Matroids representable over Modules,Electrical Network Topology and Behavioural Systems Theory [1]

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1 Abstract

In this paper we show that there are connections between the theories of Matroids, Electrical Network Topology and Behavioural Systems. We show that there are two kinds of duality in Behavioural Systems: one between a behaviour and the module generated by the rows of the coefficient matrix of its kernel representation and the other, between complementary orthogonal modules. In both cases we derive theorems where duals are built implicitly. Using one such result we develop definitions of adjoints for Behavioural Systems and prove that these adjoints have desirable properties. Finally we consider a number of fundamental problems from Topological Network Theory and the translation of these problems and their solutions to Behavioural Systems. Although our primary concern is with 1-D systems, our main results hold for N-D systems and the statements and proofs are explicitly given also for N-D systems.

2 Introduction

Behavioural Systems theory appears to have arisen in an attempt to study dynamical systems 'as they are' without forcing canonical representations on them and also, where possible, without forcing artificial partitions of the manifest variables into 'inputs/outputs' [Polderman+Willems97]. This approach can of course be profitably employed to study many classes of systems including Electrical Networks. On the other hand, there is a way of studying Electrical Networks emphasizing the manner

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in which the devices are connected together while ignoring the characteristics of devices [Narayanan86a], [Narayanan87], [Narayanan97]. Curiously, this 'Topological Network Theory' has developed in ways parallel to Behavioural Systems theory. In some cases results of the former can be applied to Behavioural Systems. The bridge between these two is the theory of matroids representable over modules [Welsh76], [Narayanan97]. In this paper we attempt to describe the connections and analogies between these theories and also indicate some applications of results developed in these other theories to Behavioural Systems. In this paper we confine ourselves mainly to 1-D (derivatives being with respect to a single variable) systems. However, our main results (Theorems 4.3 and 5.1) and their proofs go over essentially unchanged to N-D systems. So do our construction of adjoint and the validity of its properties, as given in Theorem 6.2. The minor modifications required for generalization are indicated at appropriate places. Explicit statements and proofs of the results for N-D systems are given in Appendix I and Appendix II.

3 Preliminaries

We consider two types of vectors on a finite set S:(a) *Operator vector* $f_S : S \longrightarrow \Re[\xi]$, where $\Re[\xi]$ is the collection of all real polynomials of a single variable ξ; (b) *Behaviour vector* $w_S : \Re \longrightarrow \Re^S$, where $\Re^S$ denotes the collection of all real functions on the finite set S.

Addition of vectors on a set S is defined in the usual way. For operator vectors the *scalar multiplication* λf_S is defined by $(\lambda f_S)(e) \equiv \lambda(f_S(e)), e \in S, \lambda \in \Re[\xi]$. For behaviour vectors the scalars are real numbers.

For the purpose of this paper a *module* on S would be a collection of operator vectors f_S closed under addition and scalar multiplication, where the scalars belong to $\Re[\xi]$. *Linear combination of vectors over* S using scalars from $\Re[\xi]$, *linear dependence, linear independence* etc. are defined in the usual manner. A matrix R with a finite number of rows and columns over $\Re[\xi]$ would be denoted by $R[\xi]$. The matrix $R[\xi]$ generates its *row module (column module)* through all possible linear combinations of its rows (columns) and would be called a *generator matrix* for its row module and a *column generator matrix* for its column module. If the set of rows of R form a maximally linearly independent set of vectors (basis) of a module $\mathcal{C}_S$, then R is called a *representative matrix* of $\mathcal{C}_S$. In this case the columns of R may be identified with S. As is well known, all maximally independent sets of vectors of a module $\mathcal{C}_S$ have the same cardinality. This number is called the *rank of* $\mathcal{C}_S$ and is denoted by $r(\mathcal{C}_S)$.

A matrix R over $\Re[\xi_1, \cdots, \xi_n]$ would be denoted by $R[\xi_1, \cdots, \xi_n]$. If $\mathcal{C}_S$ is a module on a finite set S over $\Re[\xi_1, \cdots, \xi_n]$, then it is known (using, for instance, the Hilbert Basis Theorem) to have a generator matrix which may have linearly dependent rows. Existence of a representative matrix is not guaranteed. However, even in this case, the size of a maximally linearly independent set of the module can be shown to be invariant and this number is called the *rank* of the module.

Two matrices $R_1[\xi_1, \cdots, \xi_n]$, $R_2[\xi_1, \cdots, \xi_n]$ are said to be *row equivalent* iff each can be derived from the other by row linear combinations, the scalars being from $\Re[\xi_1, \cdots, \xi_n]$.

In the single variable case, any matrix $R[\xi]$ can be reduced, through reversible row operations, to a row equivalent matrix

$$\begin{bmatrix} R_1[\xi] \\ 0 \end{bmatrix}$$

where rows of $R_1[\xi]$ are linearly independent. This is through the well known class of algorithms related to the Euclidean algorithm for G.C.D. of polynomials (see for instance subsection 2.5.3 of [Polderman+Willems97]) which we will call the Repeated Remainder Algorithm (RRA). If reversible column operations are also permitted, we can reduce $R_1[\xi]$ to a matrix of the *Smith Canonical Form* (SCF)

$$\begin{bmatrix} D[\xi] & 0 \\ 0 & 0 \end{bmatrix}$$

where $D[\xi]$ is a diagonal matrix, having nonzero diagonal entries with the i^{th} diagonal entry, a factor of the j^{th} diagonal entry, whenever $i < j$. For any matrix $R[\xi]$, the Smith Canonical Form can be shown to be unique.

A square matrix $T[\xi_1, \cdots, \xi_n]$ is said to be *unimodular* iff its determinant is real and nonzero. It is immediate that the inverse of such a matrix is also unimodular. In the single variable case, it is clear that the SCF of such a matrix is the identity matrix. It can be seen that reversible row (column) operations on a matrix $R[\xi_1, \cdots, \xi_n]$ are equivalent to premultiplication (postmultiplication) by a unimodular matrix. Thus, in the single variable case, a matrix can be reduced to its SCF through pre- and post- multiplication by unimodular matrices.

We would call a matrix $R[\xi_1, \cdots, \xi_n]$ *row g-unimodular* iff it has a right inverse, i.e., there exists a matrix $B[\xi_1, \cdots, \xi_n]$ such that $R[\xi_1, \cdots, \xi_n]B[\xi_1, \cdots, \xi_n]$ is the identity matrix. In the single variable case this happens iff its SCF has the form $\begin{bmatrix} I & 0 \end{bmatrix}$, I being an identity submatrix of appropriate order. If R is row g-unimodular with right inverse B, it is easy to see that

$$\begin{bmatrix} R & 0 \\ K & I \end{bmatrix}$$

is also row g-unimodular (its right inverse being

$$\begin{bmatrix} B & 0 \\ -KB & I \end{bmatrix}).$$

Column g-unimodular matrices are defined as transposes of row g-unimodular matrices. A matrix would be called *g-unimodular* iff it is row or column g-unimodular. A module $\mathcal{C}_S$ over $\Re[\xi_1, \cdots, \xi_n]$ is said to be *unimodular* iff it has a g-unimodular representative matrix, equivalently, iff it has atleast one representative matrix and all its representative matrices are g-unimodular. For any module $\mathcal{C}_S$ over $\Re[\xi]$, it is clear that all representative matrices have the same SCF. It is clear that such $\mathcal{C}_S$ is *unimodular* iff this SCF is row g-unimodular.

Let f_S, g_S be operator vectors on S. The *dot product* $< f_S, g_S >$ of f_S, g_S is defined by $< f_S, g_S > \equiv \sum_{e_i \in S} f_S(e_i).g_S(e_i)$. We say f_S, g_S are *orthogonal* to each other iff $< f_S, g_S >= 0$. The definitions of *operator vectors*,*dot product* and *orthogonality* carry over immediately to the case where the ring of scalars is $\Re[\xi_1, \cdots, \xi_n]$. Let $\mathcal{K}_S$ be a collection of operator vectors on S. Then $\mathcal{K}_S^{\perp}$ denotes the collection of all operator vectors orthogonal to members of $\mathcal{K}_S$. It is easy to see that $\mathcal{K}_S^{\perp}$ is always a module (in fact, as we prove later for the single variable case, a unimodular module) whether or not $\mathcal{K}_S$ is. If $\mathcal{K}_S^{\perp\perp} = \mathcal{K}_S$, we say that $\mathcal{K}_S$ and $\mathcal{K}_S^{\perp}$ are *complementary orthogonal.* For the case where the ring of scalars is $\Re[\xi_1, \cdots, \xi_n]$ we define a module $\mathcal{C}_S$ to be *generalized unimodular* iff $\mathcal{C}_S^{\perp\perp} = \mathcal{C}_S$. It is easy to show that a module $\mathcal{C}_S$ is generalized unimodular iff it has the property '$\alpha f_S \in \mathcal{C}_S$ implies $f_S \in \mathcal{C}_S$ whenever α is a nonzero scalar' (see Appendix I). A consequence of this result is that every unimodular module is also generalized unimodular (see Appendix I). Another consequence is that if a generalized unimodular module over $\Re[\xi_1, \cdots, \xi_n]$ is contained in another module and the two have the same rank, then they are identical. It follows trivially that if two unimodular modules over $\Re[\xi_1, \cdots, \xi_n]$ have the same rank and one is contained in the other, then they are identical. For the single variable case, we prove later that unimodularity and generalized unimodularity are identical.

A *behaviour* $\mathcal{B}_S$ (for the purpose of this paper) is the collection of all behaviour vectors (or trajectories) $w_S : \Re \longrightarrow \Re^S$ which are the infinitely differentiable ($\mathcal{C}^\infty$) solutions of linear constant coefficient differential equation of the form, $(R[\frac{d}{dt}])w_S = 0$. This equation is a 'kernel representation' for $\mathcal{B}_S$. Let $\mathcal{C}_S$ be the row module of $R[\xi]$. We say that $\mathcal{C}_S$ and $\mathcal{B}_S$ are *associated* with each other. Clearly $\mathcal{C}_S$ fixes $\mathcal{B}_S$ uniquely. By using the SCF and the fact that the collections of solutions of two different linear constant coefficient homogeneous equations of a single variable are distinct it can be shown that $\mathcal{B}_S$ also fixes $\mathcal{C}_S$ uniquely. This fact is true also in the N-D case, where $\mathcal{C}_S$ is a module on a finite set S over $\Re[\xi_1, \cdots, \xi_n]$ and $\mathcal{B}_S$ is the set of solutions of linear constant coefficient partial differential equations of the form, $(R[\frac{d}{dx_1}, \cdots, \frac{d}{dx_n}])w_S = 0$ (an expression such as ${\xi_1}^2{\xi_2}^3 w_{e_1}$ translating to $\frac{d^2}{(dx_1)^2}\frac{d^3}{(dx_2)^3}w_{e_1}$),$R[\xi_1, \cdots, \xi_n]$, being a generator matrix of $\mathcal{C}_S$ (see Theorem 2.61,[Oberst90]).

We denote the behaviour associated with $\mathcal{C}_S$ by $\mathcal{B}(\mathcal{C}_S)$ and the module associated with $\mathcal{B}_S$ by $\mathcal{C}(\mathcal{B}_S)$.

