DSP & ITS APPLICATIONS

- Signal – any function of an independent variable — typically time and typically which is encountered in real experience.

- Processing → Extraction, → Interpretation, → Modification

- Processing is done with computing devices

- Computing devices are digital computers

- ∴ we need to convert continuous time signal to digital form
⇒ ~~E~~ Sample + Quantize + Encode

- Way to look at signal
  1) Natural domain — time domain
  2) $\underbrace{\text{Other domain}}_{\text{Tranform domain}}$ — frequency domain

Tranform domain
- Fourier Transform
- Z-Transform

Goal — Disign DT Filters and realize them

{ stable, rational LSI s/m

Bridge bet^n description & realization

Z domain - - - - - → ↑

Appropriate description (Transforms)

DTFT - - - - - → ↑

Discrete time (natural domain)

Sampling - - - - - → ↑

Continuous time

↓

Normalization
(Making analysis independent of sampling rate)

{ Alias → False identity }

SAMPLING

- In process of sampling we loose some information.
- Hence we should maintain a threshold above which we should not loose information. $\Rightarrow$ Avoid Aliasing.

To avoid aliasing
- we should not loose any half cycle bet^n two samples
- If we sample a sinusoid, there should be at least two samples per cycle.
- Sampling interval < one half cycle

$$\therefore \quad T_s < \frac{T_m}{2}$$

$$\therefore \quad \frac{1}{T_s} > \frac{1}{T_m/2}$$

$$\therefore \quad f_s > 2f_m$$ $\longrightarrow$ Nyquist Sampling Thm [Nyquist-Shanon-Whittakor]

1) s/g has to be band limitted
   - It has max. freq. component of f_m
   - ie Fourier Transform is zero after freq. f_m

2) $f_s > 2f_m$

$*$ Consider $\cos(\Omega_0 t + \phi_0)$
sample it with a sampling period $= T$,
ie at $t = nT$; $n \in \mathbb{Z}$ ($\mathbb{Z} \Rightarrow$ set of integers)

$\therefore$ Sequence is

$$x[n] = \cos(\Omega_0 nT + \phi_0) \longrightarrow (1)$$

$\cos\theta = \cos(-\theta)$

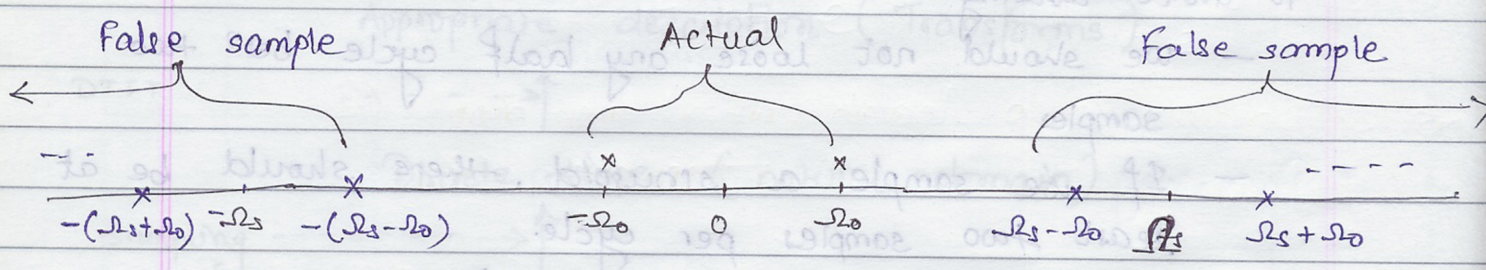
$\cos\theta = \cos(2\pi k + \theta)$, $k \in \mathbb{Z}$

$$\therefore x[n] = \cos(2\pi kn \pm (\Omega_0 nT + \phi_0))$$

$$= \cos\left[\left(\frac{2\pi k}{T} \pm \Omega_0\right) nT \pm \phi_0\right] \longrightarrow (2)$$

From eqⁿ 1 & 2 we get that
for same set of samples (amp and phase) ~~of eqⁿ can~~ with freq Ω_0 can be obtained with freq $\left(\frac{2\pi k}{T} \pm \Omega_0\right)$

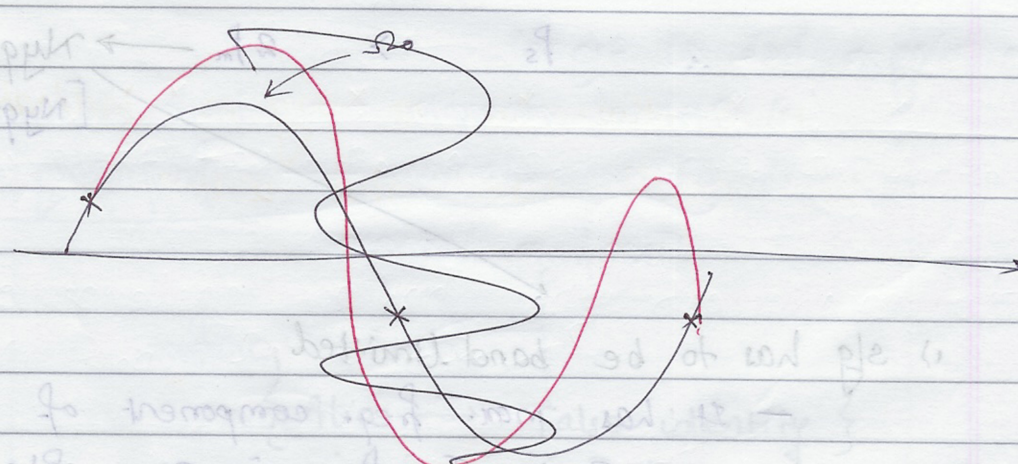
$$= \underbrace{2\pi F_s}_{\Omega_s} k \pm \Omega_0$$

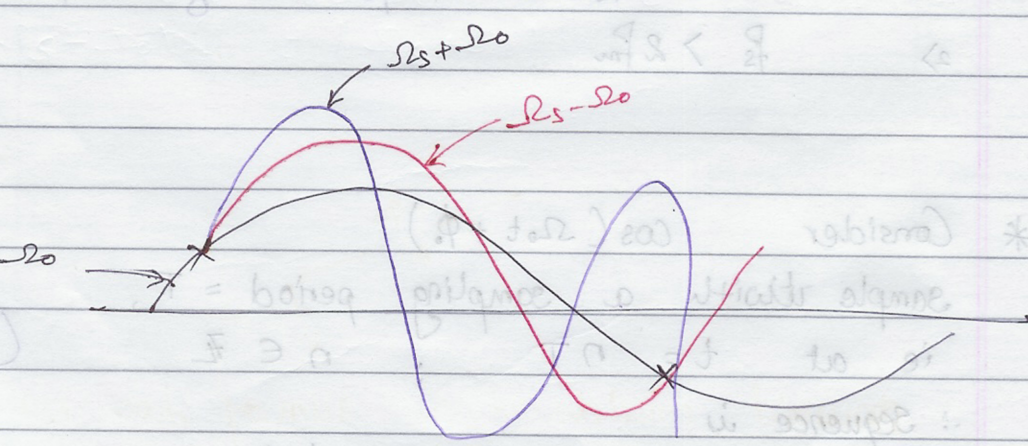


∴ we should have

$$\Omega_s - \Omega_0 > \Omega_0$$

$$\therefore \quad \Omega_s > 2\Omega_0$$





- All sinusoidal freq's of the form

$$\frac{2\pi k}{T} \pm \Omega_0 \qquad \forall k \in \mathbb{Z}$$

have the same samples $x[n]$ at that points $t = nT$ $\forall n \in \mathbb{Z}$

- Consider a sine wave

$$A_0 \cos(\Omega_0 t + \phi_0)$$

Consider a change in amplitude

$$A_1 \cos(\Omega_0 t + \phi_0)$$

$$= \frac{A_1}{A_0} \cdot A_0 \cos(\Omega_0 t + \phi_0)$$

Hence amplitude change can be considered as multiplied by constant

Consider a phase change

$$A_0 \cos(\Omega_0 t + \phi_1)$$

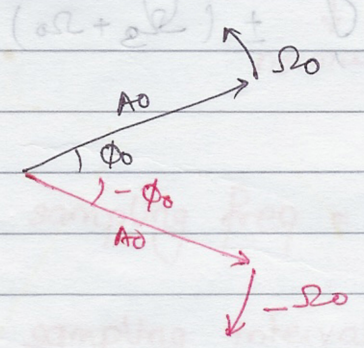
Here there is no way to indicate this change

$\Rightarrow$ We can't use sine wave

- Consider a complex phasor

$$2A_0 \cos(\Omega_0 t + \phi_0)$$

$$= A_0 e^{j(\Omega_0 t + \phi_0)} + A_0 e^{-j(\Omega_0 t + \phi_0)} \qquad \left(\because \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}\right)$$



$\therefore$ sinusoid considered as sum of two rotating phasors

Consider change in Amp

$$A_1 \cos(\Omega_0 t + \phi_0)$$

$\Rightarrow$ Changes length of rotating phasors.

Consider change in phase

$$2A_0 \cos(\Omega_0 t + \phi_1)$$

$$= A_0 e^{j(\Omega_0 t + \phi_1)} + A_0 e^{-j(\Omega_0 t + \phi_1)}$$

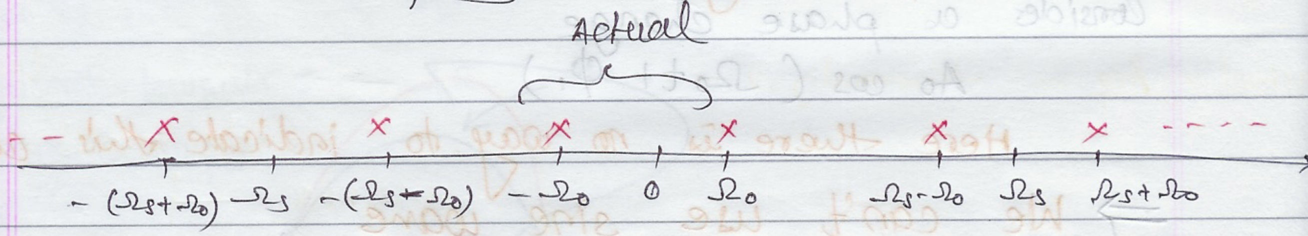
$$= A_0 e^{j(\Omega_0 t + \phi_0)} \cdot \underbrace{e^{j(\phi_1 - \phi_0)}}$$

$$+ A_0 e^{-j(\Omega_0 t + \phi_0)} \cdot \underbrace{e^{-j(\phi_1 - \phi_0)}}$$

⇒ Again can be considered as a multiplication with constant

∴ Instead of sinusoid we prefer phasors.

