Towards Optimal Grouping and Resource Allocation for Multicast Streaming in LTE [†]

Sadaf ul Zuhra*, *Student Member, IEEE,* Prasanna Chaporkar*, *Member, IEEE,*
and Abhay Karandikar*, *Member, IEEE*

Abstract—Multimedia traffic is predicted to account for **82%** of the total data traffic by the year 2020. With the increasing popularity of video streaming applications like YouTube, Netflix, Amazon Prime Video, popular video content is often required to be delivered to a large audience simultaneously. Multicast transmission can be used to cater to such applications efficiently. The common content can be transmitted to the users on the same resources resulting in considerable resource conservation. This paper proposes various schemes for efficient grouping and resource allocation for multicast transmission in LTE. The optimal grouping and resource allocation problems are shown to be NP-hard and so, we propose heuristic algorithms for both these problems. We also formulate a Simulated Annealing based algorithm to approximate the optimal resource allocation for our problem. The LP-relaxation based resource allocation proposed by us results in allocations very close to the estimated optimal.

Index Terms—Multicast, NP-hardness, Video streaming, LTE, eMBMS, Resource allocation.

I. INTRODUCTION

Multicast transmission refers to one-to-many transmission from a single source to multiple receivers simultaneously. Today's cellular communication is primarily based on unicast communication where the base station communicates with each user separately over orthogonal resources. However, using multicast, multiple users can receive content on the same resources simultaneously. It can be effectively used for applications like video streaming from popular platforms such as YouTube, Netflix and Amazon Prime, streaming of television (TV) programs, large scale software updates, news updates, weather forecasts and managing IoT devices in which the same content is required to be transmitted to a large number of devices at a time. Assigning orthogonal resources to every user in this scenario is a very inefficient manner of resource allocation. Using multicast transmission for such applications can save considerable network resources. As of September 2018, KT, Verizon, Telstra and Reliance (Jio) have already deployed Long Term Evolution (LTE) multicast services and 41 operators have invested in LTE multicast [2] in the form of trials and deployments worldwide.

For successfully implementing multicast in LTE, there are two main challenges that need to be addressed. The first is the problem of dividing User Equipments (UEs) into multicast groups. UEs in a multicast group are treated as a single entity by the evolved NodeB (eNB) and can be served using the same resources. One obvious requirement for grouping is for all the UEs in a group to require the same content. But, as we shall see, grouping all the UEs into a single group based on this criterion alone may lead to a degraded system performance due to varied channel gains experienced by the UEs. Therefore, the channel gains of the UEs also need to be considered

while grouping. The second problem to be addressed is that of allocating resources to the multicast groups. Here, we aim at minimizing the resources used for multicast transmissions so that its impact on other services is minimized.

Provision for multicast has been introduced in Release 9 [3] of the Third Generation Partnership Project (3GPP) standards by inclusion of Multimedia Broadcast Multicast Services (MBMS). Enhanced version of MBMS introduced in 3GPP Release 11 [4] is known as evolved MBMS (eMBMS). A radio frame in LTE spans 10 ms and consists of 10 sub-frames of 1 ms each. A sub-frame is made up of smaller units called Physical Resource Blocks (PRBs). A PRB is the smallest unit of allocation in LTE. eMBMS allows for point-to-multipoint transmission [5]. All UEs subscribed to a particular eMBMS service are served on common PRBs. Thus, UEs are grouped based only on the subscribed content. This makes the performance of eMBMS dependent on the channel gain of the weakest UE in the group which may lead to a degraded system performance. We consider an example to illustrate this.

Consider a sub-frame of 10 PRBs. Two UEs, U_1 and U_2 have subscribed to the same eMBMS service. Let the required rate for this service be 10^3 bits/sub-frame. When a PRB is allotted to a group of UEs, the rate at which reliable transmission can take place corresponds to the UE with the least channel gain in the group. Transmitting at a rate greater than this leads to unsuccessful decoding by the UEs with lower channel gains. Consider a state where U_1 has a good channel in all odd numbered PRBs so that as many as 10^3 bits can be transmitted in each of them. In the rest of the PRBs, U_1 can only get a maximum of 100 bits each. Similarly assume that U_2 can receive a maximum of 10^3 bits in each of the even numbered PRBs and 100 bits in the odd ones. If these UEs are grouped together, data can be transmitted at a rate corresponding to the UE with the least channel gain (and hence the least rate) in each PRB. In this case, only 100 bits can be transmitted in each PRB and to provide the required rate, all 10 PRBs will be used. On the other hand, if U_1 and U_2 are grouped separately, they can be alloted PRB 1 and PRB 2 respectively and 10^3 bits can be transmitted in each of these PRBs. Thus, the required rate for both will be satisfied in just 2 PRBs, 8 less than the previous scenario. This example shows that appropriate grouping of multicast UEs is essential for obtaining any benefit whatsoever from multicast operations.

We now briefly discuss the main challenges involved in grouping and resource allocation for multicast transmission. The main challenge in grouping UEs based on their channel gains is that, due to fast fading, channel gains experienced by the UEs keep on changing. As a result, grouping done based on channel conditions in one sub-frame may not be optimal in subsequent sub-frames. However, grouping UEs based on their instantaneous Signal-to-Noise Ratio (SNR) in every sub-frame is also not feasible as it will lead to increased control overhead due to frequent changes in grouping. Since

each multicast group is treated as a separate entity by the eNB, each group is assigned a unique eMBMS Radio Network Temporary Identifier (M-RNTI). M-RNTI of a group is used for scrambling its Downlink Control Indicator which carries the resource allocation information in LTE [6]. If grouping is changed every sub-frame, a new M-RNTI will have to be assigned and conveyed to UEs every sub-frame, leading to increased control overhead. Therefore, the grouping policies need to achieve a balance between the efficiency and robustness of grouping. In addition, grouping policies need to answer key questions like the number of groups that should be formed or the maximum number of UEs that should be placed in a group. Creating a lesser number of groups means more UEs in a single group which may result in lesser number of PRBs being used. However, as the number of UEs in a group increases, the probability that at least one UE is in deep fade also increases, leading to a degraded system performance. Thus, there is a trade-off between the number of multicast groups formed and the number of UEs in each group which needs to be balanced by a grouping policy.

Once the groups are formed, PRBs have to be allocated to the groups in each sub-frame. The aim of the resource allocation problem here, is to minimize the number of PRBs given to multicast UEs while guaranteeing a certain Quality of Service (QoS). The optimal resource allocation problem is a Binary Linear Program (BLP). BLPs are hard to solve and require significant computational power even for small input sizes. Thus, the optimal grouping and optimal resource allocation problems are non-trivial and we need to design efficient algorithms to solve them. In this paper, we address these problems and design algorithms to overcome the discussed challenges. Next, we discuss some of the relevant literature.

A. Related Literature

The literature related to multicast grouping and resource allocation can be broadly classified into the following categories:

1) Opportunistic Multicast Scheduling (OMS): Opportunistic scheduling schemes schedule UEs with the best channel conditions in a sub-frame to maximize the throughput. In [7], the authors present an optimized version of OMS that balances between multicast gain and multi-user diversity. [8] studies the use of opportunistic multicasting for maximizing spectral efficiency in Single Frequency Networks. In [9], the authors propose a frequency domain packet scheduler for MBMS that maximizes the minimum rate achievable by UEs in a PRB. It uses an adversarial framework in that it only minimizes the performance loss caused by the worst PRB assignment.

In [10], the authors propose the use of a throughput maximizing genetic algorithm for resource allocation in OFDMA multicast subject to power and fairness constraints. Power allocation is done based on the technique proposed in [11]. In [12] and [13], the authors propose solutions for radio resource management and grouping for multicast over 5G satellite systems aimed at maximizing the Aggregate Data Rate (ADR) of the system. Maximizing the ADR is also the objective function of [14] that proposes game theoretic bargaining solutions for multicast grouping and resource allocation.

Most of the literature considers only wideband CQI for grouping and resource allocation in multicast transmission. [15] is one work that explores the use of subband CQI values in multicast resource allocation. All the papers mentioned in this section seek to maximize the ADR in some way. In this paper, however, we focus on providing a certain rate to every multicast UE based on the service it is subscribed to.

2) Joint optimization for unicast and multicast: In [16] and [17], joint delivery of unicast and multicast in LTE and OFDMA systems has been considered. Policies proposed in [17] guarantee a certain rate to the UEs and use unicast transmission for serving UEs with the worst CQIs. [18] presents a generalized auction based resource allocation for multicast and unicast. In [19], the authors deal with resource allocation in eMBMS. They assume that video content is simultaneously available through unicast and eMBMS and their problem seeks to jointly optimize over grouping UEs and allocating resources to unicast and eMBMS. However, none of these papers consider the varying channel conditions of UEs over different PRBs while allocating resources. In this work, we account for the fact that the CQIs of UEs may be different in every PRB of a sub-frame.

3) Sub-group formation for multicast: In [20], the authors deal with the grouping problem for MBMS in High Speed Packet Access (HSPA) networks. They propose a grouping policy that minimizes a 'Global Dissatisfaction Index' that accounts for the difference in the maximum data rates achievable by UEs and the rates actually assigned to them. In [21], the same authors investigate the effect of pedestrian mobility on the performance of the grouping policy proposed in [20]. In [1], users are grouped based on their average SNR.

In [22], the authors propose sub-grouping and resource allocation schemes for multicast in LTE-Advanced. For resource allocation, they make use of the bargaining solutions proposed in [14]. Extension of bargaining solutions proposed in [14] to multi-carrier systems like LTE-A have also been studied in [23]. In [24], the authors have extended the work from [14] to exploit frequency selectivity for improving the spectral efficiency of multicast in LTE. [25] and [26] deal with the use of multicast in heterogeneous networks and grouping of UEs for MBMS respectively. Low complexity variations of the Subgroup Merging Scheme [27] that provide better ADR have been proposed in [28] for improving scalability.

Most of the papers mentioned in this section use the entire set of PRBs for catering to multicast transmission. We, however, aim to satisfy the multicast UEs in the minimum possible number of PRBs since an eNB has to typically support multiple other services alongside multicast sessions.

4) Multicasting of Scalable Video Coded (SVC) content: [29] and [30] deal with resource allocation for MBMS Operation On-Demand for video streams. The authors consider Quality of Experience (QoE) instead of QoS as the utility function that is maximized by the resource allocation schemes. [31] examines power efficient streaming of high quality SVC videos via MBMS. The algorithms proposed try to minimize the power consumption by sending traffic in discontinuous bursts, allowing UEs to sleep in between bursts. The algorithms proposed in [32] make use of Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [33] for taking multi-criteria decisions for sub-grouping and resource allocation. TOPSIS has also been used in [34] for comparing the performance of various multicast resource allocation schemes based on their ADR, fairness and spectral efficiency. In [35], the authors exploit multiuser diversity to improve the performance of LTE multicast. They propose resource alloca-

tion schemes for multicasting SVC video content where the base layer is transmitted to all the users and the enhancement layers are provided to certain groups according to their channel qualities. A similar work for multicasting scalable Internet protocol TV over WiMAX has been carried out in [36]. In this paper, the authors make use of opportunistic policies to transmit the enhancement video layers and improve the system utility. In [37], the authors study the use of multicast with caching for SVC and DASH videos. They optimize over content placement in caches and overlapping content requested by users is multicast to them from the nearest suitable helper. Even though SVC provides an interesting method of video encoding with various benefits, non-layered video codecs continue to be the choice for encoding videos over the Internet. Most of the popular streaming platforms like Netflix [38] and YouTube [39] use H.264/AVC or VP9 to encode their videos.

B. Main Contributions

In this paper, we address the problem of grouping and resource allocation for satisfying the rate requirement of each eMBMS UE while minimizing the resources used. This problem is of immense practical importance because the success of multicast services strongly depends on how well they can co-exist with the large number of services supported by LTE and 5G networks [40]. While eMBMS is suitable for streaming, its resource utilization has to be such that sufficient resources are available for other services simultaneously provided in the cells.

Most of the literature assumes the rate achievable by a UE to be the same in all PRBs. This assumption greatly simplifies the resource allocation problem as the identities of the PRBs are no longer important. In practice, however, the channel experienced by a UE is different for different frequency channels resulting in varying channel gains over different PRBs. In this work, we take these variations into account. A large portion of the literature including [10] and [17] claim that the grouping and resource allocation problems are 'hard to solve' or 'infeasible'. However, none of these papers present any mathematical proof of hardness of these problems. In this paper, for the first time, we present the proof of NP-hardness of both the optimal grouping and the optimal resource allocation problems. The main contributions of this paper are:
• We prove that the optimal resource allocation problem that minimizes the number of PRBs utilized while providing a certain rate to all the multicast groups is an NP-hard problem. Note that unicast is a special case of multicast and hence the same result applies for unicast communication as well.
• We prove that the optimal grouping problem is NP-hard.
• We devise a randomized scheme for estimating the optimal resource allocation. It works iteratively to end up at the optimal solution with high probability. • We propose a greedy scheme and an LP-relaxation based heuristic scheme for resource allocation to multicast groups. • We also propose a hybrid grouping policy for multicast group formation. • Through simulations, we have also compared the performance of our resource allocation schemes to that of the existing schemes.
• We have also conducted simulations with video traces from an actual video and shown the feasibility of our policies for practical usage.

Suitability for streaming: The allocation policies proposed in this paper are specially suitable for streaming services. Our policies ensure that all the users receive the video streams at a steady rate. The rate of transmission required by a steaming service depends on the kind and quality of video being streamed. In order to see a consistent streaming quality, the users must be served at a certain fixed rate. The resource allocation policies proposed in this paper achieve this objective since they serve all the users at their required rates.

The rest of this paper is organized as follows. In Section II, we discuss the problem formulation and the system model. The Simulated Annealing (SA) based randomized scheme and related results are presented in Section III. Sections IV and V discuss the proposed heuristic schemes for resource allocation and grouping respectively. We present the simulation results in Section VI and conclude in Section VIII.