A useful fact about linear constant coefficient differential equations which can also be proved using SCF is that the equation

$$(R[\frac{d}{dt}])w = f,$$

where f is a vector of $\mathcal{C}^\infty$ functions, always has a solution if $R[\xi]$ has linearly independent rows.

A behaviour $\mathcal{B}$ is said to be *controllable* iff for each $w^1, w^2 \in \mathcal{B}$ there exists a $w \in \mathcal{B}$ and a $t' > 0$ s.t. $w(t) = w^1(t)$ for $t \leq 0$ and $w(t) = w^2(t)$ for $t \geq t'$.

In the single variable case, it can be shown that $\mathcal{B}_S$ is controllable iff $\mathcal{C}(\mathcal{B}_S)$ is unimodular.

Let $\mathcal{B}_S$ be a behaviour and let S be partitioned into S_1,S_2. We say $\mathcal{B}$ is S_2- *observable* iff whenever (w_{S_1}, w_{S_2}), (w_{S_1}, w'_{S_2}) belong to $\mathcal{B}_S$, we also have $w_{S_2} = w'_{S_2}$.

Let f_S be an operator vector over $\Re[\xi]$ and let $w_S : \Re \longrightarrow \Re^S$ be a behaviour vector of $\mathcal{C}^\infty$ functions. The *operator product* $[f_S, w_S]$ of f_S, w_S is defined by

$$[f_S, w_S] \equiv \sum_{e\in S}(f_{Se}[\frac{d}{dt}])(w_{Se}(\cdot)),$$

where w_{Se} denotes the function which is the entry at the e^{th} position of w_S, and f_{Se} denotes the differential operator at the e^{th} position of f_S obtained by replacing 'ξ' by '$\frac{d}{dt}$'. We say f_S, w_S are *q-orthogonal* iff $[f_S, w_S] = 0$. The definitions of *operator product* and *q-orthogonality* carry over immediately to the case where the ring of scalars for operator vectors is $\Re[\xi_1, \cdots, \xi_n]$.

Let $\mathcal{K}_S$ be a collection of operator vectors on S. Then $\mathcal{K}_S^*$ denotes the collection of all behaviour vectors q-orthogonal to members of $\mathcal{K}_S$, i.e., the behaviour whose kernel representation is $Rw_S = 0$, where the R is a generator matrix of the set of operator vectors in $\mathcal{K}_S$.

Let $\mathcal{B}_S$ be a behaviour. Then $\mathcal{B}_S^*$ denotes the collection of all operator vectors q-orthogonal to the behaviour vectors in $\mathcal{B}_S$.

If $\mathcal{C}_S$ is a module on S then $\mathcal{C}_S^* = \mathcal{B}(\mathcal{C}_S)$ and if $\mathcal{B}_S$ is a behaviour (with a kernel representation) then $\mathcal{B}_S^* = \mathcal{C}(\mathcal{B}_S)$. Thus, as mentioned earlier, $(\mathcal{C}_S^*)^* = \mathcal{C}_S$ and $(\mathcal{B}_S^*)^* = \mathcal{B}_S$.

4 Matroids and Behavioural Systems

We now sketch some elementary ideas from the theory of matroids associated with modules. In many cases the routine proofs are omitted. Our approach, essentially, is to study a behaviour $\mathcal{B}$ in terms of the module $\mathcal{C}(\mathcal{B})$ associated with it. The operations performed on this module such as contractions and restrictions are standard in the theory of matroids represented over integral domains (but appear to be not naturally studied in module theory). A *matroid represented over a module* is the family of independent sets of column vectors of a rectangular matrix with entries from an integral domain. In our case this domain is $\Re[\xi]$. Abstract matroids are not immediately needed but later (in subsubsection **??**) we will indicate one application of abstract matroid theory to construction of minimal hybrid (partly kernel and partly image) representation of a behaviour through contraction and restriction.

Let $\mathcal{C}$ be a module on S and let $T \subseteq S$. Then the *restriction* of $\mathcal{C}$ to T is denoted by $\mathcal{C} \cdot T$ and is defined by

$$\mathcal{C} \cdot T \equiv \{f/T, f \in \mathcal{C}\}$$

and the contraction of $\mathcal{C}$ to T is denoted by $\mathcal{C} \times T$ and is defined by

$$\mathcal{C} \times T \equiv \{f/T, f \in \mathcal{C}, f/S - T = 0\}$$

Clearly both $\mathcal{C} \cdot T$ and $\mathcal{C} \times T$ are modules on T.

Let $\mathcal{B}_S \equiv \mathcal{B}(\mathcal{C}_S)$. Let $T \subseteq S$. The *contraction of* $\mathcal{B}_S$ *to* T denoted by $\mathcal{B}_S \times T$ is defined by

$$\mathcal{B}_S \times T \equiv \{w_t, (0_{S-T}, w_T) \in \mathcal{B}_S\}$$

and the *restriction of* $\mathcal{B}_S$ *to* T denoted by $\mathcal{B}_S \cdot T$ is defined by

$$\mathcal{B}_S \cdot T \equiv \{w_T, \exists w_{S-T} s.t.(w_{S-T}, w_T) \in \mathcal{B}_S\}$$

Observe that $\mathcal{B}_S \cdot T$ is obtained by projecting trajectories (w_{S-T}, w_T) to w_T. This may be regarded as equivalent to elimination of variables w_{S-T}.

To build representative matrices for restrictions and contractions from a representative matrix $R[\xi]$ of $\mathcal{C}$ on S, we first a build a row equivalent matrix of the following form (using RRA):

$$\begin{array}{cc} T & S-T \\ \end{array}$$
$$\left[\begin{array}{cc} R_{TT} & 0 \\ R_{(S-T)T} & R_{(S-T)(S-T)} \end{array}\right] \tag{1}$$

where rows of $R_{(S-T)(S-T)}$ are linearly independent. Then, R_{TT} is a representative matrix of $\mathcal{C} \times T$ and $R_{(S-T)(S-T)}$ is a representative matrix of $\mathcal{C} \cdot T$. We omit the routine proof. If $\mathcal{C}$ is unimodular (i.e., the matrix in equation 1 is row g-unimodular) then it is clear that R_{TT} is row g-unimodular (i.e., $\mathcal{C} \times T$ is unimodular) but $R_{(S-T)(S-T)}$ is not necessarily so.

An immediate consequence of the above construction is the following

Theorem 4.1 *Let $\mathcal{C}$ be a module on S and let $T \subseteq S$. Then $r(\mathcal{C}) = r(\mathcal{C} \times T) + r(\mathcal{C} \cdot (S - T)$.*

Let the module $\mathcal{C}(\mathcal{B})$ on S, associated with behaviour $\mathcal{B}$ be given. We examine the behaviours corresponding to $\mathcal{C}(\mathcal{B}) \cdot T$,$\mathcal{C}(\mathcal{B}) \times T$. We may assume that a kernel representation of $\mathcal{B}$ is available in the form

$$\left[\begin{array}{cc} R_{TT} & 0 \\ R_{(S-T)T} & R_{(S-T)(S-T)} \end{array}\right] \left[\begin{array}{c} w_T \\ w_{(S-T)} \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]. \tag{2}$$

with rows of $R_{(S-T)(S-T)}$ linearly independent.

The restriction $\mathcal{B} \cdot T$ of $\mathcal{B}$ would then have the kernel representation

$$R_{TT} w_T = 0 \tag{3}$$

To see this, suppose $(\alpha_T, \alpha_{(S-T)})$ is a member of $\mathcal{B}$. Clearly α_T would satisfy equation 3. On the other hand suppose α_T is a member of $\mathcal{B}$, then by solving the equation

$$R_{(S-T)(S-T)} w_{(S-T)} = -R_{(S-T)T} \alpha_T.$$

We can always get a member $(\alpha_T | \alpha_{S-T})$ of $\mathcal{B}$. This latter equation is solvable since $R_{(S-T)(S-T)}$ has linearly independent rows and $R_{(S-T)T}\alpha_T$ is a vector of $\mathcal{C}^{\infty}$ functions.

Next consider the contraction $\mathcal{B} \times (S-T)$ of $\mathcal{B}$.

We claim that this has the kernel representation $R_{(S-T)(S-T)} w_{S-T} = 0$. To see this, suppose $(0_T, \alpha_{S-T})$ is a member of $\mathcal{B}$. By the second row of equation 2, clearly $R_{(S-T)(S-T)}\alpha_{S-T} = 0$. On the other hand, suppose $R_{(S-T)(S-T)}\alpha_{S-T} = 0$. Then $(0_T, \alpha_{S-T})$ satisfies equation 2 and is a member of $\mathcal{B}$.

The following theorem summarizes the above discussion. Corollary 9.1 states that the result is true also for N-D systems.

Theorem 4.2 *Let $\mathcal{C}_S$ be a module on S and let $T \subseteq S$. Then*
(a) $\mathcal{B}(\mathcal{C}_S \cdot T) = (\mathcal{B}(\mathcal{C}_S)) \times T$ (b) $\mathcal{B}(\mathcal{C}_S \times T) = (\mathcal{B}(\mathcal{C}_S)) \cdot T$.

Contractions and restrictions and the modules obtained through successive application of the corresponding operations are called *minors* of the original module. We need a further generalization for our purposes which however, is not a part of standard matroid theory.

Let $\mathcal{K}_{SP}, \mathcal{K}_P$ be collections of operator vectors defined on $S \uplus P, P$ respectively. Then the *generalized minor* of $\mathcal{K}_{SP}$ with respect to $\mathcal{K}_P$, denoted by $\mathcal{K}_{SP} \longleftrightarrow \mathcal{K}_P$ is defined as follows:

$$\mathcal{K}_{SP} \longleftrightarrow \mathcal{K}_P \equiv \{f_S : \exists f_{SP} \in \mathcal{K}_{SP}, f_{SP}/S = f_S, f_{SP}/P \in \mathcal{K}_P\}$$

If $\mathcal{K}_{SP}, \mathcal{K}_P$ are modules, $\mathcal{K}_{SP} \longleftrightarrow \mathcal{K}_P$ would be a module on S. We denote by $\mathcal{K}_{SP} + \mathcal{K}_P$ the collection of all vectors $f_{SP} + (0_S, f_P)$ on $S \uplus P$. when $\mathcal{K}_{SP}, \mathcal{K}_P$ are modules, it is clear that

$$\mathcal{K}_{SP} \longleftrightarrow \mathcal{K}_P = (\mathcal{K}_{SP} + \mathcal{K}_P) \times S$$

We may define generalized minors of behaviours similarly. Let $\mathcal{B}_{SP}, \mathcal{B}_P$ be behaviours on $S \uplus P, P$ respectively. The *generalized minor* $\mathcal{B}_{SP} \longleftrightarrow \mathcal{B}_P$ of $\mathcal{B}_{SP}$ with respect to $\mathcal{B}_P$ is defined as follows:

$$\mathcal{B}_{SP} \longleftrightarrow \mathcal{B}_P \equiv \{w_S : (w_S, w_P) \in \mathcal{B}_{SP}, w_P \in \mathcal{B}_P\}$$

We now have a generalization of Theorem 4.2 which we could regard as a kind of 'Implicit Duality Theorem'.

