C of M . Q such that $C - (P \cap Q) = \{a\}$. Hence there exists a circuit C_1 of M such that $C_1 \supseteq C$ and $C_1 - P = \{a\}$. But P is closed in M. Hence we have a contradiction and therefore we conclude that $P \cap Q$ is closed in M . Q.

Q.E.D.

Using Lemma 4.6, Theorem 4.16 and Theorem 4.14 we obtain the following result.

__Theorem 4.17.__ Let M be a matroid on S. Let $P_1, P_2 \ldots P_n$ be its P-sequence. Let $M = M \times (\bigcup_{i=1}^{k} P_i) . P_k$, ($k \in \{1,2,\ldots n\}$) be such that $M_1 \neq M_u$ or M_o on P_k. Let R^i be its set of atoms and let L be the partial order on R^i (as in Definition 4.7). Let $B_1 \subseteq R^i$ and let $D_1 = \{S_a \mid S_a \leq S_d , S_d \in B_1\}$. Let $D = \{a \mid a \in S_d , S_d \in D_1\}$. Then

$$(\bigcup_{i=1}^{k-1} P_i) \cup D \text{ is closed in M and}$$

$$(\bigcup_{i=k+1}^{n} P_i) \cup P_k - D \text{ is closed in } M^* .$$

We now describe the effect of admissible functions on the sets of atoms and the partial orders on them for an arbitrary matroid, through a series of Theorems.

Theorem 4.18. Let S^i represent the set of atoms for molecular matroid M_i on S and let L^i represent the usual partial order on S^i ($i = 1,2$). Let $S^1 = S^2$ and $L^1 = L^2$ and $M_1 \vee M_2 \neq M_u$ on S. Then S^1 is the set of atoms and L^1 is the usual partial order on S^1 for the molecular matroid $M_1 \vee M_2$.

Proof. We note that $M_1 \vee M_2$ is molecular when M_1 and M_2 are molecular by Corollary 2 of Theorem 3.1. By Definition 4.7', Theorem 4.9 and Lemma 4.1, to prove this theorem we need only show that for any $T \subseteq S$, when the conditions of the theorem are satisfied,

$$d\,[(M_1 \vee M_2) \times T] = d\,(M_1 \vee M_2) \text{ iff } d(M_1) = d(M_1 \times T)$$

and

$$d(M_2) = d\,(M_2 \times T).$$

If $d(M_1) = d(M_1 \times T)$ and $d(M_2) = d(M_2 \times T)$, by Theorem 3.1 we have

$$d\,[(M_1 \vee M_2) \times T] = d\,[M_1 \vee M_2]$$

Conversely, suppose $d\,[(M_1 \vee M_2) \times T] = d\,[M_1 \vee M_2]$.

Now since $M_1 \vee M_2$ is molecular and $M_1 \vee M_2 \neq M_u$ on S, $(M_1 \vee M_2) \times T$ is molecular on T and $(M_1 \vee M_2) \times T \neq M_u$ on T. Hence we have by Corollary 1 of Theorem 1.2

$$\frac{r[M_1 \vee M_2]}{|S|} = \frac{r(M_1)}{|S|} + \frac{r(M_2)}{|S|}$$

and

$$\frac{r[(M_1 \vee M_2) \times T]}{|T|} = \frac{r(M_1 \times T)}{|T|} + \frac{r(M_2 \times T)}{|T|}$$

Since

$$\frac{r(M_1 \vee M_2)}{|S|} = \frac{r[(M_1 \vee M_2) \times T]}{|T|},$$

it follows that

$$\frac{r(M_1)}{|S|} = \frac{r(M_1 \times T)}{|T|} \quad \text{and} \quad \frac{r(M_2)}{|S|} = \frac{r(M_2 \times T)}{|T|},$$

since M_1 and M_2 are molecular.

Hence

$d(M_1) = d(M_1 \times T)$ and $d(M_2) = d(M_2 \times T)$

Q.E.D.

In the next theorem we extend Theorem 4.18 to the case where M_1 and M_2 are not molecular but aligned.

Theorem 4.19. Let M_1 and M_2 be aligned matroids on S. Let $P_1^1, P_2^1, \ldots P_n^1$ be the P-sequence of M_1 and let $P_1^2, P_2^2 \ldots P_p^2$ be the P-sequence of M_2. Let $M_1 \vee M_2$ have the P-sequence $P_1, P_2 \ldots P_m$. Then

(1) For all $P_i \neq B(M_1 \vee M_2)$ or $C(M_1 \vee M_2)$, we have $P_i = P_{j_i}^1 = P_{k_i}^2$ for some $j_i \in \{1, 2, \ldots n\}$ and some $k_i \in \{1, 2, \ldots p\}$.

(2) Further if the molecular matroids

$$M_1 \times \left(\bigcup_{x=1}^{j_1} P_x^{\,1} \right) . P_{j_1}^{\,1} \text{ and } M_2 \times \left(\bigcup_{x=1}^{k_1} P_x^{\,2} \right) . P_{k_1}^{\,2}$$

determined by $P_{j_1}^{\,1}$ and $P_{k_1}^{\,2}$ of (1) have the same set R^1 of atoms and the same partial order L on R^1, then the molecular matroid $(M_1 \vee M_2) \times \left(\bigcup_{z=1}^{i} P_z \right) . P_i$ determined by P_i of (1) has R^1 as its set of atoms and L as the usual partial order on R^1.

Proof. (1) follows directly from Theorem 3.4, and (2) follows by using Corollary 1 of Theorem 3.3, Corollary 4 of Theorem 1.1 and Theorem 4.18.

Q.E.D.

Theorem 4.20. Let M be a molecular matroid on S. Let S^1 be its set of atoms and L be the usual partial order on S^1 for M. Then the molecular matroid M^* on S has S^1 as its set of atoms and L^1(= the dual of L) as the usual partial order on S^1.

Proof. We may note that M^* is molecular by Theorem 3.5. Now if for $T \subseteq S$,

$d(M \times T) = d(M)$ it is clear that $d[M . (S-T)] = d(M)$.

Hence $d[(M \times T)^*] = d(M^*)$ and $d[(M.(S-T))^*] = d(M^*)$

i.e. $d(M^* . T) = d(M^*)$ and $d[M^* \times (S-T)] = d(M^*)$.

The theorem now follows easily by using Theorem 4.9, Lemma 4.1 and Definition 4.7'.

Q.E.D.

The next theorem is a simple consequence of Theorems 4.18 and 4.20.

<u>Theorem 4.21.</u> Let M be a molecular matroid on S. Let S^1 be its set of atoms and L be the usual partial order on S^1. If f is an admissible function such that $f(M) \neq M_u$ or M_o on S, then S^1 is the set of atoms of f(M) and L is the usual partial order on S^1 or L' (= the dual of L) is the usual partial order on S^1 for f(M) according as f is positive or negative.

We next extend Theorem 4.21 to the case where the matroid is not molecular.

<u>Theorem 4.22.</u> Let M be a matroid on S. Let $P_1, P_2 \ldots P_n$ be the P-sequence of M. Let f be an admissible function and let $P_1', P_2' \ldots P_p'$ be the P-sequence of f(M).

(1) Then for every $P_j' \neq B \,[\, f(M) \,]$ or $C \,[\, f(M) \,]$ we have $P_j' = P_{k_j}$ for some $k_j \leq n$.

(2) Further if $M \times \left(\bigcup_{i=1}^{k_j} P_i \right) \cdot P_{k_j}$ has R^1 as its set of atoms and L as the usual partial order on R^1, then

$(f(M)) \times \left(\bigcup_{i=1}^{j} P_i' \right) \cdot P_j'$ has R^1 as its set of atoms, and has L (L', the dual of L) as the usual partial order on R^1 if f is positive (negative).

<u>Proof.</u> Under the conditions of $P_j' \neq B \,[\, f(M) \,]$ or $C \,[\, f(M) \,]$, that $P_j' = P_{k_j}$ for some $k_j \leq n$, follows from Theorem 3.5.

(2) follows by using Corollary 1 of Theorem 3.6 and Theorem 4.21.

Q.E.D.

Example 2.4.1. Consider the polygon matroid M of the graph G shown in Figure 2.4.1.

It can be seen that M is a molecular matroid with $d(M) = 2$. We now wish to construct the set S^1 of atoms of M and the usual partial order L on S^1.

Choose disjoint bases b_1, b_2 for M as follows :

$$b_1 = \{1,3,5,7,9,10,12,15,17,18,21,23,25,27,29,30,33,34\}$$

$$b_2 = \{2,4,6,8,11,13,14,16,19,20,21,24,26,28,31,32,35,36\}$$

For $e \in S = E(G)$ we represent the set of all elements approachable from e by Q_e. Then

$$Q_1 = \{1,2\}$$

$$Q_3 = Q_5 = Q_7 = \{1,2,3,4,5,6,7,8\}$$

$$Q_9 = Q_{10} = Q_{12} = \{1,2,9,10,11,12,13,14\}$$

$$Q_{15} = Q_{17} = Q_{18} = \{1,2,9,10,11,12,13,14,15,16,17,18,19,20\}$$

$$Q_{21} = Q_{23} = Q_{25} = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,21,22,23,24,25,26\}$$

$$Q_{27} = Q_{21} \cup \{27,28\} \cup Q_{15}$$

$$Q_{29} = Q_{30} = Q_{33} = Q_{34} = Q_{15} \cup \{29,30,31,32,33,34,35,36\}$$

The elements of S^1 are therefore

$$a = \{1,2\}$$

$$b = \{3,4,5,6,7,8\}$$

$c = \{9,10,11,12,13,14\}$

$d = \{15,16,17,18,19,20\}$

$e = \{21,22,23,24,25,26\}$

$f = \{27,28\}$

$g = \{29,30,31,32,33,34,35,36\}$

The partial order L on S' is given by the diagram in Figure 2.4.2.

There exists a directed path from k to j ($k,j \in S^1$) iff $k > j$ in L.

Example 2.4.2. (Examples of atomic and molecular matroids)

(1) A simple example of an atomic matroid with $d(M) = p/q$, where p, q are positive integers with $p/q \geq 1$, is constructed as follows :

Let S be a set such that

P $\subseteq$ S in independent in M iff $|P| \leq q$. Then M is atomic and $d(M) = p/q$.

(2) Simple examples of graphic matroids which are atomic can be constructed simply by considering totally edge symmetric graphs - complete graphs, complete - bipartite graphs etc. For these matroids the minimum number of independent sets required to cover the underlying set of the matroid is simply the smallest integer $\geq d(M)$. A simple example of an atomic matroid that is not totally symmetric is the wheel graph on

n nodes (A polygon on (n-1) nodes and an n^{th} node that is adjacent to each of these (n-1) nodes).

Section 5 : Applications To Network Analysis

In this section we will first describe Kishi-Kajitani's solution of the problem of determination of the topological degree of freedom of an electrical network. Next we describe how our partition of molecular matroids into their sets of atoms has some bearing on the proper utilization of Kishi-Kajitani's results for solving networks by tearing. Indeed our partition for matroids of kind (2) reduces effectively to the results of T. Ohtsuki et al for graphs [Oht 1].

We need the following theorem for the proof of the main result.

Theorem 5.1. Let M be a matroid on S. Let $S_1 = A(M^2) \cup C(M^2)$ and $S_2 = B[(M^*)^2]$. Let $T_1 \subseteq S_1$, $T_2 \subseteq S_2$. Then

$$d(M \times S_1 . T_1) > 2, \quad d(M \times S_2 . T_2) \geq 2 .$$

Proof. Let $P_1, P_2 .. P_n$ be the P-sequence of M. Let $S_1 = \bigcup_{i=1}^{k} P_i$ and $S_2 = \bigcup_{i=1}^{m} P_i$. Then $P_1, P_2 ... P_k$ is the P-sequence of $M \times S_1$ and $P_1, P_2 ... P_m$ is the P-sequence of $M \times S_2$, by Corollary 1 of Theorem 2.6. By Theorem 2.7, $P_k ... P_2, P_1$ is the P-sequence of $(M \times S_1)^* = M^* . S_1$, and $P_m ... P_2, P_1$ is the P-sequence of $(M \times S_2)^* = M^* . S_2$. Now $d(M \times S_1 . P_k) > 2$ by Theorem 3.7. Hence $d(M^* . S_1 \times P_k) < 2$. Therefore by Theorem 2.6, for any

$T_1 \subseteq S_1$, $d(M^* \cdot S_1 \times T_1) < 2$ i.e. $d(M \times S_1 \cdot T_1) > 2$. Similarly, since $d(M \times S_2 \cdot P_m) \geq 2$ by Theorem 3.7, we conclude that $d(M \times S_2 \cdot T_2) \geq 2$.

Q.E.D.

<u>Corollary 1.</u> Let M be a matroid on S. Let $S_1 = A[(M^*)^2] \cup C[(M^*)^2]$, $S_2 = B(M^2)$. Let $T_1 \subseteq S_1$, $T_2 \subseteq S_2$. Then,

$$d(M^* \times S_1 \cdot T_1) > 2 \quad \text{and}$$

$$d(M^* \times S_2 \cdot T_2) \geq 2 .$$

Hence $d(M \cdot S_2 \times T_2) \leq 2$ and $d(M \cdot S_1 \times T_1) < 2$.

The following theorem is due to Kishi and Kajitani [Ki 2]. It was originally stated for graphs. The result was extended to matroids by Bruno and Weinberg [Br 2]. Here however we have used the notation followed so far in this chapter.

<u>Theorem 5.2.</u> Let M be a matroid on S. Let $S_1 = A(M^2) \cup C(M^2)$. Then if $S_2 \subseteq S$

$$r(M \times S_1) + \mu(M \cdot (S - S_1)) \leq r(M \times S_2) + \mu[M \cdot (S - S_2)].$$

The equality holds only if $S_2 \supseteq S_1$.

<u>Proof.</u> By Theorem T3

$$r[M \times S_1] + \mu[M \cdot (S-S_1)] =$$

$$= r[M \times (S_1 \cap S_2)] + r[M \times S_1 \cdot (S_1 - S_2)]$$

$$+ \mu[M \cdot (S - S_1) \cdot (S-(S_1 \cup S_2))] + \mu[M \cdot (S-S_1) \times (S_2-S_1)],$$

$$= r \lfloor M \times (S_1 \cap S_2) \rfloor + r \lfloor M \times S_1 \cdot (S_1 - S_2) \rfloor$$

$$+ \mu \lfloor M \cdot (S - (S_1 \cup S_2)) \rfloor + \mu \lfloor M \times (S_1 \cup S_2) \cdot (S_2 - S_1) \rfloor$$

by Theorem T7-(3) and T7-(4).

Similarly,

$$r \lfloor M \times S_2 \rfloor + \mu \lfloor M \cdot (S - S_2) \rfloor$$

$$= r \lfloor M \times (S_1 \cap S_2) \rfloor + r \lfloor M \times S_2 \cdot (S_2 - S_1) \rfloor$$

$$+ \mu \lfloor M \cdot (S - (S_1 \cup S_2)) \rfloor + \mu \lfloor M \times (S_1 \cup S_2) \cdot (S_1 - S_2) \rfloor .$$

Clearly it is sufficient for us to show that

$$r \lfloor M \times S_2 \cdot (S_2 - S_1) \rfloor \geq \mu \lfloor M \times (S_1 \cup S_2) \cdot (S_2 - S_1) \rfloor$$

and

$$\mu \lfloor M \times (S_1 \cup S_2) \cdot (S_1 - S_2) \rfloor > r \lfloor M \times S_1 \cdot (S_1 - S_2) \rfloor$$

Now by using Axiom System 1 and Theorem T3, we can see that

$$r \lfloor M \times S_2 \cdot (S_2 - S_1) \rfloor \geq r \lfloor M \times (S_1 \cup S_2) \cdot (S_2 - S_1) \rfloor \quad .. \quad (4)$$

Now $M \times (S_1 \cup S_2) \cdot (S_2 - S_1) = M \cdot (S - S_1) \times (S_2 - S_1)$.

Hence, by Corollary 1 of Theorem 5.1 using Theorem T7 -(10)

$$r \lfloor M \times (S_1 \cup S_2) \cdot (S_2 - S_1) \rfloor \geq \mu \lfloor M \times (S_1 \cup S_2) \cdot (S_2 - S_1) \rfloor$$

Hence,

$$r \lfloor M \times S_2 \cdot (S_2 - S_1) \rfloor \geq \mu \lfloor M \times (S_1 \cup S_2) \cdot (S_2 - S_1) \rfloor .$$

Next using the same argument as in (A) above,

$$\mu [M \times (S_1 \cup S_2) \cdot (S_1 - S_2)] \geq \mu [M \times S_1 \cdot (S_1 - S_2)]$$

By Theorem 5.1

$$\mu [M \times S_1 \cdot (S_1 - S_2)] > r [M \times S_1 \cdot (S_1 - S_2)],$$

and therefore

$$\mu [M \times (S_1 \cup S_2) \cdot (S_1 - S_2)] > r [M \times S_1 \cdot (S_1 - S_2)] \quad \ldots (1)$$

Hence

$$r [M \times S_1] + \mu [M \cdot (S - S_1)] \leq r [M \times S_2] + \mu [M \cdot (S - S_2)]$$

Also noting (1) we find that the equality clearly holds only if $S_1 - S_2 = \Phi$.