II. PROBLEM FORMULATION

We consider an LTE cell with M UEs. All UEs have subscribed to the same eMBMS service and need to be served at a rate of R bits/sec. R can be varied across sub-frames depending on the incoming rate of packets. Note that such fixed rate streaming can potentially experience fluctuating video quality since a consistent quality video stream might require a varying bit rate. However, the varying bit rate can be taken care of by the playback buffer of an application. A large enough playback buffer can compensate for the fluctuating bit rate and provide a good video quality under fixed rate streaming, while greatly simplifying the resource allocation issues. Adaptive playback buffers [41] can also be used for adapting to the changing bit rates.

We denote the number of PRBs in a sub-frame by N. Let $[n] = \{1, \ldots, n\}$ and let $|A|$ denote the cardinality of a set A. Thus, $[M]$ and $[N]$ denote the set of multicast UEs and the set of PRBs in a sub-frame, respectively. We assume that channel gains are location and time varying. Thus, each UE has different channel gains in different PRBs and also across different sub-frames. We assume block fading channel model, and hence the channel gain of a UE is assumed to remain the same during a sub-frame. Though we do not consider mobility explicitly, our approach can be extended to cases where UE positions evolve at a slower time scale than the sub-frame duration. Let $h_{iu}[t]$ denote the channel gain for UE u on i^{th} PRB in sub-frame t. $h_{iu}[t] = \overline{h}_{iu} + H_{iu}[t]$, is made up of 2 components. $\overline{h}_{iu}$ denotes the average channel gain which accounts for path loss and shadowing and is invariant across sub-frames. $H_{iu}[t]$ is the fast-fading component that varies across sub-frames. $H_{iu}[t]$'s are independent and identically distributed (i.i.d) exponential random variables. We assume that the eNB has full Channel State Information (CSI) of all the UEs. This not a restricting assumption in the current state of LTE systems where CQI can be periodically fed back to the eNB by the UEs [42]. Corresponding to the channel gain, there is a maximum supportable rate, $r_{iu}[t]$ bits/sec for UE u on i^{th} PRB in sub-frame t. Note that $r_{iu}[t]$ is determined by the Modulation and Coding Scheme (MCS) used, and thus can take finitely many values (15 as per current standards for LTE [42]). Next, we discuss grouping.

Since all multicast UEs want the same content in each sub-frame, the UEs can be grouped together and served on common PRBs. A grouping strategy Δ is defined as follows:

Definition 1. *A grouping strategy Δ, defines a partition $\{G_1^\Delta, \ldots, G_L^\Delta\}$ of $[M]$, where $G_i^\Delta \subseteq [M]$ is referred to as the ith group.*

Note that $L \leq M$. For $L = M$, we have the unicast case. Henceforth, unicast is not dealt with separately. Throughout this paper, we assume that groups once defined at the beginning of an eMBMS session cannot be changed during the session. This is done to avoid excessive control overhead that may result due to rapid changes in grouping. One can relax the assumption and allow for grouping to be potentially changed every K sub-frames, where K is large. This will allow the scheme to adapt in case of mobile networks. The minimum supportable rate for a group G_j on i^{th} PRB in sub-frame t ($r_{ij}^\Delta[t]$) is equal to the minimum of the rates achievable by its constituent members, i.e., $r_{ij}^\Delta[t] = \min_{u \in G_i^\Delta} \{r_{iu}[t]\}$. This ensures that the content received by a group can be successfully decoded by all the members. If we transmit at rates more than this, the weakest UE in the group may not be able to decode the received content successfully. Once $r_{ij}^\Delta[t]$s are obtained, we need to decide how to allot resources to each group so that the total number of PRBs used is minimized subject to giving each group at least the minimum required rate R. This is a resource allocation problem. The formal definition of a resource allocation policy is stated below.

Definition 2. *For a given grouping Δ a resource allocation policy Γ defines an assignment of PRBs to the L multicast groups, $\{\overline{V}_{1\Gamma}^\Delta, \ldots, \overline{V}_{L\Gamma}^\Delta\}$, where, $\overline{V}_{i\Gamma}^\Delta$ is the set of PRBs assigned to group i by resource allocation policy Γ under grouping Δ. The allocation Γ should be such that $\overline{V}_{i\Gamma}^\Delta \cap \overline{V}_{j\Gamma}^\Delta = \phi$ whenever $i \neq j$ and $\bigcup_{i=1}^L \overline{V}_{i\Gamma}^\Delta \subseteq [N]$.*

Resource allocation policy Γ is said to be feasible if $\sum_{j \in \overline{V}_{i\Gamma}}^\Delta r_{ij}^\Delta[t] \geq R$ for every $i \in [L]$. The other parameter used by us to characterize a resource allocation policy is the number of PRBs left unused after resource allocation in a sub-frame t, $S_\Gamma^\Delta[t] = N - |\bigcup_{i=1}^L \overline{V}_{i\Gamma}^\Delta|$. We shall now formally state our resource allocation and grouping problems.

A. Problem 1: Optimal Resource Allocation $\mathbf{B}_\Delta^\star$

Consider a fixed grouping policy Δ, and define indicators in sub-frame t as follows:

$$x_{ij}[t] = \begin{cases} 1, & \text{if PRB } j \text{ is assigned to group } i \\ 0, & \text{otherwise.} \end{cases}$$

The optimal resource allocation can then be obtained as a solution to the following BLP for every t:

$$(\mathbf{B}_\Delta^\star): \quad \min \sum_{\mathbf{j} \in [\mathbf{N}]} \sum_{\mathbf{i} \in [\mathbf{L}]} \mathbf{x_{ij}[t]},$$

$$\text{subject to:} \quad \sum_{j \in [N]} x_{ij}[t] r_{ij}^\Delta[t] \geq R, \quad \forall i \in [L], \quad (1)$$

$$\sum_{i \in [L]} x_{ij}[t] \leq 1, \quad \forall j \in [N]. \quad (2)$$

The objective function of $\mathbf{B}_\Delta^\star$ seeks to minimize the number of PRBs used in sub-frame t. Constraint (1) guarantees that the rate given to each group is atleast equal to the required rate R and (2) ensures that each PRB is given to at most one group. Note that $\mathbf{B}_\Delta^\star$ gives the optimal resource allocation for any grouping Δ. Next, we establish the hardness of $\mathbf{B}_\Delta^\star$.

TABLE I: Table of notations

Notation	Meaning
M	Number of multicast UEs
L	Number of multicast groups
N	Number of PRBs in a sub-frame
$\mathcal{N}$	Set of available PRBs in a sub-frame
$\mathcal{L}$	Set of multicast groups
$h_{iu}[t]$	Channel gain of UE u on i^{th} PRB in sub-frame t
$r_{iu}[t]$	Maximum rate supportable by UE u on i^{th} PRB in sub-frame t
Δ	Grouping strategy
G_i^Δ	i^{th} group under policy Δ
Γ	Resource allocation policy
$\overline{V}_{i\Gamma}^\Delta$	Set of PRBs assigned to G_i^Δ under policy Γ
R	Rate requirement of the multicast UEs
$S_\Gamma^\Delta[t]$	Number of PRBs left unutilized under Δ in sub-frame t using Γ
$x_{ij}[t]$	Indicator random variable that equals 1 when PRB j is assigned to group i in sub-frame t

Lemma 1. *Optimization $\mathbf{B}_\Delta^\star$ is NP-hard.*

Proof. See Appendix A for the detailed proof. ☐

B. Problem 2: Optimal Grouping $\mathbf{C}^\star$

Recall that $S_\Gamma^\Delta[t]$ denotes the number of PRBs left unutilized under grouping policy Δ in sub-frame t using resource allocation scheme Γ. Note that these PRBs can be used for other services in the system. Define,

$$\overline{S}_\Gamma^\Delta = \liminf_{T \to \infty} \frac{1}{T} \sum_{t=1}^T S_\Gamma^\Delta[t]. \quad (3)$$

Thus, $\overline{S}_\Gamma^\Delta$ is the average number of unutilized PRBs per sub-frame under grouping policy Δ and resource allocation policy Γ. The optimal grouping problem can be defined for any given resource allocation policy Γ. The definition of the optimal grouping problem is stated below:

($\mathbf{C}^\star$) : Determine the optimal grouping policy $\Delta^\star$ such that $\overline{S}_\Gamma^{\Delta^\star} \geq \overline{S}_\Gamma^\Delta$ for every Δ.

We note that determining $\overline{S}_\Gamma^\Delta$ for a general grouping Δ and resource allocation Γ itself is a very hard, if not an impossible problem. The value of $\overline{S}_\Gamma^\Delta$ depends on the combined channel states of all the UEs in various sub-frames. We show in the following result that the problem of determining $\Delta^\star$ for given Γ is NP-hard.

Lemma 2. *For a fixed Γ, the problem of determining $\Delta^\star$ is NP-hard.*

Proof. The detailed proof is given in Appendix B. ☐

Since we have proved that both optimal grouping and optimal resource allocation problems are NP-hard, no polynomial time algorithms exist for determining their optimal solutions unless P = NP. We can, however, use some intelligent heuristic schemes to obtain near optimal solutions. In the following section, we formulate an iterative randomized scheme for estimating the optimal resource allocation.

III. RANDOMIZED ALGORITHM FOR OPTIMAL RESOURCE ALLOCATION

As stated in the previous section, no polynomial time algorithm exists for determining the optimal resource allocation. We can, however, estimate the optimal solution using randomized algorithms that iteratively explore all possible solutions to

converge to the optimum. The proposed randomized scheme serves dual purpose, 1) it provides near optimal solution in much lesser computational power than that required to solve $\mathbf{B}^\star_\Delta$ and, 2) its output can be used as a benchmark for evaluating the heuristic schemes which we propose in Section IV. We now describe the randomized algorithm.

The allocation of resources in LTE is done in every sub-frame. So, for brevity, we fix a sub-frame t and omit it from notations in this section. Grouping strategy Δ impacts resource allocation via r_{ij}^Δ, which is the rate achievable by group i in PRB j. Here, we deal with resource allocation for any given Δ. So, we omit Δ from the notations as well for better readability.

The Randomized Scheme (RS) used here is based on SA, a well known Markov Chain Monte Carlo (MCMC) technique [43]. SA is a randomized algorithm used for obtaining the global optimum of a function. In SA, we construct a Markov chain on the states of the problem under consideration and transition between the states to ultimately end up at the global optimum with high probability. Here, states correspond to possible resource allocations to groups. Therefore, a state, s_d of the Discrete Time Markov Chain (DTMC) is a possible distribution of PRBs, $\{\overline{V}_{0d}, \overline{V}_{1d} \ldots, \overline{V}_{Ld}\}$ where $\overline{V}_{id}$ is the set of PRBs assigned to group G_i, $i \in [L]$ in state s_d. G_0 is a dummy group that is assigned all the unused PRBs. The state space χ corresponds to all possible PRB allocations to groups. Let ℓ_{di} denote the total rate achieved by the ith group in allocation s_d. Thus, $\ell_{di} = \sum_{j \in \overline{V}_{id}} r_{ij}$. Moreover, let q_d denote $|\{i : \ell_{di} \geq R\}|$, i.e. q_d is the number of satisfied groups in allocation s_d. In SA, each state has an associated reward that defines how good the state is. For our DTMC, we define the real valued reward function E as follows:

$$E(s_d) = |\overline{V}_{0d}| - \sum_{i=1}^{L} [R - \ell_{di}]^+ + q_d, \qquad (4)$$

where $[y]^+ = \max\{y, 0\}$ and $|\overline{V}_{0d}|$ is the number of unused PRBs in state s_d. The reward function is a monotonically increasing function of the number of satisfied groups and number of unused PRBs. It also decreases proportionally with the difference between the required and achieved rates of the groups. Thus, intuitively, maximizing E maximizes the number of unused PRBs while satisfying all the groups. We prove this formally in the next result.

Lemma 3. *Let $\mathbf{B}^\star_\Delta$ has a feasible solution and $s_{d^\star} \in \arg\max_{s_d} E(s_d)$. Define $x_{ij}^\star = 1$ if $j \in \overline{V}_{id^\star}$ and 0 otherwise. Then, $\{x_{ij}^\star\}_{i,j}$ is the optimal solution of the BLP $\mathbf{B}^\star_\Delta$.*

Proof. The detailed proof is given in Appendix C. $\square$

Thus, determining a state that maximizes the reward function is equivalent to determining the optimal solution of $\mathbf{B}^\star_\Delta$. Note that the proposed approach uses a DTMC on χ where $|\chi| = (L + 1)^N$. Recall that L is the number of groups and N is the number of PRBs available in a sub-frame. Hence, the Transition Probability Matrix (TPM) corresponding to the DTMC will have dimensions exponential in N. So, for guaranteeing computational feasibility of the proposed approach, one must ensure that the TPM need not be stored, rather, given a state, transition probability to the neighboring states can be determined in time polynomial in system parameters. Next, we elaborate how such a DTMC can be constructed.

A. DTMC Construction

Let $E^\star$ denote the maximum value of the reward function $E(\cdot)$ defined in (4), i.e. $E^\star = \max_{s_d} E(s_d)$. Suppose we construct a DTMC $\{X_n\}_{n\geq 1}$ on χ such that $P(E(X_n) = E^\star)$ tends to 1 as n tends to ∞. If we simulate this DTMC for a large enough time, say τ, the probability that the state of the DTMC at τ yields the optimal resource allocation is very close to one. Towards this end, we first define a time homogeneous DTMC $\{X_n^T\}_{n\geq 1}$ on χ. We will subsequently define the DTMC $\{X_n\}_{n\geq 1}$ with parameter T varying as a function of n. As we will see in the following sections, transition probabilities of the constructed DTMC are a function of T. Therefore, variation of T as a function of n makes $\{X_n\}_{n\geq 1}$ non time homogeneous. For defining the DTMC $\{X_n^T\}_{n\geq 1}$, it is enough to specify its TPM, which we do next.