Theorem 4.3 *Let $\mathcal{C}_{SP}, \mathcal{C}_P$ be modules on $S \uplus P, P$ respectively. Then*

$$\mathcal{B}(\mathcal{C}_{SP} \longleftrightarrow \mathcal{C}_P) = \mathcal{B}(\mathcal{C}_{SP}) \longleftrightarrow \mathcal{B}(\mathcal{C}_P)$$

i.e., $(\mathcal{C}_{SP} \longleftrightarrow \mathcal{C}_P)^* = \mathcal{C}^*_{SP} \longleftrightarrow \mathcal{C}^*_P$.

We need the following lemma for proving the above result.

Lemma 4.1 *The equation*

$$[A[\xi]]w = f(\cdot), \tag{4}$$

$f(\cdot)$ a vector with $\mathcal{C}^\infty$ functions as entries, has a solution which is a vector with $\mathcal{C}^\infty$ functions as entries iff whenever

$$\lambda^T[\xi]A(\xi) = 0,$$

we also have

$$[\lambda^T[\xi], f(\cdot)] = 0. \tag{5}$$

Proof The necessity of the condition is obvious. We prove the sufficiency. Let equation 5 hold. By reversible row operations $A[\xi]$ can be transformed to

$$\begin{bmatrix} A_1[\xi] \\ 0 \end{bmatrix},$$

where the rows of $A_1[\xi]$ are linearly independent, and the equation 4 can be transformed through the same operations to

$$\begin{bmatrix} A_1[\xi] \\ 0 \end{bmatrix} w = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

Now equation 5 implies that every entry of f_2 is zero. Since rows of $A_1[\xi]$ are linearly independent, it follows that the equation

$$(A_1[\xi])w = f_1, \tag{6}$$

always has a solution.

$\square$

Proof of Theorem 4.3

Let $\mathcal{C}_{SP}$ have the generator matrix $[A_S \quad A_P]$ and let $\mathcal{C}_P$ have the generator matrix $[\hat{A}_P]$. Then $\mathcal{C}_{SP} + \mathcal{C}_P$ has the generator matrix

$$\begin{bmatrix} A_S & A_P \\ 0 & \hat{A}_P \end{bmatrix}.$$

Let $\mathcal{B}_+$ be associated with the module $\mathcal{C}_{SP} + \mathcal{C}_P$. Clearly $\mathcal{B}(\mathcal{C}_{SP}) \leftrightarrow \mathcal{B}(\mathcal{C}_P) = \mathcal{B}_+/w_S$. In other words, $w'_S \in \mathcal{B}(\mathcal{C}_{SP}) \leftrightarrow \mathcal{B}(\mathcal{C}_P)$ iff $\exists w'_P$, s.t.

$$\begin{bmatrix} A_S & A_P \\ 0 & \hat{A}_P \end{bmatrix} \begin{matrix} w'_S \\ w'_P \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e., iff $\exists w'_P$, s.t.

$$\begin{bmatrix} A_P \\ \hat{A}_P \end{bmatrix} w'_P = \begin{pmatrix} -A_S w'_S \\ 0 \end{pmatrix}$$

i.e., (by Lemma 4.1) iff whenever

$$\begin{bmatrix} \lambda_1[\xi] & \lambda_2[\xi] \end{bmatrix} \begin{bmatrix} A_P \\ \hat{A}_P \end{bmatrix} = \begin{pmatrix} 0_P \end{pmatrix}$$

(equivalently whenever $\lambda_1[\xi]A_S \in \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$),

we also have $[\lambda_1[\xi], A_S w'_S] = 0$ (equivalently $[\lambda_1[\xi]A_S, w'_S] = 0$),

i.e., iff whenever $\lambda_1[\xi]A_S \in \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$, we also have $[\lambda_1[\xi]A_S, w'_S] = 0$,

i.e., iff $f_S \in \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ implies $[f_S, w'_S] = 0$ (since every vector in $\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ is a linear combination of the rows of A_S). Thus $\mathcal{B}(\mathcal{C}_{SP}) \leftrightarrow \mathcal{B}(\mathcal{C}_P) = \mathcal{B}(\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P)$.

$\square$

Two special cases of the above result may be mentioned. Consider the case where $\mathcal{C}_P$ is the zero module on P. In this case $\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ is obtained by restricting to S

all those vectors in $\mathcal{C}_{SP}$ which take zero value on P. This is the contraction of $\mathcal{C}_{SP}$ on P. The corresponding behaviour is merely the projection $\mathcal{B}_{SP}/w_S$, since $\mathcal{B}_P$ has no constraints on it and is therefore 'free'. Next let $\mathcal{C}_P$ be the module on P spanned by the rows of the unit matrix.In this case $\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ is obtained by restricting to S all vectors in $\mathcal{C}_{SP}$. The corresponding behaviour is the set $\{w_S : (w_S, 0_P) \in \mathcal{B}_{SP}\}$.

We note that Theorem 4.3 would hold also for those N-D systems for which Lemma 4.1 holds, since the proof depends only on that lemma. (Lemma 4.1 is in fact known to hold for the important case where the $f(\cdot, \cdots, \cdot)$ corresponding to the $f(\cdot)$ of Equation 4 and the solution of Equation 4 are $\mathcal{C}^{\infty}$ functions ([Palamodov70])). In the proof of Theorem 4.3 for the N-D case, the modules $\mathcal{K}_{SP}, \mathcal{K}_P$ would be over rings of real polynomials of many variables, the operator vector $\lambda[\xi]$, would be replaced by $\lambda[\xi_1, \cdots, \xi_n]$, the meaning of $[\lambda[\xi_1, \cdots, \xi_n], w_S]$ would be assigned in the obvious manner. Otherwise the proof for the N-D case is line by line the same as the one given here for the 1-D case. Further, by using the above argument for special cases, it follows that Theorem 4.2 is true also for the N-D case. Relevant statements and proofs may be found in Appendix II.

The next few results connect modules (over $\Re[\xi]$) related to $\mathcal{C}$ with those related to $\mathcal{C}^{\perp}$.

Theorem 4.4 *Let $\mathcal{C}$ be a module on S and let $T \subseteq S$. Let $\mathcal{C}$ have a representative matrix*

$$\begin{array}{cc} T & S-T \end{array}$$
$$\begin{bmatrix} D & 0 \end{bmatrix} G,$$

where $[D \;\; 0]$ is in the SCF and G is unimodular, then

1. *$\mathcal{C}^{\perp}$ has the representative matrix*

$$\begin{array}{cc} T & S-T \end{array}$$
$$\begin{bmatrix} 0 & I \end{bmatrix} (G^T)^{-1}.$$

2. *$\mathcal{C}^{\perp}$ is unimodular.*

3. *$r(\mathcal{C}) + r(\mathcal{C}^{\perp}) = |S|$.*

4. *$(\mathcal{C}^{\perp})^{\perp} = \mathcal{C}$ iff $\mathcal{C}$ is unimodular.*

Proof

1. We observe that $(G^T)^{-1}$ has the entries in $\Re[\xi]$ since G is unimodular and further that it is itself unimodular.

   Let $\hat{\mathcal{C}}$ have the representative matrix

$$\begin{array}{cc} T & S-T \\ \end{array}$$
$$\left[\begin{array}{cc} 0 & I \end{array}\right](G^T)^{-1}.$$

   It is easy to see that $\hat{\mathcal{C}} \subseteq \mathcal{C}^{\perp}$.

   On the other hand let (x_T, x_{S-T}) be a vector in $\mathcal{C}^{\perp}$. We must have

$$\left(\begin{array}{cc} D & 0 \end{array}\right) G \left(\begin{array}{c} x_T \\ x_{S-T} \end{array}\right) = 0.$$

   Since D is diagonal and nonsingular, it follows that

$$G \left(\begin{array}{c} x_T \\ x_{S-T} \end{array}\right) = \left(\begin{array}{c} 0 \\ y_{S-T} \end{array}\right),$$

   for some y_{S-T}. But this means that $(x_T^T | x_{S-T}^T)$ is linearly dependent on the rows of $\left[\begin{array}{cc} 0 & I \end{array}\right](G^T)^{-1}$, i.e, $(x_T^T | x_{S-T}^T) \in \hat{\mathcal{C}}$. Thus $\mathcal{C}^{\perp} \subseteq \hat{\mathcal{C}}$.

2. Since $(G^T)^{-1}$ is unimodular, $[0|I]$ is the SCF of any representative matrix of $\mathcal{C}^{\perp}$. The latter is therefore unimodular.

3. This is immediate from the first part above.

4. From the first part above it is clear that $(\mathcal{C}^{\perp})^{\perp}$ has the representative matrix $[I|0]G$. So $\mathcal{C} = (\mathcal{C}^{\perp})^{\perp}$ iff $D = I$, i.e., iff $\mathcal{C}$ is unimodular.

There is a restricted 'duality' between contraction and restriction in the sense of the following theorem.

Theorem 4.5 *Let $\mathcal{C}$ be a module on S and let $T \subseteq S$. Then*

1. $(\mathcal{C} \cdot T)^{\perp} = \mathcal{C}^{\perp} \times T$

2. If $\mathcal{C}, \mathcal{C}^{\perp} \cdot T$ are unimodular then $(\mathcal{C} \times T)^{\perp} = \mathcal{C}^{\perp} \cdot T$.

1. This is routine.

2. We have $(\mathcal{C}^{\perp} \cdot T)^{\perp} = (\mathcal{C}^{\perp})^{\perp} \times T = \mathcal{C} \times T$, since $\mathcal{C}$ is unimodular. Hence $(\mathcal{C} \times T)^{\perp} = ((\mathcal{C}^{\perp} \cdot T)^{\perp})^{\perp} = \mathcal{C}^{\perp} \cdot T$, since $\mathcal{C}^{\perp} \cdot T$ is unimodular.

5 An Implicit Duality Theorem and its Applications

In this section we consider a result which, in its original form, was about ideal transformers, but turned out to be fundamental to Topological Network Theory. We show that this result is useful in Behavioural Systems Theory, particularly in studying the mode of representation of such systems and their adjoints.
An analogue of this result [Theorem IDTB] has already been considered in the previous section.

Theorem 5.1 *(The Implicit Duality Theorem (IDT)) Let $\mathcal{C}_{SP}, \mathcal{C}_P$ be modules on $S \uplus P, P$ respectively, such that $\mathcal{C}_{SP} + \mathcal{C}_P$ is also unimodular. Then,*
(a) $(\mathcal{C}_{SP}^{\perp} \leftrightarrow \mathcal{C}_P^{\perp})^{\perp} = \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$
(b) $\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ *is unimodular.*
If, in addition, $\mathcal{C}_{SP}, \mathcal{C}_P$, $\mathcal{C}_{SP}^{\perp} + \mathcal{C}_P^{\perp}$ *are unimodular, then*
(c) $(\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P)^{\perp} = \mathcal{C}_{SP}^{\perp} \leftrightarrow \mathcal{C}_P^{\perp}$
(d) $\mathcal{C}_{SP}^{\perp} \leftrightarrow \mathcal{C}_P^{\perp}$ *is unimodular.*

The proof of this theorem requires the use of the following lemmas, of which the first is contained essentially in Theorem 4.4 and therefore has its proof omitted.