Q.E.D.

Working with M^* in place of M we obtain the following Corollary.

<u>Corollary 1.</u> Let M be a matroid on S. Let

$S_1 = A [(M^*)^2] \cup C [(M^*)^2]$. Then if $S_2 \subseteq S$

$$r [M \times (S - S_1)] + \mu [M \cdot S_1] \leq r [M \times (S - S_2)] + \mu [M \cdot S_2].$$

The equality holds only if $S_2 \supseteq S_1$.

<u>Definition 5.1.</u> Let M be a matroid on S. The 'hybrid rank' $r_H(M)$ of the matroid is defined by

$$r_H(M) = \min_{S_1 \subseteq S} [r [M \times S_1] + \mu [M \cdot (S - S_1)]].$$

Definition 5.2. Let $N = (M_V, V, G, S)$ be a generalized network. Then the 'topological degree of freedom' $f_d(N)$ of N is defined by $f_d(N) = r_H(M)$.

Let M be a matroid on S. From the discussion of Case III of Section 2 of the previous chapter it is clear that a partition of S into S_1 and $S - S_1$, such that

$$r_H(M) = r[M \times S_1] + \mu[M \cdot (S - S_1)],$$

reduces the computation required to analyse the network. However, in general, there might be a number of such partitions possible for the matroid. As the number $|E \cup C|$ (defined in the previous chapter) might vary to some extent for different partitions, it would be useful to have a method for generating all such partitions. Our next theorem shows that we have already developed such a method in this chapter.

Theorem 5.3. Let M be a matroid on S. Let $P_1, P_2, \ldots P_n$ be its P-sequence. Let k ($k \leq n$) be such that the matroid $M_1 = M \times (\bigcup_{i=1}^{k} P_i) \cdot P_k$ satisfies $d(M_1) = 2$. Let S^1 be the set of atoms of M_1 and L be the usual partial order on S^1. Now suppose

$$r_H(M) = r[M \times T] + \mu[M \cdot (S - T)],$$

$T \subseteq S$ and $T \neq A(M^2) \cup C[M^2]$). Then there exists $R^1 \subseteq S^1$ such that for every element $S_d \in R^1$ all elements $S_a \leq S_d$ in L are also members of R^1, and

$$T = A(M^2) \cup C(M^2) \cup \left\{ \bigcup_{S_a \in R^1} S_a \right\}.$$

__Proof.__ Let $S_o = A(M^2) \cup C(M^2)$. Suppose T of the Theorem is such that $T \neq S_o$. Then by Theorem 5.2, $T \supset S_o$, and by its Corollary 1, $(S - T) \supset A[(M^*)^2] \cup C[(M^*)^2]$. We have

$$r[M \times T] + \mu[M \cdot (S-T)] = r[M \times S_o] + r[M \times T \cdot (T-S_o)] + \mu[M \cdot (S-T)].$$

$$r[M \times S_o] + \mu[M \cdot (S - S_o)] = r[M \times S_o] + \mu[M \cdot (S-T)] + \mu[M \cdot (S-S_o) \times (T-S_o)]$$

$$= r[M \times S_o] + \mu[M \cdot (S-T)] + \mu[M \times T \cdot (T - S_o)].$$

Hence

$$r[M \times T \cdot (T - S_o)] = \mu[M \times T \cdot (T - S_o)]$$

Now

$$(M \times (\bigcup_{i=1}^{k} P_i) \cdot P_k) \times (T - S_o)$$

$= M \times T \cdot (T - S_o)$ by using Theorem T7-(3) and T7-(4). Hence

$M_1 \times (T - S_o) = M \times T \cdot (T - S_o)$ and therefore

$$d[M_1 \times (T - S_o)] = 2.$$

Hence,

$T - S_o = \left\{ d \mid d \text{ approachable in } M_1 \text{ from } e,\ e \in (T-S_o) \right\}$

by Lemma 4.1.

Clearly this implies the existence of R^1 with the required properties.

Q.E.D.

The above theorem indicates that the construction of S^1 for M_1 suffices to determine the set of all partitions which correspond to the hybrid rank of the matroid. We point out here that Algorithms 4.1 and 4.2, when applied to graphs whose polygon matroids are of kind (2), reduce essentially to the algorithm due to T. Ohtsuki et al [Oht 1]. Theorem 5.3 in the case of graphs should, therefore, be credited to them, though of course they have not stated their results in our notation.

Q.E.D.

Example 2.5.1. Let G be the graph of a network N. (See Figure 2.5.1). The base t_1 (tree) with respect to which we construct the matrices R and R^* being the following

$$t_1 = \{1,2,3,4,11,12,13,17,18,19,23,24,25,26,27\} .$$

We give different partitions of S corresponding to $r_H(M)$ and the corresponding values of $|E \cup C|$ below.

(1) $S_1 = \{1,2,3,4 \ldots 10\}$ $S_2 = S - S_1$

$E \cup C = \{14,15,3,4\}$, $|E \cup C| = 4$

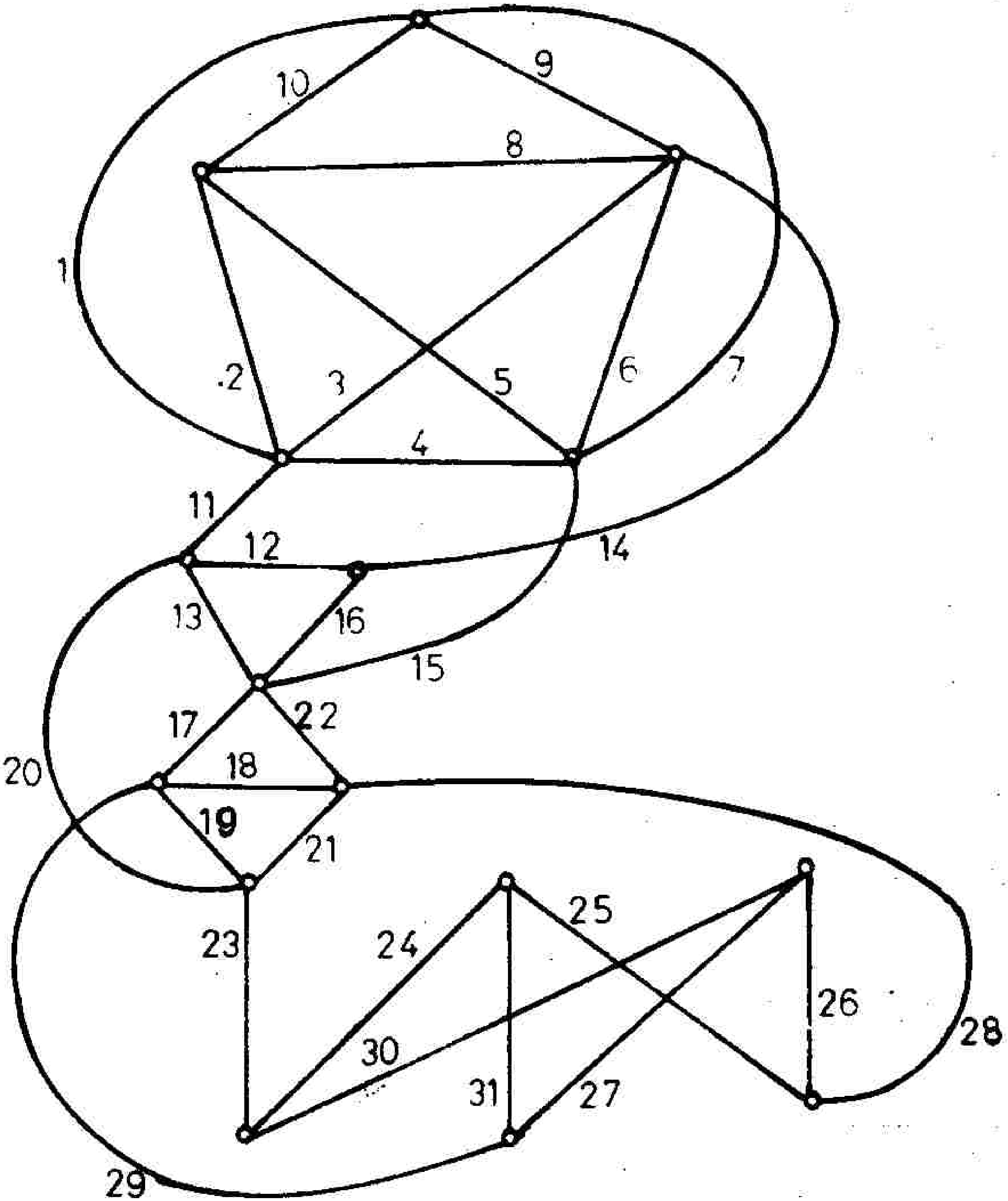


FIG. 2.5.1. THE GRAPH OF N.

(2) $S_1' = \{1,2,3,4 \ldots 16\}$ $S_2' = S - S_1'$

$EUC = \{20,13\}$. $|EUC| = 2$.

(3) $S_1'' = \{1,2,3,\ldots 22\}$ $S_2'' = S - S_1''$

$EUC = \{28,29,18,19\}$, $|EUC| = 4$.

In each of the above cases,

$$r[M \times S_1] + \mu[M.(S - S_1)] =$$

$$= r[M \times S_1'] + \mu[M.(S - S_1')] =$$

$$= r[M \times S_1''] + \mu[M.(S - S_1'')] = 14$$

$$(M = Pol[0]) .$$

Section 6 : Single Element Extension

In this section we study the relation between the P-sequence of a single element extension and the P-sequence of the original matroid.

The results obtained, however, are only partial. The methods that we use seem incapable of describing the situation completely. However such results as we have obtained seem to be of some practical use in studying the invariant sets of the new matroid in the above case in terms of the invariant sets of the old matroids.

Definition 6.1. A matroid M_e on S_e is said to be a single element extension (s.e.e.) of a matroid M on S iff

(1) $S_e = S \cup e$, $e \notin S$.

(2) $M_e \times S = M$.

Notation. (1) The letter 'e' will always be used in this section for the only element in $S_e - S$.

(2) We use δ_n for the admissible function such that

$$\delta_n(M) = \lfloor (M^{n-1})^* \vee M^* \rfloor$$

for every matroid M on S and for every positive integer n > 1. We have chosen to work with $(M^{n-1})^* \vee M^*$ in stead of $((M^{n-1})^*)^n$. This should however cause no trouble since by Theorem 3.7

$$B[((M^{n-1})^*)^n] = B[(M^{n-1})^* \vee M^*] .$$

We first prove a series of Lemmas which are needed for the proof of the main theorem.

<u>Lemma 6.1.</u> Let M be a matroid on S. Let $R \subseteq S$. Then

(1) $A[(M \times R)^n] \subseteq A[M^n]$ where n is a positive integer

(2) $B[\delta_n(M \times R)] \subseteq B[\delta_n(M)]$

where n is a positive integer greater than one.

<u>Proof.</u> (1) is obvious since $(M \times R)^n = M^n \times R$.

(2) From Corollary 4 of Theorem 1.1 we know that any set independent in $[((M^{n-1})^* . R) \vee (M^* . R)]$ is independent in $[(M^{n-1})^* \vee M^*] . R$, and hence in $((M^{n-1})^* \vee M^*)$. But

$$((M^{n-1})^* . R) \vee (M^* . R) = ((M^{n-1} \times R)^* \vee (M \times R)^*)$$

(by Theorem T7-(8))

$$= [((M \times R)^{n-1})^* \vee (M \times R)^*]$$

by Corollary 4 of Theorem 1.1 of Chapter 2. Hence

$$B(\delta_n(M \times R)) \subseteq B(\delta_n(M)) .$$

Q.E.D.

<u>Lemma 6.2.</u> Let M be a matroid on S. Let $T \subseteq S$ such that $M \times T$ has k p.d. bases. Then $T \subseteq B[\delta_n(M)]$.

<u>Proof.</u> Using Theorem 3.9 we can see that

$$B[\delta_n(M \times T)] = T$$

Since from Lemma 6.1 we have

$B[\delta_n(M \times T)] \subseteq B[\delta_n(M)]$, it follows that

$$T \subseteq B[\delta_n(M)] .$$

Q.E.D.

<u>Lemma 6.3.</u> Let M be a matroid on S and let M_e on S_e be its s.e.e. Then

$$A[M^n] = A[M_e^n] \text{ iff } e \notin A[M_e^n].$$

<u>Proof.</u> Let $T = A[M_e^n]$. From Corollary 4 of Theorem 1.1 we know that

$$(M_e \times T)^n = (M_e^n) \times T .$$

Hence $A[M_e \times T]^n = A[(M_e^n) \times T] = T$

Now if $e \notin T$, $M_e \times T = M \times T$.

Hence $A[(M_e \times T)^n] = A[(M \times T)^n] = T$.

But by Lemma 6.1,

$$A[(M \times T)^n] \subseteq A[M^n]$$

Hence $T \subseteq A[M^n]$.

But $M = M_e \times S$. Hence, $A[(M_e \times S)^n] \subseteq A[M_e^n]$

Thus $A[M^n] \subseteq T$ and it follows that

$$A[M^n] = A[M_e^n] .$$

The necessity of the condition is obvious.

Q.E.D.

<u>Lemma 6.4.</u> Let M be a matroid on S and let M_e be its s.e.e. on S_e.

Then $(A(M_e^n) - e) \subseteq B(\delta_n(M))$.

<u>Proof.</u> Let $T = A(M_e^n) - e$.

Then it is easy to see that

$M_e \times T = M \times T$ has n p.d. bases such that e does not belong to any of them.

Hence $\bar{T} \subseteq B[\delta_n(M)]$ by Lemma 6.2.

Q.E.D.

<u>Note</u> : When we write $\bar{T}$ we mean the closure of T in M_e. This convention we will follow until the end of Theorem 6.3.

Lemma 6.5. Let M be a matroid on S and M_e be its s.e.e. on S_e. Then $e \in A(M_e^n)$ iff $e \in \overline{B[\delta_n(M)]}$.

<u>Proof.</u> If $e \in A(M_e^n)$ it is clear that $e \in \overline{(A(M_e^n) - e)}$ since there exists a base b of $(M_e \times (A(M_e^n)))^n$ such that e does not belong to it.

Hence by Lemma 6.4,

$$e \in \overline{B[\delta_n(M)]} .$$

Next let $e \in \overline{B[\delta_n(M)]}$.

Let $(e \cup B[\delta_n(M)]) = T$.

Then by Lemma 3.2, $M \times (T - e)$ has n p.d. bases $b_1, b_2 \ldots b_n$. Hence $M_e \times T$ has n p.d. bases $b_1, b_2, \ldots b_n$ such that $e \notin \bigcup_{i=1}^{n} b_i$. Consider the set R of all elements accessible from e with respect to $\bigcup_{i=1}^{n} b_i$. Then it is clear that $A[(M_e \times R)^n] = R$, by using Theorem 3.9. Hence by Lemma 6.1, $R \subseteq A(M_e^n)$ and hence $e \in A(M_e^n)$.

Q.E.D.

<u>Lemma 6.6.</u> Let M be a matroid on S, and let M_e be its s.e.e. on S_e. If $e \in \overline{B[\delta_n(M)]}$, then $B[\delta_n(M_e)] - e \subseteq B[\delta_n(M)]$

<u>Proof.</u> By Lemma 6.5, if $e \in \overline{B[\delta_n(M)]}$ then $e \in A(M_e^n)$. Let $T = B[\delta_n(M_e)]$. Then $(M_e \times T)^n = M_e^n \times T$ by Corollary 4 of Theorem 1.1.

Now $A(M_e^n) = A(M_e^n \times T)$

since by Theorem 3.7, $B[\delta_n(M_e)] \supseteq A(M_e^n)$.

Hence $A(M_e^n) = A[(M_e \times T)^n]$.

Hence there exists a base b of $(M_e \times T)^n$ such that $e \notin b$. Using Lemma 3.2 we, therefore, conclude that there exists n p.d. bases $b_1, b_2 \ldots b_n$ for $(M_e \times T)$ such that $e \notin \bigcup_{i=1}^{n} b_i$.

It follows that $M_e \times (T-e) = M \times (T-e)$ has n p.d. bases $b_1, b_2 \ldots b_n$.

Hence, $B[\delta_n(M)] \supseteq T - e$ by Theorem 3.9.

Q.E.D.

<u>Lemma 6.7.</u> Let M be a matroid on S and let M_e be its s.e.e. on S_e. If $e \in \overline{A(M^n)}$, then $A(M_e^{\,n}) - e = A(M^n)$.