• Now consider sampling



⇒ Before sampling we had only two phasors with angular freq Ω_0 & $-\Omega_0$.

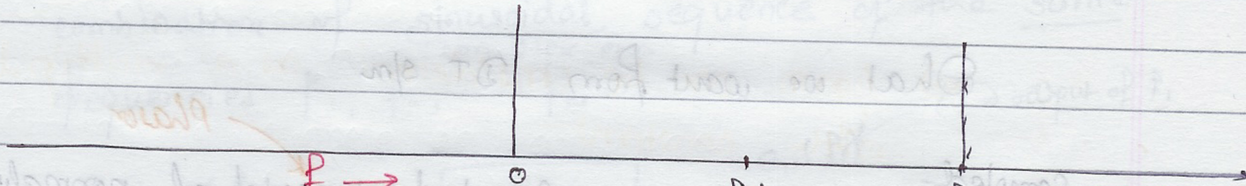
⇒ After sampling we get infinite phasors $\pm\Omega_0$, $\pm(\Omega_s - \Omega_0)$, $\pm(\Omega_s + \Omega_0)$, ...

* Spectrum of DT sig

- It is periodic with period Ω_s (as observed by sampling)

$\therefore$ We should make it independent of Ω_s

$\Rightarrow$ Normalized frequencies.



	$F \rightarrow$	0	$F_s/2$	F_s	(Hz)
	$\Omega \rightarrow$	0	$\Omega_s/2$	Ω_s	(rad/sec)
$\dfrac{\text{actual freq}}{F_s} =$	$F_{normalized} \rightarrow$	0	$1/2$	1	(Hz)
$2\pi * \dfrac{\text{actual ang freq}}{\Omega_s} =$	$\Omega_{normalized} \rightarrow$	0	π	2π	(rad/sec)

$$x[n] = A_0 \cos(\Omega_0 n T + \phi_0)$$

$$= A_0 \cos\left[2\pi\left(\frac{F_0}{1/T}\right)n + \phi_0\right]$$

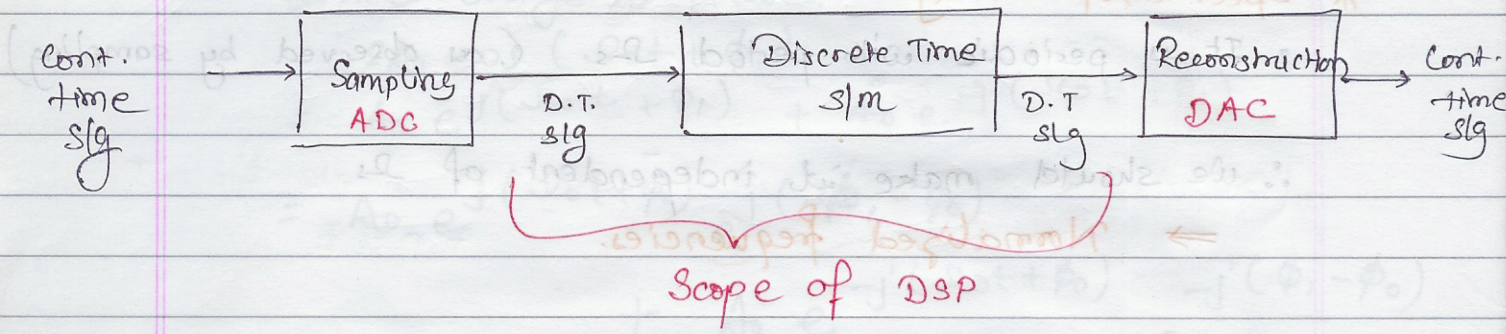
$$= A_0 \cos\left(2\pi \underbrace{\left(\frac{F_0}{F_s}\right)}_{\text{normalized freq.}} n + \phi_0\right)$$

Now, Normalized sampling freq $= \dfrac{F_s}{F_s} = \underline{1}$

$\therefore$ Normalized sampling interval $= 1/1 = \underline{1}$

* Energy of sig

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$



What we want from DT s/m

Sampled sinusoid of angular freq (ω) $\xrightarrow[\text{in}]{\text{result}}$ Sampled sinusoid of normalized angular freq (ω) [Possibly changed in amplitude and phase]

Phasor

Phasor

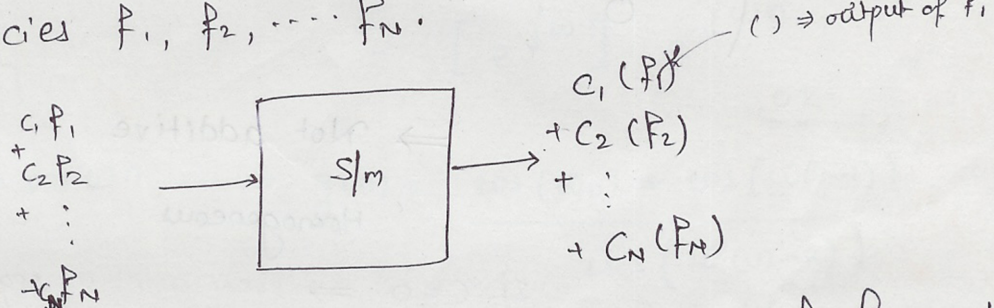
Linear s/m
- Additivity
- Homogeneity (scaling)

Properties of System →

* What we want from s/m

(i) A sinusoidal (complex exp.) sequence i/p of a given freq, to result in a sinusoidal (complex exp.) sequence o/p of the same freq.

(ii) A linear combination of such sinusoidal (complex exp) sequences of frequencies $f_1, f_2, \cdots f_N$ to result in a linear combination of sinusoidal (complex exp) sequence of the same frequencies $f_1, f_2, \cdots f_N$.



[This will help us to specify specification of freq. axis]

* Complex exponential = Phasor

① **Linearity** = Additivity + Homogeneity

Additivity

if $x_{1,2}[n] \longrightarrow y_{1,2}[n]$

then $x_1[n] + x_2[n] \longrightarrow y_1[n] + y_2[n]$

Homogeneity / scaling

if $x_1[n] \longrightarrow y_1[n]$

then $c \cdot x_1[n] \longrightarrow c \cdot y_1[n]$ $\quad \forall x, c$

c can be complex

∴ Linearity

if $x_{1,2}[n] \longrightarrow y_{1,2}[n]$

then $c_1 x_1[n] + c_2 x_2[n] \longrightarrow c_1 y_1[n] + c_2 y_2[n]$

Eg: ① $y[n] = \{x[n]\}^2$ $\Rightarrow$ Non-Linear [not Additive, not Homogeneous]

② $y[n] = \overline{x[n]}$ $\Rightarrow$ Additive

Not Homogeneous $\rightarrow$ c can be complex

$\therefore$ Non-Linear

③ $y[n] = \begin{cases} \dfrac{x[n]\,x[n-1]}{x[n-2]} & \text{if } x[n-2] \neq 0 \\ 0 & \text{if } x[n-2] = 0 \end{cases}$

$\Rightarrow$ Not additive { show by counter example

Homogeneous

If $C = 0$ $\Rightarrow$ every point in seq. is also zero

$\therefore$ TRUE

If $C \neq 0$

$\Rightarrow$ if $x(n-2) = 0$ $\Rightarrow$ $y(n) = 0$

if $x(n-2) \neq 0$

$$c\,x(n) \longrightarrow \frac{c\,x(n)\cdot \cancel{c}\,x(n-1)}{\cancel{c}\,x(n-2)}$$

$$= \frac{c\,x(n)\,x(n-1)}{x(n-2)} = c\,y(n)$$

$\therefore$ Non linear

④ $y(n) = \frac{1}{2}\{x(n) + x(n-1)\}$

$\Rightarrow$ Additive

Homogeneous

$\therefore$ Linear

⑤ $y(n) = \left(\frac{1}{2}\right)^n \{x(n) + x(n-1)\}$

$\Rightarrow$ Additive

Homogeneous

$\therefore$ Linear

next page

$\longrightarrow x(n) = \cos \omega_0 n \Rightarrow y(n) = [\cos \omega_0 n]^2 = \frac{1}{2}[1 + \cos(2\omega_0 n)]$

Not Same

$\longrightarrow x(n) = e^{j\omega n} \longrightarrow y(n) = e^{-j\omega n}$

ω $\quad$ $-\omega$

notsame

$\longrightarrow x(n) = e^{j\omega n} \longrightarrow y(n) = e^{j\omega n} \cdot e^{j\omega(n-1)} \cdot e^{-j\omega(n-2)}$

$= [e^{j\omega}] \cdot e^{j\omega n}$

<u>OK</u>

Let $x(n) = \cos \omega n \longrightarrow y(n) = \dfrac{\cos(\omega n) \cdot \cos(\omega(n-1))}{\cos(\omega(n-2))}$

Not a sinusoid

$\therefore$ Though it obeys for complex exponential but not for sinusoid.

Also it does not obey for sum of complex exponential

$\longrightarrow$ $x(n) = e^{j\omega n} \longrightarrow y(n) = \frac{1}{2}[e^{j\omega n} + e^{j\omega(n-1)}] = \frac{1}{2} e^{j\omega n}(1 + e^{-j\omega})$

<u>OK</u>

Change of amp.