Lemma 5.1 *Let $\mathcal{K}$ be a collection of vectors over $\Re[\xi]$ on S. Then $\mathcal{K}^{\perp}$ is unimodular.*

Lemma 5.2 *Let A be an $m \times n$ matrix over $\Re[\xi]$, whose column module is unimodular, b an $m \times 1$ vector over $\Re[\xi]$. Then $Ax = b$ has a solution (with x a vector over $\Re[\xi]$) iff for every $(1 \times m)$ vector λ^T over $\Re[\xi]$ we have $\lambda^T A = 0 \Rightarrow \lambda^T b = 0$.*

Proof :- $Ax = b$ has a solution iff b belongs to the column module $\mathcal{C}$ generated by columns of A,
i.e., iff $b \in \mathcal{C}^{\perp\perp}$ ($\mathcal{C} = \mathcal{C}^{\perp\perp}$ since column module of A is unimodular),
i.e., iff whenever $\lambda^T \in \mathcal{C}^{\perp}$ we also have $\lambda^T b = 0$,
i.e., iff whenever $\lambda^T A = 0$, we also have $\lambda^T b = 0$.

□

Proof of Theorem 5.1

Let $\mathcal{C}_{SP}$ have the generator matrix $[A_S \quad A_P]$ and let $\mathcal{C}_P$ have the generator matrix $[\hat{A}_P]$. Then $\mathcal{C}_{SP} + \mathcal{C}_P$ is generated by the rows of the generator matrix

$$\begin{bmatrix} A_S & A_P \\ 0 & \hat{A}_P \end{bmatrix}.$$

Further we have $\mathcal{C}_{SP} + \mathcal{C}_P$ unimodular by hypothesis. Now $x_S \in \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ iff $\exists \lambda_1, \lambda_2$ over $\Re[\xi]$ s.t.

$$\begin{bmatrix} {A_S}^T & 0 \\ {A_P}^T & \hat{A}_P^T \end{bmatrix} \begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix} = \begin{pmatrix} x_S \\ 0 \end{pmatrix}$$

i.e., (by Lemma 5.2, since column module of the above matrix is unimodular) iff whenever, for y_S, y_P over $\Re[\xi]$ we have

$$\begin{bmatrix} y_S^T & y_P^T \end{bmatrix} \begin{bmatrix} A_S^T & 0 \\ A_P^T & \hat{A}_P^T \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

we also have

$$y_S^T x_S = 0;$$

i.e., iff whenever $(y_S^T \ y_p^T) \in \mathcal{C}_{SP}^{\perp}, \quad y_P^T \in \mathcal{C}_P^{\perp}$ we also have $y_S^T x_S = 0$;
i.e., iff whenever $y_S \in \mathcal{C}_{SP}^{\perp} \leftrightarrow \mathcal{C}_P^{\perp}$ we have ${y_S}^T x_S = 0$.
Thus, $x_S \in \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ iff $x_S \in (\mathcal{C}_{SP}^{\perp} \leftrightarrow \mathcal{C}_P^{\perp})^{\perp}$. This proves (a).

The statement in (b) follows from (a) and Lemma 5.1.

(c) Suppose $\mathcal{C}_{SP}^{\perp} + \mathcal{C}_P^{\perp}$ is unimodular. We can now use (a) above and conclude that

$$(\mathcal{C}_{SP}^{\perp\perp} \leftrightarrow \mathcal{C}_P^{\perp\perp})^{\perp} = \mathcal{C}_{SP}^{\perp} \leftrightarrow \mathcal{C}_P^{\perp}$$

But $\mathcal{C}_{SP}, \mathcal{C}_P$ are also given to be unimodular and so by Theorem 4.4, $\mathcal{C}_{SP}^{\perp\perp} = \mathcal{C}_{SP}$ and $\mathcal{C}_P^{\perp\perp} = \mathcal{C}_P$.

Hence $(\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P)^{\perp} = \mathcal{C}_{SP}^{\perp} \leftrightarrow \mathcal{C}_P{}^{\perp}$.

Statement (d) follows from (c) and Lemma 5.1.

$\square$

5.1 Generalization to the N-D case

Let us define a module $\mathcal{C}_S$ on a finite set S over $\Re[\xi_1, \cdots, \xi_n]$ to be *generalized unimodular* iff $\mathcal{C}_S^{\perp\perp} = \mathcal{C}_S$, the 'dot product' and 'orthogonality' being defined just as in the single variable case. Note that any module of the form $\mathcal{K}_S^{\perp}$ is generalized unimodular. For such modules, in general, there exist no representative matrices (rows being linearly independent), but, as mentioned earlier, there always exist generator matrices with a finite set of rows with the property that every vector in the module is a linear combination of the rows of the generator matrix (which may have dependent rows). Lemmas 5.1 and 5.2 and Theorem 5.1 (the Implicit Duality Theorem) and their proofs given above are then valid for such modules replacing 'unimodular' by 'generalized unimodular' throughout the statement and proofs of the abovementioned lemmas and theorem.

5.2 An application of the Implicit Duality Theorem

We illustrate the use of the Implicit Duality Theorem through two examples. The first example is the proof of a well known result about image representation for controllable behaviours. In this case the direct proof using the existence of Smith Canonical Form is shorter. However the proof through the Implicit Duality Theorem would work even for N-D systems provided 'unimodular module' is replaced by 'generalized unimodular module', 'representative matrix' is replaced by 'generator matrix' and 'g-unimodular' is replaced by 'generator matrix of a generalized unimodular module' in the statement and proof of Theorem 5.2. The second example is on the construction of adjoints for Behavioural Systems and is considered in the next section. There one can see that some economy is indeed achieved by the use of the theorem.

Theorem 5.2 *Let $\mathcal{B}_S$ be a controllable behaviour associated with the unimodular module $\mathcal{C}_S$. Let R_2 be a representative matrix of $\mathcal{C}_S^{\perp}$. Then, $\mathcal{B}_S$ has the image representation $w_S = R_2^T \alpha$.*

Proof : Since $\mathcal{B}$ is controllable, it has a kernel representation $Rw_S = 0$ where R is g-unimodular. Let $\mathcal{C}_S$ be the module generated by the rows of R. Let $\mathcal{C}_{SP}$, $\mathcal{C}_P$ be generated by the rows of

$$\begin{array}{cc} S & P \\ \multicolumn{2}{c}{\begin{bmatrix} R & I \end{bmatrix}} \end{array}$$

and

$$\begin{array}{c} P \\ \begin{bmatrix} I \end{bmatrix}. \end{array}$$

Clearly, $\mathcal{C}_{SP} \longleftrightarrow \mathcal{C}_P = \mathcal{C}_S$. Now $\begin{bmatrix} I \end{bmatrix}$, $\begin{bmatrix} R & I \end{bmatrix}$ and $\begin{bmatrix} R & I \\ 0 & I \end{bmatrix}$ are g-unimodular (the latter since R is g-unimodular). So $\mathcal{C}_P, \mathcal{C}_{SP}$, $\mathcal{C}_{SP} + \mathcal{C}_P$ are also unimodular. Now $\mathcal{C}_P^{\perp}$ is the zero module 0_P. Hence $\mathcal{C}_{SP}^{\perp} + \mathcal{C}_P^{\perp} = \mathcal{C}_{SP}^{\perp}$ and therefore unimodular. Hence by the Implicit Duality Theorem (Theorem 5.1) we have

$\mathcal{C}_S^{\perp} = (\mathcal{C}_{SP} \longleftrightarrow \mathcal{C}_P)^{\perp} = \mathcal{C}_{SP}^{\perp} \longleftrightarrow \mathcal{C}_P^{\perp}$

Let $\mathcal{C}_S^{\perp}$ have the representative matrix R_2. Since $\mathcal{C}_S^{\perp}$ is unimodular (Lemma 5.1), R_2 is row g-unimodular. Let $\mathcal{C}_{SQ}$ be the module generated by the rows of

$$\begin{array}{cc} S & Q \\ \multicolumn{2}{c}{\begin{bmatrix} R_2 & I \end{bmatrix}.} \end{array}$$

Clearly $\mathcal{C}_{SQ}$ is unimodular and so is $C_{SQ} + \mathcal{C}_Q$, where $\mathcal{C}_Q$ is generated by the rows of

$$\begin{array}{c} Q \\ \begin{bmatrix} I \end{bmatrix}. \end{array}$$

Hence

$(\mathcal{C}_S^{\perp})^{\perp} = (C_{SQ} \longleftrightarrow \mathcal{C}_Q)^{\perp} = \mathcal{C}_{SQ}^{\perp} \longleftrightarrow \mathcal{C}_Q^{\perp} = \mathcal{C}_S.$

Now $\mathcal{C}_{SQ}^{\perp}$ has the representative matrix $(I - R_2^T)$ and $\mathcal{C}_Q^{\perp}$ is the zero module 0_Q.

Let $\mathcal{B}_{SQ}$ be defined through the equations

$$\begin{bmatrix} I & -R_2^T \end{bmatrix} \begin{bmatrix} w_3 \\ \alpha_Q \end{bmatrix} = 0 \quad (7)$$

$$(0)\alpha_Q = 0 \quad (8)$$

the second row indicating that $\mathcal{C}_Q^{\perp}$ is the zero module. Since $\mathcal{C}_S = \mathcal{C}_{SQ}^{\perp} \longleftrightarrow \mathcal{C}_Q^{\perp}$, it is clear (by Theorem 4.3) that $\mathcal{B}_{SQ} \longleftrightarrow \mathcal{B}(\mathcal{C}_Q^{\perp}) = \mathcal{B}_{SQ}.S$ is associated with $\mathcal{C}_S$. Equivalently, $\mathcal{B}$ has the image representation $w_S = R_2^T \alpha_Q$, where R_2 is a representative matrix of $\mathcal{C}_S^{\perp}$.

□

6 Adjoints for Behavioural Systems

The notion of an 'Adjoint' of a system plays a fundamental role in the study of systems described through state and output equations. Among other things, the duality that exists between controllablity and observability is best appreciated by associating them one with a given system and the other with its adjoint.

This notion does not appear to be equally natural in the context of Behavioural Systems Theory. In this section we propose some definitions for the 'adjoint' and critically examine its properties. Conventional constructions of adjoints for standard systems such as Electrical Networks may be found, for instance, in [Narayanan86b].

The following appear to be desirable while building adjoints

- the notions controllability/observability of the original system should appear to correspond to observability/controllability for the adjoints;
- the adjoint for systems in the i/s/o (input/state/output) form should correspond to the standard construction for systems in this form;
- the computational effort for building the adjoint should be very mild whatever be the original representation for the behaviour.

In addition we note that the term adjoint is conventionally used for the (linear) system for which the input/output relationship is $y' = \pm K^T u'$ if that for the original system is $y = Ku$. This property can be more conveniently stated as 'the module associated with the the adjoint behaviour should be complementary orthogonal to that associated with the original behaviour'. It would be also useful, if possible, to retain this property because of the technical advantages it confers.