<u>Proof.</u> $e \in \overline{A(M^n)}$ in M_e and hence $e \in \overline{B[\delta_n(M)]}$. Hence by Lemma 6.6, $B[\delta_n(M_e)] - e \subseteq B[\delta_n(M)]$.

By Lemma 3.2 it is possible to choose n p.d. bases for

$$M \times [B(\delta_n(M))] = M_e \times [B(\delta_n(M))] .$$

Hence $M_e \times [B(\delta_n(M_e))]$ has n p.d. bases $b_1, b_2 \ldots b_n$ such that e does not belong to any of them. Now since $e \in \overline{A(M^n)}$ all the elements accessible from $(B[\delta_n(M_e)] - \bigcup_{i=1}^{n} b_i)$ with respect to $(\bigcup_{i=1}^{n} b_i)$ are contained in $A(M^n) \cup e$.

Hence $A(M^n) \cup e \supseteq A[(M_e \times (B(\delta_n(M_e)))^n] = A(M_e^{\,n})$

Hence $A(M^n) \cup e \supseteq A(M_e^{\,n})$.

But $A(M_e^{\,n}) \supseteq A(M^n)$ by Lemma 6.1.

Hence $A(M_e^{\,n}) - e = A(M^n)$.

Q.E.D.

We now prove our main theorem.

<u>Theorem 6.1.</u> Let M be a matroid on S, and let M_e be its s.e.e. on S_e.

(1) If $e \notin \overline{A(M^n)}$ but $e \in \overline{B[\delta_n(M)]}$,

then $A(M_e^{n+1}) = A(M^{n+1})$ and

$$S_e - B[\delta_n(M_e)] = S - B[\delta_n(M)] .$$

(2) If $e \notin \overline{B[\delta_n(M)}$ but $e \in \overline{A(M^{n-1})}$,

then $A(M_e^{n}) = A(M^{n})$ and

$$S_e - A(M_e^{n-1}) = S - A(M^{n-1}) .$$

Proof. (1) If $e \notin \overline{A(M^n)}$, then $e \notin \overline{B[\delta_{n+1}(M)]}$.

Hence by Lemma 6.5 $e \notin A(M_e^{n+1})$.

Thus by Lemma 6.3, $A(M_e^{n+1}) = A(M^{n+1})$.

Since $e \in \overline{B[\delta_n(M)]}$, by Lemma 6.6 we have

$B[\delta_n(M_e)] - e \subseteq B[\delta_n(M)]$ and by Lemma 6.5 we have

$$e \in A(M_e^n) \subseteq B[\delta_n(M_e)] .$$

Hence by Lemma 6.1, it follows that

$$B[\delta_n(M_e)] - e = B[\delta_n(M)] .$$

Therefore, $S_e - B[\delta_n(M_e)] = S - B[\delta_n(M)]$.

(2) Since $e \notin \overline{B[\delta_n(M)]}$, by Lemma 6.5 we have $e \notin A(M_e^n)$.

Hence by Lemma 6.3, $A(M_e^n) = A(M^n)$.

Since $e \in \overline{A(M^{n-1})}$, by Lemma 6.7, we have,

$$A(M_e^{n-1}) - e = A(M^{n-1}) .$$

Also by Lemma 6.5, $e \in A[M_e^{n-1}]$.

Hence $S_e - A(M_e^{n-1}) = S - A(M^{n-1})$.

Q.E.D.

We can now describe the P-sequence of M_e to some extent. We collect the relevant information in the form of two theorems. First, however we state a simple lemma.

Lemma 6.8. Let M be a matroid on S and M_e be its s.e.e. on S_e . Then, if $P \subseteq S$

(1) $M_e \times P = M \times P$

and further if $S_e - P$ is closed in M_e

(2) $M_e . P = M.P.$

Proof. The lemma follows directly from the definitions of contraction, reduction and closure.

Q.E.D.

Theorem 6.2. Let M be a matroid on S and let M_e be its s.e.e. on S_e. Let $e \notin \overline{A(M^n)}$ but $e \in \overline{B[\delta_n(M)]}$.

Let $P_1, P_2 \ldots P_n$ be the P-sequence of M and let $A(M^{n+1}) = \bigcup_{i=1}^{s} P_i$

and $S - B[\delta_n(M)] = \bigcup_{i=t}^{n} P_i$.

If $P_1^1, P_2^1 \ldots P_m^1$ is the P-sequence of M_e ,

(1) When $i \in \{1,2 \ldots s\}$ then $P_i^1 = P_i$ and

$$M \times \left(\bigcup_{j=1}^{i} P_j \right) . P_i = M_e \times \left(\bigcup_{j=1}^{i} P_j^1 \right) . P_i^1 .$$

(2) When $i \in \{0,1, \ldots n-t\}$ then

$P^1_{m-i} = P_{n-i}$ and

$$M \times \left(\bigcup_{j=1}^{n-i} P_j \right) . P_{n-i} = M_e \times \left(\bigcup_{j=1}^{m-i} P_j^1 \right) , P^1_{m-i} .$$

Proof. The theorem follows from Theorem 6.1, Lemma 6.8 and and Corollary 1 of Theorem 2.6 using Theorem T7.

Q.E.D.

We can similarly prove the following theorem :

Theorem 6.3. Let M be a matroid on S and let M_e be its s.e.e. on S_e .

Let $e \notin \overline{B \lbrack \delta_n (M) \rbrack}$ but $e \in \overline{A (M^{n-1})}$.

Let $P_1, P_2 \ldots P_n$ be the P-sequence of M and let

$$A (M^n) = \bigcup_{i=1}^{s} P_i \quad \text{and} \quad S - A(M^{n-1}) = \bigcup_{i=t}^{n} P_i .$$

If $P_1^1, P_2^1 , \ldots P_m^1$ is the P-sequence of M_e

(1) When $i \in \{1,2 \ldots s\}$ then $P_i^1 = P_i$ and

$$M \times \left(\bigcup_{j=1}^{i} P_j \right) . P_i = M_e \times \left(\bigcup_{j=1}^{i} P_j^1 \right) . P_i^1 .$$

(2) When $i \in \{0, 1, \ldots n - t\}$ then

$$P^{i}_{m-i} = P_{n-i} \quad \text{and}$$

$$M \times \left(\bigcup_{j=1}^{n-i} P_j \right) . P_{n-i} = M_e \times \left(\bigcup_{j=1}^{m-i} P_j^{\,i} \right) . P^{i}_{m-i} .$$

The case where M_e on S_e is such that $M_e . S_e = M$ is treated in terms of M^* and $M_e^{\,*}$.

CHAPTER 3

APPLICATIONS OF THE MATROID UNION THEOREM

A large part of the material so far presented in this thesis revolves around the matroid union theorem. We would now like to stress the prominent role it plays in other branches of matroid theory (primarily in the theory of gammoids, transversal and base-orderable matroids). Sections 1 and 2 are concerned with transversal theory. Here we prove some well known results in transversal theory by first proving the corresponding results for the union of two matroids one of which has rank one. In Section 3 we prove some new results on gammoids and base orderable matroids.

Convention : We have defined matroid union and matroid intersection only for matroids on the same set. If we wish to consider matroids on different sets we proceed as follows :

In order to define consistently the union $M_1 \vee M_2$ of matroids M_1 and M_2 in this case we look upon M_1 and M_2 as matroids defined on $S_1 \cup S_2$ but $S_2 - S_1$ as a set of loops of M_1 and $S_1 - S_2$ as a set of loops of M_2. However, to define the intersection $M_1 \wedge M_2$ in this case we treat $S_2 - S_1$ as a set of coloops of M_1 and $S_1 - S_2$ as a set of coloops of M_2.

Section 1 : A Special Case

In this section we consider the case of the union of two matroids one of which has rank one. Mostly we are interested in the permissible structural changes in the matroid of rank one in order that the matroid union might be invariant.

Theorem 1.1. Let M_1 and M_2 be two matroids defined on S, with $r(M_2) = 1$ and $r(M_1 \vee M_2) = r(M_1) + 1$.
Let $C(M_2) = S_1$ and $S - S_1 = S_2$.
Let M_2' be a matroid of rank one on S with $C(M_2') = S_1'$.
(i) Let $S_1' \supseteq S_1$. Then $M_1 \vee M_2 = M_1 \vee M_2'$ iff $(S_1' - S_1)$ is independent in $M_1 . S_2$.
(ii) Let $S_1' \subseteq S_1$. Then $M_1 \vee M_2 = M_1 \vee M_2'$ iff $(S_1 - S_1')$ is a set of coloops for $(M_1 \vee M_2) \times S_1$.

Proof. Let $M_3 = M_1 \vee M_2$.
(i) Let $S_1' - S_1$ be independent in $M_1 . S_2$ and let b be any base of M_3.

Then $b = b_1 \cup b_2$ where b_1 is a base of M_1 and b_2 is a base of M_2. We know that b_2 is a singleton. If $b_2 \not\subseteq (S_1'-S_1)$, then b is obviously a base of $M_1 \vee M_2'$. Now let $b_2 = \{a\}$ and suppose $a \in (S_1' - S_1)$. Now $a \notin b_1$ since $r(M_3) = r(M_1) + 1$. Hence $a \cup b_1$ contains a circuit say C of M_1. Suppose $C \subseteq S_1'$. Then $C \cap S_2 = C \cap (S_1' - S_1)$ is dependent in $M_1 . S_2$. This is a contradiction. Hence $C \nsubseteq S_1'$ and $C \cap (S - S_1') \neq \Phi$.

Let $d \in C \cap (S - S_1')$.

Now $b = [(b_1 - d) \cup a] \cup d$.

By Theorem T1 we know that $[(b_1 - d) \cup a]$ is a base of M_1. Since $d \in (S - S_1')$ we have d as a base of M_2'. Hence b is a base of $M_1 \vee M_2'$. Thus every base of $M_1 \vee M_2$ is a base for $M_1 \vee M_2'$. That every base of $M_1 \vee M_2'$ is a base of M_3 is clear since $S_1' \supseteq S_1$.

Hence $M_3 = M_1 \vee M_2'$.

Suppose $S_1' - S_1$ is not independent in $M_1 \cdot S_2$. Then there exists a circuit C of $M_1 \cdot S_2$ such that $C \subseteq (S_1' - S_1)$. Let $a \in C$. Now $(C - a)$ is independent in $M_1 \cdot S_2$. Choose a base b_1 of M_1 such that $b_1 \cap S_2$ is a base of $M_1 \cdot S_2$ and $b_1 \supseteq (C - a)$. Let $b_2 = \{a\}$. Then b_2 is obviously a base of M_2. Hence $b = b_1 \cup b_2$ is a base of M_3. Suppose b is a base of $M_1 \vee M_2'$. Then $b = b_1' \cup b_2'$, where $b_2' = \{d\}$, $d \in (S - S_1')$ and $b_1' = (b_1 - d) \cup a$ is a base of M_1. But if b_1' is a base of M_1, then d belongs to the fundamental circuit C_1 of a with respect to the base b_1 of M_1. But clearly $C_1 \cap S_2 = C$. Hence $d \in (S_1' - S_1)$. This is a contradiction. Therefore b is not a base of $M_1 \vee M_2'$ and $M_1 \vee M_2' \neq M_3$.

(ii) If $S_1' \subseteq S_1$ we can reverse the roles of M_2' and M_2 in (i) above. We then have $M_3 = M_1 \vee M_2'$ iff $(S_1 - S_1')$ is independent in $M_1 \cdot (S - S_1')$. Hence $M_3 = M_1 \vee M_2'$ iff $(S_1 - S_1')$ is independent in $M_1 \cdot (S - S_1') \times (S_1 - S_1')$ i.e. iff $(S_1 - S_1')$ is independent in $M_1 \times S_1 \cdot (S_1 - S_1')$ [by the use of Theorem T7 - 4] i.e. iff $(S_1 - S_1')$ is a set of coloops in $M_1 \times S_1$.

But $M_3 \times S_1 = (M_1 \times S_1) \vee (M_2 \times S_1) = M_1 \times S_1$, by Corollaries 1,4 of Theorem 1.1 of Chapter 2. Hence $M_3 = M_1 \vee M_2'$ iff $(S_1 - S_1')$ is a set of coloops in $M_3 \times S_1$.

Q.E.D.

<u>**Theorem 1.2.**</u> Let M_1, M_2 be matroids on S with $r(M_2) = 1$. Let $M_3 = M_1 \vee M_2$. If $S_1 = C(M_2)$, then S_1 is closed in M_3.

Proof.
$$M_3 \times S_1 = (M_1 \times S_1) \vee (M_2 \times S_1)$$
$$= M_1 \times S_1$$

(using Corollaries 1,4 of Theorem 1.1 of Chapter 2).

Let $(S - S_1) = S_2$. Now let C be a circuit of M_3 such that $C \cap S_2 = \{a\}$. Then $C \cap S_1$ is independent in $M_3 \times S_1$ and therefore in $M_1 \times S_1$ and hence in M_1. But $a \in S_2$ and hence $\{a\}$ is a base for M_2. Hence $((C \cap S_1) \cup a)$ is independent in M_3. Thus C is independent in M_3 - which is a contradiction. Hence there exists no circuit C of M_3 such that $C \cap S_2 = \{a\}$ and therefore S_1 is closed in M_3.

Q.E.D.

<u>**Theorem 1.3.**</u> Let M_1, M_2, M_2' be matroids on S with

$r(M_2) = r(M_2') = 1$ and $M_1 \vee M_2 = M_1 \vee M_2'$. Let $C(M_2) = S_1$, $C(M_2') = S_1'$. If M_2'' is the matroid of rank one on S such that $C(M_2'') = (S_1 \cap S_1')$, then

$$M_1 \vee M_2'' = M_1 \vee M_2 = M_1 \vee M_2' .$$

<u>Proof.</u> It is easy to see that any base of $M_1 \vee M_2$ is a base of $M_1 \vee M_2''$. Let b be any base of $M_1 \vee M_2''$. Then $b = b_1 \cup b_2$ where b_1 is a base of M_1 and $b_2 = \{a\}$, $a \in [S - (S_1 \cap S_1')]$.

But this means $a \in (S - S_1)$ or $a \in (S - S_1')$. Suppose $a \in (S - S_1)$. Then clearly b is a base of $M_1 \vee M_2$. If $a \in (S - S_1')$ then b is a base of $M_1 \vee M_2'$. But $M_1 \vee M_2' = M_1 \vee M_2$. Hence b is a base of $M_1 \vee M_2$. Thus every base of $M_1 \vee M_2''$ is a base of $M_1 \vee M_2$ and $M_1 \vee M_2 = M_1 \vee M_2'' = M_1 \vee M_2$

Q.E.D.

The following Corollary is obvious.

<u>Corollary 1.</u> Let M_1, M_2 be matroids on S, with $r(M_2) = 1$. There exists a unique matroid M_2' of rank one on S such that

(1) $M_1 \vee M_2' = M_1 \vee M_2$

(2) If M_2'' is any matroid of rank one on S such that $M_1 \vee M_2'' = M_1 \vee M_2$ then $C(M_2') \subseteq C(M_2'')$.

<u>Definition 1.1.</u> Let M_1 be a matroid on S and M_2 be a matroid of rank one on S such that if M_2' is any matroid of rank one on S

such that if M_2' is any matroid of rank one on S with $M_1 \vee M_2 = M_1 \vee M_2'$, then $C(M_2') \subsetneq C(M_2)$. We then say that M_2 is maximal with respect to M_1. [We note that Corollary 1 of Theorem 1.3 states that the maximal matroid M_2 with respect to M_1 such that $M_1 \vee M_2 = M_1 \vee M_2'$ is unique].

Definition 1.2. Let M_1 be a matroid on S and M_2 be a matroid of rank one on S such that if M_2' is any matroid of rank one on S with $M_1 \vee M_2 = M_1 \vee M_2'$, then $C(M_2) \subsetneq C(M_2')$. We then say that M_2 is minimal with respect to M_1. But this minimal matroid is not necessarily unique.

Theorem 1.4. Let M_1, M_2 be matroids on S with M_2 of rank one such that M_2 is minimal with respect to M_1. Then $S - C(M_2)$ is a bond of $M_1 \vee M_2$.

Proof. By Theorem 1.2, $C(M_2)$ is closed in $M_1 \vee M_2$. By Theorem 1.1 and the minimality of M_2 it follows that $C(M_2)$ contains a base of M_1. Then $r[(M_1 \vee M_2) \times C(M_2)] =$
$= r[M_1 \vee M_2] - 1$.

Since $C(M_2)$ is closed in $M_1 \vee M_2$, it now follows from Theorem T10 that $S - C(M_2)$ is a bond of $M_1 \vee M_2$.

Q.E.D.

Theorem 1.5. Let M_1, M_2, M_2' be matroids on S with $r(M_2) = r(M_2') = 1$ and $M_1 \vee M_2 = M_1 \vee M_2'$. Also let M_2 and M_2' be minimal with respect to M_1.
Then $|S - C(M_2)| = |S - C(M_2')|$.