$= |\frac{1}{2}(1 + e^{-j\omega})|$

$= |\frac{1}{2} e^{-j\frac{\omega}{2}}(e^{+j\omega/2} + e^{-j\frac{\omega}{2}})|$

$= |\cos(\omega/2)|$

Change of phase

$= -\omega/2$

$x(n) = \cos \omega n \longrightarrow y(n) = \frac{1}{2}[\cos(\omega n) + \cos \omega(n-1)] = \frac{1}{2} 2\cos\frac{\omega}{2} \cdot \cos \omega(n - 1/2)$

$= \left(\cos\frac{\omega}{2}\right) \cdot \cos\{\omega(n - 1/2)\}$ <u>OK</u>

$\therefore$ amp change $= \cos \omega/2$ $\quad$ phase change $= -\omega/2$

Here both sinusoid and complex exp. result in same amp. change and same phase change. $\Rightarrow$ Obeys properties (i) and (ii) also

$$\rightarrow y(n) = \left(\tfrac{1}{2}\right)^n \left[x(n) + x(n-1)\right]$$

$$e^{j\omega} \longrightarrow y(n) = \left(\tfrac{1}{2}\right)^n \left[e^{j\omega n} + e^{j\omega(n-1)}\right]$$

$$=$$

$$\cos \omega n \longrightarrow y(n) = \left(\tfrac{1}{2}\right)^n \left[\cos \omega n + \cos(\omega(n-1))\right]$$

$$= \left(\tfrac{1}{2}\right)^n 2 \cdot \cos \frac{\omega}{2} \cos \omega(n - 1/2)$$

$$= \left(\tfrac{1}{2}\right)^{n-1} \cos \frac{\omega}{2} \cos \omega(n - 1/2)$$

$\Rightarrow$ Here it disobeys property (i)

Here we are multipling with a exp. term which results that the output is not sinusoidal though input is sinusoidal

This is because this s/m is not

Shift-Invarient

(2) Shift-Invarience

A system is said to be shift invarient iff

$$x[n] \longrightarrow y[n]$$

$$\Rightarrow x[n-D] \longrightarrow y[n-D] \quad \forall x, D$$

$$D \in \mathbb{Z}$$

Eg:

(1) $y[n] = \{x[n]\}^2 \longrightarrow SI$

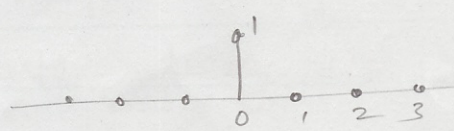
(2) $y[n] = \overline{x[n]}$

(3) $y[n] = \begin{cases} \dfrac{x[n]\,x[n-1]}{x[n-2]} \\ 0 \end{cases}$

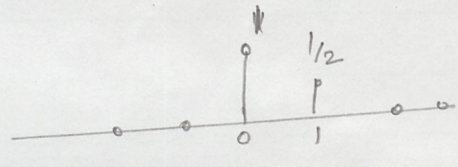
(4) $y(n) = \frac{1}{2}\{x[n] + x[n-1]\} \longrightarrow$ Shift Invarient

(5) $y[n] = \left(\frac{1}{2}\right)^n \{x[n] + x[n-1]\} \longrightarrow$ Shift Varient

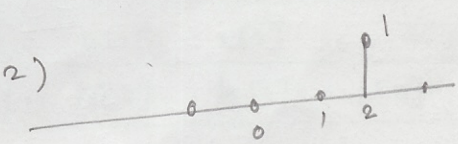
Eg : $x(n) = \delta(n)$



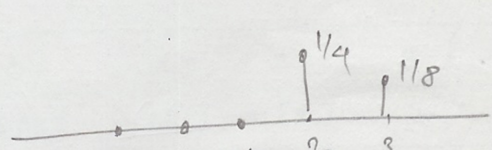
$\therefore y(n) =$



Now consider
$x(n) = \delta(n-2)$



$y_1(n)$



$y_1(n) \neq y(n-2)$

Not Shift Invarient

∴ If we want that we should satisfy property (i) and (ii) we should have s/m

1) Linear (Additive + Homogeneous)

2) Shift Invarient

Excersis: find all eight combination of Additive / Homogeneous / Shift Invarient

- $\delta(n) \longrightarrow \boxed{\text{LSI}} \longrightarrow h(n)$

  unit impulse → $\delta(n)$ Impulse response → $h(n)$

- Any sequence can be written as sum of weighted impulses as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k]$$

- $$x[n] \xrightarrow{\text{LSI}} y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] \rightarrow \text{Convolution}$$

$$= x[n] * h[n]$$

∴ For any LSI s/m if we know the impulse response of s/m we can find o/p $y[n]$ for any input $x[n]$ as convolution of $x[n]$ with $h[n]$

⇒ LSI s/m is completely characterized by its impulse response $h[n]$

Eg:-

Let $n = 3$

k	... −3	−2	−1	0	1	2	3 ...
	...	$x[-2]$	$x[-1]$	$x[0]$	$x[1]$	$x[2]$	...
	...	$h[5]$	$h[4]$	$h[3]$	$h[2]$	$h[1]$	$h[0]$

Let $n = 4$

k	−3	−2	−1	0	1	2	3 ...
	...	$x[-2]$	$x[-1]$	$x[0]$	$x[1]$	$x[2]$	...
	...	$h[6]$	$h[5]$	$h[4]$	$h[3]$	$h[2]$	...

⟹ Convolution is operation

1) Fold
2) Shift
3) Multiply & ~~And~~ Add

*** Convolution is commutative**

ie $x[n] * h[n] = h[n] * x[n]$

or $\sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\cdot x[n-k]$

Response of an LSI s/m to a complex exponential

$$x[n]=A_0 e^{j(\omega_0 n+\phi_0)} \longrightarrow \boxed{\begin{array}{c} LSI \\ h[n] \end{array}} \longrightarrow y[n]$$

$$\therefore\ y[n] = x[n] * h[n] = h[n] * x[n]$$

$$= \sum_{k} h[k]\cdot x[n-k]$$

$$= \sum_{k} h[k]\cdot A_0 e^{j(\omega_0(n-k)+\phi_0)}$$

$$= \underbrace{A_0\cdot e^{j(\omega_0 n+\phi_0)}}_{x[n]} \cdot \underbrace{\left\{\sum_{k} h[k]\cdot e^{-j\omega_0 k}\right\}}_{\text{Constant for particular freq}}$$

$$= H(\omega_0)$$

$$= x[n]\cdot H(\omega_0)$$

$\Rightarrow$ If $|H(\omega_0)| < \infty$

then a complex exponential input $A_0 e^{j(\omega_0 n+\phi_0)}$ has resulted in complex exponential ~~at~~ output of the same frequency scaled by $H(\omega_0)$

ie freq is ~~sa~~ same

only Amplitude and phase have changed

change in amplitude $= |H(\omega_0)|$

change in phase $= \angle H(\omega_0)$

Here we need $H(\omega_0)$ to converge for $\underbrace{-\pi < \omega_0 < \pi}$

since $H(\omega_0)$ is periodic with period 2π

$$H(\omega_0) = \sum_k h[k] \cdot e^{-j\omega_0 k}$$

$$\therefore H(\omega_0+2\pi) = \sum_k h[k] \cdot e^{-j(\omega_0+2\pi)k}$$

$$= \sum_k h[k] \cdot e^{-j\omega_0 k} \cdot \underbrace{e^{-j2\pi k}}_{=1}$$

$$= \sum_k h[k] \cdot e^{-j\omega_0 k}$$

② Response of LSI s/m to $\underbrace{\text{sum of}}$ complex exponential

since s/m is linear

$\Rightarrow$ It will satisfy

③ Response to sinusoid $\longrightarrow$

Sinusoid is sum of two exponential phaser

$\Rightarrow$ It will satisfy *

~~Note~~

Note $\rightarrow$

① If $h[n]$ is real $\Rightarrow$ $h[n] = \overline{h[n]}$

then $\boxed{H(-\omega_0) = \overline{H(\omega_0)}}$

$\Rightarrow$ at $-\omega_0$, mag $= |H(\omega_0)|$

phase $= -\angle H(\omega_0)$

$\Rightarrow$ Mag. response is symmetric

phase Response is antisymmetric

(4) Response to sum of sinusoids

s/m is linear

$\Rightarrow$ It will satisfy

$$A_1 \cos(\omega_0 n + \phi_0) = \frac{A_1}{2}\left[e^{j(\omega_0 n + \phi_0)} + e^{-j(\omega_0 n + \phi_0)}\right]$$

$$\therefore y[n] = H(\omega_0) \cdot \frac{A_1}{2} e^{j(\omega_0 n + \phi_0)} + H(-\omega_0) \cdot \frac{A_1}{2} e^{-j(\omega_0 n + \phi_0)}$$

$$= H(\omega_0) \cdot \frac{A_1}{2} e^{j(\omega_0 n + \phi_0)} + \overline{H(\omega_0)} \frac{A_1}{2} e^{-j(\omega_0 n + \phi_0)}$$

$$= 2\,\mathrm{Re}\left\{ H(\omega_0) \frac{A_1}{2} e^{j(\omega_0 n + \phi_0)} \right\}$$

$(x+jy) + (x-jy) = 2x$
$= 2[\text{Real Part}]$

$$= 2\,\mathrm{Re}\left\{ \underset{|H(\omega_0)| \measuredangle H(\omega_0)}{\underset{\downarrow}{H(\omega_0)}} \frac{A_1}{2} \left(\cos(\omega_0 n + \phi_0) + j\sin(\omega_0 n + \phi_0)\right)\right\}$$

$$= A_1 \underbrace{|H(\omega_0)|} \cdot \cos(\omega_0 n + \phi_0 + \underbrace{\measuredangle H(\omega_0)})$$

Indicates change in amplitude

Indicates change in phase.