Let behaviour $\mathcal{B}$ have the kernel representation

$$\begin{pmatrix} R_w & R_l \end{pmatrix} \begin{matrix} w \\ l \end{matrix} = 0$$

Define the adjoint $\mathcal{B}^{adj1}$ as having the kernel representation

$$\begin{bmatrix} I & -R_w^T \\ 0 & -R_l^T \end{bmatrix} \begin{matrix} \hat{w} \\ \hat{l} \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Observe that, while $w, \hat{w}$ have the same number of entries $l, \hat{l}$ may not have. It is easy to see that

$$\begin{gathered} \mathcal{B} \text{ is } l\text{- observable (i.e.,} R_l \text{ is column g-unimodular)} \\ \equiv \mathcal{B}^{adj1} \text{ is controllable (since } R_l^T \text{ is row g-unimodular, the matrix} \\ \begin{bmatrix} I & -R_w^T \\ 0 & -R_l^T \end{bmatrix} \\ \text{is row g-unimodular) and} \\ \mathcal{B} \text{ is controllable (i.e., } (R_w \; R_l) \text{ is row g-unimodular)} \\ \equiv \mathcal{B}^{adj1} \text{ is } \hat{l} \text{ - observable.} \end{gathered}$$

We further have (through the use of the Implicit Duality Theorem)

Theorem 6.1 :- *If behaviour $\mathcal{B}$ is controllable*

$$\mathcal{C}(\mathcal{B}/w) = (\mathcal{C}(\mathcal{B}^{adj1}/\hat{w}))^{\perp}.$$

If $\mathcal{B}$ is l-observable then

$$(\mathcal{C}(\mathcal{B}/w))^{\perp} = \mathcal{C}(\mathcal{B}^{adj1}/\hat{w})$$

This theorem is a restricted version (by taking the y variables to be absent) of Theorem 6.2 proved below. Its proof is therefore omitted.

Note that the adjoint is essentially, but not exactly, unique. If $(R_w R_l)$ is replaced by a matrix obtained by unimodular row transformation, then in the adjoint $\hat{w}$ would change through the inverse transformation. Similarly the adjoint of the adjoint would be essentially the original behaviour but would not be identical to it. We show now that this definition captures the usual i/s/o adjoint nicely.

If the original system is in the i/s/o form we have

(taking $w^T = (u^T \quad y^T), x = l$),

$$\begin{bmatrix} -B & 0 & (\xi I - A) \\ -D & I & -C \end{bmatrix} \begin{matrix} u \\ y \\ x \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The adjoint then is

$$\begin{bmatrix} I & 0 & B^T & D^T \\ 0 & I & 0 & -I \\ 0 & 0 & -(\xi I - A)^T & C^T \end{bmatrix} \begin{matrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{l}_1 \\ \hat{l}_2 \end{matrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The variables $\hat{w}_2, \hat{l}_2$ are seen to be identical using the second row. The constraints on $\hat{w}_1, \hat{w}_2, \hat{l}_1$ are

$$\begin{bmatrix} I & D^T & B^T \\ 0 & C^T & -(\xi I - A)^T \end{bmatrix} \begin{matrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{l}_1 \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Clearly this is the usual adjoint of a system in i/s/o form.

Next let us consider the behaviour given in the input, output, latent variable form.

Let the behaviour $\mathcal{B}$ have the kernel representation with linearly independent rows

$$\begin{bmatrix} R_{1u} & R_{1l} & 0 \\ R_{2u} & R_{2l} & I \end{bmatrix} \begin{array}{c} u \\ l \\ y \end{array} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Observe that practical systems, when they have clearly defined inputs and outputs, very often have this form. The latent variables need not however correspond to the state variables.

Let the adjoint $\mathcal{B}^{adj2}$ be defined through the representation

$$\begin{bmatrix} I & -R_{1u}^T & -R_{2u}{}^T \\ 0 & -R_{1l}^T & -R_{2l}^T \end{bmatrix} \begin{array}{c} \hat{u} \\ \hat{l} \\ \hat{y} \end{array} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Observe that, while $u, \hat{u}$, and $y, \hat{y}$ have the same number of entries, $l, \hat{l}$ may not have.

We observe that $\mathcal{B}/ul$ has the kernel representation

$$\begin{bmatrix} R_{1u} & R_{1l} \end{bmatrix} \begin{array}{c} u \\ l \end{array} = 0$$

and that $\mathcal{B}^{adj2}/\hat{y}\hat{l}$ has the kernel representation

$$\begin{bmatrix} -R_{1l}^T & -R_{2l}^T \end{bmatrix} \begin{array}{c} \hat{y} \\ \hat{l} \end{array} = 0.$$

We note that the original behaviour is invariant if in the kernel representation, a linear combination of the first set of rows is added to the second set of rows. It follows that the adjoint is not uniquely defined but depends on the original representation.
However, the adjoint does have the following attractive properties which can be proved through the Implicit Duality Theorem (Theorem 5.1).

Theorem 6.2 *1. $\mathcal{B}/ul$ is controllable $\equiv \mathcal{B}^{adj2}$ is $\hat{l}$-observable.*

2. $\mathcal{B}$ is l-observable $\equiv \mathcal{B}^{adj2}/\hat{y}\hat{l}$ is controllable.

3. *If $\mathcal{B}/ul$ is controllable then* $(\mathcal{C}(\mathcal{B}^{adj2}/\hat{u}\hat{y}))^{\perp} = \mathcal{C}(\mathcal{B}/uy)$.

4. *If $\mathcal{B}$ is l-observable then* $(\mathcal{C}(\mathcal{B}/uy))^{\perp} = \mathcal{C}(\mathcal{B}^{adj2}/\hat{u}\hat{y})$.

Proof:- LHS and RHS of part (1) are equivalent to the condition that $(R_{1u}R_{1l})$ is g-unimodular and LHS and RHS of part (2) are equivalent to the condition that $(R_{1l}^T R_{2l}^T)$ is g-unimodular.

We now prove parts (3) and (4).

Using additional variables l' the kernel representation of $\mathcal{B}$ may be re-written as follows:

$$\begin{pmatrix} R_{1u} & 0 & I & 0 \\ R_{2u} & R_{2l} & 0 & I \end{pmatrix} \begin{matrix} u \\ l \\ l' \\ y \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9}$$

$$\begin{bmatrix} 0 & -R_{1l} & I & 0 \end{bmatrix} \begin{matrix} u \\ l \\ l' \\ y \end{matrix} = 0 \tag{10}$$

The kernel representation of $\mathcal{B}^{adj2}$ may be rewritten as follows:-

$$\begin{pmatrix} I & 0 & -R_{1u}^T & -R_{2u}^T \\ 0 & I & 0 & -R_{2l}^T \end{pmatrix} \begin{matrix} \hat{u} \\ \hat{l} \\ \hat{l}' \\ \hat{y} \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

$$\begin{bmatrix} 0 & I & R_{1l}^T & 0 \end{bmatrix} \begin{matrix} \hat{u} \\ \hat{l} \\ \hat{l}' \\ \hat{y} \end{matrix} = 0 \tag{12}$$

Let $\mathcal{C}_{VLL'Y}$, be the module spanned by the rows of the coefficient matrix in Equation

9, $\mathcal{C}_{LL'}$ be the module spanned by the rows of $[-R_{1l}I]$ in Equation 10. It can be seen that $\mathcal{B}/uy = \mathcal{B}(\mathcal{C}_{VLL'Y}) \longleftrightarrow \mathcal{B}(\mathcal{C}_{LL'})$. By Theorem 4.3 it follows that $\mathcal{C}(\mathcal{B}/uy) = \mathcal{C}(\mathcal{B}(\mathcal{C}_{VLL'Y}) \longleftrightarrow \mathcal{B}(\mathcal{C}_{LL'})) = \mathcal{C}_{VLL'Y} \longleftrightarrow \mathcal{C}_{LL'}$.

Clearly ${\mathcal{C}_{VLL'Y}}^{\perp}$ is spanned by the rows of the coefficient matrix in equation 11 and $\mathcal{C}^{\perp}_{LL'}$ is spanned by the rows of (IR^T_{1l}) in Equation 12.

It can be seen that $\mathcal{B}^{adj2}/uy = \mathcal{B}(\mathcal{C}^{\perp}_{VLL'Y}) \longleftrightarrow \mathcal{B}(\mathcal{C}^{\perp}_{LL'})$. By Theorem 4.3 and the fact that there is a unique module associated with a behaviour,it follows that $\mathcal{C}(\mathcal{B}^{adj2}/uy) = \mathcal{C}(\mathcal{B}(\mathcal{C}^{\perp}_{VLL'Y}) \longleftrightarrow \mathcal{B}(\mathcal{C}^{\perp}_{LL'})) = \mathcal{C}^{\perp}_{VLL'Y} \longleftrightarrow \mathcal{C}^{\perp}_{LL'}$.

Clearly $\mathcal{C}_{LL'}, \mathcal{C}_{VLL'Y}, {\mathcal{C}_{LL'}}^{\perp}, {\mathcal{C}_{VLL'Y}}^{\perp}$ are unimodular modules since they have representative matrices with full rank identity submatrices.
The coefficient matrix for Equations 9, 10, taken together is g-unimodular if $(R_{1u}R_{1l})$ is g-unimodular. If this holds, by the Implicit Duality Theorem (Theorem 5.1)

$$(\mathcal{C}^{\perp}{}_{VLL'Y} \leftrightarrow \mathcal{C}^{\perp}{}_{LL'})^{\perp} = \mathcal{C}_{VLL'Y} \leftrightarrow \mathcal{C}_{LL'}.$$

Since $\mathcal{C}_{VLL'Y} \leftrightarrow \mathcal{C}_{LL'}$ is the module associated with $\mathcal{B}/uy$ and ${\mathcal{C}_{VLL'Y}}^{\perp} \leftrightarrow {\mathcal{C}_{LL'}}^{\perp}$ is the module associated with $\mathcal{B}^{adj2}/\hat{u}\hat{y}$, it follows that

$$(\mathcal{C}(\mathcal{B}^{adj2}/\hat{u}\hat{y}))^{\perp} = \mathcal{C}(\mathcal{B}/uy).$$

Similarly, the coefficient matrix for Equations 11, 12, taken together is g-unimodular if $(-R^T_{1l} - R^T_{2l})$ is g-unimodular. If this holds, by IDT, we have

$$(\mathcal{C}^{\perp\perp}_{VLL'Y} \leftrightarrow \mathcal{C}^{\perp\perp}_{LL'})^{\perp} = \mathcal{C}^{\perp}_{VLL'Y} \leftrightarrow \mathcal{C}^{\perp}_{LL'}$$

i.e. we have (by the unimodularity of $\mathcal{C}_{VLL'Y}, \mathcal{C}_{LL'}$)

$$(\mathcal{C}_{VLL'Y} \leftrightarrow \mathcal{C}_{LL'})^{\perp} = \mathcal{C}^{\perp}_{VLL'Y} \leftrightarrow \mathcal{C}^{\perp}_{LL'}$$

$$\text{i.e., } (\mathcal{C}(\mathcal{B}/uy))^{\perp} = \mathcal{C}(\mathcal{B}^{adj2}/\hat{u}\hat{y}).$$

□

Finally we verify that the i/s/o adjoint does fit into the present definition. Let the original behaviour $\mathcal{B}$ have the representation.

$$\begin{bmatrix} -B & (\xi I - A) & 0 \\ -D & -C & I \end{bmatrix} \begin{matrix} u \\ x \\ y \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then $\mathcal{B}^{adj2}$ has the representation

$$\begin{bmatrix} I & B^T & D^T \\ 0 & -(\xi I - A)^T & C^T \end{bmatrix} \begin{matrix} \hat{u} \\ \hat{x} \\ \hat{y} \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which is identical to the usual i/s/o adjoint.