Proof. Let M_2'' be a matroid of rank one such that

$$C(M_2'') = (C(M_2)) \cap (C(M_2')).$$

Then by Theorem 1.3, $M_1 \vee M_2 = M_1 \vee M_2''$. Hence by Theorem 1.1 $(C[M_2] - C[M_2''])$ and $(C[M_2'] - C[M_2''])$ are bases in $M_1 \cdot [S - C(M_2'')]$. Hence it follows that $|S - C(M_2)| = |S - C(M_2')|$.

Q.E.D.

Theorem 1.6. Let M_1, M_2 be matroids on S with $r(M_2) = 1$. Let M_2 be maximal with respect to M_1. Let ρ be the rank function of $M_1 \vee M_2$. Suppose M_2' is any matroid of rank one on S such that $M_1 \vee M_2 = M_1 \vee M_2'$. Then

$$|S - C(M_2')| = |S - C(M_2)| - [\rho(C(M_2')) - \rho(C(M_2))].$$

Proof. By Theorem 1.1, $C(M_2') - C(M_2)$ is a set of coloops for $[(M_1 \vee M_2) \times C(M_2')]$. The theorem is now immediate.

Q.E.D.

Corollary 1. Let M_1, M_2 be matroids on S, with M_2 a matroid of rank one. Let M_2 be maximal with respect to M_1. Let ρ be the rank function of $M_1 \vee M_2$. Now if M_2' is any matroid of rank one on S such that $M_1 \vee M_2 = M_1 \vee M_2'$, then $(S - C(M_2'))$ has the maximum cardinality among all subsets of $(S - C(M_2))$ whose complements have rank $\rho[C(M_2')]$ in $M_1 \vee M_2$.

Proof. If A is any subset of $S - C(M_2)$ such that $\rho(S-A) = \rho[C(M_2')]$, then clearly

$$|S - A| \geq |C(M_2)| + [\, r(S-A) - r(C[M_2])\,].$$

Therefore,

$$|A| \leq |S - C(M_2)| - [\, r[C(M_2')] - r[C(M_2)]\,]$$

The result is now immediate from Theorem 1.6.

Q.E.D.

Theorem 1.7. Let M_1, M_2 be matroids on S with $r(M_2) = 1$. Let M_2 be maximal with respect to M_1. Then $C(M_2)$ is closed in $M_1 \vee M_2$ and $(M_1 \vee M_2) \times (C[M_2])$ has no coloops.

Proof. By Theorem 1.2, $C(M_2)$ is closed in $M_1 \vee M_2$. Suppose S_3 is a set of coloops for $(M_1 \vee M_2) \times [C(M_2)]$. Then $M_1 \vee M_2 = M_1 \vee M_2'$ where M_2' is a matroid of rank one on S and $C(M_2') = C(M_2) - S_3$. This contradicts the maximality of M_2 with respect to M_1. Hence the theorem holds.

Q.E.D.

Theorem 1.8. Let M_1, M_1', M_2, M_2', M_2'' be matroids on S, with M_2, M_2', M_2'' being matroids of rank one. Further let M_2, M_2' be maximal with respect to M_1, M_1' respectively and

$$M_1 \vee M_2 = M_1' \vee M_2' = M_1 \vee M_2'' = M_1' \vee M_2''.$$

Then $M_2 = M_2'$.

Proof. From Theorem 1.1 it is clear that both $C(M_2)$ and $C(M_2')$ are obtained from $C(M_2'')$ by removing the coloops of $(M_1 \vee M_2) \times [C(M_2'')]$. Hence $M_2 = M_2'$.

Q.E.D.

<u>Theorem 1.9.</u> Let M_1, M_2 be matroids on S with $r(M_2) = 1$. Let M_2 be maximal with respect to M_1. Let M_2' be any matroid of rank one on S.

Then $M_1 \vee M_2 = M_1 \vee M_2'$ iff

(1) $C(M_2') \supseteq C(M_2)$ and

(2) $M_1 \times [C(M_2')]$ has $C(M_2') - C(M_2)$ as its set of coloops.

<u>Proof.</u> Let us denote $C(M_2)$ by S_1 and $C(M_2')$ by S_1'. By the definition of maximality of M_2 with respect to M_1 and by Theorem 1.1 we have,

$M_1 \vee M_2 = M_1 \vee M_2'$ iff (a) $S_1' \supseteq S_1$ and

(b) $(M_1 \vee M_2') \times S_1'$ has $S_1' - S_1$ as its set of coloops.

But $(M_1 \vee M_2') \times S_1' = M_1 \times S_1' \vee M_2' \times S_1'$ and $M_2' \times S_1'$ is the matroid M_0 on S_1'.

Hence $(M_1 \vee M_2') \times S_1' = M_1 \times S_1'$.

Thus the theorem follows.

Q.E.D.

Section 2 : Transversal Matroids

We will now prove some results on the structure of transversal matroids using the theorems of last section. Most existing proofs of these results do not make explicit use of the matroid union theorem (al though implicit use is often made). Our aim is to show that matroid union theorem makes these results appear quite natural. We have, however, not attempted to be exhaustive. Except for Theorem 2.6 all the results stated in this section are quite well known. As usual we consider only finite matroids.

The following definition serves to distinguish between families and sets. The need for distinguishing between families and sets is clear since the elements of a family may 'repeat'.

Definition 2.0. Let E and I be sets. Let $\mathcal{U} : I \to E$ be a mapping, and write $\mathcal{U}(i) = x_i$ for all $i \in I$. We call $\mathcal{U}$ a family of elements of E indexed by I. In what follows E will usually be a collection of subsets of a set S and I a finite subset of natural numbers. We will denote a family $\mathcal{U}$ as $(A_i : i \in I)$ or as $(A_{i_1}, A_{i_2} \ldots A_{i_n})$ where $i_j \in I$ for all $j \in \{1,2 \ldots n\}$. We say that an element x of E belongs to $\mathcal{U}$ or is an element of $\mathcal{U}$ iff $x_i = x$ for some $i \in I$. Since usually the elements of $\mathcal{U}$ are subsets of some set S we will speak of A_i as a subset of (or in) $\mathcal{U}$. If $\mathcal{U} = (A_i : i \in I)$ is a family and $I' \subseteq I$ we will call the family $\mathcal{U}' = (A_i : i \in I')$ a subfamily of $\mathcal{U}$.

Definition 2.1. Let $\mathfrak{U} = (A_i : i \in \{1,2 \ldots n\})$ be a finite family of subsets of a finite set S. A subset T of S is a transversal of $\mathfrak{U}$ iff there exists a bijection $\psi : T \to \{1,2 \ldots n\}$ such that $x \in A_{\psi(x)}$ for all $x \in T$. We call a subset R of S a partial transversal of $\mathfrak{U}$ iff R is a transversal of a subfamily of $\mathfrak{U}$. A subset R of S is a maximal partial transversal of $\mathfrak{U}$ iff R is a partial transversal of $\mathfrak{U}$ but R is not a proper subset of any partial transversal of $\mathfrak{U}$.

Unless otherwise stated all families and all sets considered henceforth in this section are finite.

In the next theorem we show that the collection of transversals of a family of subsets of a set S gives rise to a matroid.

The theorem is due to Edmonds and Fulkerson [Ed 3].

Theorem 2.1. Let $\mathfrak{U}$ be a family of subsets of a set S. Let I be the collection of all partial transversals of $\mathfrak{U}$. Then (S,I) is a matroid on S.

Proof. Let $\mathfrak{U} = (A_1, A_2 \ldots A_n)$, A_i being subsets of S. Let $M(A_i)$ for all $i \in \{1,2 \ldots n\}$ denote the matroid on S such that

(1) $r[M(A_i)] = 1$.

(2) $A_i = S - C[M(A_i)]$.

Consider the matroid $M = M(A_1) \vee \ldots \vee M(A_n)$. Any set R independent in M is the union of bases

$$b_{i_1} \cup b_{i_2} \cup \ldots \cup b_{i_r}$$

where $b_{i_1} = \{a_{i_1}\}$ is a base of $M(A_{i_1})$

$\vdots$

$b_{i_r} = \{a_{i_r}\}$ is a base of $M(A_{i_r})$.

Clearly R is a partial transversal of $\mathfrak{A}$.

Now consider any partial transversal R of $\mathfrak{A}$.

Let $R = \{a_{i_1}, a_{i_2} \ldots a_{i_k}\}$

where $a_{i_1} \in A_{i_1}, \ldots a_{i_k} \in A_{i_k}$.

Then $R = \{a_{i_1}\} \cup \ldots \cup \{a_{i_k}\}$.

Since $\{a_{i_1}\}$ is a base of $M(A_{i_1})$

$\vdots$

$\{a_{i_k}\}$ is a base of $M(A_{i_k})$,

we conclude that R is independent in M.

Q.E.D.

<u>Definition 2.2.</u> A matroid on S is a transversal matroid iff the class of its independent sets coincides with the class of

partial transversals of some family $\mathfrak{U}$ of subsets of S. Such a matroid, we will call the transversal matroid of $\mathfrak{U}$.

<u>Note</u> :- Our definition of a transversal matroid permits the presence of loops in the matroid. For instance, let $\mathfrak{U} = (A_1, A_2 \ldots A_n)$ be a family of subsets of a set S such that $S - (\bigcup_{i=1}^{n} A_i) \neq \Phi$. Then the transversal matroid of $\mathfrak{U}$ will be defined on S and therefore would have $(S - (\bigcup_{i=1}^{n} A_i))$ as the set of all its loops.

The following Corollary is now obvious.

<u>Corollary 1 of Theorem 2.1.</u> (D.J.A. Welsh)

A matroid is a transversal matroid iff it can be expressed as the union of matroids of rank at most one.

We need the following Lemma for the proof of the next theorem. We omit the obvious proof for the Lemma.

<u>Lemma 2.1.</u> Let M_1, M_2 be matroids defined on S. Then $r(M_1 \vee M_2) = r(M_1)$ implies

(1) $S - C(M_2) \subseteq B(M_1)$

(2) $M_1 \vee M_2 = M_1$.

<u>Theorem 2.2.</u> Let $\mathfrak{U} = (A_1, A_2 \ldots A_n)$ be a family of subsets of S. Let M be its transversal matroid on S. Then if $r(M)=k \leq n$, there exists a subfamily $\mathfrak{U}_1$ of $\mathfrak{U}$ which has M as its transversal matroid and has k subsets.

Also $k < n$ only if M has coloops.

Proof. $M = M(A_1) \vee \ldots \vee M(A_n)$.

By the use of Lemma 2.1 we can see that there exists a subfamily $\mathfrak{U}_1 = (A_{i_1}, A_{i_2} \ldots A_{i_k})$ such that

$$M = M(A_{i_1}) \vee \ldots \vee M(A_{i_k}) .$$

This proves the first part of the theorem. Now let $k < n$.

We know that there exists $\mathfrak{U}_1 = (A_{i_1}, \ldots A_{i_k})$ such that

$$M = M(A_{i_1}) \vee \ldots \vee M(A_{i_k})$$

Hence

$$M = M(A_{i_1}) \vee \ldots \vee M(A_{i_k}) \vee M(A_{i_{k+1}}) \vee \ldots \vee M(A_{i_n}) .$$

Hence $M = M \vee (M(A_{i_{k+1}}) \vee \ldots \vee M(A_{i_n}))$.

By Lemma 2.1 this is possible only if

$\bigcup_{i=k+1}^{n} A_i$ is a set of coloops for M.

Q.E.D.

<u>Notation</u> :- 1. To represent the transversal matroid of the family $\mathfrak{U} = (A_1, A_2 \ldots A_n)$ of subsets of a set S, we may write $M(A_1, A_2 \ldots A_n)$ or $M(\mathfrak{U})$ the matroid being defined on S. $S - \bigcup_{i=1}^{n} A_i$ is the set of all loops of $M(\mathfrak{U})$.

2. We will express the transversal matroid M repeatedly as the union of matroids of rank at most one. The transversal matroid M on S such that $M = M(A_1, A_2 \dots A_n)$ will often be expressed as

$$M(A_1) \vee \dots \vee M(A_n),$$

where $M(A_i)$ for all $i \in \{1, 2 \dots n\}$ is the matroid of rank at most one such that

$$C[M(A_i)] = S - A_i$$

The meaning of $M(A_i)$ would, therefore, depend on the set of definition of the transversal matroid M.

<u>Theorem 2.3.</u> Let $\mathfrak{U} = (A_1, A_2 \dots A_n)$ be a family of subsets of a set S. Let M be its transversal matroid. Let $r(M) = n$. If $\mathfrak{U}_1 = (A_{i_1}, \dots A_{i_k})$ is a subfamily of $\mathfrak{U}$ such that

$$\left| \bigcup_{j=1}^{k} A_{i_j} \right| = k \quad , \text{ then}$$

$\bigcup_{j=1}^{k} A_{i_j}$ is a set of coloops for M.

<u>Proof.</u> Let $\mathfrak{U}_1 = (A_{i_1}, \dots A_{i_k})$ be a subfamily of $\mathfrak{U}$ such that

$$\left| \bigcup_{j=1}^{k} A_{i_j} \right| = k \quad .$$

$$M = M(A_{i_1}) \vee \dots \vee M(A_{i_k}) \vee M(A_{i_{k+1}}) \dots \vee M(A_{i_n}).$$

Let $M_1 = M(A_{i_1}) \vee \ldots \vee M(A_{i_k})$,

$$M_2 = M(A_{i_{k+1}}) \vee \ldots \vee M(A_{i_n})$$

Now $r(M) = n$.

Since $r[M(A_{i_j})] = 1$ for all $j \in \{1,2 \ldots n\}$, it follows that $r[M(A_{i_1}) \vee \ldots \vee M(A_{i_k})] = k$.

But $\left|\bigcup_{j=1}^{n} A_{i_j}\right| = k$. Then $\bigcup_{j=1}^{k} A_{i_j} = B(M_1)$.

Hence $B(M) \supseteq \bigcup_{j=1}^{k} A_{i_j}$.

Q.E.D.

__Definition 2.3.__ Let M be a transversal matroid on S. Let $\mathfrak{U} = (A_1, A_2 \ldots A_n)$ be a family of subsets of S. Then $\mathfrak{U}$ is a presentation of M iff (1) $r(M) = n$ and (2) $M = M(\mathfrak{U})$.

__Definition 2.4.__ A presentation $\mathfrak{U} = (A_1, A_2 \ldots A_n)$ of the transversal matroid M is minimal iff there exists no presentation $\mathfrak{U}_1 = (B_1, B_2 \ldots B_n)$ of M such that

(1) $B_i \subseteq A_i$ for all $i \in \{1,2 \ldots n\}$.

(2) $B_k \subset A_k$ for some $k \in \{1,2 \ldots n\}$.

__Definition 2.5.__ A presentation $\mathfrak{U} = (A_1, A_2 \ldots A_n)$ of the transversal matroid M is maximal iff there exists no presentation $\mathfrak{U}_1 = (B_1, B_2 \ldots B_n)$ of M such that

(1) $B_i \supseteq A_i$ for all $i \in \{1,2 \ldots n\}$

(2) $B_k \supset A_k$ for some $k \in \{1,2 \ldots n\}$.

<u>Definition 2.6.</u> Let M be a matroid on S. We say $F \subseteq S$ is a flat of M iff F is closed. We say F is a coloop free flat iff F is a flat and M X F has no coloops.

<u>Theorem 2.4.</u> Let M be a transversal matroid on S of rank n. Let F be a closed set of M such that M X F has no coloops and r (M X F) = p. Let $\mathfrak{U} = (A_1, A_2 \ldots A_n)$ be a presentation of M. Then $\mathfrak{U}$ has a subfamily $\mathfrak{U}_1 = (A_{i_1} , A_{i_2} , \ldots A_{i_{n-p}})$ such that

(1) $\bigcup_{j=1}^{n-p} A_{i_j} = S - F$

and

(2) Every set of the family $\mathfrak{U} - \mathfrak{U}_1$ intersects F.

<u>Proof.</u> Consider M X F.

By Corollary 4 of Theorem 1.1 of Chapter 2 we have

$$M \text{ X } F = (M (A_1) \text{ X } F) \vee \ldots \vee (M(A_n) \text{ X } F)$$

$$= M (A_1 \cap F) \vee \ldots \vee M (A_n \cap F) .$$

Hence M X F is the transversal matroid of the family of sets $\mathfrak{U}_2 = (A_1 \cap F, \ldots A_n \cap F)$. But M X F has no coloops. Hence n - p of the subsets $A_i \cap F$ are void by Theorem 2.2. If the void sets are $A_{i_1} \cap F, \ldots A_{i_{n-p}} \cap F$, consider the subfamily $\mathfrak{U}_1 = (A_{i_1} , A_{i_2} \ldots A_{i_{n-p}})$ of $\mathfrak{U}$. Clearly $\mathfrak{U}_1$ satisfies condition 2 and $\bigcup_{j=1}^{n-p} A_{i_j} \subseteq (S - F)$.