(*) We want $H(\omega_0)$ to converge for DTFT to hold true

$\therefore$ Let us now consider the subclass of LSI s/m where $H(\omega_0)$ converges for all ω_0.

$$H(\omega_0) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega_0 n}$$

Consider
$$|H(\omega_0)| = \left| \sum_{n} h[n]\, e^{-j\omega_0 n} \right|$$

$$\leq \sum_{n} \left| h[n]\, e^{-j\omega_0 n} \right|$$

$\therefore |\alpha + \beta| \leq |\alpha| + |\beta|$

$$\leq \underbrace{\sum_{n} |h[n]|}_{\text{Absolute Sum of impulse response}}$$

$|e^{-j\omega_0 n}| = 1$

This is a sufficient condition $\Rightarrow$ $H(\omega_0)$ will converge if absolute sum of impulse responce is finite

* What we say about the LSI s/m when $\sum_{n} |h[n]| < \infty$

$$x[n] \longrightarrow \boxed{\text{LSI s/m}}_{h[n]} \longrightarrow \sum_{k} h[k]\cdot x[n-k] = y[n]$$

Now

$$|y[n]| = \left| \sum_{k} h[k]\cdot x[n-k] \right|$$

$$\leq \sum_{k} |h[k]\cdot x[n-k]|$$

$$\leq \sum_{k} |h[k]| \cdot |x[n-k]|$$

Suppose ~~x~~ i/p is bounded ie $|x[n]| \leq M_x < \infty \;\; \forall n$

$$\therefore |y[n]| \leq \sum_{k} |h[k]| \cdot M_x$$

$$\leq M_x \sum |h[k]|$$

also $\sum_{n} |h[n]| < \infty$

$\Rightarrow$ output $y[n]$ is also bounded

$\Rightarrow$ Bounded i/p gives bounded o/p

$\Rightarrow$ s/m is <u>BIBO Stable</u>

<u>Stable s/m</u>

A stable s/m is one for which every bounded input produces a bounded output.

Eg ① $y[n] = \{x(n)\}^2$

$\Rightarrow$ stable s/m

$|x[n]| \leq M_x < \infty \longrightarrow |y[n]| \leq M_y = M_x^2 < \infty$

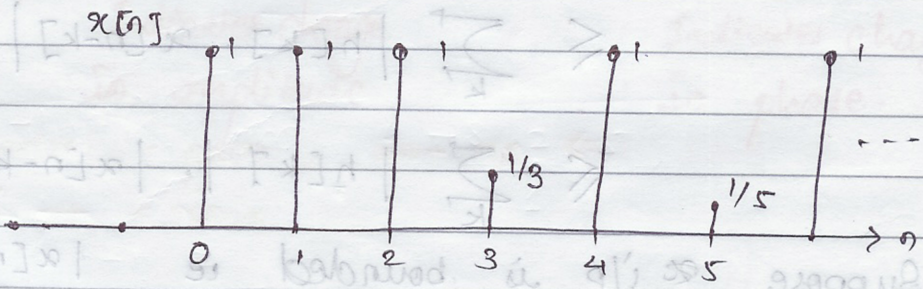
② $y[n] = \begin{cases} \dfrac{x[n]}{x[n-1]} & \text{if } x[n-1] \neq 0 \\ 0 & \text{else} \end{cases}$

as $x[n-1] \longrightarrow 0$

$y[n] \longrightarrow \infty$

$\Rightarrow$ Unstable s/m

eg $x[n] = \begin{cases} 0 & \forall n < 0 \\ 1 & n = 0 \\ 1 & n > 0, \text{ even} \\ 1/n & n > 0, \text{ odd} \end{cases}$



$\Rightarrow$ Bounded i/p

n	-1	0	1	2	3	4	5	6	
o/p $y[n]$	0	0	1	1	1/3	3	1/5	5	1/7 ...

③ $y[n] = x[n] \cdot x[n-1]$

$\Longrightarrow$ stable s/m

④ $h[n] = (1/2)^n u[n]$

$$\sum_n |h[n]| = \sum_{n=-\infty}^{\infty} \left| (1/2)^n u(n) \right|$$

$$= \sum_{n=0}^{\infty} (1/2)^n$$

$$= \frac{1}{1 - 1/2}$$

$$= \underline{\underline{2}}$$

$\Rightarrow$ stable

⑤ $h[n] = 2^n u[n]$

$$\sum_n |h[n]| = \infty$$

$\Longrightarrow$ Unstable

Exersice — Show that (i) Additivity
(ii) Homogenity
(iii) Shift Invarience
(iv) Stability

} all are independent

Eg s/m is ~~stable~~ bounded + I/p is Bounded

$\longrightarrow$ ~~still o/p is unbounde~~ still s/m is unstable

$h[n] = u[n]$

$x[n] = u[n]$

$y[n] = x[n] * h[n] = (1+n) u[n] \Longrightarrow$ unbounded.

Prob

① Find $y[n] = x[n] * h[n]$, $x[n] = u[n]$, $h[n] = u[n]$

Graphically —

k		-2	-1	0	1	2		
$x[k]$	...	0	0	1	1	1	....	
$h[k]$	...	0	0	1	1	1	....	
$h[-k]$	...	1	1	1	0	0	....	$\Rightarrow y[0] = 1$
$h[1-k]$	...	1	1	1	1	0	....	$\Rightarrow y[1] = 2$
$h[2-k]$	...	1	1	1	1	1	0 ...	$\Rightarrow y[2] = 3$
$h[-1-k]$	...	1	0	0	0	0		$\Rightarrow y[-1] = 0$
$h[-2-k]$	... 1	0	0	0	0	0	...	$\Rightarrow y[-2] = 0$

$\therefore\ y[n] = \{\ldots 0, 0, \underset{\uparrow}{1}, 2, 3, \ldots\}$

$= (n+1)u[n]$

Algebrically —

$$y[n] = \sum_k x[k]\,h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} u[k]\cdot u[n-k]$$

$$u[k] = \begin{cases} 1 & \forall\, n \geq 0 \\ 0 & \text{else} \end{cases}$$

$$u[n-k] = \begin{cases} 1 & \forall\, n-k \geq 0 \Rightarrow k \leq n \\ 0 & \text{else} \end{cases}$$

$$= \left[\sum_{k=0}^{n} (1)\right] u[n]$$

$$= (n+1)u[n]$$

Exercise ① $\{\underset{\substack{\uparrow\\0}}{3}, -1, 7\}$ and $\{-8, \underset{\substack{\uparrow\\1}}{2}, 5\}$

	3	−1	7
→ −8	−24	+8	−56
2	6	−2	14
5	15	−5	35

$y[n] = \{-24, \underset{\uparrow}{14}, -43, 9, 35\}$

* $x[n]$ is non-zero only for $N_1 \leq n \leq N_2$
and $h[n]$ is non-zero only for $N_3 \leq n \leq N_4$
$\rightarrow$ $y[n]$ is non-zero only for $\underline{?}$

$\longrightarrow$

Product is non-zero when

$$N_1 \leq k \leq N_2 \quad \text{—(1)}$$

and

$$N_3 \leq n-k \leq N_4 \quad \text{—(2)}$$

Adding

$$N_1 + N_3 \leq n \leq N_2 + N_4$$

$\Rightarrow$ $y[n]$ is non-zero only for $(N_1+N_4) \leq n \leq N_2 + N_4$

*
(1) right sided * right sided $\Rightarrow$ convolution is right sided
(2) left " * left $\Rightarrow$ " " left "
(3) right " * left $\Rightarrow$ " both "
(4) lef " * right $\Rightarrow$ " both "
(5) one seq. finite * other infinite $\Rightarrow$ Not necessary of infinite length

eg: $x[n] = u(n)$, $h[n] = \{1, -1\}$
$\Rightarrow$ o/p $= y[n] = \delta[n]$

(6) Both are infinite $\Rightarrow$

* If an LSI s/m has an absolutely summable impulse response, the LSI s/m is stable
$\rightarrow$ This is a sufficient condition for stability
Is ~~of~~ ab-solute summability necessary for stability?

Proof

Let an LSI s/m have ~~its~~ impulse response $h[n]$

$$\therefore y[n] = \sum_k x[k] \cdot h[n-k]$$

Consider

$$y[0] = \sum_k x[k] \cdot h[-k]$$

Define

$$x[n] = \begin{cases} \dfrac{\overline{h[-n]}}{|h[-n]|} & \text{if } h[-n] \neq 0 \\ 0 & \text{else} \end{cases}$$

$$\therefore \; y[0] = \sum_{\substack{k=-\infty \\ h[k]\neq 0}}^{\infty} h[k] \cdot \frac{\overline{h[k]}}{|h[k]|}$$

$$= \sum_{\substack{k=-\infty \\ h[k]\neq 0}}^{\infty} \frac{|h[k]|^2}{|h[k]|}$$

$$= \sum_{k=-\infty}^{\infty} |h[k]|$$

If the s/m is stable, then $y[0]$ must be finite because $x[n]$ is finite i.e. $x[n]$ is bounded

$$\Rightarrow \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

$\Rightarrow$ Absolute Summability is a necessary condition

Theorem An LSI system is stable if and only if its impulse responce is absolutely summable

The kind of s/m that we want to design is
1) Additive
2) Homogeneous
3) Shift-invarient
4) Stable

Example

$$y[n] = \frac{1}{2}\{x[n] + x[n+1]\} \Rightarrow \text{linear, SI}$$

$$\therefore h[n] = \frac{1}{2}\{\delta[n] + \delta[n+1]\}$$

$$\sum |h[n]| = 1/2 + 1/2 = 1 \Rightarrow \text{stable}$$

[Note → If the s/m is not LSI then above approach can't be used, as impulse response characterizes only LSI s/m.