1) Neighboring States: Consider any state $s_d \in \chi$. A state $s_{d'}$ is a neighbor of s_d if it can be obtained from s_d using one of the following actions:

• *Swap* (A_1): Swap takes two PRBs j_1 and j_2 from groups i_1 and i_2 respectively and assigns j_1 to i_2 and j_2 to i_1. Only allocation to groups i_1 and i_2 are changed through this action. Mathematically, $s_{d'}$ is obtained from s_d using swap if:
1) $j_1 \in \overline{V}_{i_1 d}$ and $j_2 \in \overline{V}_{i_2 d}$,
2) $\overline{V}_{id'} = \overline{V}_{id}$ for all $i \neq i_1, i_2$ and
3) $\overline{V}_{i_1 d'} = (\overline{V}_{i_1 d} \backslash \{j_1\}) \cup \{j_2\}$, $\overline{V}_{i_2 d'} = (\overline{V}_{i_2 d} \backslash \{j_2\}) \cup \{j_1\}$.

• *Drop* (A_2): The drop action takes a PRB j_1 from a group i_1 ($i_1 \neq 0$) and assigns it to group G_0. Here, only allocation of groups i_1 and 0 is changed by dropping the PRB j_1. Mathematically, $s_{d'}$ is obtained from s_d using drop if:
1) $j_1 \in \overline{V}_{i_1 d}$,
2) $\overline{V}_{id'} = \overline{V}_{id}$ for all $i \neq i_1, 0$ and
3) $\overline{V}_{i_1 d'} = \overline{V}_{i_1 d} \backslash \{j_1\}$, $\overline{V}_{0d'} = \overline{V}_{0d} \cup \{j_1\}$.

• *Add* (A_3): The add action takes a PRB j_1 from $\overline{V}_{0d}$ and assigns it to a group $i_1 \neq 0$. Here, only allocation of groups i_1 and 0 is changed by assigning the PRB j_1 to group i_1. Mathematically, $s_{d'}$ is obtained from s_d using add if:
1) $j_1 \in \overline{V}_{0d}$,
2) $\overline{V}_{id'} = \overline{V}_{id}$ for all $i \neq i_1, 0$ and
3) $\overline{V}_{i_1 d'} = \overline{V}_{i_1 d} \cup \{j_1\}$, $\overline{V}_{0d'} = \overline{V}_{0d} \backslash \{j_1\}$.

Note that the neighboring relation defined here is symmetric in nature. This is proved in the following result.

Lemma 4. *The neighboring relation of the DTMC $\{X_n^T\}_{n\geq 1}$ is symmetric. Moreover, if transition from s_d to $s_{d'}$ occurs due to a swap action, then transition from $s_{d'}$ to s_d can also take place using a swap action only. Similarly, if transition to s_d from $s_{d'}$ occurs due to add (drop, respectively), the transition from $s_{d'}$ to s_d can only result from drop (add, respectively).*

Proof. To prove the required result, we need to show that if a state $s_{d'}$ is a neighbor of the state s_d, then, s_d is also a neighbor of $s_{d'}$. Since neighbors are defined using three different actions, we consider the following cases separately:

• **Swap:** Consider that $s_{d'}$ is obtained from s_d by swapping PRBs j_1 and j_2 belonging to groups i_1 and i_2 respectively. From definition of the swap action, $\overline{V}_{id'} = \overline{V}_{id} \; \forall \; i \neq i_1, i_2$, $\overline{V}_{i_1 d'} = (\overline{V}_{i_1 d} \backslash \{j_1\}) \cup \{j_2\}$ and $\overline{V}_{i_2 d'} = (\overline{V}_{i_2 d} \backslash \{j_2\}) \cup \{j_1\}$. Now, let us see if s_d can be obtained from $s_{d'}$. Say PRBs j_1 and j_2 are picked for swapping in $s_{d'}$. In $s_{d'}$, $j_1 \in \overline{V}_{i_2 d'}$ and

$j_2 \in \overline{V}_{i_1 d'}$. For the resulting state $s_{d''}$, we have:

$$\overline{V}_{id''} = \overline{V}_{id'} = \overline{V}_{id}, \forall\ i \neq i_1, i_2,$$
$$\overline{V}_{i_1 d''} = (\overline{V}_{i_1 d'} \setminus \{j_2\}) \cup \{j_1\} = \overline{V}_{i_1 d},$$
$$\overline{V}_{i_2 d''} = (\overline{V}_{i_2 d'} \setminus \{j_1\}) \cup \{j_2\} = \overline{V}_{i_2 d}.$$

Therefore, $\overline{V}_{id''} = \overline{V}_{id}$ for all i which implies that $s_{d''} \equiv s_d$. So, s_d is also a neighbor of $s_{d'}$ and can be obtained from $s_{d'}$ using a swap action only.

• **Add:** Consider that $s_{d'}$ is obtained from s_d by adding PRB j_1 to group i_1. Then, from the definition of the add action, $\overline{V}_{id'} = \overline{V}_{id}$ for all $i \neq i_1, 0$, $\overline{V}_{i_1 d'} = \overline{V}_{i_1 d} \cup \{j_1\}$ and $\overline{V}_{0d'} = \overline{V}_{0d} \setminus \{j_1\}$. Now, let us see if state s_d can be obtained from $s_{d'}$. Say PRB j_1 is picked for a drop action in $s_{d'}$. Note that in $s_{d'}$, $j_1 \in \overline{V}_{i_1 d'}$. For the resulting state $s_{d''}$, we have:

$$\overline{V}_{id''} = \overline{V}_{id'}, \forall\ i \neq i_1, 0,$$
$$\overline{V}_{i_1 d''} = \overline{V}_{i_1 d'} \setminus \{j_1\} = \overline{V}_{i_1 d},$$
$$\overline{V}_{0d''} = \overline{V}_{0d'} \cup \{j_1\} = \overline{V}_{0d}.$$

Therefore, $\overline{V}_{id''} = \overline{V}_{id}$ for all i which implies that $s_{d''} \equiv s_d$. So, s_d is also a neighbor of $s_{d'}$ and can be obtained from $s_{d'}$ using a drop action only.

• **Drop:** The proof for drop action is very similar to that for add. It can be shown in the same manner that if $s_{d'}$ is obtained from s_d using a drop action, s_d can be obtained from $s_{d'}$ using an add action and so s_d is also a neighbor of $s_{d'}$. □

We now define the TPM.

2) Transition Probability Matrix: Let $p_{dd'}$ denote the probability that the DTMC transitions to $s_{d'}$ in the next step from the current state s_d. The transition happens in two steps. 1) In state s_d, we first randomly choose one of the three actions A_1, A_2 or A_3 and then randomly choose a neighboring state s_{d_p} that can be obtained from s_d by performing the chosen action. The state s_{d_p} is referred to as the proposed next state. 2) Based on the reward values $E(s_d)$ and $E(s_{d_p})$, the proposed transition from s_d to s_{d_p} is either accepted, i.e. $s_{d'} = s_{d_p}$ or rejected, i.e. $s_{d'} = s_d$. We discuss these steps in detail below.

• Step 1: In this step, one of the three actions is picked. Since different actions lead to different sets of potential neighboring states, we will use $s_{d_{A_i}}$ to denote a state that can be obtained from s_d by performing action $A_i, i \in \{1, 2, 3\}$. Probability of picking every action is different. Action A_1 is picked with probability (w.p.) $\beta_{dd_{A_1}} = \frac{1}{3}$, A_2 is picked w.p. $\beta_{dd_{A_2}} = \frac{2}{3} \times \frac{N - |\overline{V}_{0d}|}{L(|\overline{V}_{0d}|+1) + (N - (|\overline{V}_{0d}|+1))}$ and A_3 is picked w.p. $\beta_{dd_{A_3}} = \frac{2}{3} \times \frac{L|\overline{V}_{0d}|}{L|\overline{V}_{0d}| + (N - |\overline{V}_{0d}|)}$. With the remaining probability, the state of the DTMC remains unchanged. A_3 corresponds to the add action and so, is chosen with a probability directly proportional to the number of unused PRBs and the number of multicast groups. Therefore, for greater number of groups and unused PRBs, the algorithm is more likely to choose the add action. Similarly, for greater number of used PRBs, the algorithm is more likely to choose the drop action.

Now we explain how one of the neighboring states is chosen for potential transition given the chosen action. If the chosen action is A_1, the two PRBs to be swapped, j_1 and j_2 are chosen uniformly at random from $[N]$. The swap of j_1 and j_2 is then performed as discussed in Section III-A1. For A_2, the PRB to be dropped, j_1 is picked uniformly at random from $[N] \setminus \overline{V}_{0d}$ and dropped as discussed in Section III-A1. Similarly for A_3, a group i_1 is picked uniformly at random

from $[L]$ and a PRB to be added to it, j_1 is chosen uniformly at random from $\overline{V}_{0d}$. The addition of j_1 to i_1 is then done as discussed in Section III-A1. In the next step, we discuss how the transition probabilities are finally determined.

• Step 2: Let $s_{d'}$ denote the state chosen for transition. If $s_{d'}$ has reward greater than or equal to that of s_d, the DTMC transitions to $s_{d'}$. Otherwise, transition to $s_{d'}$ takes place w.p. $e^{(-(E(s_d) - E(s_{d'}))/T)}$. Thus, probability that the DTMC will transition to $s_{d'}$ is $\alpha_{dd'} = \min\left(1, e^{(-(E(s_d) - E(s_{d'}))/T)}\right)$. T is a parameter commonly known as 'temperature'. $T > 0$ is fixed and $\{X_n^T\}_{n \geq 1}$ is the corresponding time homogeneous DTMC.

$s_{d_{A_1}}, s_{d_{A_2}}$ and $s_{d_{A_3}}$ denote the states resulting from s_d due to A_1, A_2 and A_3 respectively. Then the corresponding transition probabilities take the following form :

$$p_{dd_{A_1}} = \beta_{dd_{A_1}} \times \frac{1}{N(N-1)} \times \alpha_{dd_{A_1}}, \tag{5}$$

$$p_{dd_{A_2}} = \beta_{dd_{A_2}} \times \frac{1}{N - |\overline{V}_{0d}|} \times \alpha_{dd_{A_2}}, \tag{6}$$

$$p_{dd_{A_3}} = \beta_{dd_{A_3}} \times \frac{1}{|\overline{V}_{0d}|} \frac{1}{L} \times \alpha_{dd_{A_3}}, \tag{7}$$

$$p_{dd'} = 0, \text{if } s_{d'} \text{ is not a neighbor of } s_d. \tag{8}$$

Note that (5), (6), (7) and (8) completely describe the TPM. $p_{dd_{A_1}}$ is the probability of transitioning to $s_{d_{A_1}}$ from s_d. In (5), $\beta_{dd_{A_1}}$ is the probability of picking action A_1, the second term, $\frac{1}{N(N-1)}$ accounts for choosing 2 PRBs for swapping and $\alpha_{dd_{A_1}}$ is the probability with which the DTMC transitions to the resulting state $s_{d_{A_1}}$. Thus, $p_{dd_{A_1}}$ is the overall probability of transitioning to state $s_{d_{A_1}}$ from s_d. Similarly, in (6) and (7), $\beta_{dd_{A_2}}$ and $\beta_{dd_{A_3}}$ are the probabilities of picking A_2 and A_3 respectively, $\frac{1}{N - |\overline{V}_{0d}|}$ is the probability of choosing one of the allocated PRBs for dropping, $\frac{1}{|\overline{V}_{0d}|} \frac{1}{L}$ is the probability of choosing a certain PRB for addition from $\overline{V}_{0d}$ times the probability of picking a certain group to which the PRB can be assigned, $\alpha_{dd_{A_2}}$ and $\alpha_{dd_{A_3}}$ are the probabilities with which the DTMC transitions to the selected states $s_{d_{A_2}}$ and $s_{d_{A_3}}$ respectively. In (8), $p_{dd'} = 0$ because the DTMC cannot jump from s_d to a state that is not a neighbor of s_d.

In the randomized scheme here, we aim to simulate this DTMC with these transition probabilities. The steps involved in the randomized scheme are presented in the form of a pseudo-code in Algorithm 1. Note that the TPM of the DTMC is not being stored in this algorithm and the transition probabilities defined above can be determined in polynomial time. Thus, the TPM satisfies all the conditions stated earlier for computational feasibility of the algorithm. In the next result, we prove certain important properties of the DTMC.

Lemma 5. *The constructed DTMC $\{X_n^T\}_{n \geq 1}$ is finite, aperiodic and irreducible for every $T \in (0, \infty)$.*

Proof. The DTMC is finite because total number of possible resource allocations is $(L + 1)^N$. The DTMC has self loops which makes it aperiodic. The DTMC can transition from any state s_d to any other state $s_{d'}$ by first dropping all the used PRBs into G_0 by choosing the drop action repeatedly. Then, PRBs can be added one by one according to the assignment in $s_{d'}$ by choosing the add action repeatedly. Thus, there is at least one finite length path from any state s_d to any other state $s_{d'}$. Hence, the DTMC is irreducible. □

Having established that the DTMC is finite, aperiodic and irreducible, it is guaranteed to have a unique steady state

---

Algorithm 1: Algorithm for the Randomized Scheme

Input: Rates $r_{ij} \forall\, i \in [L]$ and $j \in [N]$, $max\_iter = 10^5$

1 Initialize: s_0, initial random allocation state
2 $s_d \leftarrow s_0$
3 **for** $n = 1 : max\_iter$ **do**
4    $s_{d'} \leftarrow s_d$
5    $T \leftarrow \frac{1}{\log(n)}$
6   Pick action A_1, A_2 or A_3 w.p. $\beta_{dd_{A_1}}$, $\beta_{dd_{A_2}}$ and $\beta_{dd_{A_3}}$ respectively
7   **if** action=A_1 **then**
8   Pick any two PRBs, $j_1, j_2 \in [N]$. Say,
     $j_1 \in \overline{V}_{i_1 d'}$ & $j_2 \in \overline{V}_{i_2 d'}$
9      $\overline{V}_{i_1 d'} = \overline{V}_{i_1 d'} \setminus \{j_1\} \cup \{j_2\}$,
     $\overline{V}_{i_2 d'} = \overline{V}_{i_2 d'} \setminus \{j_2\} \cup \{j_1\}$
10   **else if** action=A_2 **then**
11   Pick a PRB, $j \in \cup_{i=1}^{L} \overline{V}_{id'}$. Say, $j \in \overline{V}_{id'}$
12      $\overline{V}_{id'} = \overline{V}_{id'} \setminus \{j\}$, $\overline{V}_{0d'} = \overline{V}_{0d'} \cup \{j\}$
13   **else**
14   Pick any $j \in \overline{V}_{0d'}$ and any $i \in \{1,2,\dots,L\}$
15      $\overline{V}_{id'} = \overline{V}_{id'} \cup \{j\}$, $\overline{V}_{0d'} = \overline{V}_{0d'} \setminus \{j\}$
16   **end**
17    $s_d \leftarrow s_{d'}$, if $E(s_{d'}) \geq E(s_d)$
18    $s_d \leftarrow s_{d'}$ w.p. $e^{(-(E(s_d)-E(s_{d'}))/T)}$, otherwise
19 **end**
20 s_d is the proposed resource allocation

---

distribution. In the following result, we determine this steady state distribution.