Let, if possible, $a \in S - F - \left(\bigcup_{j=1}^{n-p} A_{i_j} \right)$.

Let $F_1 = F \cup a$. Consider $M \times (F \cup a)$. $r(M \times F_1) \geq p$.

But $M \times F_1 = M(A_1 \cap F_1) \vee \ldots \vee M(A_n \cap F_1)$ and

$A_{i_j} \cap F_1 = \phi$ for all $j \in \{1, 2 \ldots n - p\}$.

Hence $r(M \times F_1) \leq p$.

Hence $r(M \times F_1) = r(M \times F)$.

This contradicts the fact that F is closed.

Thus $\bigcup_{j=1}^{n-p} A_{i_j} = (S - F)$.

Q.E.D.

<u>Corollary 1.</u> Let $\mathcal{U} = (A_1, A_2 \ldots A_n)$ be a family of subsets of S and let M be its transversal matroid. Let B be the set of all coloops of M and let $|B| = k$. Then there exists a subfamily

$$\mathcal{U}_1 = \left(A_{i_1}, \ldots A_{i_k}\right) \text{ of } \mathcal{U} \text{ such that}$$

$$\bigcup_{j=1}^{k} A_{i_j} = B.$$

<u>Proof.</u> Let $r(M) = p$.

By Theorem 2.2 there exists a subfamily $\mathcal{U}_2$ of $\mathcal{U}$ such that $M(\mathcal{U}_2) = M$ and $\mathcal{U}_2 = (B_1, B_2 \ldots B_p)$. Now (S-B) is closed in M and

$r[M \times (S-B)] = p - k$. Hence by Theorem 2.4, $\mathfrak{A}_2$ (and therefore $\mathfrak{A}$) has a subfamily $\mathfrak{A}_1 = (A_{i_1}, \ldots A_{i_k})$ such that $\bigcup_{j=1}^{k} A_{i_j} = B$.

Q.E.D.

<u>Corollary 2.</u> [Brualdi-Mason [Brl 2]].

Let M be a transversal matroid on S with rank n. Let k be any integer with $1 \leq k \leq n$. Then M has at most $\binom{n}{k}$ coloop free flats.

<u>Proof.</u> Let $\mathfrak{A} = (A_1, \ldots A_n)$ be a presentation of B.

For any coloop free flat of rank k we have a subfamily $\mathfrak{A}_1$ of $\mathfrak{A}$ such that

(1) $\mathfrak{A}_1$ has (n-k) subsets

(2) $\bigcup_{A_i \in \mathfrak{A}_1} A_i = (S - F)$.

and

(3) If $A_i \in \mathfrak{A} - \mathfrak{A}_1$, $A_i \cap F \neq \Phi$.

Since $\mathfrak{A}_1$ can be chosen in atmost $\binom{n}{k}$ ways the result follows.

Q.E.D.

Hall's Theorem can now be proved quite easily.

<u>Theorem 2.5.</u> (Philip Hall [PH 1]). The family $\mathfrak{A} = (A_1, A_2, \ldots A_n)$ of subsets of S possesses a transversal iff $\mathfrak{A}$ satisfies Hall's condition i.e. iff for each subset

$$J = \{i_1, i_2, \dots i_k\} \text{ of } \{1, 2 \dots n\},$$

$$\left| \bigcup_{j=1}^{k} A_{i_j} \right| \geq |J| .$$

<u>Proof.</u> The theorem is obviously true if $n = 1$. Let us assume the theorem to be true for $n < q$. Let $\mathfrak{A} = (A_1, A_2, \dots A_q)$. If $\mathfrak{A}$ has a transversal it is obvious that $\mathfrak{A}$ satisfies Hall's condition. Conversely let $\mathfrak{A}$ satisfy Hall's condition and let

$$\mathfrak{A}_1 = (A_1, A_2 \dots A_{q-1}) . \text{ Then}$$

$M(\mathfrak{A}) = M(\mathfrak{A}_1) \vee M(A_q)$. Let B be the set of all coloops of $M(\mathfrak{A}_1)$. Then by Corollary 1 of Theorem 2.4, there exists a subfamily $\mathfrak{A}_2 = (A_{i_1}, A_{i_2} \dots A_{i_p})$ such that $|B| = p$ and $B = \bigcup_{j=1}^{p} A_{i_j}$.

From Lemma 2.1 it is clear that $\mathfrak{A}$ has a transversal iff $r[M(\mathfrak{A})] = q$. Now $\mathfrak{A}_1$ clearly satisfies Hall's condition and therefore by the induction assumption, has a transversal. Hence $\mathfrak{A}$ has a transversal iff

$$r[M(\mathfrak{A})] = r[M(\mathfrak{A}_1)] + 1 ,$$

i.e. iff $A_q \nsubseteq B[M(\mathfrak{A}_1)]$. But if $A_q \subseteq B[M(\mathfrak{A}_1)]$, then $\left| (\bigcup_{j=1}^{p} A_{i_j}) \cup A_{i_q} \right| < p + 1$. But this contradicts the fact that $\mathfrak{A}$ satisfies Hall's condition. Hence $A_q \nsubseteq B[M(\mathfrak{A}_1)]$ and $\mathfrak{A}$ has a transversal.

Q.E.D.

Theorem 2.6. Let $\mathcal{U} = (A_1, A_2, \ldots A_n)$ be a family of subsets of S. Let $\mathcal{U}$ be a presentation of the transversal matroid M on S. Let $F \subseteq S$. If for every $J \subseteq \{1, 2, \ldots n\}$ such that $A_i \cap (S-F) \neq \phi$ for all $i \in J$ we have

$$\left| \bigcup_{i \in J} (A_i \cap (S-F)) \right| \geq |J| ,$$

then M . F is a transversal matroid.

Proof. Let the subfamily of $\mathcal{U}$ corresponding to the subsets which intersect (S - F) be $\mathcal{U}_1 = (B_1, B_2 \ldots B_q)$. It is clear that $\mathcal{U}_2 = (B_1 \cap (S-F), \ldots B_q \cap (S-F))$ satisfies Hall's condition and hence has a transversal. Now by Corollary 4 of Theorem 1.1 of Chapter 2 we have

$$M \times (S-F) = (M(A_1) \times (S-F)) \vee \ldots \vee (M(A_n) \times (S-F))$$

$$= M(A_1 \cap (S-F)) \vee \ldots \vee (M(A_n \cap (S-F))$$

$$= M(B_1 \cap (S-F)) \vee \ldots \vee M(B_q \cap (S-F)) \vee \underbrace{M_o \ldots \vee M_o}_{(n-q) \text{ times}}$$

the matroid M_o being on (S - F).

But $\mathcal{U}_2$ has a transversal.

Hence $r[M \times (S-F)] = r[M(B_1 \cap (S-F)] + \ldots +$

$$+ r[M(B_q \cap (S-F)] +$$

$$+ \underbrace{r(M_o) + \ldots + r(M_o)}_{(n-q) \text{ times}}$$

By corollary 2 of Theorem 1.3 of Chapter 2 it follows that

$$M \cdot F = M(A_1) \cdot F \vee \dots \vee M(A_n) \cdot F .$$

Clearly $M(A_i) \cdot F$ for all $i \in \{1, 2 \dots n\}$ are matroids of rank atmost one. Hence $M \cdot F$ is a transversal matroid.

Q.E.D.

Corollary 1. (Brualdi-Mason) [Brl 2]

Let M be a transversal matroid on S. Let $F \subseteq S$ and suppose $M \times (S - F)$ is coloop free. Then $M \cdot F$ is a transversal matroid.

Proof. Let $\mathcal{U} = (A_1, A_2 \dots A_n)$ be a presentation of M. Let $\mathcal{U}_1 = (B_1, B_2 \dots B_k)$ be the subfamily of $\mathcal{U}$ composed of all the sets in $\mathcal{U}$ which intersect $(S - F)$. Now by the use of Corollary 4 of Theorem 1.1 of Chapter 2 we can see that

$$M \times (S-F) = M(B_1 \cap (S-F)) \vee \dots \vee M(B_k \cap (S-F)).$$

Since $M \times (S-F)$ has no coloops, we conclude that

$$r[M \times (S-F)] = \sum_{i=1}^{k} r[M(B_i \cap (S-F))]$$

$$= k .$$

Thus the family $\mathcal{U}_1 = (B_1, B_2 \dots B_k)$ has a transversal. It is now easy to see that Theorem 2.6 is applicable and hence that M.F is a transversal matroid.

Q.E.D.

Theorem 2.7. (Brualdi-Dinolt) [Brl 1]

Let M be the transversal matroid $M(A_1, A_2 \dots A_n)$ on S. Then $S - A_i$ is a flat of M, $(1 \leq i \leq n)$.

Proof. $M(A_1, A_2, \dots A_n) = M(A_1, A_2 \dots A_{i-1}, A_{i+1}, \dots A_n)$

$\vee M(A_i)$

The result now follows from Theorem 1.2.

Q.E.D.

The next two theorems are among the more important results on the structure of transversal matroids. They follow quite easily from Theorem 1.1.

Theorem 2.8. [Bondy and Welsh [Bo 1]]

Let M be the transversal matroid $M(A_1, \dots A_n)$ of rank $r \leq n$ on S. Let $Y \subseteq (S - A_i)$, for some fixed i $(1 \leq i \leq n)$. Then $M = M(A_1, \dots, A_i \cup Y, \dots A_n)$ iff Y is a set of coloops of

$M \times (S - A_i) = M(A_1 - A_i, A_2 - A_i, \dots A_n - A_i)$.

Proof. $M = M(A_1, \dots, A_{i-1}, A_{i+1}, \dots A_n) \vee M(A_i)$.

Hence by Theorem 1.1, we have,

$M = M(A_1, \dots A_{i-1}, A_{i+1}, \dots A_n) \vee M(A_i \cup Y)$

iff Y is a set of coloops for $M \times (S - A_i)$. But by Corollary 4 of Theorem 1.1 of Chapter 2 it is easy to see that

$$M \times (S-A_i) = M(A_1 - A_i) \vee \ldots \vee M(A_n - A_i)$$

$$= M(A_1 - A_i, \ldots, A_n - A_i)$$

Thus the Theorem follows.

Q.E.D.

<u>Corollary 1.</u> Let $\mathfrak{U} = (A_1, A_2 \ldots A_n)$ be a maximal presentation of the transversal matroid M on S. Then $(S-A_i)$ is a coloop free flat of M for all $i \in \{1,2, \ldots, n\}$.

<u>Proof.</u> The result follows from Theorem 2.7 and Theorem 2.8. It can also be seen directly from Theorem 1.7.

Q.E.D.

<u>Theorem 2.9.</u> [Bondy and Welsh [Bo 1]]

Let $\mathfrak{U} = (A_1, A_2, \ldots A_n)$ be family of subsets of S. Let M be the transversal matroid of $\mathfrak{U}$ and let $r(M) = q$. Let $Y \subseteq A_i$ for a fixed $i \in \{1,2, \ldots, n\}$. Then,

$M = M[A_1, \ldots, A_i - Y, \ldots A_n]$ iff there exists a maximum partial transversal D of $\mathfrak{U}_2 = (A_1, \ldots A_{i-1}, A_{i+1}, \ldots A_n)$ such that $D \cap A_i$ has minimum cardinality and $Y \subseteq D \cap A_i$.

<u>Proof.</u> We have,

$$M = M(A_1, \ldots A_{i-1}, A_{i+1}, \ldots A_n) \vee M(A_i).$$

By Theorem 1.1, we know that

$M = M(A_1, \ldots A_{i-1}, A_{i+1}, \ldots A_n) \vee M(A_i - Y)$ iff Y is independent in $M(A_1, \ldots A_{i-1}, A_{i+1}, \ldots A_n) \cdot A_i$ i.e. iff there exists a base D of $M(A_1, \ldots A_{i-1}, A_{i+1}, \ldots A_n)$ such that $|D \cap A_i|$ is a minimum and $Y \subseteq D \cap A_i$. [This follows easily from Theorem T3]. The result follows immediately.

Q.E.D.

The next theorem describes the characteristics of a minimal presentation.

<u>Theorem 2.10.</u> [Bondy and Welsh [Bo 1]]

Let M be a transversal matroid on S and let $\mathfrak{U} = (A_1, A_2, \ldots A_n)$ be a presentation of M. If A_i for some i ($1 \leq i \leq n$), is not a bond of M, then there exists $x \in A_i$ such that $M = M(A_1, \ldots A_{i-1}, A_i - x, A_{i+1} \ldots A_n)$. Hence finally, if $\mathfrak{U}_1 = (C_1, C_2, \ldots C_n)$ is a minimal presentation of M, then C_i ($1 \leq i \leq n$) are bonds of M and are necessarily distinct.

<u>Proof.</u> $M = M(A_1, \ldots A_{i-1}, A_{i+1}, \ldots A_n) \vee M(A_i)$.

By Theorem 1.4, $M(A_i)$ is minimal with respect to $M(A_1, \ldots A_{i-1}, A_{i+1} \ldots A_n)$ only if A_i is a bond of M. Hence if A_i is not a bond of M, there exists at least one element $x \in A_i$ such that $M = M(A_1, \ldots A_{i-1}, A_i - x, A_{i+1}, \ldots A_n)$. This proves the first part of the Theorem. By repeated application of this result, it follows that $\mathfrak{U} = (C_1, C_2 \ldots C_n)$ would be a minimal presentation of M only if C_i ($1 \leq i \leq n$) are bonds of M. If $C_i = C_j$ ($i \neq j$), it is easy to see that

$$M \times (S-C_i) = M(C_1 - C_i, \ldots, C_n - C_i)$$

$$= M(C_1-C_i, \ldots C_{i-1}-C_i, C_{i+1}-C_i, \ldots C_{j-1}-C_i,$$

$$C_{j+1}-C_i, \ldots C_n - C_i)$$

Hence $r[M \times (S-C_i)] \leq n-2$. This contradicts the fact that C_i is a bond of M. Hence the C_i ($1 \leq i \leq n$) are necessarily distinct.

Q.E.D.

While there may be many minimal presentations of a transversal matroid there is a unique maximal presentation. The following theorem due to Bondy [Bo 2] states this. We omit the proof since a proof in keeping with the rest of this section would turn out to be identical to that given by Bondy.

<u>Theorem 2.11.</u> Let M be a transversal matroid on S, and let $\mathcal{U}_1 = (A_1, A_2, \ldots A_n)$ and $\mathcal{U}_2 = (B_1, B_2 \ldots B_n)$ be two maximal presentations of M. Then there exists a permutation $\begin{pmatrix} 1, 2 \ldots\ldots n \\ i_1, i_2 \ldots\ldots i_n \end{pmatrix}$ such that $A_{i_j} = B_j$ for all $j \in \{1, 2, \ldots n\}$.

<u>Corollary 1.</u> Let $\mathcal{U} = (A_1, A_2 \ldots A_n)$ be a presentation of the transversal matroid M on S. Let $(B_1, B_2 \ldots B_n)$ be a family of subsets of S such that for every i, $(1 \leq i \leq n)$, $M(B_i)$ is the maximal matroid with respect to

$M(A_1, \ldots A_{i-1}, A_{i+1}, \ldots A_n)$ such that

$M(A_1, A_2 \ldots A_n) = M(A_1, \ldots A_{i-1}, A_{i+1}, \ldots A_n) \vee M(B_i)$.

Then $\mathfrak{U}_1 = \left(B_1, B_2, \ldots B_n\right)$ is the maximal presentation of M.

<u>Proof.</u> By Theorem 1.7, $(S - B_i)$ $(1 \leq i \leq n)$ is a coloop free flat of $M(B_1, B_2, \ldots B_n)$. Hence by Theorem 1.1, it follows that if $\mathfrak{U}_1 = (B_1, B_2, \ldots B_n)$ is a presentation of M it is the maximal presentation of M. We will now show that $\mathfrak{U}_1$ is a presentation of M. We know that $(B_1, A_2, \ldots A_n)$ is a presentation of M. Suppose $(B_1, B_2, \ldots B_{k-1}, A_k \ldots A_n)$ is a presentation of M. We then have

$$M(B_1, \ldots B_{k-1}, A_{k+1}, \ldots A_n) \vee M(A_k) = M(A_1, \ldots A_{k-1}, A_{k+1} \ldots A_n) \vee M(A_k).$$

If $D \subseteq S$ is such that $M(D)$ is maximal with respect to $M(B_1, \ldots B_{k-1}, A_{k+1}, \ldots A_n)$ such that

$$M(B_1, \ldots B_{k-1}, D, A_{k+1}, \ldots A_n) = M(A_1, \ldots A_n),$$

then by Theorem 1.8, $D = B_k$. Hence $(B_1, B_2 \ldots B_k, A_{k+1}, \ldots A_n)$ is a presentation of M. It thus follows by induction that $(B_1, \ldots B_n)$ is a presentation of M.

Q.E.D.