Another Approach

$$|y[n]| = \left|\frac{1}{2}\{x[n] + x[n+1]\}\right|$$

$$\leq \frac{1}{2}|x[n]| + \frac{1}{2}|x[n+1]|$$

$$\leq M_x$$

$\Rightarrow$ BIBO stable

* CAUSALITY → (cause & effect)

Informally, a causal s/m is one where o/p at 'n' depends only on the i/p upto 'n' and possibly the o/p upto (n-1)

Eg: 1) $y[n] = x[n] + x[n-1] + y[n-1]$ → causal & stable

2) $y[n] = \begin{cases} \frac{x[n]}{x[n-1]} & \text{if } x[n-1] \neq 0 \\ 0 & \text{if } x[n-1] = 0 \end{cases}$ ⇒ Causal, unstable & non-linear s/m

iii) $y[n] = x[n] \cdot x[n-1]$

$\rightarrow$ Causal & stable ~~linear~~

iv) $y[n] = n\,x[n]$

$\rightarrow$ causal, unstable, linear

Exercise * show that additivity & homogeneity are equivalent to

if $\;\forall\; x_{1,2}[n] \longrightarrow y_{1,2}[n]$

then $\alpha x_1[n] + \beta x_2[n] \longrightarrow \alpha y_1[n] + \beta y_2[n]$

$\forall\; \alpha, \beta, x_1, x_2$

* Show that linearity, Causality, stability are independe

* FORMAL DEFINATION OF CAUSALITY

Consider two input sequences

$x_1[n]$ & $x_2[n]$

Let $\quad x_1[n] = x_2[n] \quad \forall\; n \le n_0$

(Note - Nothing is specified for $n > n_0$)

Then in a causal s/m corresponding o/p $y_1[n]$ and $y_2[n]$ are

$$y_1[n] = y_2[n] \quad \forall\; n \le 0$$

$$\forall\; x_1, x_2, n_0$$

* Causality of LSI s/m

$$y[n] = \sum_k x[k] \cdot h[n-k]$$

Let for eg:

$$h[n] = 6, \underset{\uparrow}{2}, -8$$

Then $y[n] = 6x[n+1] + 2x[n] - 8x[n-1]$

$6x[n+1]$ make s/m non-causal

Theorem

A ~~LSI~~ LSI s/m is causal iff its impulse response $h[n] = 0 \quad \forall\, n < 0$

Challenging Problem :-

Compute several examples of s/m whic are not ~~linear~~ LSI but where the impulse response $\neq 0$ $\forall\, n < 0$, but the s/m is causal.

For the moment, we are satisfied that, we should design s/m which are LSI, stable and causal.

* Memoryless s/m is always causal.

for LSI s/m $h[n]$,

$$H(\omega_0) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega_0 n}$$

Property - $H(\omega_0)$ converges for "almost all" $-\pi \leq \omega \leq \pi$

"almost all" means except for finite no. of isolated points in the finite interval. $[-\pi, \pi]$

[ ] ⇒ closed interval
i.e limits are also included

( ) ⇒ open interval
i.e limits are not included

( ] ⇒ upper limit not included
lower limit included

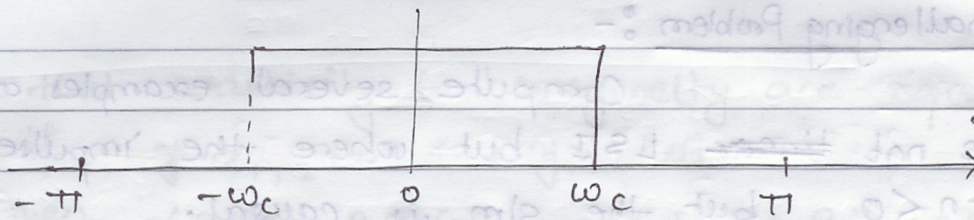
[ ) ⇒ upper limit included
lower limit not included

Stable s/m always follow this property for all ω_0 in $[-\pi, \pi]$

A (Linear) filter is an LSI s/m where ~~m~~ $H(\omega_0)$ converges for almost all ω_0 in $[-\pi, \pi]$

Eg:

Ideal LPF



Normalized Angular Frequency

$\Downarrow$

$\left[\frac{f \times 2\pi}{f_s}\right]$

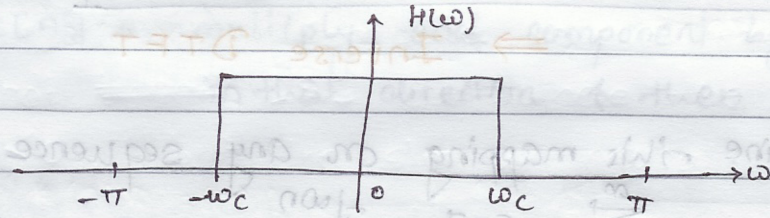
Ideal low pass filter is one where $H(\omega_0)$ converges to 1 for $(-\omega_c, \omega_c)$ & 0 for $[-\pi, -\omega_c)$ and $(+\omega_c, \pi]$

Filters

- A filter is an LSI s/m where $H(\omega) = \sum_{n=-\infty}^{\infty} h[n] \cdot e^{-j\omega n}$

converges for almost all ω.

Eg: Ideal Low pass Filter



We want to ~~con~~ $H(\omega)$ to convege from $-\pi$ to π but we leave out $\omega = -\omega_c$ and $\omega = \omega_c$ points

In particular, a stable LSI s/m is a filter
$H(\omega)$ = frequency response of the filter.

$h[n] \longrightarrow H(\omega)$

sequence — is a periodic f^n of ω with period 2π

- For Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \cdot e^{jn\frac{2\pi}{T}t} \quad \text{—— (1)}$$

and
$$x_n = \frac{1}{T}\int_T x(t) \cdot e^{-j\frac{2\pi}{T}nt}\, dt$$

$\Rightarrow$ Fourier series is mapping from periodic f^n to a sequence of Harmonics.

- $$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

Put $m = -n$

$$= \sum_{m=-\infty}^{\infty} h[-m] \cdot e^{jm\omega}$$

$$= \sum^{\infty} h[-m] \cdot e^{jm \cdot \frac{2\pi}{2\pi}\omega} \quad \text{—— (2)}$$

from ① & ②

$$h[-m] = \frac{1}{2\pi} \int_{2\pi} H(\omega)\, e^{-jm\omega}\, d\omega$$

or

$$\boxed{h[m] = \frac{1}{2\pi} \int_{2\pi} H(\omega) \cdot e^{jm\omega}\, d\omega} \quad \text{———} \quad (*)$$

⇒ Inverse DTFT

- we can define this mapping on any sequence $x[n]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n} \quad \text{———} \quad (*)$$

If $x[n]$ is absolutely summable the $X(\omega)$ is certainly defined.

$$\therefore \; x[n] \underset{\text{I.DTFT}}{\overset{\text{DTFT}}{\longleftrightarrow}} X(\omega)$$

—x—

* For sig $x[n]$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k]$$

$\delta[n-k]$ is the basis and there are such contabally infinite basis vectors.

Smillarly $e^{j\omega n}$ is abasis

* Consider two complex vectors $[a_1, a_2]$ and $[b_1; b_2]$

then $[a_1, a_2] \cdot [b_1, b_2] = a_1 \bar{b}_1 + a_2 \bar{b}_2$

dot product
or Inner product

* For eq^n ⊛

$X(\omega)$ is the inner product of the sequence $x[n]$ and $e^{j\omega n}$.

$\Rightarrow$ $X(\omega)$ is projection of $x[n]$ in the direction of $e^{j\omega n}$

* From eq ⊛

$x[n]$ = multiply each component by unit vector ($= 1/2\pi$) in that direction & then add up (Here we integrate)

* Consider

$$x[n] \longrightarrow y[n] = x(n) * h(n)$$

also
$$x[n] \xleftrightarrow{DTFT} X(\omega)$$
$$h[n] \xleftrightarrow{\;''\;} H(\omega)$$
$$y[n] \xleftrightarrow{\;''\;} Y(\omega)$$

then
$$Y(\omega) = X(\omega) \cdot H(\omega)$$

Proof

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(n) \cdot h(n-k)$$

$$\therefore \text{DTFT}(y(n)) = \text{DTFT}\left[\sum_{k=-\infty}^{\infty} x[k] \cdot h(n-k)\right]$$

$$\therefore Y(\omega) = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]\right] e^{-j\omega n}$$

Put $n-k = m$

$$= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k] \cdot h[m]\, e^{-j\omega(m+k)}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot \left[\sum_{m=-\infty}^{\infty} h[m] \cdot e^{-j\omega m}\right] \cdot e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot e^{-j\omega k} \cdot H(\omega)$$

$$= H(\omega) \cdot \sum_{k=-\infty}^{\infty} x[k] \cdot e^{-j\omega k}$$

$$= H(\omega) \cdot X(\omega)$$

$$= X(\omega) \cdot H(\omega)$$

Hence proved.