It is worth noting that the above definition of adjoint, although representation dependent, does agree, essentially, with the usual definition of adjoint for Electrical Networks (such as the one in [Narayanan86b]). For instance, the adjoint does have outputs/inputs in place of the original inputs/outputs, a current/voltage input becomes a voltage/ current output and a current/voltage output becomes a voltage/ current input. Further, if the original network has device characteristic

$$\begin{bmatrix} R_1[\xi] & R_2[\xi] \end{bmatrix} \begin{matrix} i \\ v \end{matrix} = 0,$$

the adjoint has device characteristic

$$\begin{bmatrix} R_1^T[\xi] \\ R_2^T[\xi] \end{bmatrix} \alpha = \begin{pmatrix} i \\ v \end{pmatrix},$$

where α is free. One is tempted to suggest that a truly representation independent definition of adjoint for Behavioural Systems is perhaps impossible. We may have to plug in the fact that we are dealing with a certain type of system (such as Electrical Networks).

6.0.1 Generalization to the N-D case

We observe that our definition of adjoint, and the statement and proof of Theorem 6.2, hinge on the following:

- The fact that (through the kernel representation) there is a unique module $\mathcal{C}(\mathcal{B})$ associated with a behaviour $\mathcal{B}$ (which is known to have a kernel representation) and a unique behaviour $\mathcal{B}(\mathcal{C})$ associated with a module $\mathcal{C}$.

- The validity of Theorem 4.3

- The validity of the Implicit Duality Theorem (Theorem 5.1).
- If R is row g-unimodular then

$$\begin{bmatrix} R & 0 \\ K & I \end{bmatrix}$$

is also row g-unimodular

As we have pointed out earlier in Section 3, in the remarks after Theorem 4.3 and in Subsection 5.1, all the above are known to be true for N-D systems. There is, however, some freedom in the choice of definition of 'N-D controllable' and 'N-D l-observable' corresponding to working with unimodular or generalized unimodular modules. For both these choices, the definitions of adjoint, and the statement and proof of Theorem 6.2, go through for the N-D case as we show in Appendix II.

7 Conclusion

In this paper we have tried to establish connections between the theories of Matroids, Electrical Network Topology and Behavioural Systems. Among other things we have shown that standard matroid operations such as contraction and restriction and their generalization from Topological Network Theory are useful in studying Behavioural Systems. We have proved two theorems of the 'implicit duality ' type and their applications. One of these applications is for the construction of 'adjoints of Behavioural Systems'.

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8 Appendix I

In this appendix we prove the elementary results on modules over $\Re[\xi_1, \cdots, \xi_n]$ stated without proof in Section 3. We assume throughout that S is a finite set. Throughout we will deal with vectors and modules over a commutative ring with a unit but without zero divisors. We denote an identity matrix by I. We define the *rank* of a module $\mathcal{C}_S$ to be the size of a maximally independent set of vectors in it and denote it by $r(\mathcal{C}_S)$. It is well known that this quantity is well defined.

Lemma 8.1 *Let $\mathcal{C}_S$ be a module and let R be a matrix whose rows form a maximally linearly independent set of vectors of $\mathcal{C}_S$. then, if $f_S \in \mathcal{C}_S$ we must have αf_S linearly dependent on rows of R, for some nonzero scalar α.*

Proof: The lemma is immediate from the definition of the matrix R.

Lemma 8.2 *Let T be a square matrix with entries from a commutative ring with a unit but without zero divisors.Then*
there exists a matrix Q such that $TQ = QT = (det(T))I$;
T has linearly independent rows (columns) iff $det(T)$ is not zero.

We omit the standard proof.

Lemma 8.3 *Let $\mathcal{C}_S$ be a module. Then $r(\mathcal{C}_S) + r(\mathcal{C}_S^{\perp}) = |S|$.*

Proof: Let R be a matrix whose rows form a maximally linearly independent set of vectors of $\mathcal{C}_S$. Clearly $\mathcal{C}_S^{\perp}$ is the collection of all vectors orthogonal to the rows of R. Thus $\mathcal{C}_S^{\perp}$ is the solution space of the equation $Rx = 0$. Without loss of generality, we may assume that R can be partitioned into $(R_{11}\ R_{12})$, where R_{11} is square and invertible. Thus $(x_1^T\ x_2^T)$ is in $\mathcal{C}_S^{\perp}$ iff $R_{11}x_1 = -R_{12}x_2$. Let Q be such that $R_{11}Q = QR_{11} = det(R_{11})I$. Let $(y_1^T\ y_2^T) \equiv (det(R_{11}))(x_1^T\ x_2^T)$, where $(x_1^T\ x_2^T)$ is a vector on S. Clearly $(y_1^T\ y_2^T) \in \mathcal{C}_S^{\perp}$ iff $y_1 = -QR_{12}x_2$ and $y_2 = (det(R_{11}))x_2$, for arbitrary values of x_2. Since the number of columns in R_{12} is $|S| - r(\mathcal{C}_S)$, the result now follows.

□

Lemma 8.4 *A module $\mathcal{C}_S$ is generalized unimodular iff $\alpha f_S \in \mathcal{C}_S$ implies $f_S \in \mathcal{C}_S$ whenever α is a nonzero scalar.*

Proof: Suppose $\mathcal{C}_S$ is generalized unimodular. Then if αf_S is orthogonal to $\mathcal{C}_S^{\perp}$ and α is nonzero it follows that f_S is also orthogonal to $\mathcal{C}_S^{\perp}$ and is therefore in $\mathcal{C}_S^{\perp\perp}$, i.e, in $\mathcal{C}_S$.
On the other hand, suppose $\mathcal{C}_S$ has the specified property. Since $r(\mathcal{C}_S) = r(\mathcal{C}_S^{\perp\perp})$, it follows that $f_S \in \mathcal{C}_S^{\perp\perp}$ implies that for some nonzero $\alpha, \alpha f_S \in \mathcal{C}_S$, i.e., that $f_S \in \mathcal{C}_S$. Since $\mathcal{C}_S^{\perp\perp} \supseteq \mathcal{C}_S$ it follows that $\mathcal{C}_S^{\perp\perp} = \mathcal{C}_S$.

□

Lemma 8.5 *Every unimodular module is generalized unimodular.*

Proof: Let $\mathcal{C}_S$ be a unimodular module. Suppose R is a g-unimodular matrix and is a representative matrix of $\mathcal{C}_S$ and B is its right inverse. The row module of B^T intersects $\mathcal{C}_S^{\perp}$ only in the zero vector. On the other hand $r(B^T) + r(C_S^{\perp}) = |S|$. So any vector orthogonal to $\mathcal{C}_S^{\perp}$ as well as to rows of B^T must necessarily be the zero vector. Let $\alpha f_S \in \mathcal{C}_S, \alpha \neq 0$. If f_S is the zero vector, it is in $\mathcal{C}_S$. So let f_S be nonzero. Clearly $f_S B$ is not zero. Consider $f_S BR - f_S$. This vector is orthogonal to $\mathcal{C}_S^{\perp}$ as well as to rows of B^T and is therefore the zero vector. Thus $f_S BR = f_S$ and therefore, since $(f_S B)R \in \mathcal{C}_S$, it follows that $f_S \in \mathcal{C}_S$.

□

9 Appendix II

In this appendix we state and prove the generalizations of Theorems 4.3, 5.1, 6.2.

As stated before in Section 3 , let us define a module $\mathcal{C}_S$ on a finite set S over $\Re[\xi_1, \cdots, \xi_n]$ to be *generalized unimodular* iff $\mathcal{C}_S^{\perp\perp} = \mathcal{C}_S$, the 'dot product' and 'orthogonality' being defined just as in the single variable case. Note that if a module has a representative matrix which has a full rank identity submatrix, then it is easy to see that it is generalized unimodular. (If $(I\ K)$ generates $\mathcal{C}$ then $(-K^T\ I)$ generates $\mathcal{C}^\perp$ and hence $(I\ K)$ generates $\mathcal{C}^{\perp\perp}$). Let $\mathcal{B}_S$ be the set of solutions of linear constant coefficient partial differential equations of the form, $(R[\frac{d}{dx_1}, \cdots, \frac{d}{dx_n}])w_S = 0$,$R[\xi_1, \cdots, \xi_n]$, being a generator matrix of $\mathcal{C}_S$. Then as mentioned before $\mathcal{B}_S$ and $\mathcal{C}_S$ fix each other uniquely and are said to be associated with each other (see Theorem 2.61,[Oberst90]). We may denote $\mathcal{B}_S$ by $\mathcal{B}(\mathcal{C}_S)$ and $\mathcal{C}_S$ by $\mathcal{C}(\mathcal{B}_S)$. As before, the *operator product* $[f_S, w_S]$ of f_S, w_S is defined by

$$[f_S, w_S] \equiv \sum_{e\in S}(f_{Se}[\frac{d}{dx_1}, \cdots, \frac{d}{dx_n}])(w_{Se}(\cdot)),$$

where w_{Se} denotes the function which is the entry at the e^{th} position of w_S, and f_{Se} denotes the differential operator at the e^{th} position of f_S obtained by replacing '$\xi_1, \cdots, \xi_n$' by '$\frac{d}{dx_1}, \cdots, \frac{d}{dx_n}$'.

The statement and proof of the generalization of Theorem 4.3 are identical to those of itself, Lemma 4.1 being interpreted as being stated for N-D systems ([Palamodov70]). For completeness however, we repeat these.

Theorem 9.1 *Let $\mathcal{C}_{SP}, \mathcal{C}_P$ be modules on $S \uplus P, P$ respectively. Then*

$$\mathcal{B}(\mathcal{C}_{SP} \longleftrightarrow \mathcal{C}_P) = \mathcal{B}(\mathcal{C}_{SP}) \longleftrightarrow \mathcal{B}(\mathcal{C}_P)$$

i.e., $(\mathcal{C}_{SP} \longleftrightarrow \mathcal{C}_P)^* = \mathcal{C}_{SP}^* \longleftrightarrow \mathcal{C}_P^*$.

We need the following lemma due to Palamodov [Palamodov70] for proving the above result.