Theorem 1. *For any fixed $T > 0$, the steady state distribution of the DTMC $\{X_n^T\}_{n \geq 1}$ is given by*

$$\pi_d^T = \frac{e^{E(s_d)/T}}{\sum_{s_d} e^{E(s_d)/T}} \forall\, s_d \in \chi. \quad (9)$$

Proof. To prove the required, we show that the transition probabilities in (9) satisfy $\pi_d^T p_{dd'} = \pi_{d'}^T p_{d'd}$ for every $s_d, s_{d'}$. This will imply that the DTMC is reversible [44] and has steady state distribution $\pi_d^T = \frac{e^{E(s_d)/T}}{\sum_{s_d} e^{E(s_d)/T}}, \forall\, s_d \in \chi$.

Suppose s_d and $s_{d'}$ are not neighboring states, then $p_{dd'} = p_{d'd} = 0$. Hence, the required follows trivially. Thus, it suffices to consider the case when s_d and $s_{d'}$ are neighbors. If s_d and $s_{d'}$ are neighbors, there are three possibilities, that $s_{d'}$ is obtained from s_d by 1) swap action, 2) drop action or 3) add action. We consider each case separately:

• **Swap:** If the transition from s_d to $s_{d'}$ occurs due to a swap action, then $p_{dd'}$ and $p_{d'd}$ take the form given by (5). For $E(s_d) \geq E(s_{d'})$ we have:

$$\frac{e^{E(s_d)/T}}{\sum_{d \in \chi} e^{E(s_d)/T}} \frac{1}{3} \frac{1}{N(N-1)} e^{-(E(s_d)-E(s_{d'}))/T}$$
$$= \frac{e^{E(s_{d'})/T}}{\sum_{d \in \chi} e^{E(s_d)/T}} \frac{1}{3} \frac{1}{N(N-1)},$$

which is true. Therefore, the given π_d^T satisfies $\pi_d^T p_{dd'} = \pi_{d'}^T p_{d'd}$ for the swap action. This can be similarly shown for $E(s_d) < E(s_{d'})$ as well.

• **Add:** If the transition from s_d to $s_{d'}$ occurs due to an add action, $p_{dd'}$ and $p_{d'd}$ will be given by (7) and (6) respectively.

For $E(s_d) \geq E(s_{d'})$, we have:

$$\frac{2\pi_d^T}{3\left(L|\overline{V}_{0d}| + (N - |\overline{V}_{0d}|)\right)} e^{-(E(s_d)-E(s_{d'}))/T}$$
$$= \frac{2\pi_{d'}^T}{3\left(L(|\overline{V}_{0d'}|+1) + (N - (|\overline{V}_{0d'}|+1))\right)}. \quad (10)$$

Since $s_{d'}$ is obtained from s_d by an add action, $|\overline{V}_{0d}| = |\overline{V}_{0d'}| + 1$ which means that $L|\overline{V}_{0d}| + (N - |\overline{V}_{0d}|) = L(|\overline{V}_{0d'}|+1) + (N - (|\overline{V}_{0d'}|+1))$ in (10). So, (10) becomes:

$$\pi_d^T e^{-(E(s_d)-E(s_{d'}))/T} = \pi_{d'}^T,$$
$$\implies \frac{e^{E(s_d)/T}}{\sum_d e^{E(s_d)/T}} e^{-(E(s_d)-E(s_{d'}))/T} = \frac{e^{E(s_{d'})/T}}{\sum_d e^{E(s_d)/T}},$$

which is true. Therefore, the given π_d^T satisfies $\pi_d^T p_{dd'} = \pi_{d'}^T p_{d'd}$ for the add action. This can be similarly shown for $E(s_d) < E(s_{d'})$ as well.

• **Drop:** If the transition from s_d to $s_{d'}$ occurs due to a drop action, $p_{dd'}$ and $p_{d'd}$ will be given by (6) and (7) respectively. Also, in this case, $|\overline{V}_{0d'}| = |\overline{V}_{0d}| + 1$. Following the same steps as for the add action, it can be shown that the given π_d^T satisfies $\pi_d^T p_{dd'} = \pi_{d'}^T p_{d'd}$ for the drop action as well.

Therefore, we conclude that the steady state distribution of the DTMC is $\pi_d^T = \frac{e^{E(s_d)/T}}{\sum_{s_d} e^{E(s_d)/T}} \forall\, s_d \in \chi$. ☐

For a fixed T, the DTMC is time homogeneous with steady state distribution π_d^T as shown in Theorem 1. When T varies as a function of time n, the DTMC is no longer time homogeneous and the steady state distribution cannot be determined in the same manner. We require this non time homogeneous DTMC $\{X_n\}_{n \geq 1}$ to end up in a reward maximizing state. In the following theorem, we show that this does indeed happen.

Theorem 2. *For the non time homogeneous DTMC $\{X_n\}_{n \geq 1}$, $\lim_{n \to \infty} P(X_n = s_d)$ exists, call it π_d. Moreover, $\pi_d = \lim_{T \to 0} \pi_d^T$. Specifically,*

$$\pi_d = \begin{cases} 1/|\arg\max_d E(s_d)|, & \forall\, d \in \arg\max_d E(s_d), \\ 0, & otherwise. \end{cases} \quad (11)$$

Thus, π_d is a uniform distribution over the optimal resource allocation states.

Proof. This follows directly from Theorem 1 of [45]. ☐

We mentioned the parameter T above, while discussing the TPM. Now, we elaborate its significance in more detail. SA involves an exploration versus exploitation trade-off. It achieves a balance between exploration and exploitation through this parameter T. T is kept very high in the beginning so that the algorithm can explore a large number of states quickly. As the time index increases, T goes on decreasing and so does the likelihood of transitioning to lower reward states. $T = 1/\log(n)$, n being the time index is the optimal cooling schedule [45]. This form of T ensures that the algorithm escapes local optima faster and ends up at the global optimum as T goes to 0. Specifically, by varying T, we can achieve the required $\lim_{n \to \infty} P(E(X_n) = E^\star) = 1$. In the next section, we compare the results of the RS with the optimal solution obtained by solving the BLP $\mathbf{B}_\Delta^\star$ for small input sizes.

TABLE II: Performance comparison of RS and BLP

No. of groups	RS	BLP	% Error
2	93.53	96	2.57
3	90.05	94	4.2
4	86.54	91	4.9

B. Performance comparison of the RS and the BLP

The optimal resource allocation can be obtained by solving BLP $\mathbf{B}_{\Delta}^{\star}$ from Section II. BLPs, as mentioned before, are inherently hard to solve. They can however be solved for small input sizes. Using the computing power at our disposal (Intel i7, 2.90 GHz quad-core processor with 16 GB RAM), we were able to obtain a solution of $\mathbf{B}_{\Delta}^{\star}$ for an input size of up to 4 groups. Note that the search space scales as $(L+1)^N$ where L is the number of groups and N is the number of PRBs in a sub-frame. So, even for 4 groups and 100 PRBs, the search space consists of 5^{100} states which is why the BLP fails to give a solution for more than 4 groups. The outputs of the BLP and the RS for up to 4 groups, averaged over 100 different channel conditions are tabulated in Table II. As we can see, the output of the RS is close (difference in number of PRBs saved $< 5\%$) to the optimal obtained by solving the BLP.

The randomized scheme works iteratively to obtain an optimal solution and so, it is not guaranteed to converge within a sub-frame of 1 ms. We require resource allocation schemes that can output a near optimal solution (if not optimal) every sub-frame. We now present two such heuristic schemes.

IV. HEURISTIC SCHEMES FOR RESOURCE ALLOCATION

In this section, we propose two heuristic schemes for allocating PRBs to multicast groups. The first one allocates PRBs greedily and the second makes use of Linear Programming (LP) relaxation. Resource allocation in LTE is done every sub-frame. So, for brevity, we fix a sub-frame t and omit it from notations in this section. Grouping strategy Δ impacts resource allocation via r_{ij}^{Δ}, which is the rate achievable by group i in PRB j. Our aim is to propose resource allocation for any given Δ. So, we fix Δ and omit it from the notations as well.

A. Greedy Allocation

Algorithm 2: Greedy Resource Allocation Scheme

Input: Rates r_{ij} for all $i \in [L]$ and $j \in [N]$
1 Initialize: $\mathcal{N} = [N]$, $\mathcal{L} = [L]$ and $x_{ij} = 0$ for every i, j
2 **while** $\mathcal{N} \cap \mathcal{L} \neq \phi$ **do**
3   Assign $(i^{\star}, j^{\star}) = \arg\max_{(i,j) \in \mathcal{N} \times \mathcal{L}} r_{ij}$
4     $x_{i^{\star}j^{\star}} \leftarrow 1$, $\mathcal{N} \leftarrow \mathcal{N} \setminus \{j^{\star}\}$
5   **if** $\sum_{j \in [N]} x_{i^{\star}j} r_{i^{\star}j} \geq R$ **then**
6        $\mathcal{L} \leftarrow \mathcal{L} \setminus \{i^{\star}\}$
7   **end**
8 **end**

The pseudo code for this scheme is given in Algorithm 2. Here, $\mathcal{N}$ and $\mathcal{L}$ denote the unallocated PRBs and the groups whose rate requirements are not yet satisfied, respectively. These quantities are updated every iteration and are monotone non-increasing. The algorithm terminates when either of the two sets becomes empty. In each iteration, the algorithm determines indices $i^{\star}$ and $j^{\star}$ from $\mathcal{L}$ and $\mathcal{N}$, respectively, that correspond to the maximum r_{ij}. PRB $j^{\star}$ is allotted to group $i^{\star}$ and is removed from $\mathcal{N}$. Also, if the total sum rate on all the allotted PRBs to $i^{\star}$ is greater than or equal to the requirement R, then $i^{\star}$ is also removed from $\mathcal{L}$. Next

iteration starts with the new values of $\mathcal{N}$ and $\mathcal{L}$. Note that $\mathcal{N}$ is monotone decreasing, thus, the algorithm terminates in at most N iterations. At the termination, if only $\mathcal{N} = \phi$ and $\mathcal{L}$ is non-empty, then the greedy resource allocation scheme fails to output a feasible resource allocation, else variables x_{ij}'s yield the required resource allocation. The resource allocation thus obtained is inherently fair as the algorithm provides the minimum required rate R to all the UEs.

B. LP-relaxation Based Allocation

Recall that the optimal resource allocation can be obtained as a solution to the BLP $\mathbf{B}_{\Delta}^{\star}$. BLPs are hard to solve and cannot be solved in reasonable time except for very small input sizes. A standard approach is to do LP-relaxation of the BLP i.e., relax the binary variables (in our case, x_{ij}s) to take values in the interval $[0, 1]$. The resulting LP can be solved in polynomial time. Let $\tilde{x}_{ij}$ for all i, j denote the optimal solution of the relaxed LP. Now, $\tilde{x}_{ij}$s are real numbers and we need to convert them to binary values without violating the constraints of $\mathbf{B}_{\Delta}^{\star}$. To do this, we use a greedy algorithm (Algorithm 3) similar to the one used in Section IV-A. In each iteration, PRB j is assigned to an unsatisfied group i if it has the largest value of $\tilde{x}_{ij}$ for that PRB. This is intuitive, as a higher value of $\tilde{x}_{ij}$ means that group i was assigned a larger share of PRB j by the LP. Note that the resource allocation obtained using this scheme is inherently fair as the algorithm ensures that the rate R is provided to all the UEs. We shall refer to this scheme as the LPr scheme from this point onwards.

Algorithm 3: Rounding off algorithm for LP-relaxation

Input: $\tilde{x}_{ij}$ for all $i \in [L]$ and $j \in [N]$
1 Initialize: $\mathcal{N} = [N]$, $\mathcal{L} = [L]$ and $x_{ij} = 0$ for every i, j
2 **while** $\mathcal{N} \cap \mathcal{L} \neq \phi$ **do**
3   Assign $(i^{\star}, j^{\star}) = \arg\max_{(i,j) \in \mathcal{N} \times \mathcal{L}} \tilde{x}_{ij}$
4     $x_{i^{\star}j^{\star}} \leftarrow 1$, $\mathcal{N} \leftarrow \mathcal{N} \setminus \{j^{\star}\}$
5   **if** $\sum_{j \in [N]} x_{i^{\star}j} r_{i^{\star}j} \geq R$ **then**
6        $\mathcal{L} \leftarrow \mathcal{L} \setminus \{i^{\star}\}$
7   **end**
8 **end**

---

1) Performance Comparison of RS and LPr: In order to compare the performance of the LPr scheme to that of the RS, we simulate an LTE cell with all the multicast UEs requiring the same content from the eNB. PRBs are allocated to the UEs using the RS as well as the LPr scheme. We gradually increase the number of UEs in the cell starting from 10 UEs and go up to 100. For each of the resulting 10 scenarios, the PRB allocation is done for 100 different fading variations using both the schemes. The average number of PRBs saved is used as a measure for performance comparison. The results of the simulations are plotted in Fig. 1a. Each point in the curves has been obtained by averaging over 100 different channel gain variations. Note that all the groups achieved the required rates at all points in the two curves. Both the algorithms show a similar trend as the number of UEs in the cell increases. Even though the RS saves more PRBs throughout, the ratio of the number of PRBs saved by the RS to the number of PRBs saved by the LPr scheme is no more than 1.25.