⇒ Convolution in time domain = Multiplication in freq. domain

10/08/06

$$x[n] \xrightarrow{DTFT} X(\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

$$X(\omega) \xrightarrow{IDTFT} x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} \, d\omega$$

$$x(n) \longrightarrow \boxed{\begin{matrix} LSI \text{ s/m} \\ h(n) \end{matrix}} \longrightarrow y(n) = x(n) * h(n)$$

$$X(\omega) \qquad H(\omega) \qquad Y(\omega) = X(\omega) \cdot H(\omega)$$

Note — to get $X(\omega)$ (or $H(\omega)$) we should have $x(n)$ (or $h(n)$) absolutely summable

If $x(n)$ (or $h(n)$) are not absolutely sumable then $X(\omega)$ (or $H(\omega)$) are guarenteed else $X(\omega)$ (or $H(\omega)$) are not guarenteed.

eg:

$x[n] = 2^n u[n] \longrightarrow$ not absolutely sumable

⇒ $X(\omega)$ diverges for all ω!

$$X(\omega) = \sum_{n=-\infty}^{\infty} 2^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(2 \cdot e^{-j\omega}\right)^n$$

$$= \infty.$$

Properties of DTFT

① Convolution in 'n' translates to multiplication in 'w'

② If $x[n] \xleftrightarrow{DTFT} X(\omega)$

then $\overline{x[n]} \longleftrightarrow \overline{X(-\omega)}$

Proof $DTFT\{\overline{x[n]}\} = \sum_{n=-\infty}^{\infty} \overline{x[n]} e^{-j\omega n} = \overline{\sum_{n=-\infty}^{\infty} x[n] e^{j\omega n}}$

$$= \overline{\sum_{n} x[n] e^{-j(-\omega)n}} = \overline{X(-\omega)}$$

Corollary: If $x[n] = \overline{x[n]} \Rightarrow$ Real seq.

then $X(\omega) = \overline{X(-\omega)}$

$\Rightarrow$ Mag response is symmetric

phase response is antisymm

③ Time Reversal

$$x[n] \xleftrightarrow{DTFT} X(\omega)$$

$$x[-n] \longleftrightarrow X(-\omega)$$

Proof:

$$DFT\{x[-n]\} = \sum_n x[-n] e^{-j\omega n}$$

Put $m = -n$

$$= \sum_m x[m] e^{j\omega m}$$

$$= \sum_m x[m] e^{-j(-\omega) m}$$

$$= X(-\omega)$$

$\Rightarrow$ Role of Rotating phasors is interchanged

④ Linearity

$$x_{1,2}[n] \xleftrightarrow{DTFT} X_{1,2}[\omega] \text{ If it exists}$$

then $\alpha x_1[n] + \beta x_2[n] \xleftrightarrow{DTFT} \alpha X_1(\omega) + \beta X_2(\omega)$

$\forall \alpha, \beta, x_1, x_2$

Exercise ① x_1, x_2 do not have DTFT but $x_1 + x_2$ does?

(Prove) ② Inverse DTFT is also linear?

⑤ Multiplication in time

$$x[n] \longleftrightarrow X(\omega)$$

$$y[n] \longleftrightarrow Y(\omega)$$

$$x[n] \cdot y[n] \longleftrightarrow X(\omega) \circledast Y(\omega)$$

Proof:

$$DTFT[x[n] \cdot y[n]] = \sum_n x[n] y[n] e^{-j\omega n}$$

$$= \sum_n x[n] \left\{ \frac{1}{2\pi} \int_{2\pi} Y(\lambda) e^{j\lambda n} d\lambda \right\} e^{-j\omega n}$$

$$= \frac{1}{2\pi} \int_{2\pi} Y(\lambda) \left\{ \sum_n x[n] e^{-j(\omega - \lambda) n} \right\} d\lambda$$

$$= \frac{1}{2\pi} \int_{2\pi} Y(\lambda) \cdot X(\omega - \lambda)\, d\lambda$$

$$= \underbrace{\text{Periodic} \quad \text{Periodic}}_{\text{Periodic convolution}}$$

↳ convolution is found only for a period of 2π

otherwise $\int_{-\infty}^{\infty}$ will always diverge

$$= X(\omega) \circledast Y(\omega)$$

eg: $x[n] = \left(\frac{1}{2}\right)^n u[n] = y[n]$

$$\therefore X(\omega) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n]\, e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} e^{-j\omega}\right)^n$$

$$= \frac{1}{1 - \frac{1}{2} e^{-j\omega}} = Y(\omega)$$

$\therefore \quad x[n][y] = \left(\frac{1}{4}\right)^n u[n] \Rightarrow \text{DTFT}\left\{x[n] \cdot y[n]\right\} = \frac{1}{1 - \frac{1}{4} e^{-j\omega}}$

$$\therefore \text{DTFT}\left\{x[n] \cdot y[n]\right\} = X(\omega) * Y(\omega)$$

$$= \frac{1}{2\pi} \int_{2\pi} \left[\frac{1}{1 - \frac{1}{2} e^{-j\lambda}}\right] \cdot \left\{\frac{1}{1 - \frac{1}{2} e^{-j(\omega - \lambda)}}\right\} d\lambda$$

Corollary $\quad x[n] \cdot \overline{y[n]} \xrightarrow{DTFT} X(\omega) \circledast \overline{Y(\omega)}$

$$= \frac{1}{2\pi} \int_{2\pi} X(\lambda) \cdot Y(\lambda - \omega) \, d\lambda$$

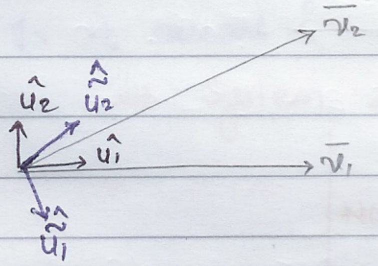
In particular, write this equivalence for $\omega = 0$

$$\left\{ \sum_{n=-\infty}^{\infty} x[n] \cdot \overline{y[n]} \, e^{-j\omega n} \right\}_{\omega=0} = \frac{1}{2\pi} \int_{2\pi} X(\lambda) \cdot \overline{Y(\lambda)} \, d\lambda$$

$$\underbrace{\sum_{n=-\infty}^{\infty} x[n] \, \overline{y[n]}}_{\text{Dot product of } x[n] \text{ and } y[n]} = \underbrace{\frac{1}{2\pi} \int_{2\pi} X(\lambda) \cdot \overline{Y(\lambda)} \, d\lambda}_{\text{Dot product of } X(\omega) \text{ and } Y[\omega]}$$

Parseval's Theorem

$$\sum_{n=-\infty}^{\infty} x[n] \cdot \overline{y[n]} = \frac{1}{2\pi} \int_{2\pi} X(\lambda) \cdot \overline{Y(\lambda)} \, d\lambda$$



Let $\hat{u}_1$ & $\hat{u}_2$ be basis then

$\bar{v}_1 = K_{11} \hat{u}_1 + K_{12} \hat{u}_2$

$v_2 = K_{21} \hat{u}_1 + K_{22} \hat{u}_2$

and if $\hat{\tilde{u}}_1$ and $\hat{\tilde{u}}_2$ are basis then

$\bar{v}_1 = \tilde{K}_{11} \hat{\tilde{u}}_1 + \tilde{K}_{12} \hat{\tilde{u}}_1$

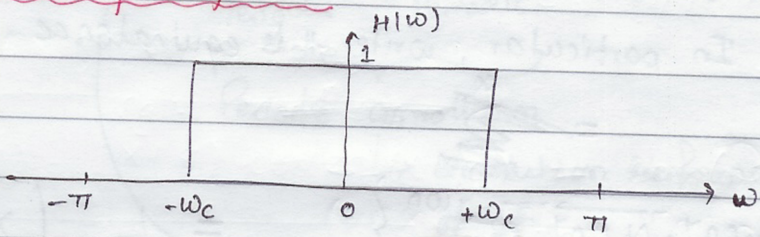
$v_2 = \tilde{K}_{21} \hat{\tilde{u}}_1 + \tilde{K}_{22} \hat{\tilde{u}}_2$

and $\bar{v}_1 \cdot \bar{v}_2 = K_{11} K_{21} + K_{12} \cdot K_{22}$

$= \tilde{K}_{11} \tilde{K}_{21} + \tilde{K}_{12} \tilde{K}_{22}$

IDEAL FILTERS

① Ideal Low pass filter —



$$\therefore\ H(\omega) = \begin{cases} 1 & -\omega_c < \omega < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Using Inverse DTFT

$$\therefore\ h[n] = \frac{1}{2\pi}\int_{2\pi} H(\omega)\, e^{+j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

When $n = 0$

$$h[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega$$

$$= \frac{2\omega_c}{2\pi}$$

$$= \frac{\omega_c}{\pi}$$

When $n \neq 0$

$$h[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c}$$

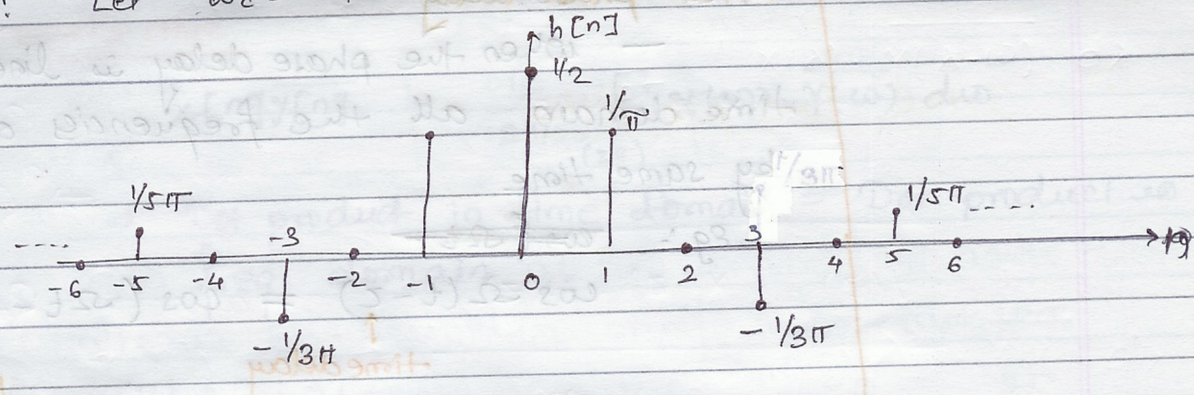
$$= \frac{1}{n\pi}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right]$$

$$= \frac{\sin(\omega_c n)}{n\pi}$$

$$\Rightarrow\ \boxed{h[n] = \begin{cases} \omega_c/\pi & n = 0 \\ \dfrac{\sin(\omega_c n)}{n\pi} & n \neq 0 \end{cases}}$$

for ideal low pass filter

Eg : Let $\omega_c = \pi/2$



1) To check stability

$$\sum |h[n]| = \frac{1}{2} + 2\left[\frac{1}{\pi} + \frac{1}{3\pi} + \frac{1}{5\pi} + \cdots\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right]$$

Exersice

$$= \frac{1}{2} + \frac{2}{\pi}\left[1 + \underbrace{\frac{1}{3} + \frac{1}{5}}_{>1/2} + \underbrace{\frac{1}{7} + \cdots}_{4\text{ terms},\ >1/2}\right]$$

$$= \underline{\underline{\infty}}$$

$\Rightarrow$ System (Filter) is unstable

2) Is it causal ?