Lemma 9.1 *The equation*

$$[A[\xi_1, \cdots, \xi_n]]w = f(\cdot), \tag{13}$$

$f(\cdot)$ a vector with $\mathcal{C}^\infty$ functions as entries, has a solution which is a vector with $\mathcal{C}^\infty$ functions as entries iff whenever

$$\lambda^T[\xi_1,\cdots,\xi_n]A[\xi_1,\cdots,\xi_n]=0,$$

we also have

$$[\lambda^T[\xi_1,\cdots,\xi_n],f(\cdot)]=0. \tag{14}$$

Proof of Theorem 9.1

Let $\mathcal{C}_{SP}$ have the generator matrix $[A_S \quad A_P]$ and let $\mathcal{C}_P$ have the generator matrix $[\hat{A}_P]$. Then $\mathcal{C}_{SP}+\mathcal{C}_P$ has the generator matrix

$$\begin{bmatrix} A_S & A_P \\ 0 & \hat{A}_P \end{bmatrix}.$$

Let $\mathcal{B}_+$ be associated with the module $\mathcal{C}_{SP}+\mathcal{C}_P$. Clearly $\mathcal{B}(\mathcal{C}_{SP}) \leftrightarrow \mathcal{B}(\mathcal{C}_P) = \mathcal{B}_+/w_S$. In other words, $w'_S \in \mathcal{B}(\mathcal{C}_{SP}) \leftrightarrow \mathcal{B}(\mathcal{C}_P)$ iff $\exists w'_P$, s.t.

$$\begin{bmatrix} A_S & A_P \\ 0 & \hat{A}_P \end{bmatrix} \begin{matrix} w'_S \\ w'_P \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e., iff $\exists w'_P$, s.t.

$$\begin{bmatrix} A_P \\ \hat{A}_P \end{bmatrix} w'_P = \begin{pmatrix} -A_S w'_S \\ 0 \end{pmatrix}$$

i.e., (by Lemma 9.1) iff whenever

$$\begin{bmatrix} \lambda_1[\xi_1,\cdots,\xi_n] & \lambda_2[\xi_1,\cdots,\xi_n] \end{bmatrix} \begin{bmatrix} A_P \\ \hat{A}_P \end{bmatrix} = \begin{pmatrix} 0_P \end{pmatrix}$$

(equivalently whenever $\lambda_1[\xi_1,\cdots,\xi_n]A_S \in \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$),

we also have $[\lambda_1[\xi_1,\cdots,\xi_n], A_S w'_S]=0$ (equivalently $[\lambda_1[\xi_1,\cdots,\xi_n]A_S, w'_S]=0$),

i.e., iff whenever $\lambda_1[\xi_1,\cdots,\xi_n]A_S \in \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$, we also have $[\lambda_1[\xi_1,\cdots,\xi_n]A_S, w'_S] = 0$,

i.e., iff $f_S \in \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ implies $[f_S, w'_S]=0$ (since every vector in $\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ is a linear combination of the rows of A_S). Thus $\mathcal{B}(\mathcal{C}_{SP}) \leftrightarrow \mathcal{B}(\mathcal{C}_P) = \mathcal{B}(\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P)$.

$\square$

We now have the following useful corollary.

Corollary 9.1 *Let $\mathcal{C}_{SP}$ be a module on $S \uplus P$. Then*
(a) $\mathcal{B}(\mathcal{C}_{SP} \cdot S) = (\mathcal{B}(\mathcal{C}_{SP})) \times S$ (b) $\mathcal{B}(\mathcal{C}_{SP} \times S) = (\mathcal{B}(\mathcal{C}_{SP})) \cdot S$.

This result follows from Theorem 9.1 by noting that $\mathcal{C}_{SP} \cdot S = (\mathcal{C}_{SP} \leftrightarrow I_P)$, where I_P is the module of all vectors on P over $\Re[\xi_1, \cdots, \xi_n]$, that $\mathcal{C}_{SP} \times S = (\mathcal{C}_{SP} \leftrightarrow O_P)$, where O_P is the module consisting of the zero vector on P over $\Re[\xi_1, \cdots, \xi_n]$, that if $\mathcal{B}(\mathcal{C}_{SP})$ is a collection of vectors of the form (w_S, w_P), then $(\mathcal{B}(\mathcal{C}_{SP})) \times S$ is the collection of all vectors (w_S), where $(w_S, 0_P)$ is in $\mathcal{B}(\mathcal{C}_{SP})$ and $(\mathcal{B}(\mathcal{C}_{SP})) \cdot S = \mathcal{B}(\mathcal{C}_{SP})/w_S$ (i.e., the collection of all vectors w_S, where there exists w_P such that (w_S, w_P) is in $\mathcal{B}(\mathcal{C}_{SP})$). It is also necessary to remember that $\mathcal{B}(I_P)$ is the singleton $\{(0_P)\}$ and $\mathcal{B}(O_P)$ is the collection of all vectors (w_P) over $\Re[\xi_1, \cdots, \xi_n]$.

Next, let us define a module $\mathcal{C}_S$ over $\Re[\xi_1, \cdots, \xi_n]$ to be 'generalized unimodular' iff $\mathcal{C}_S^{\perp\perp} = \mathcal{C}_S$. It is clear that $\mathcal{C}_S^{\perp}$ is always generalized unimodular. We now state and prove the generalization of Theorem 5.1.

Theorem 9.2 *(The Implicit Duality Theorem (IDTND)) Let $\mathcal{C}_{SP}, \mathcal{C}_P$ be modules on $S \uplus P, P$ respectively, such that $\mathcal{C}_{SP} + \mathcal{C}_P$ is generalized unimodular. Then,*
(a) $(\mathcal{C}_{SP}^{\perp} \leftrightarrow \mathcal{C}_P^{\perp})^{\perp} = \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$
(b) $\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ is generalized unimodular.
If, in addition, $\mathcal{C}_{SP}, \mathcal{C}_P$, $\mathcal{C}_{SP}^{\perp} + \mathcal{C}_P^{\perp}$ are generalized unimodular, then
(c) $(\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P)^{\perp} = \mathcal{C}_{SP}^{\perp} \leftrightarrow \mathcal{C}_P^{\perp}$
(d) $\mathcal{C}_{SP}^{\perp} \leftrightarrow \mathcal{C}_P^{\perp}$ is generalized unimodular.

The proof of this theorem requires the use of the following lemma.

Lemma 9.2 *Let A be an $m \times n$ matrix over $\Re[\xi_1, \cdots, \xi_n]$, whose column module is generalized unimodular, b an $m \times 1$ vector over $\Re[\xi_1, \cdots, \xi_n]$. Then $Ax = b$ has a solution (with x a vector over $\Re[\xi_1, \cdots, \xi_n]$) iff for every $(1 \times m)$ vector λ^T over $\Re[\xi_1, \cdots, \xi_n]$ we have $\lambda^T A = 0 \Rightarrow \lambda^T b = 0$.*

Proof :- $Ax = b$ has a solution iff b belongs to the column module $\mathcal{C}$ generated by columns of A,
i.e., iff $b \in \mathcal{C}^{\perp\perp}$ ($\mathcal{C} = \mathcal{C}^{\perp\perp}$ since column module of A is generalized unimodular),

i.e., iff whenever $\lambda^T \in \mathcal{C}^\perp$ we also have $\lambda^T b = 0$,

i.e., iff whenever $\lambda^T A = 0$, we also have $\lambda^T b = 0$.

$\square$

Proof of Theorem 9.2

Let $\mathcal{C}_{SP}$ have the generator matrix $[A_S \quad A_P]$ and let $\mathcal{C}_P$ have the generator matrix $[\hat{A}_P]$. Then $\mathcal{C}_{SP} + \mathcal{C}_P$ is generated by the rows of the generator matrix

$$\begin{bmatrix} A_S & A_P \\ 0 & \hat{A}_P \end{bmatrix}.$$

Further we have $\mathcal{C}_{SP} + \mathcal{C}_P$ generalized unimodular by hypothesis. Now $x_S \in \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ iff $\exists \lambda_1, \lambda_2$ over $\Re[\xi_1, \cdots, \xi_n]$) s.t.

$$\begin{bmatrix} {A_S}^T & 0 \\ {A_P}^T & \hat{A}_P^T \end{bmatrix} \begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix} = \begin{pmatrix} x_S \\ 0 \end{pmatrix}$$

i.e., (by Lemma 9.2, since column module of the above matrix is generalized unimodular) iff whenever, for y_S, y_P over $\Re[\xi_1, \cdots, \xi_n]$) we have

$$\begin{bmatrix} y_S^T & y_P^T \end{bmatrix} \begin{bmatrix} A_S^T & 0 \\ A_P^T & \hat{A}_P^T \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

we also have

$$y_S^T x_S = 0;$$

i.e., iff whenever $(y_S^T \; y_p^T) \in \mathcal{C}_{SP}^\perp, \quad y_P^T \in \mathcal{C}_P^\perp$ we also have $y_S^T x_S = 0$;

i.e., iff whenever $y_S \in \mathcal{C}_{SP}^\perp \leftrightarrow \mathcal{C}_P^\perp$ we have ${y_S}^T x_S = 0$.

Thus, $x_S \in \mathcal{C}_{SP} \leftrightarrow \mathcal{C}_P$ iff $x_S \in (\mathcal{C}_{SP}^\perp \leftrightarrow \mathcal{C}_P^\perp)^\perp$. This proves (a).

The statement in (b) follows from (a) and the definition of generalized unimodularity.

(c) Suppose $\mathcal{C}_{SP}^\perp + \mathcal{C}_P^\perp$ is generalized unimodular. We can now use (a) above and conclude that

$$(\mathcal{C}_{SP}^{\perp\perp} \leftrightarrow \mathcal{C}_{P}^{\perp\perp})^{\perp} = \mathcal{C}_{SP}^{\perp} \leftrightarrow \mathcal{C}_{P}^{\perp}$$

But $\mathcal{C}_{SP}, \mathcal{C}_{P}$ are also given to be generalized unimodular and so $\mathcal{C}_{SP}^{\perp\perp} = \mathcal{C}_{SP}$ and $\mathcal{C}_{P}^{\perp\perp} = \mathcal{C}_{P}$.

Hence $(\mathcal{C}_{SP} \leftrightarrow \mathcal{C}_{P})^{\perp} = \mathcal{C}_{SP}^{\perp} \leftrightarrow \mathcal{C}_{P}{}^{\perp}$.

Statement (d) follows from (c) and the definition of generalized unimodularity.

$\square$

We now describe explicitly, our construction of adjoints for N-D systems. As can be seen, this construction is not perfectly general but is valid only for those with a kernel representation of a special kind. Later we indicate how to carry out the construction for N-D systems using generalized unimodularity rather than unimodularity.

Let the behaviour $\mathcal{B}$ have the kernel representation, with the **coefficient matrix having linearly independent rows**:

$$\begin{bmatrix} R_{1u} & R_{1l} & 0 \\ R_{2u} & R_{2l} & I \end{bmatrix} \begin{matrix} u \\ l \\ y \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Let the adjoint $\mathcal{B}^{adj2}$ be defined through the representation

$$\begin{bmatrix} I & -R_{1u}^{T} & -R_{2u}{}^{T} \\ 0 & -R_{1l}^{T} & -R_{2l}^{T} \end{bmatrix} \begin{matrix} \hat{u} \\ \hat{l} \\ \hat{y} \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Observe that, while $u, \hat{u}$, and $y, \hat{y}$ have the same number of entries, $l, \hat{l}$ may not have.

Using (Corollary 9.1), we observe that $\mathcal{B}/ul$ has the kernel representation

$$\begin{bmatrix} R_{1u} & R_{1l} \end{bmatrix} \begin{matrix} u \\ l \end{matrix} = 0$$

and that $\mathcal{B}^{adj2}/\hat{y}\hat{l}$ has the kernel representation

$$\left[\begin{array}{cc} -R_{1l}^T & -R_{2l}^T \end{array} \right] \begin{array}{c} \hat{y} \\ \hat{l} \end{array} = 0.$$

We note that the original behaviour is invariant if in the kernel representation, a linear combination of the first set of rows is added to the second set of rows. It follows that the adjoint is not uniquely defined but depends on the original representation.