We next prove a simple but useful Lemma, needed in the proof of Theorem 2.12.

Lemma 2.2. Let $\mathfrak{u} = (A_1, A_2, \dots A_n)$ and $\mathfrak{u}' = (B_1, B_2 \dots B_n)$ be two presentations of a transversal matroid M on S such that $A_i \subseteq B_i$ for all $i \in \{1, 2, \dots n\}$. Then

$$\mathfrak{u}_i = \left(A_1, A_2, \dots A_{i-1}, B_i, A_{i+1}, \dots A_n\right)$$

is a presentation of M.

Proof. It is easy to see that any set independent in $M(A_1, A_2 \dots A_n)$ is independent in $M(A_1, \dots A_{i-1}, B_i, A_{i+1}, \dots A_n)$ and that any set independent in $M(A_1, \dots A_{i-1}, B_i, A_{i+1}, \dots A_n)$ is independent in $M(B_1, B_2 \dots B_n)$. Since

$$M(A_1, A_2, \dots A_n) = M(B_1, B_2 \dots B_n) ,$$

the Lemma is immediate.

Q.E.D.

Theorem 2.12. Let M be a transversal matroid on S. Let $\mathfrak{u} = (B_1, B_2, \dots B_n)$ be a maximal presentation of M and $\mathfrak{u}_1 = (A_1, A_2 \dots A_n)$ be any presentation of M with $A_i \subseteq B_i$ $(1 \leq i \leq n)$. Then if $A_k \subseteq B_j$ and $A_j \subseteq B_k$ $(k \neq j)$ we have $B_k = B_j$.

Proof. Since the maximal presentation is unique, we clearly need consider only the maximal presentation of M obtained as in Corollary 1 of Theorem 2.11 i.e. $M(B_i)$ is the maximal matroid with respect to $M(A_1, \dots A_{i-1}, A_{i+1}, \dots A_n)$ such that

$$M = M(A_1, \dots A_{i-1}, A_{i+1}, \dots A_n) \vee M(B_i) . \quad (1 \leq i \leq n) .$$

Now let $\mathfrak{U}' = (D_1, D_2 \ldots D_n)$ be a family of subsets of S such that $D_i = B_i$ for all $i \in (\{1,2 \ldots n\} - \{j, k\})$.

$$D_k = B_j \text{ ; } D_j = B_k .$$

Clearly $\mathfrak{U}'$ is a presentation of M and $D_i \supseteq A_i$ ($1 \le i \le n$). Hence by Lemma 2.2 we have,

$$\begin{aligned} M (A_1, \ldots A_n) &= M (A_1, \ldots\ldots\ldots\ldots A_{j-1}, D_j, A_{j+1}, \ldots A_n) \\ &= M (A_1, \ldots A_{j-1}, A_{j+1} \ldots A_n) \vee M (D_j) \\ &= M (A_1, \ldots A_{j-1}, A_{j+1}, \ldots A_n) \vee M (B_j) \end{aligned}$$

But $M(B_j)$ is maximal with respect to $M(A_1, \ldots A_{j-1}, A_{j+1}, \ldots A_n)$ and hence by Corollary 1 of Theorem 1.3, $D_j \subseteq B_j$. Similarly we can show that $D_k \subseteq B_k$. Hence $B_j = B_k$.

Q.E.D.

The following Corollary is now obvious.

<u>Corollary 1.</u> Let M be a transversal matroid on S and let $\mathfrak{U} = (B_1, B_2, \ldots B_n)$ be the maximal presentation of M. Let $\mathfrak{U}_1 = (A_1, A_2, \ldots A_n)$ be any presentation of M such that $A_i \subseteq B_i$ for all $i \in \{1,2, \ldots n\}$. Then if $M (D_i)$ is the maximal matroid with respect to $M(A_1, \ldots A_{i-1}, A_{i+1}, \ldots A_n)$ such that $M (A_1, A_2, \ldots A_n) = M(A_1, \ldots A_{i-1}, A_{i+1}, \ldots A_n) \vee M(D_i)$ for all $i \in \{ 1,2, \ldots n\}$, then

$$D_i = B_i \text{ for all } i \in \{1,2, \ldots n\} .$$

Theorem 2.13. [Brualdi - Dinolt [Brl 1]]

Let M be a transversal matroid on S and let $\mathfrak{U} = (B_1, B_2, \ldots B_n)$ be the maximal presentation of M. Suppose $\mathfrak{U}_1 = (A_1, A_2 \ldots A_n)$ is any presentation of M such that $A_i \subseteq B_i$ and $r[M \times (S - A_i)] = k_i$ $(1 \leq i \leq n)$. Then $|A_i|$ is the maximum cardinality of all subsets of B_i whose complement has rank k_i $(1 \leq i \leq n)$ in M. In particular if also $\mathfrak{U}_2 = (A_1', A_2' \ldots A_n')$ is any presentation of M such that $r[M \times (S - A_i')]$ with $A_i' \subseteq B_i$ $(1 \leq i \leq n)$, then $|A_i| = |A_i'|$ $(1 \leq i \leq n)$.

Proof. Since $M(A_1, A_2, \ldots A_n) = M(B_1, B_2 \ldots B_n)$ and $A_i \subseteq B_i$ $(1 \leq i \leq n)$, we have by Lemma 2.8,

$$M(A_1, \ldots A_i, \ldots A_n) = M(A_1, \ldots A_{i-1}, B_i, A_{i+1}, \ldots A_n)$$
$$\text{for all } i \in \{1, 2 \ldots n\}$$

Hence

$$M(A_1, \ldots A_{i-1}, A_{i+1}, \ldots A_n) \vee M(A_i)$$
$$= M(A_1, \ldots A_{i-1}, A_{i+1}, \ldots A_n) \vee M(B_i).$$

Using Corollary 1 of Theorem 2.12 it is clear that $M(B_i)$ is the maximal matroid with respect to $M(A_1, \ldots A_{i-1}, A_{i+1}, \ldots A_n)$ such that

$M(A_1, A_2 \ldots A_n) = M(A_1, \ldots A_{i-1}, A_{i+1}, \ldots A_n) \vee M(B_i)$

for all $i \in \{1, 2 \ldots n\}$. The theorem now follows immediately from Corollary 1 of Theorem 1.6.

Q.E.D.

Section 3 : Gammoids, Base Orderable Matroids and Series-parallel Networks

In this section we show that the operation of series connection is a special case of the matroid union operation. This enables us to show that the classes of binary gammoids, binary base orderable matroids and series parallel networks are identical.

The following results in this section are believed to be new.

Theorem 3.7 and its corollaries, Theorem 3.10, Theorem 3.12 and its corollary 2, Theorem 3.13 and its corollary 2, Theorem 3.16 and its corollary, Theorems 3.17 and 3.18, proof of Theorem 3.19, Theorems 3.20 and 3.21.

Notation. (1) M(S) will denote the matroid M on S.

(2) If F_1 and F_2 are two families of subsets of S and T respectively, then $F_1 \times F_2$ will represent the family of all subsets $C \cup D$ where C is a member of F_1 and D is a member of F_2.

(3) We will denote the class of all bases of the matroid M by I (M).

(4) $\Gamma(N)$ denotes an oriented graph (without loops) with a finite set of nodes N and edges Ed (Γ). When the set of nodes is clear from the context we will simply use Γ instead of $\Gamma(N)$.

An edge e of $\Gamma(N)$ with a positive end n_1 and a negative end n_2 will be represented by the ordered pair (n_2, n_1) (i.e. directed from n_2 towards n_1).

We now give a list of definitions.

Definition 3.1. A pointed matroid is an ordered pair (M,p) where M is a matroid on a set S and $p \in S$.

Let (M, p) be a pointed matroid defined on a set S with $p \in S$. Then we define dual of (M,p) as $(M,p)^* = (M^*,p)$, contraction of (M,p) to $T \subseteq S$ (with $p \in T$) as

$$(M,p) \times T = (M \times T,p) .$$

Reduction of (M,p) to $T \subseteq S$ (with $p \in T$) as

$$(M,p) . T = (M . T, p) .$$

Definition 3.2. Given a pointed matroid (M,p) defined on S and a family F of subsets of M, we define F'(M) to be the subfamily of F whose members do not contain the base point p and F''(M) to be the family of all those subsets H of S-p such that $(H \cup p) \in F$.

Definition 3.3. Given two pointed matroids (M_1,p) and (M_2,p) defined on S and T respectively with $S \cap T = \{p\}$, the series connection $S\,[(M_1,p), (M_2, p)]$ is the pointed matroid (M_S,p) defined on $S \cup T$, such that its class of bases, $I(M_S)$, is defined by the following :

$$I'(M_S) = I'(M_1) \times I(M_2)$$

$$I''(M_S) = [\, I''(M_1) \times I'(M_2) \,] \cup [\, I'(M_1) \times I''(M_2) \,].$$

<u>Definition 3.4.</u> We define the parallel connection $P[(M_1,p), (M_2, p)] \equiv (M_p, p)$ of (M_1,p) and (M_2,p) in terms of the class $I(M_p)$ of its bases as follows :

$$I'(M_p) = [\, I'(M_1) \times I''(M_2) \,] \cup [\, I''(M_1) \times I'(M_2) \,]$$

$$I''(M_p) = I''(M_1) \times I''(M_2) .$$

Definitions 3.3 and 3.4 are taken from [Bry 1].

<u>Definition 3.5.</u> Let M_1 and M_2 be matroids on S_1 and S_2 respectively. Then M_1 and M_2 are said to be isomorphic iff there exists a bijection $\sigma : S_1 \to S_2$ such that for any $b \subseteq S_1$, b is a base of M_1 iff $\sigma(b)$ is a base of M_2. The bijection σ is called an isomorphism from M_1 onto M_2.

The pointed matroids (M_1, p_1) , (M_2,p_2) are isomorphic iff there exists an isomorphism from M_1 onto M_2 such that $\sigma(p_1) = p_2$.

<u>Definition 3.6.</u> The operation ($\vee$) will be called the matroid union operation while the operation ($\wedge$) will be called the matroid intersection operation. These operations have been defined earlier in Definition 1.2 and Corollary 2 of Theorem 1.1 of Chapter 2. We further define the union and intersection of pointed matroids as

$$(M_1, p) \vee (M_2, p) = (M_1 \vee M_2, p)$$

and

$$(M_1, p) \wedge (M_2, p) = (M_1 \wedge M_2, p)$$

where M_1 and M_2 are matroids defined on the same set S.

<u>Definition 3.7.</u> A matroid M is said to be base orderable iff for any two bases b_1 and b_2 of M there exists a bijection $\sigma : b_1 \to b_2$ such that $(b_2 \cup d - \sigma(d))$ and $(b_1 \cup \sigma(d) - d)$ are both bases for all d in b_1.

<u>Definition 3.8.</u> A matroid is said to be strongly base orderable iff for any two bases b_1 and b_2 of M there exists a bijection $\sigma : b_1 \to b_2$ such that $(b_2 \cup A - \sigma(A))$ and $(b_1 \cup \sigma(A) - A)$ are both bases for all subsets A contained in b_1.

Definitions 3.7 and 3.8 are taken from [Brl 7]. Clearly every strongly base orderable matroid is base orderable. The following is an example (due to Prof. A.W. Ingleton [Ing 1]) of a base orderable matroid that is not strongly base orderable.

<u>Example 3.3.1.</u> Let $S = \{a_1, a_2, a_3, a_4, d_1, d_2, d_3, d_4\}$.
Let M be the matroid on S with $r(M) = 4$ and with the following as the circuits of cardinality less than 5.

$\{a_1, d_1, d_2, d_4\}$, $\{a_1, a_3, a_4, d_3\}$, $\{a_2, d_1, d_2, d_3\}$,
$\{a_2, a_3, a_4, d_4\}$, $\{a_1, a_2, d_3, d_4\}$.

M can be shown to be base orderable but for the pair of bases $\{a_1, a_2, a_3, a_4\}, \{d_1, d_2, d_3, d_4\}$ there exists no bijection satisfying the conditions of Definition 3.8.

<u>Definition 3.9.</u> A path P in the oriented graph $\Gamma(N)$ is an alternating sequence of distinct nodes n_i and edges (n_i, n_{i+1}) of Γ:

$$P = n_1 . (n_1,n_2) . n_2 . (n_2,n_3) . n_3 ....$$

beginning and ending with a node.

We admit the degenerate path $P = n_1$.

<u>Definition 3.10.</u> We use In.P, Nod.P and Ter.P to denote the initial node, set of nodes and the terminal node of P, respectively.

<u>Definition 3.11.</u> $\tilde{P}$ will denote a finite family of paths and In $\tilde{P}$ etc. will denote the set of initial nodes etc. of the members of $\tilde{P}$.

<u>Definition 3.12.</u> A family $\tilde{P}$ of paths is pairwise-node-disjoint (p.n.d.) iff Nod. $P_i \cap$ Nod. $P_j = \phi$ for all paths $P_i \neq P_j$ in P.

<u>Definition 3.13.</u> A set of nodes A is said to be linked in $\Gamma(N)$ to a set of nodes B iff there is a p.n.d. family of paths $\tilde{P}$ with In.$\tilde{P}$ = A and Ter $\tilde{P}$ = B.

<u>Definition 3.14.</u> For each $n \in N$, the star of n in $\Gamma(N)$ is the set $St(n) \equiv \{ x \mid x \in N, (n,x) \in Ed(\Gamma) \}$.

We state the following theorem due to Brualdi [Brl 7] without proof.

<u>Theorem 3.1.</u> Let $\Gamma(N)$ be an oriented graph. Let $M_1(N)$ be a matroid. Then subsets of N which are linked in Γ to independent sets of $M_1(N)$ form the independent sets of a matroid M(N).

We will say that M(N) is induced by $M_1(N)$ through Γ.

<u>Definition 3.15.</u> Let M(N) be a matroid induced by $M_1(N)$ through $\Gamma(N)$. If $M_1(N)$ has a unique basis then M(N) is called a strict gammoid.

<u>Definition 3.16.</u> A matroid M(Y) is a gammoid iff there exists a strict gammoid $M_1(N)$ such that $Y \subseteq N$ and $(M_1(N)) \times Y = M(Y)$.

Definitions 3.15 and 3.16 are due to Mason. In [Ma 4] he has made a detailed study of gammoids and strict gammoids and has shown, among other things, that gammoids form the 'closure' class of strict gammoids under the operations of contraction, reduction and dualization. The following theorem due to Ingleton and Piff [Ing 2], stated here without proof, forms a natural complement to Mason's work.

<u>Theorem 3.2.</u> Strict gammoids and transversal matroids are dual to each other. We can now prove the following simple result :

<u>Theorem 3.3.</u> (a) Matroid union of transversal matroids is a transversal matroid.

(b) Matroid intersection of strict gammoids is a strict gammoid.

<u>Proof.</u> (a) follows directly from Corollary 1 of Theorem 2.1 and (b) follows from the definition of matroid intersection, Theorem 3.2 and (a) above.

The following simple results (Theorems 3.4, 3.5) are due to Mason [Ma 4].

<u>Theorem 3.4.</u> Every transversal matroid is a gammoid.

<u>Proof.</u> Let M be a transversal matroid, with a presentation $\mathcal{U} = (A_1, A_2 \dots A_n)$, defined on $S \supseteq \bigcup_{i=1}^{n} A_i$. Consider the oriented graph Γ on the set of nodes $\{a_1, a_2 \dots a_n\} \cup S$ ($S \cap \{a_1, a_2 \dots a_n\} = \Phi$), all its edges having their positive ends in $\{a_1, a_2 \dots a_n\}$ and negative ends in S and such that $a_i \in St(x)$ iff $x \in A_i$. Let $M_1(S \cup \{a_1, a_2 \dots a_n\})$ be the matroid with $\{a_1, a_2 \dots a_n\}$ as its unique basis. Let M_2 be the strict gammoid induced by M_1 through Γ. Then it is easy to see that $M = M_2 \times S$. Hence M is a gammoid.

<u>Theorem 3.5.</u> Reduction of a strict gammoid is a strict gammoid.

<u>Proof.</u> Let M be a strict gammoid on S. Let $Y \subseteq S$. We need to show that $M \cdot Y$ is a strict gammoid. By Theorem 3.2 M^* is a transversal matroid. Now $M^* \times Y$ is a transversal

matroid by Corollary 4 of Theorem 1.1 of Chapter 2 and Corollary 1 of Theorem 2.1 of this chapter. Since by Theorem T7-(9), $(M^{*} \times Y) = (M \cdot Y)^{*}$ it follows that $M \cdot Y$ is a strict gammoid.

Q.E.D.

<u>Corollary 1.</u> The contraction, reduction and dual of a gammoid are gammoids.

<u>Proof.</u> Let M be a gammoid on S. Then there exists a strict gammoid M_1 on some $N \supseteq S$ such that $M_1 \times S = M$. Hence $M \times T = M_1 \times T$ for every $T \subseteq S$. Hence every contraction of M, is a gammoid.