2) Time Comparison of RS and LPr: Recall that, in LTE, the allocation of PRBs is done every sub-frame. Since a sub-frame spans only 1 ms in time, it is important for whatever

TABLE III: Time taken in seconds to run RS and LPr

No. of groups	RS	LPr	Ratio
6	0.082	0.015	5.47
7	0.086	0.019	4.53
8	0.089	0.021	4.24
9	0.097	0.017	5.71
10	0.096	0.018	5.33

resource allocation scheme we employ, to be time efficient as well. We now do a time comparison of RS and LPr schemes.

The RS is an iterative algorithm and cannot be guaranteed to converge within the span of a sub-frame. While simulating the RS in this paper, we perform 10^5 iterations. However, for the time comparison here, we will first see how the reward of the current state of the RS changes as a function of the number of iterations. Fig. 1b illustrates the change in the reward of the current state of the RS as a function of the number of iterations for different number of UEs in the cell. We can observe from the figure that the output saturates well before 2000 iterations in each curve. So, for the sake of time comparison with the LPr scheme, we consider the time taken by just 2000 iterations of the RS. Table III illustrates the time taken by the RS and the LPr scheme for different number of groups in the cell. The time taken is averaged over 200 different channel gains. We observe that the RS takes about 5 times more time to run than the LPr scheme even with just 2000 iterations. Note that in practice, depending upon the system, we might need to run the algorithm for a much larger number of iterations.

From the performance and time comparisons of the LPr and the RS, we conclude that LPr performs nearly as well as the RS in 5 times lesser duration than the RS. Thus, the LPr scheme is a suitable resource allocation scheme for practical implementation. In the next section, we present a heuristic scheme for the grouping of UEs for multicast transmission.

V. HEURISTIC SCHEME FOR GROUPING

We proved in Section II that obtaining the optimal grouping strategy $\Delta^\star$ that maximizes $\overline{S}^{\Delta^\star}$ is NP-hard. Indeed, even quantifying $\overline{S}^\Delta$ for a given Δ is a very difficult task as the channel gains and hence the rates vary over time. This is because obtaining the optimal resource allocation in a given sub-frame itself is NP-hard. However, even if some genie provides us with the value $\overline{S}^\Delta$ for any given Δ, determining the optimal $\Delta^\star$ is still NP-hard. Hence, in this section, we present the following heuristic algorithm for grouping.

A. Hybrid Grouping Policy

For grouping, eNB fixes the SNR thresholds for groups and then UEs are assigned to various groups based on their average SNR values. 3GPP standards for LTE [42] define 15 CQI values, 15 being the best and 1 being the worst. In keeping
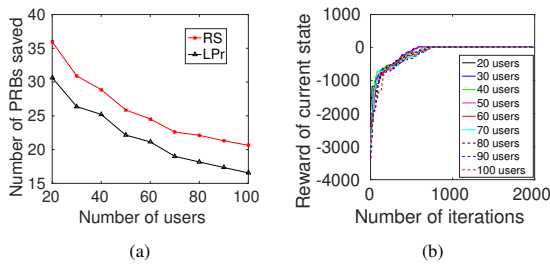

Fig. 1: a) Number of PRBs saved under LPr and RS, b) Variation of the reward of the state of RS with the increasing number of iterations

with the number of CQI values, we fix the number of grouping intervals to be 15. In LTE, a range of SNR values get mapped to a CQI value [42] (many to one map). Let the minimum SNR that can be mapped to a CQI value c be denoted by $\mathrm{SNR}_{\min}(c)$. We define a threshold corresponding to CQI c at such a level that with a large probability (0.9), the instantaneous SNR of every UE in the group will stay above or at $\mathrm{SNR}_{\min}(c)$. Specifically, a threshold $D(c)$ is defined such that,

$$P\{\mathrm{SNR} \geq \mathrm{SNR}_{\min}(c)|\mathrm{SNR}_{\mathrm{avg}} = D(c)\} = 0.9. \quad (12)$$

To compute $D(c)$, we need the distribution of SNR which depends on the distribution of $h_{iu}[t]$. $H_{iu}[t]$ (the fast-fading component of $h_{iu}[t]$ as defined in Section II) are i.i.d exponential with mean 1. Given that the average SNR is equal to $D(c)$, the distribution of the instantaneous SNR is exponential with parameter $D(c)$. Therefore, (12) can be written as:

$$e^{-\frac{\mathrm{SNR}_{min}(c)}{D(c)}} = 0.9, \quad (13)$$

$$\implies D(c) = \frac{\mathrm{SNR}_{\min}(c)}{\log(10/9)}. \quad (14)$$

Now that the thresholds have been defined, UEs are classified into groups based on their average SNR values. UEs with average SNR greater than or equal to $D(15)$ are classified as Group 1 and those with SNR below $D(2)$ are grouped into Group 15. UEs with average SNR between $D(14)$ and $D(15)$ are put into Group 2 and so on. Thus, Group 1 (Group 15) corresponds to the UEs with the best (worst) channel.

As group sizes grow, the probability that one or more UEs in a group will experience a poor channel increases. Therefore, the performance of the grouping scheme may start worsening with increasing group sizes. To prevent this, we propose a second layer of grouping. If the number of UEs in a group exceeds a certain maximum value, it is further divided into smaller groups. We fix the maximum group size such that all UEs in a group experience a good channel in at least 10% of the PRBs in a sub-frame. Since the thresholds have been set so that the instantaneous SNR of a UE remains above $\mathrm{SNR}_{min}(c)$ with probability 0.9 and the channels are independent across UEs and across PRBs, this probability is given by:

$$p = \sum_{j=\lfloor 0.1N \rfloor}^{N} \binom{N}{j} 0.9^{kj}(1 - 0.9^k)^{(N-j)}, \quad (15)$$

where k is the group size. We need this probability to be large. For example, in a 20 MHz LTE system with $N = 100$, $p = 0.9452$ for $k = 18$. So, we fix the maximum group size for this system at 18 and whenever a group grows beyond 18 UEs, the group is split into smaller groups of 18 UEs or less. Note that p is a monotonically decreasing function of k.

After the UEs are divided into groups, the rate for a particular group is set at the value corresponding to the weakest UE in the group. Once the achievable rate for each group is determined using the 3GPP mappings [42], PRB allocation is done according to the resource allocation schemes discussed in the previous section.

B. Complexity

The greedy allocation algorithm has a complexity of $\mathcal{O}(LN^2)$ and the LP relaxation based allocation has a complexity of $\mathcal{O}(LN^2 + LN)$. The hybrid grouping policy has a complexity of $\mathcal{O}(M)$.

VI. SIMULATION RESULTS

A. Simulation Settings

Our simulation setup comprises an LTE cell of radius 375 m in accordance with 3GPP simulation parameters for macro cells [46]. We have used the MATLAB [47] LTE simulator designed in [48] to conduct our simulations. LTE specific physical layer conditions have been created using 3GPP channel models [46]. SNR to CQI and CQI to rate mapping has also been done according to 3GPP specifications [46].

An eNB located at the center of the cell multicasts the eMBMS content to all the multicast groups. Rate requirement for each UE (R) is taken to be 1 Mbps. The UEs are distributed uniformly at random within the cell and are grouped using the hybrid grouping policy proposed in Section V. Groups are formed at the beginning of eMBMS session and remain unchanged during the session. For dividing the UEs into groups, we need to determine the average SNR received at the UEs. For calculating the average SNR, we use shadowing and path loss models as per 3GPP specifications [46]. The channel gain of each UE may be different for different PRBs. The channel gains are determined by: 1): Path Loss, 2): Shadowing and 3): Multipath due to reflections from the surrounding environment. After the grouping is done, PRBs are allocated using the policies proposed in Section IV. Resource allocation policies make use of the instantaneous SNR values for taking the allocation decisions. For determining the instantaneous SNR values, we also take Rayleigh fading into account. We compare the performance of the proposed schemes with unicast transmission. The performance of the resource allocation policies is also compared to that of the widely used Proportional Fair (PF) policy [10], [11], [19]. Parameters relevant to our simulations are given in Table IV.

For a given grouping and resource allocation, their performance is affected by two sources of randomness, (1) channel variations around mean on account of fast fading and (2) average channel gain variations on account of node positions. We evaluate the performance of our schemes by averaging over these two sources of randomness. Towards this end, we consider 100 random UE placements and the performance of each placement is evaluated and averaged over 1000 sub-frames with different channel gains. In addition to unicast and the proposed grouping policy, we also simulate a random grouping where each UE is placed in one of 10 groups uniformly at random. Under the proposed hybrid grouping policy, a maximum of 15 groups can be formed. However, the actual number of groups formed will depend upon the average SNR of the users. The average number of groups formed during the simulations is given in Table V.

B. Results

Fig. 2a and Fig. 2c illustrate plots of the number of PRBs saved against the number of UEs in the cell for greedy and LPr schemes respectively. Observations from these plots are: a) Unicast performs the worst and is unable to support more than 20 UEs. b) Random grouping is able to support up to 30 UEs successfully. The number of PRBs saved rapidly decreases to 0 beyond 30 UEs. c) With hybrid grouping, greedy allocation saves greater than 10 PRBs for up to 70 UEs. Using LPr saves around 20 PRBs even for 100 UEs.

TABLE IV: System Simulation parameters [46]

Parameters	Values
System bandwidth	20 MHz
Center frequency	2 GHz
eNB cell radius	375 m
Path loss model	L = 128.1 + 37.6 log 10(d), d in kilometers
Shadowing	Log Normal Fading with 10 dB standard deviation
White noise power density	−174 dBm/Hz
eNB noise figure	5 dB
eNB transmit power	46 dBm
PRB width	180 kHz
Number of PRBs	100 per sub-frame
ITU path loss model	ITU-R M.2135-1 [49]

Fig. 2b and Fig. 2d illustrate the number of sub-frames (out of 1000) for which the allocations are rendered infeasible for greedy and LPr schemes respectively. We observe that: a) Unicast and random grouping quickly become completely infeasible beyond 40 UEs for greedy scheme and beyond 50 for LPr. b) Using greedy allocation, the number of infeasible cases for hybrid grouping is zero for up to 40 UEs. Using LPr, the allocation is always feasible. We have also conducted all these simulations for $R = 2$ Mbps. The corresponding results are plotted in Fig. 3. We observe a similar relative performance of the policies as that for $R = 1$ Mbps. Fig. 4a and Fig. 4b illustrate the number of PRBs saved at the eNB for different UE placements. For every M, 100 different UE placements have been considered. Out of these, 90% closest to the mean have been plotted as a scatter plot. The conclusions from these plots are: a) For varying UE placements, the number of PRBs saved at the eNB is between ± 5 PRBs around the mean number of PRBs saved for all the schemes. b) Overall trend of the number of PRBs saved as the UE count increases is the same as that observed in Fig. 2a and Fig. 2c.

In addition to QoS, it is also important to guarantee a good QoE in video streaming. QoE is known to be a function of various QoS parameters of the network [50]. The QoE of a video stream primarily depends on the delay, delay jitter and the packet loss rate in the network [51], [52]. To study the impact of our policies on the QoE of users, we evaluate their performance using data from an actual video stream. For this purpose, we have used an H.264 encoded video of Star Wars IV (obtained from (http://trace.eas.asu.edu)) [53]. For transmitting this video stream, the required rate R is changed every sub-frame according to the requirement of the video frame being transmitted. Under our policies, packets of the video are transmitted as soon as they arrive. As a result, the access network does not induce any additional delay and jitter in the video stream. The users only experience the delay incurred due to the core network. Therefore, the QoE of the users is not degraded by our policies. Fig. 6 shows the histogram of the number of PRBs saved while transmitting the frames of this video. The proposed resource allocation policies are able to meet the requirements of the video stream in far lesser number of resources than unicast transmission.

These simulation results clearly establish the superiority of the proposed algorithms for use in multicast systems. The hybrid grouping policy provides a significant advantage over unicast and random grouping. The poor performance of random grouping reinforces the importance of efficient grouping algorithms. It clearly shows that if users are thrown together without considering their channel conditions, multicast may

TABLE V: Average number of groups formed

No. of UEs	No. of groups	No. of UEs	No. of groups
10	5.39	20	6.94
30	7.75	40	7.96
50	8.45	60	8.39
70	8.66	80	8.77
90	8.77	100	9

not provide any advantage over unicast. Among the proposed resource allocation policies, LPr does better than the greedy policy. It satisfies the users in lesser number of PRBs and successfully meets their requirements in every sub-frame. Next, we compare the performance of LPr with the widely used PF policy.

For comparing LPr to PF, we consider a scenario with both multicast and unicast UEs in the system. We use system throughput and user rate satisfaction as metrics for the comparison. In order to compare our scheme with PF, we first allocate the required number of PRBs to multicast groups using LPr. Since the rate requirements of the unicast UEs are not fixed, allocation to the unicast UEs is done using the PF policy.

In Fig. 5a, we see the average sum throughout provided by LPr and PF allocations. The figure shows that our scheme results in a significantly better system throughput. Even though it uses a chunk of PRBs to satisfy the rate requirement of the multicast UEs, LPr policy still provides a better overall throughput. In Fig. 5b, we plot the percent unsatisfied multicast groups as a function of the number of users in the system. The PF policy nearly always fails to meet the requirements of the multicast users even with no constraint on the amount of resources it can use. While PF schemes work well in a unicast only scenario, they are not suitable for rate constrained streaming systems that require a certain rate to be provided to the subscribers in every sub-frame. Grouping users according to the hybrid grouping policy and allocating resources using the LPr policy provides the best performance in terms of resource utilization and user satisfaction.

VII. GENERALIZATIONS

In this paper, we have primarily focused on grouping and resource allocation for multicast streaming for non-layered video coding such as H.264/AVC in LTE. However, the proposed policies can also be used in several other more general scenarios and also in 5G networks. We discuss some of these generalizations in this section.