Let us say that a behaviour $\mathcal{B}(\mathcal{C})$ is *N-D controllable* iff $\mathcal{C}$ is unimodular and that $\mathcal{B}(\mathcal{C})$ is *N-D l-observable* iff in a generator matrix of $\mathcal{C}$, the columns corresponding to the variables l generate a unimodular module. Then the adjoint does have the following attractive properties which can be proved through the Implicit Duality Theorem (IDTND) (Theorem 9.2).

Theorem 9.3 *1. $\mathcal{B}/ul$ is N-D controllable $\equiv \mathcal{B}^{adj2}$ is N-D $\hat{l}$-observable.*

2. $\mathcal{B}$ is N-D l-observable $\equiv \mathcal{B}^{adj2}/\hat{y}\hat{l}$ is N-D controllable.

3. If $\mathcal{B}/ul$ is N-D controllable then $(\mathcal{C}(\mathcal{B}^{adj2}/\hat{u}\hat{y}))^{\perp} = \mathcal{C}(\mathcal{B}/uy)$.

4. If $\mathcal{B}$ is N-D l-observable then $(\mathcal{C}(\mathcal{B}/uy))^{\perp} = \mathcal{C}(\mathcal{B}^{adj2}/\hat{u}\hat{y})$.

Proof:- LHS and RHS of part (1) are equivalent to the condition that $(R_{1u}R_{1l})$ is a representative matrix of a unimodular module and LHS and RHS of part (2) are equivalent to the condition that $(R_{1l}^T R_{2l}^T)$ is a representative matrix of a unimodular module.

We now prove parts (3) and (4).

Using additional variables l' the kernel representation of $\mathcal{B}$ may be re-written as follows:

$$\begin{pmatrix} R_{1u} & 0 & I & 0 \\ R_{2u} & R_{2l} & 0 & I \end{pmatrix} \begin{array}{c} u \\ l \\ l' \\ y \end{array} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{15}$$

$$\begin{bmatrix} 0 & -R_{1l} & I & 0 \end{bmatrix} \begin{array}{c} u \\ l \\ l' \\ y \end{array} = 0 \tag{16}$$

The kernel representation of $\mathcal{B}^{adj2}$ may be rewritten as follows:-

$$\begin{pmatrix} I & 0 & -R_{1u}^T & -R_{2u}^T \\ 0 & I & 0 & -R_{2l}^T \end{pmatrix} \begin{array}{c} \hat{u} \\ \hat{l} \\ \hat{l}' \\ \hat{y} \end{array} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{17}$$

$$\begin{bmatrix} 0 & I & R_{1l}^T & 0 \end{bmatrix} \begin{array}{c} \hat{u} \\ \hat{l} \\ \hat{l}' \\ \hat{y} \end{array} = 0 \tag{18}$$

Let $\mathcal{C}_{VLL'Y}$, be the module spanned by the rows of the coefficient matrix in Equation 15, $\mathcal{C}_{LL'}$ be the module spanned by the rows of $[-R_{1l}I]$ in Equation 16. It can be seen that $\mathcal{B}/uy = \mathcal{B}(\mathcal{C}_{VLL'Y}) \longleftrightarrow \mathcal{B}(\mathcal{C}_{LL'})$. By Theorem 9.1, and by the fact that the module associated with $\mathcal{B}$ is unique, it follows that $\mathcal{C}(\mathcal{B}/uy) = \mathcal{C}(\mathcal{B}(\mathcal{C}_{VLL'Y}) \longleftrightarrow \mathcal{B}(\mathcal{C}_{LL'})) = \mathcal{C}_{VLL'Y} \longleftrightarrow \mathcal{C}_{LL'}$.

Clearly $\mathcal{C}_{VLL'Y}{}^{\perp}$ is spanned by the rows of the coefficient matrix in equation 17 and $\mathcal{C}_{LL'}^{\perp}$ is spanned by the rows of (IR_{1l}^T) in Equation 18.

It can be seen that $\mathcal{B}^{adj2}/\hat{u}\hat{y} = \mathcal{B}(\mathcal{C}_{VLL'Y}^{\perp}) \longleftrightarrow \mathcal{B}(\mathcal{C}_{LL'}^{\perp})$. By Theorem 9.1 and the fact that there is a unique module associated with a behaviour which has a kernel representation, it follows that $\mathcal{C}(\mathcal{B}^{adj2}/\hat{u}\hat{y}) = \mathcal{C}(\mathcal{B}(\mathcal{C}_{VLL'Y}^{\perp}) \longleftrightarrow \mathcal{B}(\mathcal{C}_{LL'}^{\perp})) = \mathcal{C}_{VLL'Y}^{\perp} \longleftrightarrow \mathcal{C}_{LL'}^{\perp}$.

Clearly $\mathcal{C}_{LL'}, \mathcal{C}_{VLL'Y}, \mathcal{C}_{LL'}{}^{\perp}, \mathcal{C}_{VLL'Y}{}^{\perp}$ are generalized unimodular modules since they have representative matrices with full rank identity submatrices.

The coefficient matrix for Equations 15, 16, taken together is generates a unimodular (and therefore a generalized unimodular) module if $(R_{1u}R_{1l})$ generates a unimodular module. If this holds, by the Implicit Duality Theorem (IDTND)(Theorem 9.2)

$$(\mathcal{C}^{\perp}{}_{VLL'Y} \leftrightarrow \mathcal{C}^{\perp}{}_{LL'})^{\perp} = \mathcal{C}_{VLL'Y} \leftrightarrow \mathcal{C}_{LL'}.$$

Since $\mathcal{C}_{VLL'Y} \leftrightarrow \mathcal{C}_{LL'}$ is the module associated with $\mathcal{B}/uy$ and $\mathcal{C}_{VLL'Y}{}^{\perp} \leftrightarrow \mathcal{C}_{LL'}{}^{\perp}$ is the module associated with $\mathcal{B}^{adj2}/\hat{u}\hat{y}$, it follows that

$$(\mathcal{C}(\mathcal{B}^{adj2}/\hat{u}\hat{y}))^{\perp} = \mathcal{C}(\mathcal{B}/uy).$$

Similarly, the coefficient matrix for Equations 17, 18, taken together generates a unimodular (and therefore a generalized unimodular) module if $(-R_{1l}^{T} - R_{2l}^{T})$ generates a unimodular module. If this holds, by IDTND, we have

$$(\mathcal{C}^{\perp\perp}_{VLL'Y} \leftrightarrow \mathcal{C}^{\perp\perp}_{LL'})^{\perp} = \mathcal{C}^{\perp}_{VLL'Y} \leftrightarrow \mathcal{C}^{\perp}_{LL'}$$

i.e. we have (by the generalized unimodularity of $\mathcal{C}_{VLL'Y}, \mathcal{C}_{LL'}$)

$$(\mathcal{C}_{VLL'Y} \leftrightarrow \mathcal{C}_{LL'})^{\perp} = \mathcal{C}^{\perp}_{VLL'Y} \leftrightarrow \mathcal{C}^{\perp}_{LL'}$$

$$\text{i.e., } (\mathcal{C}(\mathcal{B}/uy))^{\perp} = \mathcal{C}(\mathcal{B}^{adj2}/\hat{u}\hat{y}).$$

□

Let us now consider the situation where 'generalized unimodularity' replaces 'unimodularity'. We notice that our construction of adjoints for N-D systems using unimodularity hinges on the following:

- The fact that (through the kernel representation) there is a unique module $\mathcal{C}(\mathcal{B})$ associated with a behaviour $\mathcal{B}$ (which is known to have a kernel representation) and a unique behaviour $\mathcal{B}(\mathcal{C})$ associated with a module $\mathcal{C}$.
- The validity of Theorem 9.1
- The validity of the Implicit Duality Theorem (Theorem 9.2).
- If R is row g-unimodular then
$$\begin{bmatrix} R & 0 \\ K & I \end{bmatrix}$$
is also row g-unimodular.

We therefore have to appropriately modify only the last condition.The following lemma supplies what is needed.

Lemma 9.3 *If R is the generator matrix of a generalized unimodular module then*

$$\begin{bmatrix} R & 0 \\ K & I \end{bmatrix}$$

is also the generator matrix of a generalized unimodular module.

Proof: By Lemma 8.4, we need to show that if $\alpha f, \alpha \neq 0$, is linearly dependent on the rows of

$$\begin{bmatrix} R & 0 \\ K & I \end{bmatrix},$$

then so is f. Let $f \equiv (f_1, f_2)$ corresponding to the column partition of the above matrix. We then have $\alpha f_1 = y^T R + z^T K$ and $z^T I = \alpha f_2$, for suitable y, z and nonzero α. Clearly we need only show that for some q, we have $f_1 = q^T R + f_2 K$. Now, $\alpha(f_1 - f_2 K) = y^T R$. Since R is the generator matrix of a generalized unimodular module it follows that $(f_1 - f_2 K) = q^T R$ for some q^T. Thus $f_1 = q^T R + f_2 K$ as required.

□

We can begin by assuming that the original behaviour $\mathcal{B}$ has the kernel representation

$$\begin{bmatrix} R_{1u} & R_{1l} & 0 \\ R_{2u} & R_{2l} & I \end{bmatrix} \begin{matrix} u \\ l \\ y \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where R_{1u} R_{1l} generates a generalized unimodular module. Then we could define the adjoint behavior $\mathcal{B}^{adj3}$ to have the kernel representation

$$\begin{bmatrix} I & -R_{1u}^T & -{R_{2u}}^T \\ 0 & -R_{1l}^T & -R_{2l}^T \end{bmatrix} \begin{matrix} \hat{u} \\ \hat{l} \\ \hat{y} \end{matrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Further we need to define 'N-D generalized controllable' behaviours to be those whose associated module is generalized unimodular and to define a behaviour $\mathcal{B}$ to be 'N-D generalized l- observable' iff the corresponding columns in a generator matrix of the associated module generate a generalized unimodular module. This definition of observability is ofcourse quite unsatisfactory but, under the above definitions,the following desirable properties of 'adjoint' turn out to be true

- $\mathcal{B}/ul$ is N-D generalized controllable $\equiv \mathcal{B}^{adj3}$ is N-D generalized $\hat{l}$-observable.

- $\mathcal{B}$ is N-D generalized l-observable $\equiv \mathcal{B}^{adj3}/\hat{y}\hat{l}$ is N-D generalized controllable.

- If $\mathcal{B}/ul$ is N-D generalized controllable then $(\mathcal{C}(\mathcal{B}^{adj3}/\hat{u}\hat{y}))^{\perp} = \mathcal{C}(\mathcal{B}/uy)$.

- If $\mathcal{B}$ is N-D generalized l-observable then $(\mathcal{C}(\mathcal{B}/uy))^{\perp} = \mathcal{C}(\mathcal{B}^{adj3}/\hat{u}\hat{y})$.

The proof is a line by line translation of that of Theorem 9.3, using 'generalized unimodularity' in place of 'unimodularity', 'N-D generalized controllable' in place of 'N-D controllable' and 'N-D generalized l-observable' in place of 'N-D l-observable'.