$\Rightarrow$ System is not causal

Note $\rightarrow$ A non-causal s/m can be made causal if it has finite no. of terms on -ve side

$$x[n] \rightarrow \underbrace{\boxed{h[n]} \rightarrow \boxed{h_1[n]}}_{h'[n]} \rightarrow y[n]$$

	6	-4	4	2	0	1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
1	6	-4	4	2	0	1

$h[n] = \{6 \ -4 \ \underset{\uparrow}{4} \ 2 \ 0 \ 1\}$

$h_1[n] = \{\underset{\uparrow}{0} \ 0 \ 1\}$

$\Rightarrow h'[n] = \{\underset{\uparrow}{6}, -4, 4, 2, 0, 1\} = h[n-2]$

$\hookrightarrow$ It is now causal

Also $H_1(\omega) = 1 \cdot e^{-j\omega 2}$

$\Rightarrow |H_1(\omega)| = 1$

$\angle H_1(\omega) = -2\omega$

$\rightarrow$ It will not affect mag. response of $H(\omega)$ but only adds a linear phase delay of -2ω

Linear phase delay

— when the phase delay is linear, in time domain all the frequencies are delayed by same time

eg :- ~~cos Ωt~~

$$\cos \Omega(t-\underset{\uparrow}{\tau}) = \cos(\Omega t - \Omega\underset{\uparrow}{\tau})$$

time delay   phase delay

- PHASE RESPONSE IS OFEN A "NECESSARY EVIL"
- Linear phase is the least of the evils

★ Parseval's thm

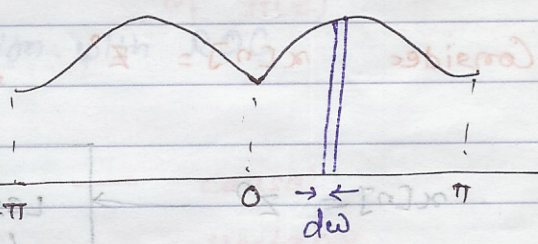
$$\sum_{n=-\infty}^{\infty} x[n]\overline{y[n]} = \frac{1}{2\pi}\int_{(2\pi)} X(\omega)\cdot\overline{Y(\omega)}\,d\omega$$

⇒ Dot product in time domain = Dot product in freq. domain

If $x[n] = y[n]$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{(2\pi)} |X(\omega)|^2\,d\omega$$

⇓ Energy in sequence $x[n]$ ⇓ Energy in spectrum of $X(\omega)$



$\Longrightarrow |X(\omega)|^2 = ESD$
(Energy Spectral Density)

★ Why can we not use the ideal filter directly?

1. ~~that~~ Unstable
2. Not causal [for infinite terms]
   ↳ It cannot be made causal by delaying
3. Irrational → Cannot be implemented with limited resources (elements)
   i.e. The equation describing this LSI s/m cannot be realized with a finite amount of computation per (unit) output ~~s/m~~ sample.

Z - TRANSFORM

$$e^{j\omega n} \longrightarrow \boxed{\begin{matrix}\text{LSI s/m} \\ \text{(subclass)}\end{matrix}} \longrightarrow H(\omega)\cdot e^{j\omega n}$$

stable

When $e^{j\omega n}$ is fed in to LSI s/m (stable) comes out in same form multiplied by constant ($= H(\omega)$)
i.e. $e^{j\omega n}$ is "own" seq. of LSI s/m

"own" $\equiv$ "Eigen"

$\Rightarrow$ $e^{j\omega n}$ = Eigen sequence
& $H(\omega)$ = Eigen value

Consider $x[n] = z^n$, where $z = e^{j\omega}$

$$x[n] = z^n \longrightarrow \boxed{\begin{matrix}\text{LSI s/m} \\ (h[n])\end{matrix}} \longrightarrow y[n] = x[n] * h[n]$$

$$y[n] = \sum_{k=\infty}^{\infty} x[k]\cdot h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\cdot x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\cdot z^{(n-k)}$$

$$= z^n \underbrace{\left[\sum_{k=-\infty}^{\infty} h[k]\cdot z^{-k}\right]}$$

Infinite sumation $\Rightarrow$ may or may not converge

If it converges, it is a f^n 'z'. We then call it "Z-transform" of $h[n]$

$$\therefore \quad \underset{\text{sequence}}{h[n]} \xrightarrow{\text{Z-tranform}} H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

$$= \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

may converge for some region of the complex plane z

Region in the z-plane where it converges: REGION OF CONVERGENCE of $H(z)$

$\Rightarrow \quad z^{n}$ = Eigen sequence

$H(z)$ = Eigen value $\longrightarrow$ in Region of convergence of $H(z)$

$\Rightarrow$ z-transform is expression with ROC

Eg

(1) $h[n] = 2^{n} u[n]$ $\longrightarrow$ causal

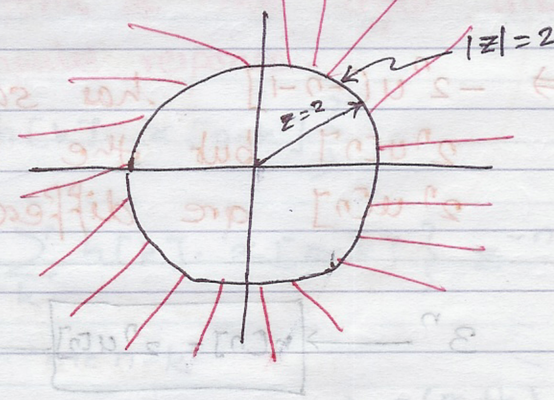
: unstable

$$\therefore H(z) = \sum_{n=-\infty}^{\infty} 2^{n} u[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} 2^{n} z^{-n} = \sum_{n=0}^{\infty} (2z^{-1})^{n}$$

$$= \frac{1}{1-2z^{-1}} \qquad \text{if } |2z^{-1}| < 1$$

$$= \frac{z}{z-2} \qquad \text{ie } |z| > 2$$



Consider $H(z) = \dfrac{1}{1-2z^{-1}}$ $\quad |2z^{-1}| > 1$

$\Rightarrow |z| < 2$

$\Rightarrow \left|\frac{1}{2}z^{-1}\right| < 1$

$$\therefore H(z) = \frac{1}{1-2z^{-1}} = \frac{1/2\, z}{1/2\, z - 1}$$

$$= \frac{(-1/2\, z)}{1-\left(\frac{1}{2}z\right)}$$

$$= \left(-\frac{1}{2}z\right) \cdot \sum_{k=0}^{\infty} \left(1/2\, z\right)^k$$

$$= (-1/2\, z)\left(1 + \frac{1}{2}z + \left(\frac{1}{2}\right)^2 z^2 + \left(\frac{1}{2}\right)^3 z^3 + \cdots\right)$$

Taylor series

$$= -\frac{1}{2}z - \left(\frac{1}{2}\right)^2 z^2 - \left(\frac{1}{2}\right)^3 z^3 - \cdots$$

$$\Rightarrow h[n] = \cancel{\left\{0, \frac{-1}{2}\right\}} \left\{\cdots -\left(\frac{1}{2}\right)^3, -\left(\frac{1}{2}\right)^2, -\frac{1}{2}, \underset{\uparrow}{0}\right\}$$

$$= \left\{\cdots -2^{-3}, -2^{-2}, -2^{-1}, \underset{\uparrow}{0}\right\}$$

$$= -2^n u[-n-1]$$

$\Rightarrow -2^n u[-n-1]$ has same z-transform $H(z)$ as that of $2^n u[n]$ but the ROC of $-2^n u[-n-1]$ and $2^n u[n]$ are different

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Eg $\quad 3^n \longrightarrow \boxed{h[n] = 2^n u[n]} \longrightarrow \dfrac{1}{1-2(3)^{-1}} 3^n \Rightarrow$ converges

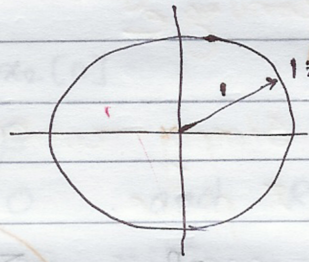
$(1/3)^n$ ———————— diverges

$h[n] \xleftrightarrow{z} H(z) \Rightarrow$ System function
(Transfer function)

Relation bet^n Z transform and DTFT

$z^n \qquad e^{j\omega n}$

$\Rightarrow \quad z = e^{j\omega}$: Z transform



$|z| = 1 \longrightarrow$ DTFT

$\Rightarrow$ If we want to find DTFT, ROC of Z transform should contain $|z| = 1$

Eg: $(1/2)^n u(n) \longrightarrow$ Has Z Tr $\longrightarrow |z| > 1/2$
$\Downarrow$
Has DTFT

$(2)^n u(n) \longrightarrow$ Has Z Tr $\longrightarrow |z| > 2$
$\downarrow$
Does not contain $|z| = 1$
$\Rightarrow$ Does not have DTFT

Theorem: If they exist, the Z transform of o/p of an LSI s/m is product of the Z transform of the input, and the Z-transform of the impulse response.

$$\vec{y}[n] = \left\{ \sum_k h[k] \cdot x[n-k] \right\}$$

$$\therefore \quad Y(z) = \sum_{n=-\infty}^{\infty} \left\{ \sum_k h[k] \cdot x[n-k] \right\} z^{-n}$$

Put $m = n-k$

$$= \sum_m \left\{ \sum_k h[k] \cdot x[m] \right\} z^{-(m+k)}$$

$$= \sum_k h[k] \left\{ \sum_m x[m] z^{-m} \right\} z^{-k} \quad ;$$

$$= \sum_k h[k] \cdot X(z) \cdot z^{-k} \quad ; \; R_x$$

$$\therefore \quad Y(z) = X(z) \cdot \sum_{k} h[k] z^{-k} \quad ; \quad R_x$$

$$= X(z) \cdot H(z) \quad ; \quad R_x \text{ and } R_h$$

atleast $\Rightarrow R_x \cap R_h$

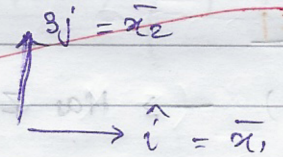
$$\therefore \quad \boxed{Y(z) = X(z) \cdot H(z)} \quad \text{atleast } R_x \cap R_h$$

or

$$H(z) = \frac{Y(z)}{X(z)}$$

Let $z = re^{j\theta}$

$$\therefore X(z) = \sum_{n} x[n] \, z^{-n} = \sum_{n} x[n] \cdot r^{-n} e^{-j\theta n}$$



Let $\bar{x}_1$ and $\bar{x}_2$ be two ⊥ar vector but they are not unit vector

⇒ Any vector now can be expressed as dot product of $\bar{x}_1$ & $\bar{x}_2$

⇒ form the basis

similarly $\delta(n-k)$ form a basis

Now mag. of vector is square root of dot. product of vector with itself

S'lly in the mag of impulse ⇒ mag. square is energy.