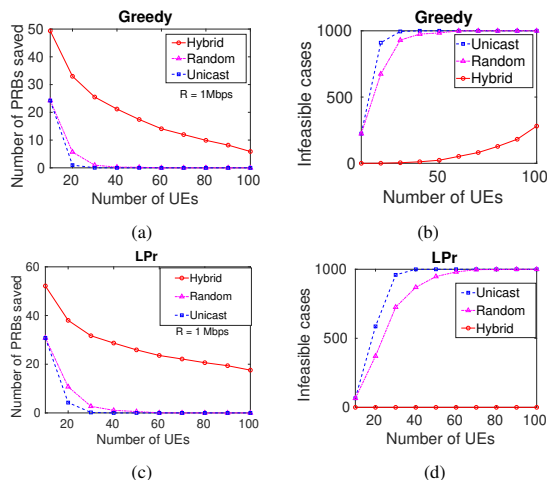• *Heterogeneous quality demands*: The users subscribed to the same eMBMS service may want to see different qualities of the same video stream. Some users may want ultra HD quality while others may prefer a lower quality video for a lesser price. This heterogeneity of user demands can be handled by treating the users who require the same quality as a separate group with a specific rate requirement. The proposed allocation policies can be used as is for such a system.

• *Rate adaptation*: The proposed policies can also be used in streaming systems with rate adaptation as long as the rate adaptation takes place on a slower time scale than a sub-frame (1 ms). When the rate requirements change, the grouping of users can be changed accordingly. As discussed in Section II, we can allow for the groups to change every K sub-frames, where K is large. Even if rate adaptation occurs on the order of a few seconds, K would be of the order of a few 1000 sub-frames and the proposed policies can still be used.

• *SVC*: When SVC is used for encoding the streaming content, different sets of users may require a different number of enhancement layers of the video. The base layer, however, needs to be transmitted to all the users. The algorithms proposed in
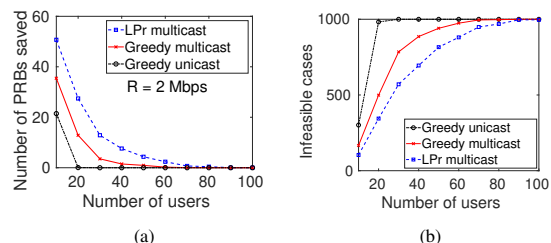

Fig. 3: Performance evaluation of resource allocation policies (R = 2 Mbps): a) number of PRBs saved b) number of infeasible cases
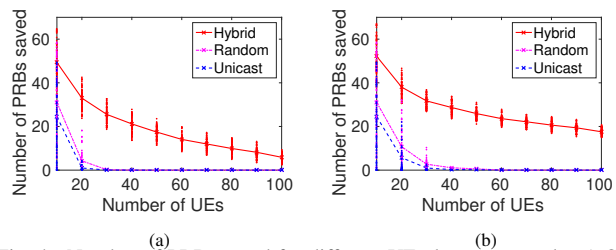

Fig. 4: Number of PRBs saved for different UE placements under a) Greedy scheme b) LPr scheme
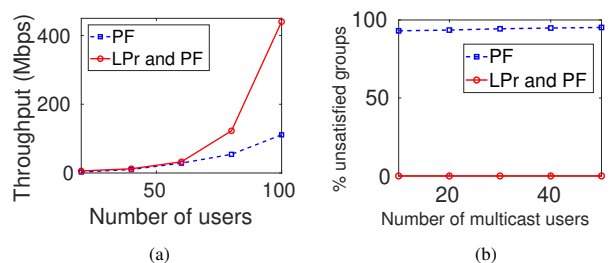

Fig. 5: Comparison of LPr scheme with PF for a) average system throughput and b) unsatisfied groups
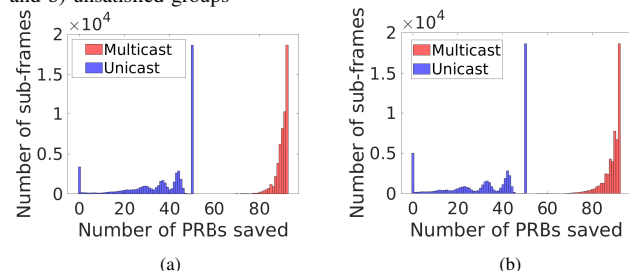

Fig. 6: Histogram of number of PRBs saved for a real-time video stream under a) LPr b) Greedy

Fig. 2: Performance evaluation of proposed policies

this paper can then be used for transmitting the base layer to all the subscribed users and separate algorithms can then be used for opportunistically transmitting the enhancement layers to the groups with good channel conditions [35], [36].

• *5G*: Even though the policies proposed in this paper have been discussed in the context of an LTE system, the policies can be used in next generation 5G systems as well. The proposed hybrid grouping policy makes use of the SNR to CQI mappings to define the grouping thresholds. Similar mappings are also defined in 5G [54] which can be similarly used to define the SNR thresholds. The proposed resource allocation policies are also technology agnostic. The bandwidth in 5G is also divided into PRBs [55] and the proposed policies can be used to determine resource allocation for 5G in the same manner.

Thus, the algorithms proposed in this paper can be easily adapted for video multicasting in a wide variety of systems.

VIII. CONCLUSIONS AND FUTURE WORK

In this paper, we have addressed the problems of grouping and resource allocation in multicast transmission. We have formulated these problems with the aim of minimizing the number of PRBs needed to cater to multicast services. In this paper, for the very first time, we have proved that the optimal grouping and the optimal resource allocation problems are NP-hard and therefore, no polynomial time algorithms exist for determining their optimal solutions. We have proposed a randomized scheme (RS) that works iteratively for estimating the optimal resource allocation. The output of the RS provides a benchmark for performance evaluation of heuristic resource allocation schemes. We have proposed two efficient heuristics for resource allocation, a greedy and an LPr scheme. LPr results in feasible resource allocations that save nearly as many PRBs as that saved by RS in about one-fifth the time taken by RS. We have proposed a hybrid grouping policy for multicast group formation as well. Using extensive simulations, we have shown that using the proposed policies for grouping and resource allocation results in significant resource conservation. The proposed schemes can act as an enhancement to eMBMS. These enhancements will not only improve the performance of eMBMS but will also make its multicast operations more flexible and versatile.

As a future direction, it will be interesting to evaluate the effect of the proposed policies on the QoE of multicast users. To do so, we need to use the proposed policies for video transmissions over an actual LTE system. This will require implementing the policies in an LTE testbed wherein we can transmit various kinds of videos to a group of users and record their quality of experience.

IX. ACKNOWLEDGEMENT

APPENDIX

A. Proof of Lemma 1

The optimal resource allocation problem $\mathbf{B}_{\boldsymbol{\Delta}}^{\star}$ was defined in Section II. Since $\mathbf{B}_{\boldsymbol{\Delta}}^{\star}$ is an optimization problem, in order to prove that it is NP-hard, we must show the corresponding decision problem to be NP-complete. The decision problem corresponding to $\mathbf{B}_{\boldsymbol{\Delta}}^{\star}$ (denoted by $\mathbf{B}_{\mathbf{D}}^{\star}$) is defined as follows:

$\mathbf{B}_{\mathbf{D}}^{\star}$: Does there exist an assignment of binary variables $\{x_{ij}\}$, $i \in [L]$ and $j \in [N]$ such that (1) and (2) of $\mathbf{B}_{\boldsymbol{\Delta}}^{\star}$ are satisfied?

$\mathbf{B}_{\mathbf{D}}^{\star}$ determines whether or not there exists a feasible solution of $\mathbf{B}_{\boldsymbol{\Delta}}^{\star}$. In order to prove that $\mathbf{B}_{\boldsymbol{\Delta}}^{\star}$ is an NP-hard problem, it is sufficient to show that $\mathbf{B}_{\mathbf{D}}^{\star}$ is NP-complete. We prove the NP-completeness of $\mathbf{B}_{\mathbf{D}}^{\star}$ by reduction from a version of the 3-partition problem (3P) defined below [56]:

• **Input:** A set Y, of $P = 3m$ positive integers, $\{\rho_1, \rho_2, \ldots, \rho_P\}$ such that $\frac{B}{4} < \rho_k < \frac{B}{2}$ for every $\rho_k \in$ Y and $\sum_{k=1}^{P} \rho_k = mB$.

• **Problem:** Can we obtain a disjoint partition of Y, $\{Y_1, Y_2, \ldots, Y_m\}$ such that $\sum_{\rho_k \in Y_i} \rho_k = B$ and $|Y_i| = 3$ for every $Y_i, i \in \{1, 2, \ldots, m\}$ and $\bigcup_{i=1}^{m} Y_i =$ Y?

• **Output:** If the problem is feasible, output a suitable partition of Y, else output that the problem is infeasible.

The 3P problem is known to be NP-complete [56]. Next, we show the NP-completeness of $\mathbf{B}_{\mathbf{D}}^{\star}$ by reduction from 3P.

Theorem 3. $\mathbf{B}_{\mathbf{D}}^{\star}$ *is an NP-complete problem.*

Proof. In order to prove that $\mathbf{B}_{\mathbf{D}}^{\star}$ is NP-complete, we first need to show that $\mathbf{B}_{\mathbf{D}}^{\star}$ belongs to the class NP. Given a certificate for $\mathbf{B}_{\mathbf{D}}^{\star}$, we can verify in polynomial time whether or not it is a solution by checking if it satisfies the requirements stated in constraints (1) and (2) of $\mathbf{B}_{\boldsymbol{\Delta}}^{\star}$. This can be done in $O(LN)$ computations. Therefore, $\mathbf{B}_{\mathbf{D}}^{\star} \in$ NP.

Having proved that $\mathbf{B}_{\mathbf{D}}^{\star} \in$ NP, we now need to reduce 3P to an instance of $\mathbf{B}_{\mathbf{D}}^{\star}$ in polynomial time. The pseudo-code for the algorithm used for the said reduction is presented in Algorithm 4. Note that, to define an instance of $\mathbf{B}_{\mathbf{D}}^{\star}$, we need to state the number of groups, number of available PRBs, rate requirement of groups (R) and the rates that can be achieved by the groups in every PRB. These are defined in lines 1 through 4 of Algorithm 4 respectively. The reduction in Algorithm 4 can be accomplished in $O(N)$ computations.

We now show that a solution for $\mathbf{B}_{\boldsymbol{\Delta}}^{\star}$ gives us a solution for 3P as well. Assume that there exists a polynomial time algorithm for solving $\mathbf{B}_{\boldsymbol{\Delta}}^{\star}$. If we try to solve $\mathbf{B}_{\boldsymbol{\Delta}}^{\star}$ using this algorithm, it will either give us a feasible solution or tell us that $\mathbf{B}_{\boldsymbol{\Delta}}^{\star}$ is infeasible. We will now show how each of these outputs can be mapped to a corresponding solution for 3P.

Algorithm 4: Pseudo-code for reducing 3P to $\mathbf{B}_{\mathbf{D}}^{\star}$

Input: 3-partition problem with set Y, of $P = 3m$ positive integers, $\{\rho_1, \rho_2, \ldots, \rho_P\}$ such that $\frac{B}{4} < \rho_k < \frac{B}{2} \ \forall \ \rho_k \in$ Y and $\sum_{k=1}^{P} \rho_k = mB$
Output: An instance of $\mathbf{B}_{\mathbf{D}}^{\star}$ with

1   $L \leftarrow m$
2   $N \leftarrow P$
3   $R \leftarrow B$
4   $r_{ik} = r_k \leftarrow \rho_k \ \forall \ k \in \{1, 2, \ldots, P\} \ , \ i \in \{1, 2, \ldots, m\}$

Say that the algorithm gives us a feasible solution for $\mathbf{B}_{\boldsymbol{\Delta}}^{\star}$. Let the feasible solution be a matrix of binary values $[\tilde{x}_{ij}]_{i,j}$ for $i \in [L]$ and $j \in [N]$. The corresponding solution for 3P can be obtained from this solution in polynomial time as follows:

For every $i \in [m]$, $Y_i = \{\rho_j : \tilde{x}_{ij} = 1\}$.

The solution thus obtained gives us a feasible solution for 3P as well. To prove this, we need to prove that:

• The solution results in a disjoint partition of Y, $\{Y_1, Y_2, \ldots, Y_m\}$.

- $\sum_{k\in Y_i}\rho_k = B$, for every i.
- $|Y_i| = 3$ for every i.

We shall prove these by contradiction as follows:

1) Let's first show that the resulting solution is a disjoint partition on Y. Suppose not. Then, one of the following two things must happen:

a) there exists Y_i and Y_k such that $Y_i \cap Y_k \neq \phi$ or,

b) there exists some k such that $\rho_k \notin \bigcup_i Y_i$.

If 1a is true and there exist Y_i and Y_k such that $Y_i \cap Y_k \neq \phi$, it means that:

$$\exists \ j \in [P] \text{ such that}, \tilde{x}_{ij} = 1 \text{ and } \tilde{x}_{kj} = 1,$$
$$\implies \sum_l \tilde{x}_{lj} \geq 2,$$

which violates constraint (2) of $\mathbf{B}_\Delta^\star$. This means that $[\tilde{x}_{ij}]_{i,j}$ is not a feasible solution of $\mathbf{B}_\Delta^\star$ which is a contradiction. Therefore, $Y_i \cap Y_k = \phi$ for every i and $k \in [m]$.

If 1b is true and there exists $k \in [P]$, such that $\rho_k \notin \bigcup_i Y_i$, it means that $\tilde{x}_{ik} = 0$ for every i. But, we have a feasible solution of $\mathbf{B}_\mathbf{D}^\star$ which guarantees that the rate requirement of every group is satisfied. So,

$$\sum_{k\in Y_i}\rho_k \geq B, \ \forall \ i \in [m],$$
$$\implies \sum_{j=1, j\neq k}^{P}\rho_j \geq mB, \implies \sum_{j=1}^{P}\rho_j > mB,$$

which is a contradiction. Hence, 1b cannot be true. Hence, the resulting solution will be a partition on Y.