Now for every $T \subseteq S$ we have,

$$M \cdot T = M_1 \times S \cdot T = M_1 \cdot (N - (S - T)) \times T$$

But by Theorem 3.5, $M_1 \cdot (N - (S-T))$ is a strict gammoid and hence it follows that $M \cdot T$ is a gammoid. Now

$$M^{*} = (M_1 \times S)^{*} = M_1^{*} \cdot S .$$

But M_1^{*} is a transversal matroid by Theorem 3.2 and therefore by Theorem 3.4 is a gammoid. Hence $M_1^{*} \cdot S = M^{*}$ is a gammoid.

Q.E.D.

<u>Theorem 3.6.</u> Let M be a matroid on S. Let T be a separator of M. Then M is a gammoid iff $M \times T$ and $M \times (S-T)$ are gammoids.

Proof. Let T be a separator of M and M X T, M X (S-T) be gammoids. Let M_1 and M_2 be strict gammoids on sets P and Q such that P $\supseteq$ T, Q $\supseteq$ (S-T), P $\cap$ Q = Φ, M_1 X T = M X T and M_2 X (S-T) = M X (S-T). Let M_3 be the matroid on P $\cup$ Q having P as a separator such that M_3 X P = M_1 and M_3 X Q = M_2. It is easy to see that M_3 can be induced by a matroid on P $\cup$ Q having a unique basis and is therefore a strict gammoid. Now clearly M_3 X S has T as a separator and M_3 X S X T = M_1 X T = M X T and M_3 X S X (S-T) = M_2 X (S-T) = M X (S-T). Hence M_3 X S = M and M is therefore a gammoid. If M is a gammoid by Corollary 1 of Theorem 3.5, M X T and M X (S-T) are gammoids.

Q.E.D.

Corollary 1. Let M be a matroid on S and let M_1 be a matroid obtained from M by adding loops and coloops. Then M_1 is a gammoid iff M is a gammoid.

Proof. The result follows immediately from Theorem 3.6 when we note that any matroid on a singleton set is a gammoid.

Q.E.D.

Theorem 3.7. Let Γ_1 be an oriented graph on the set of nodes N_1. Let M_2 be a gammoid on N_1. Then M_2 induces a gammoid M_3 on N_1 through Γ_1 .

Proof. Since M_2 is a gammoid, there exists a strict gammoid $M_1(N)$ such that (1) $M_1(N)$ is induced by a matroid M(N) having $N_o \subseteq N$ as its unique basis, through an oriented graph Γ.

(2) $N_1' \subseteq N$.

(3) There exists a bijection $f : N_1 \to N_1'$ such that for any $A \subseteq N_1$, $f(A)$ is independent in $M_1(N) \times N_1'$ iff A is independent in $M_2(N_1)$ i.e. $M_1(N) \times N_1'$ and $M_2(N_1)$ are isomorphic.

We will take $N \cap N_1 = \phi$. Now we construct an oriented graph Γ' on the set of nodes $K = N \cup N_1$ as follows : First construct the oriented graph Γ on N with N_1 as a set of isolated vertices. Now for each node $d \in N_1$ we add an oriented edge $(d, f(d))$ from d to $f(d) \in N_1'$. Let $M_4(K)$ be the matroid having N_o as its unique basis and let $M_5(K)$ be the strict gammoid induced on K through Γ' by $M_4(K)$. Now it is easy to see that a set A is independent in $M_5(K) \times N_1$ iff $f(A)$ is independent in $M_5(K) \times N_1'$.

Since $M_5(K) \times N_1' = M_1(N) \times N_1'$ is isomorphic to $M_2(N_1)$, it follows that $M_2(N_1) = M_5(K) \times N_1$. We next construct the following oriented graph on K. First construct Γ' on K. Then construct Γ_1 on N_1. We shall call the resulting oriented graph on K, Γ''.

Let $M_6(K)$ be the strict gammoid on K induced by $M_4(K)$ through Γ''. Consider $M_6(K) \times N_1$. It is clear that a subset A of N_1 is independent in $M_6(K) \times N_1$ iff it is linked to an independent set of $M_5(K) \times N_1$ by p.n.d. paths in Γ. Thus $M_6(K) \times N_1 = M_3(N_1)$ and $M_3(N_1)$ is a gammoid.

Q.E.D.

Corollary 1. If M_1 and M_2 are gammoids on S, $M_1 \vee M_2$ is a gammoid.

Proof. Let M_1 and M_2 be matroids on S. Let S_1 and S_2 be sets disjoint from S and from each other such that $|S_1| = |S_2| = |S|$. Let M_3 be the matroid on $S_1 \cup S_2$ such that

(1) S_1 is a separator of M_3.

(2) The matroids $M_3 \times S_1$ and M_1 are isomorphic.

(3) The matroids $M_3 \times S_2$ and M_2 are isomorphic.

Then by Theorem 3.6 it is clear that M_3 is a gammoid. Let f_1 be a bijection from S_1 to S and f_2 a bijection from S_2 to S. Now construct the oriented graph Γ on the set of nodes $S \cup S_1 \cup S_2$ as follows : For each $d \in S$ add oriented edges $(d, f_1(d))$ and $(d, f_2(d))$.

Let M_4 be the matroid on $S \cup S_1 \cup S_2$ such that $M_4 \times (S_1 \cup S_2) = M_3$ and $S \subseteq C(M_4)$. Let $M_5(S \cup S_1 \cup S_2)$ be the matroid induced by M_4 through Γ. Then it is easy to see that $M_5 \times S = M_1 \vee M_2$. Also M_4 is a gammoid by Corollary 1 of Theorem 3.6, and so is M_5 by Theorem 3.7. Hence by Corollary 1 of Theorem 3.6, $M_5 \times S = M_1 \vee M_2$ is a gammoid.

Q.E.D.

Corollary 2. If M_1 and M_2 are gammoids, $M_1 \wedge M_2$ is a gammoid.

Proof. The result follows from the definition of matroid intersection, Corollary 1 of Theorem 3.6 and Corollary 1 above.

Q.E.D.

We now give some of the corresponding simple results for (strongly) base orderable matroids.

First we state the following obvious theorem :

Theorem 3.8. The contraction, reduction and dual of a (strongly) base orderable matroid is (strongly) base orderable.

Theorem 3.9. Union of (strongly) base orderable matroids is (strongly) base orderable.

Proof. We will prove the theorem for strongly base orderable matroids and omit the case of base orderable matroids since its proof is similar.

Let M_1, M_2 be strongly base orderable matroids and $M = M_1 \vee M_2$. Clearly it is sufficient for our purpose to consider only the case where M has no coloops. Then by Corollary 1 of Theorem 1.2 of Chapter 2 it follows that if b is any base of M, $b = b_1 \cup b_2$, where b_1 is a base of M_1 and b_2 is a base of M_2. Let b^1, b^2 be two bases of M, with $b^1 = b_1^{\,1} \cup b_2^{\,1}$ and $b^2 = b_1^{\,2} \cup b_2^{\,2}$, where $b_1^{\,1}$, $b_1^{\,2}$ are bases of M_1 and $b_2^{\,1}$, $b_2^{\,2}$ are bases of M_2. Since $M_1(M_2)$ is strongly base orderable we have a bijection $f_1(f_2)$ from $b_1^{\,1}$ $(b_2^{\,1})$ to $b_1^{\,2}(b_2^{\,2})$ such that for any $A \subseteq b_1^{\,1}$ ($D \subseteq b_2^{\,1}$), $(b_1^{\,1} \cup f_1(A) - A)((b_2^{\,1} \cup f_2(D)-D)))$ is a base for $M_1(M_2)$ and

$(b_1^{\,2} \cup A - f_1(A))$ $((b_2^{\,2} \cup D - f_2(D)))$ is a base for $M_1(M_2)$.

Let $f : b^1 \rightarrow b^2$ be the bijection such that $f/b_1^{\,1} = f_1$, $f/b_2^{\,1} = f_2$

Then if $E \subseteq b^1$

$(b^1 \cup f(E) - E)$

$= (b'_1 \cup f_1 (E \cap b'_1) - (E \cap b'_1)) \cup (b'_2 \cup f_2(E \cap b'_2) - (E \cap b'_2))$,

and hence is a base of M. Similarly one can show that $b^2 \cup E - f(E)$ is a base of M.

Q.E.D.

The following Corollary is now obvious :

Corollary 1. If M_1 and M_2 are (strongly) base orderable $M_1 \wedge M_2$ is (strongly) base orderable.

Definition 3.17. A matroid M on S is said to be minimal non (strongly) base orderable iff

(1) M is not (strongly) base orderable, and

(2) every minor of M on every $T \subset S$ is (strongly) base orderable.

Theorem 3.10. (1) Every minimal non base orderable matroid is atomic of kind (2).

(2) Every minimal non strongly base orderable matroid is also atomic of kind (2).

Proof. We prove only the Case (2) for minimal non-strongly base orderable (m.n.s.b.) matroid, the other case (1) being similar.

Let M be a m.n.s.b. matroid. Then there exist two bases b_1 and b_2 of M such that there exists no bijection $f : b_1 \rightarrow b_2$ for which $b_1 \cup f(A) - A$ and $b_2 \cup A - f(A)$ are bases for every $A \subseteq b_1$. Hence it follows that $M \times (b_1 \cup b_2)$ is non-strongly base orderable. Therefore $b_1 \cup b_2 = S$. Since M is a m.n.s. b. matroid so must M^* be by the use of Theorem 3.8. It can therefore be seen using the above argument for M^* that b_1 and b_2 are disjoint bases.

Now let $T \subset S$ be such that $M \times T$ is a matroid of kind (2). Clearly $M . (S-T)$ is also a matroid of kind (2) and both $M . T$ and $M . (S - T)$ are strongly base orderable. Now $b_1 \cap T$, $b_2 \cap T$ are bases of $M \times T$ and $b_1 \cap (S-T)$, $b_2 \cap (S - T)$ are bases of $M . (S - T)$. Hence there exist bijections $f_1 : b_1 \cap T \rightarrow b_2 \cap T$ and $f_2 : b_1 \cap (S - T) \rightarrow b_2 \cap (S - T)$ such that for all $A \subseteq b_1 \cap T$, $((b_1 \cap T) \cup f_1(A) - A)$ and $((b_2 \cap T) \cup A - f_1(A))$ are bases of $M \times T$ and for all $D \subseteq b_1 \cap (S - T)$ $((b_1 \cap (S-T)) \cup f_2(D) - D)$ and $((b_2 \cap (S-T)) \cup D - f_2(D))$ are bases of $M . (S-T)$.

Let $f : b_1 \rightarrow b_2$ be the bijection such that $f/b_1 \cap T = f_1$ and $f/b_1 \cap (S-T) = f_2$. Then for all $E \subseteq b_1$, $(b_1 \cup f(E) - E)$ and $(b_2 \cup E - f(E))$ are bases of M by Theorem T3. This contradicts the fact that M is m.n.s.b. Hence M must be atomic of kind (2).

Q.E.D.

We state the following theorem due to Mason [Ma 4] without proof.

Theorem 3.11. Every gammoid is a strongly base orderable matroid.

We will now show that series connection is a special case of the matroid union operation and dually, parallel connection is a special case of the matroid intersection operation.

Theorem 3.12. Let (M_1, p), (M_2, p) be two pointed matroids on S_1, S_2 respectively, with $S_1 \cap S_2 = \{p\}$. Let $(\bar{M}_1, p)$ [$(\bar{M}_2, p)$] be the pointed matroid on $S_1 \cup S_2$ with its contraction to $S_1(S_2)$ being (M_1, p) [(M_2, p)] and with $(S_2 - S_1)$ [$(S_1 - S_2)$] as a set of loops. Then

$$S[(M_1, p), (M_2, p)] = (\bar{M}_1, p) \vee (\bar{M}_2, p) .$$

Proof. Let $S[(M_1, p), (M_2, p)] \equiv (M, p)$.

(i) Let b be a base of (M, p) such that $b \in I'(M)$ i.e. $p \notin b$. Then by Definition 3.3, $b = b_1 \uplus b_2$, where b_1 is a base of $(\bar{M}_1, p)$ and b_2 is a base of $(\bar{M}_2, p)$. Hence b is a base of $(\bar{M}_1, p) \vee (\bar{M}_2, p)$.

(ii) Let b be a base of (M, p) such that $(b - p) \in I''(M)$ i.e. $p \in b$. Then by Definition 3.3, $b - p = b_1 \uplus b_2$ where b_1 (b_2) is a base in $(\bar{M}_1, p)$ [$(\bar{M}_2, p)$] and $b_2 \cup p$ $(b_1 \cup p)$ is a base in $(\bar{M}_2, p)$ [$(\bar{M}_1, p)$].
Hence b is a base of $(\bar{M}_1, p) \vee (\bar{M}_2, p)$.

Conversely let b be a base of $(\bar{M}_1,p) \vee (\bar{M}_2,p)$. Again we consider two cases.

(i) $p \notin b$. Then

$b = b_1 \cup b_2$, where b_1 is a base of $(\bar{M}_1, p)$ and b_2 is a base of $(\bar{M}_2, p)$. Hence by Definition 3.3, $b \in I'(M)$.

(ii) $p \in b$. Then

$b = b_1 \cup b_2$ with $p \in b_1$ say.

Then $b_1 \cup b_2 - p$ belongs to $I''(M)$ by Definition 3.3 and hence b is a base of (M,p).

Q.E.D.

<u>Corollary 1.</u> Series connection of pointed transversal matroids is a pointed transversal matroid.

<u>Proof.</u> This is a consequence of Theorem 3.12 and Theorem 3.3.

Q.E.D.

<u>Corollary 2.</u> Series connection of pointed gammoids is a pointed gammoid.

<u>Proof.</u> This is a consequence of Theorem 3.12 and Corollary 1 of Theorem 3.7.

Q.E.D.

<u>Corollary 3.</u> Series connection of pointed (strongly) base orderable matroids is a pointed (strongly) base orderable matroid.

<u>Proof.</u> This is a consequence of Theorem 3.12 and Theorem 3.9.

Q.E.D.

We state the following lemma due to Brylawski [Bry 1] without proof.

<u>Lemma 3.1.</u> Let (M_1,p), (M_2,p) be pointed matroids defined on S_1, S_2 respectively such that $S_1 \cap S_2 = \{p\}$.

Then $P[(M_1,p), (M_2,p)] = [S[(M_1,p)^*, (M_2,p)^*]]^*$.

<u>Theorem 3.13.</u> Let (M_1,p), (M_2,p) be two pointed matroids defined on S_1, S_2 respectively with $S_1 \cap S_2 = \{p\}$. Let $(\bar{\bar{M}}_1,p)\,[\bar{\bar{M}}_1,p)]$ be the pointed matroid which has $(M_1,p)\,[(M_2,p)]$ as its contraction to $S_1\,[S_2]$ and $(S_2-S_1)\,[(S_1-S_2)]$ as a set of coloops. Then

$$P[(M_1,p), (M_2,p)] = (\bar{\bar{M}}_1, p) \wedge (\bar{\bar{M}}_2,p).$$

<u>Proof.</u> The theorem follows from the definition of intersection operation, Theorem 3.11 and Lemma 3.1.

Q.E.D.

<u>Corollary 1.</u> Parallel connection of pointed strict gammoids is a pointed strict gammoid.

<u>Proof.</u> This is a consequence of Theorem 3.13 and Theorem 3.3.

Q.E.D.

<u>Corollary 2.</u> Parallel connection of pointed gammoids is a pointed gammoid.

<u>Proof.</u> This is a consequence of Theorem 3.13 and Corollary 2 of Theorem 3.7.

Q.E.D.

Corollary 3. Parallel connection of pointed (strongly) base orderable matroids is a pointed (strongly) base orderable matroid.

Proof. This is a consequence of Theorem 3.13 and Corollary 1 of Theorem 3.9.

Q.E.D.

The following definition is due to Bixby [Bix 1].

Definition 3.18. Let M be a matroid on S. M is decomposable iff there exist pointed matroids (M_1,p), (M_2,p) on sets S_1, S_2 respectively, $|S_1|$, $|S_2| \geq 3$, $S_1 \cap S_2 = \{p\}$ such that, if $P[(M_1,p), (M_2,p)] = (M_3, p)$ then

$$M = M_3 \times (S_1 \cup S_2 - p) \quad , \text{ or equivalently,}$$

if $S[(M_1,p), (M_2,p)] = (M_4, p)$ then

$$M = M_4 \cdot (S_1 \cup S_2 - p) \quad .$$

Definition 3.19. Let M be a matroid on S and let $R \cup T = S$. Let ϱ be the rank function of M. We define $\xi(M; R,T)$ as

$$\xi(M ; R,T) = \varrho(R) + \varrho(T) - r(M) + 1$$

and $\lambda(M)$ as

$$\lambda(M) = \inf \ \xi(M; R,T) \quad ,$$ where the infimum is taken with respect to sets R and T satisfying the following conditions

$R \cup T = S$ and $\varrho(R) + \varrho(T) - |R| \leq r(M) - 1$,

$\varrho(R) + \varrho(T) - |T| \leq r(M) - 1$, with the convention that $\xi(M; \Phi, S) = +\infty$.