This is a genralized dot product with added weighting by r^{-n}

⇒ Y(z) is dot product of component of x in Z-direction with component of h in Z direction.

Properties of Z-transform

① Linearity

$$x_{1,2}[n] \xleftrightarrow{Z} X_{1,2}(z) \quad ; \quad ROC\ R_{1,2}$$

then

$$\alpha x_1[n] + \beta x_2[n] \xleftrightarrow{Z} \alpha X_1(z) + \beta X_2(z)$$

Roc : atleast $R_1 \cap R_2$

Eg:

① Let $x_1[n] = x_2[n]$

$\alpha = -\beta$

$\Rightarrow$ Z tr = 0 and ROC : Entire Z-plane

② $x_1[n] = 2^n u[n]$

$x_2[n] = \delta[n+5] - 2^n u[n]$ $\Big\}$ $\alpha = \beta = 1$

$|z| > 2$

$\underbrace{\infty > |z| > 2}$

intersection: $\infty > |z| > 2$

$\Rightarrow$ Ztr = z^5 and ROC : $|z| < \infty$

② Time Reversal

$$x[n] \xleftrightarrow{Z} X(z) \quad ; \quad R_x$$

then

$$x[-n] \xleftrightarrow{Z} X(1/z) \quad ; \quad 1/R_x$$

$$= X(z^{-1}) \quad ; \quad z^{-1} \in R_x$$

eg:

$|z| > 2$

$\therefore |z^{-1}| > 2$

$\therefore |z| < 1/2$

③ Complex conjugate

$$x[n] \xleftrightarrow{Z} X(z) \quad ; \quad R_x$$

$$\overline{x[n]} \xleftrightarrow{Z} \overline{X(\bar{z})} \quad ; \quad \bar{z} \in R_x$$

$\rightarrow$

$$\sum_n \overline{x[n]}\, z^{-n} = \overline{\sum_n x[n]\, \bar{z}^{-n}} = \overline{\sum_n x[n] (\bar{z})^{-n}}$$

$$= \overline{X(\bar{z})} \quad ; \quad \bar{z} \in R_x$$

* $\sum_n x[n] \cdot z^{-n}$

$x[n] = 2^n u[n]$

$2^n |z|^{-n}$

⇒ Here we have multiplied 2^n by a decaying eqⁿ & forced it to converge

mag. of $|z|$ will decide whether $X(z)$ converge or diverge ⇐

* R : Simply connected disk in the z plane.

* Z tr. is analytic in ROC

④ Multiplication by n / Differentiation

$$x[n] \xleftrightarrow{z} X(z) \quad ; R$$

$$n\,x[n] \longleftrightarrow -z\frac{d(X(z))}{dz} \quad ; R$$

→

$$X(z) = \sum_n x[n]\, z^{-n}$$

$$\therefore \frac{d}{dz}(X(z)) = \sum_n (-n)\, x[n] \cdot z^{-n-1}$$

$$\therefore -z \cdot \frac{d}{dz} X(z) = \sum_n \{n\, x[n]\} z^{-n}$$

$$= n\, x[n]$$

(5) Time shifting

$$x[n] \xleftrightarrow{Z} X(z)\ ,\ R$$

then $x[n-n_0] \xleftrightarrow{Z} z^{-n_0} \cdot X(z)\ ,\ R \mp$ {possibly 0 and/or ∞}

Eg: > $x[n] = \delta(n) \Rightarrow X(z) = 1$ $\quad R_x =$ Entire z plane

then $x[n-3] = \delta(n-3) \Rightarrow X(z) = z^{-1}$ $\quad R_x: |z| < \infty$

INVERSE Z-TRANSFORM

- $X(z) = \dfrac{1}{1-\beta z^{-1}}$ $\quad |z| > |\beta| \implies x[n] = \beta^n u[n]$

- $\dfrac{z^{\pm 3}}{1-\beta z^{-1}} \longrightarrow \beta^{n \pm 3} \cdot u[n \pm n]$

- $\dfrac{(+z)\beta z^{-2}}{(1-\beta z^{-1})^2} = \dfrac{\beta z^{-1}}{(1-\beta z^{-1})} \longrightarrow n\,\beta^n\,u[n]$

- $\dfrac{1}{(1-\beta z^{-1})^2} \longrightarrow \cancel{-\frac{1}{\beta}(n+2)\beta^{n+2}\,u[n+2]}$

$$\longrightarrow \frac{(n+1)}{\beta} \cdot \beta^{n+1}\ u[n+1]$$

$$= (n+1)\,\beta^n\,u[n]$$

$\because u[n+1] = u[n]$

HW Use an inductive process to obtain the inverse Z transform of

$$\frac{1}{(1-\beta z^{-1})^M}\ ;\quad M \text{ integer},\ M \geq 1$$

with ROC: (i) $|z| > |\beta|$

(ii) $|z| < |\beta|$

* $H(z) = P(z) + \dfrac{N(z^{-1})}{D(z^{-1})}$

$P(z)$ = is a finite series in integer powers of z

$N(z^{-1})$, $D(z^{-1})$ are poly in z^{-1} such that deg. $N <$ deg D.

~~$P(z)$~~

- Inverse z-transforme of $P(z)$ can be obtained directly from def.
- $\dfrac{N(z^{-1})}{D(z^{-1})}$ can be decomposed using partial fraction & then we can find their inverse z transform

Fundamental thm of algebra:

$D(z^{-1})$ can be decomposed as

$$K_D(1-\alpha_1 z^{-1})^{M_1} \cdots (1-\alpha_Q z^{-1})^{M_Q}$$

$$\therefore \frac{N(z^{-1})}{D(z^{-1})} = \frac{N(z^{-1})}{K(1-\alpha_1 z^{-1})^{M_1}(\cdots\cdots)(\quad)}$$

$$= \frac{A_{11}}{1-\alpha_1 z^{-1}} + \cdots + \frac{A_{1M_1}}{(1-\alpha_1 z^{-1})^{M_1}} + \cdots$$

$H(z) = P(z) + \dfrac{N(z^{-1})}{D(z^{-1})}$ $\Rightarrow$ Rational z-transform

* Example of an irrational z-transform

$$e^{z^{-1}} \quad |z| > 0$$

$\rightarrow$ Using Taylor series expansion

$$e^{z^{-1}} = \sum_{n=0}^{\infty} \frac{(z^{-1})^n}{n!} \quad ; \quad 0! = 1$$

$$= 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots$$

$\therefore$ Inverse z transform is

$$= \frac{1}{n!} u[n] \qquad ; 0! = 1$$

$$\Rightarrow e^{z} \xrightarrow{z^{-1}} -\frac{1}{n!} u[-n-1]$$

Importance of Rational s/m

$$H(z) = z^{D} \frac{N(z)}{D(z)} , R$$

sol^n of $N(z) = 0 \Longrightarrow$ zeros of $H(z)$

sol^n of $D(z) = 0 \Rightarrow$ poles of $H(z)$

- Poles decide the nature of the inverse Z transform.

Let H(z) be the system f^n of an LSI s/m

$$h[n] = \sum_{l=1}^{M} P_l[n] \alpha_l^{n} \text{ (right/left sided dep. on R)}$$

Deg $= M_l - 1$

- Zero decide the ~~specification~~ specific contribution of poles.

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Examples of Inverse Z-transform

① $$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})^2(1-\frac{1}{3}z^{-1})} \qquad \frac{1}{3} < |z| < \frac{1}{2}$$

$$= \frac{\frac{1}{(1-\frac{1}{2}(3)^2)}}{1-\frac{1}{3}z^{-1}} + \left(\frac{A z^{-1}+B}{(1-\frac{1}{2}^{-1})^2}\right)$$

numerator

$$= 2A_1\left(\frac{1}{2}z^{-1}\right) + B - 2A_1 + 2A_1$$

$$= \frac{-2A_1\left(1-\frac{1}{2}z^{-1}\right) + B + 2A_1}{(1-\frac{1}{2}z^{-1})^2}$$

$$= \frac{-2A_1}{(1-\frac{1}{2}z^{-1})} + \frac{B+2A_1}{(1-\frac{1}{2}z^{-1})^2}$$

$$= \frac{\alpha_1}{1-\frac{1}{2}z^{-1}} + \frac{\alpha_2}{(1-\frac{1}{2}z^{-1})^2} + \frac{\alpha_3}{(1-\frac{1}{3}z^{-1})}$$

$$= \alpha_3 (1/3)^n u(n) + \left[\alpha_1 (1/2)^n + \alpha_2 n (1/2)^n\right] u(-n-1)$$

Rational s/m: A rational s/m is an linear shift invarience system, where impulse response has a rational Z-transform (System function exists and is rational)

Ideal filters are irrational !!

HW Challenging problem: Prove that ideal filters are irrational.

Hint! Over what region can a rational f^n be zero or constan!