2) We now show that $\sum_{k\in Y_i}\rho_k = B \ \forall \ i$. Suppose not. Since $[\tilde{x}_{ij}]_{i,j}$ is a feasible solution of $\mathbf{B}_\Delta^\star$, we have, $\sum_{k\in Y_i}\rho_k \geq B$, for every $i \in [m]$. Let's say that at least one of these is a strict inequality. That is, there exists $l \in [m]$ such that $\sum_{k\in Y_l}\rho_k > B$. This implies that $\sum_{i=1}^{P}\rho_i > mB$, which is a contradiction. Therefore, we will have $\sum_{k\in Y_i}\rho_k = B$, for every i.

3) Next, we prove that $|Y_i| = 3$ for every Y_i. Let's suppose, for the sake of contradiction, that one subset, Y_k has less than 3 elements. Since the rate requirement of every group is B, we have, $\sum_{\rho_i\in Y_k}\rho_i \geq B$. Also, from the problem definition of 3P, we have, $\rho_i < \frac{B}{2}$. Since Y_k can have a maximum of 2 members, we get, $\sum_{\rho_i\in Y_k}\rho_i < B$ which is in contradiction to $\sum_{\rho_i\in Y_k}\rho_i \geq B$ above. Thus, Y_k cannot have less than 3 elements. Therefore, $|Y_i| = 3$ for every Y_i, $i \in [m]$.

We have now established that a feasible solution of $\mathbf{B}_\Delta^\star$ gives us a feasible solution of 3P as well. All that is left to complete the proof is to show that if $\mathbf{B}_\Delta^\star$ turns out to be infeasible, then, 3P is infeasible as well. We prove this by contradiction :

Let's assume that 3P has a feasible solution even when $\mathbf{B}_\Delta^\star$ is infeasible. This means that, there exists a disjoint partition of Y, $\{Y_1, \ldots, Y_m\}$ such that, $\sum_{\rho_k\in Y_i}\rho_k = B$ and $|Y_i| = 3$ for every Y_i, $i \in \{1, 2, \ldots, m\}$. This solution can be mapped to a corresponding solution for $\mathbf{B}_\Delta^\star$ as follows:

$$x_{ij} = \begin{cases} 1, & \text{if } \rho_j \in Y_i, \\ 0, & \text{otherwise.} \end{cases}$$

So, for every i, we have:

$$\sum_{j=1}^{N} x_{ij}r_{ij} = \sum_{\rho_j\in Y_i} r_j = \sum_{\rho_j\in Y_i} \rho_j = B = R.$$

Also, since Y_i's form a disjoint partition of Y, we will have, $\sum_{i=1}^{N} x_{ij} \leq 1$ for every j. This means that $[x_{ij}]_{i,j}$ is a feasible solution for $\mathbf{B}_\Delta^\star$ which is a contradiction. Therefore, 3P has to

be infeasible every time $\mathbf{B}_\Delta^\star$ is infeasible.

Thus, a polynomial time solution for $\mathbf{B}_\mathbf{D}^\star$ results in a polynomial time solution for 3P as well which is not possible unless P = NP. Therefore, there is no polynomial time algorithm for solving the optimal resource allocation problem $\implies \mathbf{B}_\mathbf{D}^\star$ is an NP-complete problem. $\square$

Corollary 1. $\mathbf{B}_\Delta^\star$ *is an NP-hard problem.*

Proof. The proof follows from Theorem 3. Since the decision version of $\mathbf{B}_\Delta^\star$ is NP-complete, $\mathbf{B}_\Delta^\star$ is an NP-hard problem. $\square$

B. Proof of Lemma 2

The optimal grouping problem $\mathbf{C}^\star$ was defined in Section II. Before addressing the hardness of the optimal grouping problem, we wish to point out that, given a grouping policy, Δ, calculating $\overline{S}^\Delta$ in polynomial time may itself be hard. Computing $\overline{S}^\Delta$ is non-trivial even when the channels are independent across UEs. We prove the NP-hardness of $\mathbf{C}^\star$ by reduction from the Set Cover problem which is an NP-complete problem [57] and is defined as follows [57]:

- **Input:** Set Cover takes as input a universe, $\mathcal{U} = \{u_1, \ldots, u_m\}$ containing m elements and a set $\mathcal{S} = \{S_1, \ldots, S_n\}$ of subsets of $\mathcal{U}$ such that $\cup_{j=1}^{n} S_j = \mathcal{U}$.
- **Problem:** Any collection of subsets from $\mathcal{S}$ form a set cover if their union is equal to the universe. The Set Cover problem is required to determine the smallest such collection of subsets.
- **Output:** The output is the smallest collection of subsets that form a set cover.

Next, we show that $\mathbf{C}^\star$ is NP-hard by reduction from Set Cover.

Proof. To prove that $\mathbf{C}^\star$ is NP-hard, we first need to show that $\mathbf{C}^\star$ belongs to the class NP. Given a certificate for $\mathbf{C}^\star$, we can verify in polynomial time whether or not it is a solution by checking if it satisfies the requirements stated in Definition 1. This can be done in $O(L^2)$ computations. Therefore, $\mathbf{C}^\star \in$ NP.

We now prove that $\mathbf{C}^\star$ is NP-hard by reducing Set Cover to an instance of $\mathbf{C}^\star$. The pseudo-code for the algorithm used for this reduction is given in Algorithm 5. The reduction can be accomplished in $O(MN)$ computations. We define the total number of multicast UEs to be m and number of PRBs in a sub-frame to be n. k^{th} UE in $\mathbf{C}^\star$ maps to the variable u_k in Set Cover. Let r_{max} denote the maximum rate achievable in any PRB. The rate achievable by a UE k in PRB j, r_{kj} is defined to be equal to r_{max} if $u_k \in S_j$ and equal to 0 otherwise. We define the rate requirement of the groups $R < r_{max}$.

Let us now assume that there exists a polynomial time algorithm for solving $\mathbf{C}^\star$. Using this algorithm to solve $\mathbf{C}^\star$ will output some grouping $\{G_1, \ldots, G_l\}$. We now show how to map this output to a solution for Set Cover in polynomial time. Since the rate achievable by a UE in any PRB can either be r_{max} or 0, all the UEs that are grouped together will be able to achieve r_{max} in some PRB and the number of PRBs needed to satisfy the groups will be exactly l because 1 PRB will be sufficient for providing the required rate. Let the n_i^{th} PRB be that PRB for group G_i. Hence, $u_k \in G_i, r_{kn_i} = r_{max} \implies u_k \in S_{n_i}$. Therefore, the corresponding solution for the Set Cover problem is $\{S_{n_1}, \ldots, S_{n_l}\}$. By the definition of a grouping we have $\cup_{i=1}^{l} G_i = [m] \implies \cup_{i=1}^{l} S_{n_i} = \mathcal{U}$. Therefore, the resulting solution is a valid set cover of $\mathcal{U}$.

We now show that this is indeed the smallest such collection that covers the universe $\mathcal{U}$. Suppose that this is not true.

Then, there exists a collection of subsets from $\mathcal{S}$ smaller than l that forms a set cover. Let's denote this optimal solution as $\mathcal{S}' = \{S_{n'_1}, \ldots, S_{n'_z}\}, z < l$. We can then construct the following grouping from this set cover: $G_1 = S_{n'_1}, G_2 = S_{n'_2} \setminus S_{n'_1}, G_3 = S_{n'_3} \setminus \cup_{j=1}^2 S_{n'_j}, \ldots, G_z = S_{n'_z} \setminus \cup_{j=1}^k S_{n'_j}$. Since $\mathcal{S}'$ is a set cover of $\mathcal{U}$, we have $\cup_{i=1}^z G_i = [M]$ and by construction of the groups, $\cap_{i=1}^z = \phi$. Hence, $\{G_1, \ldots, G_z\}$ is a valid grouping. Moreover, the number of PRBs needed to satisfy the UEs under this grouping is $z < l$ which is a contradiction to the grouping $\{G_1, \ldots, G_l\}$ being the optimal solution of $\mathbf{C}^\star$. Therefore, $\{S_{n_1}, \ldots, S_{n_l}\}$ is the optimal solution of the Set Cover problem.

Thus, a polynomial time solution for $\mathbf{C}^\star$ results in a polynomial time solution for Set Cover as well which is not possible unless P = NP. Therefore, there is no polynomial time algorithm for solving $\mathbf{C}^\star$ i.e. $\mathbf{C}^\star$ is an NP-hard problem. $\square$

Algorithm 5: Pseudo-code for reducing Set Cover to $\mathbf{C}^\star$

Input: Set Cover problem with a universe $\mathcal{U} = \{u_1, \ldots, u_m\}$ of m variables and set $\mathcal{S} = \{S_1, \ldots, S_n\}$ of subsets of $\mathcal{U}$ such that $\cup_{i=1}^n S_i = \mathcal{U}$,
Output: An instance of $\mathbf{C}^\star$
1 $M \leftarrow m, N \leftarrow n, k^{\text{th}}$ UE $\leftarrow u_k$
2 $r_{kj} = \begin{cases} r_{max}, & \text{if } u_k \in S_j, \\ 0, & \text{otherwise.} \end{cases}$

C. Proof of Lemma 3

Proof. We have $s_{d^\star} \in \arg\max_{s_d} E(s_d)$ i.e. $E(s_{d^\star}) \geq E(s_d)$ for every $s_d \in \chi$. The solution for the BLP $\mathbf{B}_\Delta^\star$ corresponding to the state $s_{d^\star}, \{x_{ij}^\star\}_{i,j}$ is obtained as follows:

$$x_{ij}^\star = \begin{cases} 1, & \forall\ j \in \overline{V}_{id^\star}, \\ 0, & \text{otherwise.} \end{cases}$$

In LTE, the rates achievable in a PRB are discrete and can take 15 different values corresponding to the 15 possible CQI values [42]. The minimum rate that can be provided in a single PRB is 16 kbps. We will denote this by r_{min}. Since the value of $E(.)$ depends on the value of R, two cases arise:
• $R \leq r_{min}$: In this case, we can satisfy all groups by allocating a single PRB to every group. This is a trivial case and so, it is sufficient to consider the case with $R > r_{min}$.
• $R > r_{min}$: Before proving that $\{x_{ij}^\star\}_{i,j}$ is the optimal solution of $\mathbf{B}_\Delta^\star$, we will first show that $\{x_{ij}^\star\}_{i,j}$ is a feasible solution of $\mathbf{B}_\Delta^\star$. Suppose $\{x_{ij}^\star\}_{i,j}$ is not a feasible solution of $\mathbf{B}_\Delta^\star$. This means, that there exists $i \in [L]$ such that $\sum_{j=1}^N x_{ij}^\star r_{ij} < R$. Then the reward of $s_{d^\star}$ will be:

$$E(s_{d^\star}) = (N - \sum_{i \in [L]} \sum_{j \in [N]} x_{ij}^\star) - \sum_{i=1}^L [R - \ell_{d^\star i}]^+ + q_{d^\star}. \quad (16)$$

Note that $q_{d^\star} < L$ because $\{x_{ij}^\star\}_{i,j}$ is infeasible. Depending on the value of $\sum_{i \in [L]} \sum_{j \in [N]} x_{ij}^\star$, two cases arise:
1) $\sum_{i \in [L]} \sum_{j \in [N]} x_{ij}^\star < N$: For this case, consider a state s_d obtained from $s_{d^\star}$ by allotting one of the PRBs, $j' \in \overline{V}_{0d^\star}$ to one of the unsatisfied groups i'. On allocating j' to i', one of two things can happen:
• Rate requirement of the group i' is satisfied: This means that $q_d = q_{d^\star} + 1$. The reward of the resulting s_d will be:
$$E(s_d) = E(s_{d^\star}) + (R - \ell_{d^\star i'}).$$

Since group i' was unsatisfied in state $s_{d^\star}, (R - \ell_{d^\star i'}) > 0$. Therefore, $E(s_d) > E(s_{d^\star})$ which is a contradiction because $E(s_{d^\star}) \geq E(s_d)$ for every $s_d \in \chi$.
• Rate requirement of the group i' is not satisfied: In this case, the reward of the state s_d will be:
$$E(s_d) = E(s_{d^\star}) - 1 + (\ell_{di'} - \ell_{d^\star i'}).$$

Here, $(\ell_{di'} - \ell_{d^\star i'})$ is the additional rate provided to group i' by the PRB j' which is why it can be no less than r_{min}. Since $r_{min} > 1, E(s_d) > E(s_{d^\star})$ which is a contradiction.
2) $\sum_{i \in [L]} \sum_{j \in [N]} x_{ij}^\star = N$: Here, the reward of $s_{d^\star}$ is:
$$E(s_{d^\star}) = q_{d^\star} - \sum_{i=1}^L [R - \ell_{d^\star i}]^+.$$

Since $\mathbf{B}_\Delta^\star$ is feasible, let $s_{d'}$ be a state corresponding to a feasible solution $\{x_{ij}\}_{i,j}$. The reward of $s_{d'}$ will be:
$$E(s_{d'}) = (N - \sum_{i \in [L]} \sum_{j \in [N]} x_{ij}) + L > E_{s_{d^\star}},$$

which is a contradiction.