M is said to be n-connected (where n is a positive integer) if $\lambda(M) \geq n$.

The above definition is due to Tutte [Tu 10].

The following theorems due to Bixby [Bix 1] are stated here without proof.

<u>Theorem 3.14.</u> Let M be a matroid on S. Then M is not 3-connected iff M is decomposable or M has a separator $T \subset S$.

<u>Theorem 3.15.</u> Let (M_1, p), (M_2, p) be pointed matroids on sets S_1 and S_2 respectively with $S_1 \cap S_2 = \{p\}$, and $|S_1|, |S_2| \geq 3$. Let $(M_3, p) = S[(M_1, p), (M_2, p)]$ and let $M = M_3 . (S_1 \cup S_2 - p)$.

Let C be a circuit of M such that $C \cap S_1 \neq \Phi$ and $C \cap S_2 \neq \Phi$ and let $d \in C \cap S_1$, $f \in C \cap S_2$. Then there exists an isomorphism σ_1 from M_1 to $M \times (S_1 \cup C) . (S_1 \cup f)$ such that $\sigma_1(p) = f$ and an isomorphism σ_2 from M_2 to $M \times (S_2 \cup C) . (S_2 \cup d)$ such that $\sigma_2(p) = d$.

We are now in a position to prove the following simple result.

<u>Theorem 3.16.</u> Let $\mathcal{M}$ be a class of matroids closed under the operations of matroid union, contraction and reduction, and addition of loops and coloops. Let the matroid M on S be a minimal non-member of $\mathcal{M}$ (i.e. $M \notin \mathcal{M}$ but if M_1 on $T \subset S$ is a minor of M, $M_1 \in \mathcal{M}$). Then

M is 3-connected.

<u>Proof.</u> Let M be a minimal non-member of $\mathcal{M}$. If M is not 3-connected, M either (1) has a separator $T \subset S$ or (2) M can be expressed as

$$M = M_3 \cdot (S_1 \cup S_2 - p) \text{ where } |S_1|, |S_2| \geq 3$$

and $(M_3, p) = S[(M_1, p), (M_2, p)]$.

If M has a separator $T \subset S$ it is easy to see that $M = (\overline{M \times T}) \vee (\overline{M \times (S-T)})$. But $M \times T$, $M \times (S-T)$ are members of $\mathcal{M}$ by the minimality of M. Hence M is a member of $\mathcal{M}$ since $\mathcal{M}$ is closed under matroid union. This is a contradiction. Hence M has no separator $T \subset S$. Hence Case (2) holds i.e.

$M = M_3 \cdot (S_1 \cup S_2 - p)$, where M_3, S_1, S_2 are as defined above. By Theorem 3.15 it is clear that M_1 and M_2 are isomorphic to minors of M defined on proper subsets of S and hence by the minimality of M are members of $\mathcal{M}$. But then $\bar{M}_1 \vee \bar{M}_2$ is a member of $\mathcal{M}$. ($\bar{M}_1$, $\bar{M}_2$ defined as in Theorem 3.12).

Now by Theorem 3.12, $M_3 = \bar{M}_1 \vee \bar{M}_2$. Since $\mathcal{M}$ is closed under contraction and reduction it follows that $M = M_3 \cdot (S_1 \cup S_2 - p)$ is a member of $\mathcal{M}$. This is a contradiction. We, therefore, conclude that M is 3-connected.

Q.E.D.

Corollary 1. Minimal non-gammoids and minimal non-(strongly) base orderable matroids are 3-connected.

Proof. The result follows from Theorem 3.16, Corollary 1 of Theorem 3.5, Corollary 1 of Theorem 3.7, Theorem 3.8 and Theorem 3.19.

Q.E.D.

Series Parallel Networks :

We define series - parallel networks slightly different from [Bry 1]. Our aim here is to give a definition which corresponds exactly to the definition of series - parallel networks given in [Sea 1].

Definition 3.20. Let (C_2,p) be the pointed matroid on $\{d,p\}$ such that $\{d,p\}$ is a circuit. Let T (C_2,p) be the class of all pointed matroids obtained by repeated application of the series or parallel connection operations on pointed matroids each of which is isomorphic to (C_2,p).

A matroid is a series - parallel network iff it is a connected minor of a matroid M_1 such that the pointed matroid (M_1,p) $\in$ T (C_2,p).

It is clear that (C_2, p) is a pointed gammoid since it obviously is a pointed transversal matroid. Also by Corollary 2 of Theorem 3.12 and Corollary 2 of Theorem 3.13 series or parallel connection of pointed gammoids is a pointed gammoid.

We, therefore, have the following theorem :

Theorem 3.17. The class T (C_2,p) is contained in the class of pointed gammoids.

But every minor of a gammoid is a gammoid. Hence

Theorem 3.18. Every series-parallel network is a gammoid.

Let K_4 denote the polygon matroid of the complete graph on 4 nodes. We now state the following well known results without proof.

Lemma 3.2. [Bry 1]. A matroid is a series-parallel network iff it is binary and does not contain K_4 as a minor.

Lemma 3.3 [Ma 4]. K_4 is not base-orderable and hence not a gammoid.

Lemma 3.4. No base orderable matroid can contain K_4 as a minor.

Proof. This is a consequence of Lemma 3.3 and Theorem 3.8.

Q.E.D.

Theorem 3.19. Every binary base orderable matroid is a series-parallel network.

Proof. This is a consequence of Lemma 3.2 and Lemma 3.4.

Q.E.D.

Theorem 3.20. A binary matroid is a gammoid iff it is a series-parallel network.

Proof. This is a consequence of Theorem 3.11, Theorem 3.18 and Theorem 3.19.

Q.E.D.

Theorem 3.21. The classes of binary gammoids, binary (strongly) base orderable matroids and series-parallel networks are identical.

Proof. This is a consequence of Theorem 3.11, Theorem 3.19 and Theorem 3.20.

Q.E.D.

That (strongly) base orderable matroids and series - parallel networks are identical is proved in [Bo 1] and [Des 1]. The result for binary gammoids and the proof for the above result are new.

P-sequences of Series and Parallel Connections

Definition 3.22. Let M_1 , M_2 be matroids on S_1, S_2 respectively where $S_1 \cap S_2 = \phi$. Then the direct sum $M_1 \oplus M_2$ is the matroid M_3 on $S_1 \cup S_2$ such that $M_3 \times S_i = M_i$ ($i = 1,2$) and S_i are separators of M_3.

Definition 3.23. Let M_1, M_2 be matroids on S_1, S_2, $(S_1 \cap S_2 = \phi)$ respectively. Let

M_1 have the P-sequence $\left\{ P_i^1 \right\}_{i=1}^{n} = P_1^1 , \ldots P_n^1$

M_2 have the P-sequence $\left\{P_j^2\right\}_{j=1}^{m} = P_1^2, \dots P_m^2$.

Let β be the function on the class of subsets

$$F = \left\{P_1^1, P_2^1 \dots P_n^1, P_1^2, \dots P_m^2\right\}$$ defined as

follows :

$$\beta(P_i^1) = d \left[M_1 \times \left(\bigcup_{k=1}^{i} P_k^1 \right) . P_i^1 \right] \text{ for all } i \in \{1,2 \dots n\}.$$

$$\beta(P_j^2) = d \left[M_2 \times \left(\bigcup_{q=1}^{j} P_q^1 \right) . P_j^2 \right] \text{ for all } j \in \{1,2 \dots m\}.$$

Let the range of β be $\left\{\alpha_1, \alpha_2 \dots \alpha_s\right\}$, the α_i being arranged in descending order.

Let $\left\{P_i\right\}_{i=1}^{s} = P_1, P_2 \dots P_s$ be the sequence of sets obtained by defining for every $i \in \{1,2 \dots, s\}$, P_i to be the union of all sets T from the P-sequence of M_1 and M_2 such that $\beta(T) = \alpha_i$.

Then we denote the sequence of sets

$$\left\{P_i\right\}_{i=1}^{s} \text{ by } \left\{P_i^1\right\}_{i=1}^{n} \oplus \left\{P_i^2\right\}_{i=1}^{m}$$

The following theorem is an easy consequence of Theorem 1.6 and Theorem 3.7 of Chapter 2.

Theorem 3.22. Let M_1, M_2 be matroids on sets S_1, S_2 respectively, where $S_1 \cap S_2 = \phi$.

Let M_1 have the P-sequence $\left\{P_i^1\right\}_{i=1}^{n}$ and

M_2 have the P-sequence $\left\{P_i^2\right\}_{i=1}^{m}$.

Then $M_1 \oplus M_2$ has the P-sequence $\left\{P_i^1\right\}_{i=1}^{n} \oplus \left\{P_i^2\right\}_{i=1}^{m}$.

We now describe, partially, the P-sequence of the series and parallel connections of two matroids in terms of the P-sequences of these matroids, through Theorem 3.22.

Theorem 3.23. Let (M_1, p), (M_2, p) be pointed matroids on sets S_1, S_2 respectively such that $S_1 \cap S_2 = \{p\}$.

Let $(M_3, p) = S[(M_1, p), (M_2, p)]$.

Let t be the least positive integer such that

$$p \notin \Lambda(M_1^t) \text{ and } p \notin \Lambda(M_2^t).$$

Let $P_1^1, P_2^1 \ldots P_r^1$ be the P-sequence of the matroid

$$[M_1 \times \Lambda(M_1^t)] \oplus [M_2 \times \Lambda(M_2^t)].$$

Then if $P_1, P_2 \ldots P_n$ is the P-sequence of M_3, we have

$$P_1^1 = P_1, \ldots P_r^1 = P_r.$$

<u>Proof.</u> We define ($\bar{M}_1$, p) and ($\bar{M}_2$, p) as in Theorem 3.12. Then for all positive integers j we have

$$(M_3)^j = [\ \bar{M}_1 \vee \bar{M}_2\ ]^j = (\ \bar{M}_1)^j \vee (\ \bar{M}_2\ )^j \ .$$

Hence,

$$A[\ M_3{}^j\ ] \subseteq A[\ (\ \bar{M}_1)^j\ ] \cap A[\ (\ \bar{M}_2\ )^j\ ]$$

Let

$$T_1 = A\ (\ M_1{}^t\ )\ ,\quad T_2 = A\ (\ M_2{}^t\ )\ .$$

Then it follows from the above that

$$A\ (\ M_3{}^t\ ) \subseteq T_1 \cup T_2 \ .$$

Now since $p \notin T_1 \cup T_2$ it is not difficult to see that

$$M_3 \times (\ T_1 \cup T_2\ ) = (\ M_1 \times T_1\ ) \oplus (\ M_2 \times T_2\ )\ .$$

By using Theorem 1.6 of Chapter 2 it follows that

$$A[\ M_3{}^j \times (\ T_1 \cup T_2\ )\ ] = A[\ M_1{}^j \times T_1\ ] \cup A[\ M_2{}^j \times T_2\ ]$$

for every positive integer j.

Hence

$$A[\ M_3{}^t \times (\ T_1 \cup T_2\ )\ ] = A[\ M_1{}^t \times T_1\ ] \cup A[\ M_2{}^t \times T_2\ ] =$$

$$= T_1 \cup T_2\ .$$

But by Lemma 6.1 of Chapter 2,

$$A [M_3^t \times (T_1 \cup T_2)] \subseteq A [M_3^t] .$$

Hence

$$A (M_3^t) \supseteq T_1 \cup T_2 ,$$

and thus $A (M_3^t) = T_1 \cup T_2$.

The theorem is now immediate from Corollary 1 of Theorem 2.6, of Chapter 2 taking note of the fact that

$$M_3 \times (T_1 \cup T_2) = (M_1 \times T_1) \oplus (M_2 \times T_2).$$

Q.E.D.

Since

$$P [(M_1, p), (M_2, p)] = [S [(M_1, p)^* , (M_2, p)^*]]^*$$

we have the following obvious Corollary :

<u>Corollary 1.</u> Let (M_1, p), (M_2, p) be pointed matroids on sets S_1, S_2 respectively such that $S_1 \cap S_2 = \{p\}$.

Let $(M_3, p) = P [(M_1, p), (M_2, p)]$

and let $(M_1, p)^* = (M_4, p)$, $(M_2, p)^* = (M_5, p)$.

Let t be the least positive integer such that

$$p \notin A (M_4^t) \text{ and } p \notin A (M_5^t).$$

Let $P_1^{\,1}, P_2^{\,1} \ldots P_r^{\,1}$ be the P-sequence of the matroid

$$[\, M_4 \times A(M_4^{\,t})\,] \oplus [\, M_5 \times A(M_5^{\,t})\,]$$

Then if $P_1, P_2 \ldots P_n$ is the P-sequence of M_3 we have

$P_{n-r+1} = P_r^{\,1} \ldots P_n = P_1^{\,1}$.

CONCLUSION

This thesis is concerned with certain fundamental problems in Network Analysis and with the problems together with their extensions that these have given rise to in matroid theory.

Chapter 1 gives a fairly rigorous and self-contained description of Kron's Diakoptics for electrical networks. Also given is a new method of Network Analysis which can be looked upon as a natural complement to Kron's theory.

Chapter 1 also serves to highlight a fundamental problem in Network Analysis viz. that of constructing a partition for a graph (matroid) that corresponds to the 'topological degree of freedom' or 'hybrid rank' of the graph (matroid). One such partition is Kishi-Kajitani's 'Principal Partition'. Chapter 2 is concerned with a certain natural extension of the 'Principal Partition' for a matroid, that can be regarded in a sense as the finest possible. Our treatment is based almost completely on the matroid union theorem of Edmonds and Nashwilliams.

Chapter 3 deal with applications of the matroid union theorem to transversal matroids, gammoids and base orderable matroids.

Areas for Future Development

It would be interesting to carry out a systematic study of atomic matroids, especially with respect to subsets of a set S invariant under the automorphisms of an atomic matroid on S. This would be of great use in developing reasonably efficient algorithms (at least for some special cases) for generating all the invariant sets of a matroid. In this context we make the following conjecture :

Conjecture

Let M be a matroid on S. Let $P_1, P_2 \ldots P_n$ be subsets of S such that

(1) $P_1 \cup P_2 \quad \ldots \quad \cup P_n = S$

(2) $M \times \bigcup_{j \in J} P_i \cdot P_j$ is a totally symmetric matroid (i.e. the only set invariant under its automorphisms is P_j) for every $J \subseteq \{1, 2 \ldots n\}$ and every $j \in J$.

Then, if $T \subseteq S$ invariant under the automorphisms of M, $T = \bigcup_{i \in K} P_i$ where K is some subset of $\{1, 2 \ldots n\}$.

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ACKNOWLEDGEMENT

I am deeply indebted to my guide Professor M.N. Vartak, Head, Department of Mathematics, Indian Institute of Technology, Bombay, for his supervision of my thesis. Professor Vartak has been kind, patient and understanding throughout. Most of the results contained in this thesis are not only in line with his thinking but actually due directly to his influence. My co-guide, Dr. R.E. Bedford, Professor, Department of Electrical Engineering, Indian Institute of Technology, Bombay, has helped in clarifying some of my rather confused ideas. I am very grateful to him for his many kindnesses. I have had some useful discussions with Dr. M.V. Hariharan, Professor, Department of Electrical Engineering, I.I.T. Bombay. These have helped me a great deal especially in writing the chapter on Diakoptics.

To the Department of Electrical Engineering and the Department of Mathematics, I.I.T. Bombay, which have permitted me to wander somewhere between Electrical Engineering and Mathematics, and

to my friends in Hostel 2 who have made life interesting

I am very thankful.

(H. NARAYANAN)

Dept. of Electrical Engineering,
Indian Institute of Technology,
Bombay-400076.

February, 1974.

R E S U M E

This thesis is concerned with certain fundamental problems in Network Analysis and their extensions to matroid theory. The problems in Network Analysis that we have dealt with are connected with efficient methods of solving electrical networks. The related problem in matroid theory is to construct a partition, for any general matroid, that corresponds to the hybrid rank of the matroid. Such a partition is the 'Principal Partition' of Kishi and Kajitani. One of our main aims in this thesis has been to construct the finest possible natural refinement of this partition and study in detail its relation to the matroid union theorem (of Edmonds and Nashwilliams). The ultimate pieces into which we break up a matroid in this manner are called 'atomic' matroids in this thesis. Since our partition is invariant under automorphisms of the matroid and can be easily manipulated theoretically, (see for instance Section 6 of Chapter 2) a study of the invariant sets of an 'atomic' matroid should throw some light on the invariant sets of any general matroid. This may be one of the themes for future work.

In a separate chapter we have explored applications of the matroid union theorem to other branches of matroid theory such as the theory of transversals, gammoids and base orderable matroids.