Therefore, $\{x_{ij}^\star\}_{i,j}$ has to be a feasible solution of $\mathbf{B}_\Delta^\star$. All we need to complete the proof is to show that $\{x_{ij}^\star\}_{i,j}$ is also an optimal solution of $\mathbf{B}_\Delta^\star$. We show this as follows:

Suppose $\{x_{ij}^\star\}_{i,j}$ is not an optimal solution of $\mathbf{B}_\Delta^\star$. Let's denote the optimal solution of $\mathbf{B}_\Delta^\star$ by $\{\overline{x}_{ij}\}_{i,j}$ and the corresponding resource allocation state by $s_{\overline{d}}$. Since $\{x_{ij}^\star\}_{i,j}$ is not the optimal solution, we will have, $\sum_{i \in [L]} \sum_{j \in [N]} x_{ij}^\star > \sum_{i \in [L]} \sum_{j \in [N]} \overline{x}_{ij}$ The reward of $s_{\overline{d}}$ will be:

$$E(s_{\overline{d}}) = (N - \sum_{i \in [L]} \sum_{j \in [N]} \overline{x}_{ij}) + L,$$
$$\implies E(s_{\overline{d}}) > (N - \sum_{i \in [L]} \sum_{j \in [N]} x_{ij}^\star) + L = E(s_{d^\star}),$$

which is a contradiction. Therefore, $\{x_{ij}^\star\}_{i,j}$ is an optimal solution of the BLP $\mathbf{B}_\Delta^\star$. $\square$

REFERENCES

[1] S. u. Zuhra, P. Chaporkar, and A. Karandikar, "Efficient Grouping and Resource Allocation for Multicast Transmission in LTE," in *IEEE WCNC*, pp. 1–6, 2017.
[2] "LTE-Broadcast (eMBMS) Market Update," *Available: {https://gsacom.com/paper/lte-broadcast-embms-market-update-2/}*, 2018.
[3] H. Velde, O. Hus, and M. Baker, "Broadcast Operation," in *LTE-The UMTS Long Term Evolution From Theory to Practice*, pp. 293–305, Chichester: John Wiley & Sons Ltd, 2 ed., 2011.
[4] "http://www.3gpp.org/specifications/releases/69-release-11."
[5] "3GPP TS 23.246 : *Multimedia Broadcast/Multicast Service (MBMS); Architecture and functional description*, v.14.1.0 Rel. 14," 2017.
[6] "3GPP TS 36.212 : *LTE; Evolved Universal Terrestrial Radio Access (E-UTRA); Multiplexing and channel coding*, v.14.4.0 Rel. 14," 2017.
[7] T. P. Low, M. O. Pun, Y. W. P. Hong, and C. C. J. Kuo, "Optimized opportunistic multicast scheduling (OMS) over wireless cellular networks," *IEEE Trans. Wireless Commun.*, vol. 9, no. 2, pp. 791–801, 2010.
[8] P. Polacek, T.-Y. Yang, and C.-W. Huang, "Opportunistic multicasting for single frequency networks," *Wireless Communications and Mobile Computing*, vol. 16, no. 15, pp. 2253–2262, 2016.
[9] S. Lu, Y. Cai, L. Zhang, J. Li, P. Skov, C. Wang, and Z. He, "Channel-Aware Frequency Domain Packet Scheduling for MBMS in LTE," in *IEEE VTC Spring*, pp. 1–5, 2009.
[10] N. Sharma and A. Madhukumar, "Genetic algorithm aided proportional fair resource allocation in multicast OFDM systems," *IEEE Trans. Broadcast.*, vol. 61, no. 1, pp. 16–29, 2015.
[11] I. C. Wong, Z. Shen, B. L. Evans, and J. G. Andrews, "A low complexity algorithm for proportional resource allocation in OFDMA systems," in *IEEE SIPS*, pp. 1–6, 2004.
[12] G. Araniti, A. Orsino, J. Cosmas, A. Molinaro, and A. Iera, "A low computational-cost subgrouping multicast scheme for emerging 5G-satellite networks," in *IEEE BMSB*, pp. 1–6, 2016.
[13] A. Orsino, G. Araniti, P. Scopelliti, I. Gudkova, K. Samouylov, and A. Iera, "Optimal subgroup configuration for multicast services over 5G-satellite systems," in *IEEE BMSB*, pp. 1–6, 2017.

[14] L. Militano, M. Condoluci, G. Araniti, and A. Iera, "Bargaining solutions for multicast subgroup formation in LTE," in *IEEE VTC Fall*, pp. 1–5, 2012.

[15] A. de la Fuente, G. Femenias, F. Riera-Palou, and A. G. Armada, "Sub-band CQI Feedback-Based Multicast Resource Allocation in MIMO-OFDMA Networks," *IEEE Trans. Broadcast.*, 2018.

[16] J. F. Monserrat, J. Calabuig, A. Fernandez-Aguilella, and D. Gomez-Barquero, "Joint delivery of unicast and E-MBMS services in LTE networks," *IEEE Trans. Broadcast.*, vol. 58, no. 2, pp. 157–167, 2012.

[17] D. Lee, J. So, and S. R. Lee, "Power allocation and subcarrier assignment for joint delivery of unicast and broadcast transmissions in OFDM systems," *Journal of Communications and Networks*, vol. 18, no. 3, pp. 375–386, 2016.

[18] S. ul Zuhra, P. Chaporkar, and A. Karandikar, "Auction Based Resource Allocation and Pricing for Heterogeneous User Demands in eMBMS," in *IEEE WCNC*, 2019.

[19] J. Chen, M. Chiang, J. Erman, G. Li, K. K. Ramakrishnan, and R. K. Sinha, "Fair and optimal resource allocation for LTE multicast (eMBMS): Group partitioning and dynamics," in *IEEE INFOCOM*, pp. 1266–1274, 2015.

[20] G. Araniti, V. Scordamaglia, A. Molinaro, A. Iera, G. Interdonato, and F. Span, "Optimizing point-to-multipoint transmissions in high speed packet access networks," in *IEEE MSB*, pp. 1–5, 2011.

[21] G. Araniti, M. Condoluci, and A. Iera, "Adaptive multicast scheduling for HSDPA networks in mobile scenarios," in *IEEE BMSB*, pp. 1–5, 2012.

[22] L. Militano, D. Niyato, M. Condoluci, G. Araniti, A. Iera, and G. M. Bisci, "Radio resource management for group-oriented services in LTE-A," *IEEE Trans. Veh. Technol.*, vol. 64, no. 8, pp. 3725–3739, 2015.

[23] L. Militano, M. Condoluci, G. Araniti, and A. Iera, "Multicast service delivery solutions in LTE-Advanced systems," in *IEEE ICC*, pp. 5954–5958, 2013.

[24] M. Condoluci, G. Araniti, A. Molinaro, A. Iera, and J. Cosmas, "On the impact of frequency selectivity on multicast subgroup formation in 4G networks," in *IEEE BMSB*, pp. 1–6, 2013.

[25] T. Liu, H. Xia, and C. Feng, "A QoS-based multi-rate multicast scheme over heterogeneous cellular network," in *ISWCS*, pp. 292–296, 2016.

[26] G. Araniti, M. Condoluci, L. Militano, and A. Iera, "Adaptive Resource Allocation to Multicast Services in LTE Systems," *IEEE Trans. Broadcasting.*, vol. 59, no. 4, pp. 658–664, 2013.

[27] C. Tan, T. Chuah, and S. Tan, "Adaptive multicast scheme for OFDMA-based multicast wireless systems," *Electronics Letters*, vol. 47, no. 9, pp. 570–572, 2011.

[28] G. Araniti, M. Condoluci, A. Molinaro, A. Iera, and J. Cosmas, "Low complexity subgroup formation in LTE systems," in *IEEE BMSB*, pp. 1–6, 2013.

[29] R. Kaliski, C. C. Chou, H. Y. Meng, and H. Y. Wei, "Dynamic Resource Allocation Framework for MooD (MBMS Operation On-Demand)," *IEEE Trans. Broadcast.*, vol. 62, no. 4, pp. 903–917, 2016.

[30] H.-Y. Meng, C.-C. Chou, R. Kaliski, and H.-Y. Wei, "An on-demand QoE resource allocation algorithm for multi-flow LTE eMBMS," in *WOCC*, pp. 93–97, 2015.

[31] O. Karimi, J. Liu, and Z. Wang, "Power-Efficient Resource Utilization in Cellular Multimedia Multicast," in *IEEE MSN*, pp. 134–143, 2015.

[32] A. Orsino, P. Scopelliti, and M. Condoluci, "A Multi-Criteria Approach for Multicast Resource Allocation over LTE and Beyond Cellular Systems," in *European Wireless Conference*, pp. 1–6, VDE, 2016.

[33] C.-L. Hwang, Y.-J. Lai, and T.-Y. Liu, "A new approach for multiple objective decision making," *Computers & operations research*, vol. 20, no. 8, pp. 889–899, 1993.

[34] G. Araniti, M. Condoluci, A. Orsino, A. Iera, A. Molinaro, and J. Cosmas, "Evaluating the performance of multicast resource allocation policies over LTE systems," in *IEEE BMSB*, pp. 1–6, 2015.

[35] M. Condoluci, G. Araniti, A. Molinaro, and A. Iera, "Multicast resource allocation enhanced by channel state feedbacks for multiple scalable video coding streams in LTE networks," *IEEE Trans. Veh. Technol.*, vol. 65, no. 5, pp. 2907–2921, 2016.

[36] C.-W. Huang, S.-M. Huang, P.-H. Wu, S.-J. Lin, and J.-N. Hwang, "OLM: Opportunistic layered multicasting for scalable IPTV over mobile WiMAX," *IEEE Trans. Mobile Comput.*, vol. 11, no. 3, pp. 453–463, 2012.

[37] D. Jiang and Y. Cui, "Analysis and Optimization of Caching and Multicasting for Multi-Quality Videos in Large-Scale Wireless Networks," *arXiv preprint arXiv:1807.00260*, 2018.

[38] "More Efficient Mobile Encodes for Netflix Downloads, Available: {https://medium.com/netflix-techblog/more-efficient-mobile-encodes-for-netflix-downloads-625d7b082909}," 2016.

[39] "VP9: Faster, better, buffer-free YouTube videos, Available: {https://youtube-eng.googleblog.com/2015/04/vp9-faster-better-buffer-free-youtube.html}," 2015.

[40] G. Araniti, M. Condoluci, P. Scopelliti, A. Molinaro, and A. Iera, "Multicasting over emerging 5G networks: Challenges and perspectives," *IEEE Netw.*, vol. 31, no. 2, pp. 80–89, 2017.

[41] Wanqing Tu and Weijia Jia, "Adaptive playback buffer for wireless streaming media," in *IEEE ICON*, pp. 191–195, 2004.

[42] "3GPP TS 36.213 : *Evolved Universal Terrestrial Radio Access (E-UTRA); Physical layer procedures*, v.14.5.0 Rel. 14," 2018.

[43] S. M. Ross, "Markov Chain Monte Carlo Methods," in *Simulation*, pp. 271–302, Academic Press, 5 ed., 2013.

[44] M. Harchol-Balter, "Time-Reversibility and Burke's Theorem," in *Performance Modeling and Design of Computer Systems: Queueing Theory in Action*, ch. 16, Cambridge University Press, 1st ed., 2013.

[45] B. Hajek, "Cooling schedules for optimal annealing," *Mathematics of operations research 13.2*, pp. 311–329, 1988.

[46] "3GPP TR 36.931 : *Evolved Universal Terrestrial Radio Access (E-UTRA); Radio Frequency (RF) requirements for LTE Pico Node B*, v.15.0.0 Rel. 15," 2018.

[47] "https://in.mathworks.com/products/matlab/."

[48] M. Mehta, *Radio Resource and Mobility Management Techniques in Heterogeneous Cellular Network*. PhD thesis, Dept. Elect. Eng., IIT Bombay, Mumbai, India, 2014.

[49] M. Series, "Guidelines for evaluation of radio interface technologies for IMT-Advanced," *Report ITU*, vol. 638, 2009.

[50] T. Begluk, J. B. Husić, and S. Baraković, "Machine learning-based qoe prediction for video streaming over LTE network," in *International Symposium INFOTEH-JAHORINA (INFOTEH)*, pp. 1–5, 2018.

[51] M. Vaser and S. Forconi, "QoS KPI and QoE KQI relationship for LTE video streaming and VoLTE services," in *International Conference on Next Generation Mobile Applications, Services and Technologies*, pp. 318–323, 2015.

[52] S. Aroussi, T. Bouabana-Tebibel, and A. Mellouk, "Empirical QoE/QoS correlation model based on multiple parameters for VoD flows," in *IEEE GLOBECOM*, pp. 1963–1968, 2012.

[53] P. Seeling and M. Reisslein, "Video transport evaluation with H. 264 video traces," *IEEE Commun. Surveys Tut.*, vol. 14, no. 4, pp. 1142–1165, 2011.

[54] "3GPP TS 38.214 : *5G; NR; Physical layer procedures for data*, v.15.5.0 Rel. 15," 2019.

[55] "3GPP TS 38.211 : *5G; NR; Physical channels and modulation*, v.15.5.0 Rel. 15," 2019.

[56] M. R. Garey and D. S. Johnson, "Using NP-Completeness to Analyze Problems," in *Computers and Intractability A guide to the theory of NP-completeness*, vol. 29, ch. 4, W.H Freeman and Company, 2002.

[57] R. M. Karp, "On the computational complexity of combinatorial problems," *Networks*, vol. 5, no. 1, pp. 45–68, 1975.

Sadaf ul Zuhra is currently a research scholar in the department of electrical engineering at IIT Bombay, India. She received her B.Tech degree from NIT Srinagar, India, in 2014. Her research interests are resource allocation and scheduling, video streaming, queueing theory and wireless multicast.

Prasanna Chaporkar received the MS degree from the Faculty of Engineering, Indian Institute of Science, Bangalore, India, in 2000, and the PhD degree from the University of Pennsylvania, Philadelphia, Pennsylvania, in 2006. He was a ERCIM post-doctoral fellow with ENS, Paris, France, and NTNU, Trondheim, Norway. Currently, he is an associate professor in the Indian Institute of Technology, Mumbai. His research interests include resource allocation, stochastic control, queueing theory, and distributed systems and algorithms.

Abhay Karandikar is currently the Director of IIT Kanpur (on leave from IIT Bombay). He is also a Member (part-time) of Telecom Regulatory Authority of India (TRAI). In IIT Bombay, he served as Institute Chair Professor in the Electrical Engineering Department, the Dean (Faculty Affairs) from 2017 to 2018 and the Head of the Electrical Engineering Department from 2012 to 2015. Prof. Karandikar is the founding member of Telecom Standards Development Society, India (TSDSI), India's standards body for telecom. He was the Chairman of TSDSI from 2016 to 2018. His research interests include resource allocation in wireless networks, Software defined networking, frugal 5G and rural broadband. Detailed biography can be found at https://www.ee.iitb.ac.in/~karandi/.