Filter Bank Design using a Factorization of Parameterized Half Band Filters

Bhushan Patil∗ Pushkar Patwardhan £ and Vikram Gadre †

***Abstract*— This paper presents a new method for the design of a bi-orthogonal filter bank with improved frequency response. We design the filter bank using the modified polynomial factorization approach. It is shown that, filters with greater degree of freedom can be designed by increasing the order of the remainder polynomial.We also present the Chebyshev polynomial approach for symmetric factorization of the remainder polynomial, which gives symmetric filters. Results show that the frequency response of the filter can be improved using higher order remainder polynomials keeping the same regularity.**

Keywords:- Filter bank, Polynomial factorization, Chebyshev polynomial.

I. Introduction

Filter banks are an extremely valuable tool for signal processing applications. It is always desirable to have filters with some valuable properties like perfect reconstruction (PR), linear phase and good frequency response [2]. To design filter banks with a flat pass-band is always a challenging task. In this paper, we use a variable order product polynomial to improve the pass-band behavior of the designed filter keeping the same regularity. We show that the design approach gives selective parameters, which can be used to tune the frequency domain behavior of the designed filter. We also present the Chebyshev polynomial approach for symmetric factorization of the remainder polynomial to design symmetric filters. The remainder of this paper is organized as follows.

In Section II, we briefly review the two band filter bank with the polynomial factorization design approach.We illustrate the design approach with the standard 9/7 filter used in the JPEG2000 standards [6]. In Section III, the design of the filter bank using a higher order remainder polynomial is presented. It is shown that the frequency response of the filter can be improved by increasing order of remainder polynomial keeping the regularity same. In Section IV, we present the algorithm for symmetric factorization of the remainder polynomial to design symmetric filters. Section V concludes this paper.

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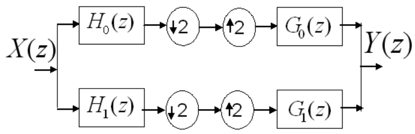

Fig. 1. Block diagram of two band filter bank

II. Background

Perfect reconstruction filter banks have been extensively studied [1],[3]. In this section we review the two band filter bank. Figure 1 shows the two channel filter bank. The output of the filter bank can be expressed as,

$$Y(z) = 1/2[G_0(z)H_0(z) + G_1(z)H_1(z)]X(z)$$
$$+1/2[G_0(z)H_0(-z) + G_1(z)H_1(-z)]X(-z)$$

where X(z) is the input signal and Y(z) is the reconstructed output signal.$H_0(z), H_1(z)$ are the analysis filters and $G_0(z), G_1(z)$ are the synthesis filters. Necessary and sufficient conditions for perfect reconstruction are,

$$H_0(z)G_0(z) + H_1(z)G_1(z) = Cz^{-D} \tag{1}$$

$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0 \tag{2}$$

Equation (1) gives the constraint for perfect signal reconstruction and equation (2) gives the constraint for alias cancellation. The constraint in (2) can be satisfied by the following choice of the synthesis filters

$$G_0(z) = H_1(-z), G_1(z) = -H_0(-z) \tag{3}$$

For the class of synthesis filters discussed in this paper, perfect reconstruction of the input signal requires that

$$\triangle(z) = P(z) - P(-z) = 2z^{-m} \qquad (4)$$

where,$P(z) = H_0(z)H_1(-z)$
It can be observed that condition (4) is satisfied if $P(z)$ is a half band polynomial[3]. A half band polynomial has only even non zero coefficients except for the middle coefficient. The generalized representation of a half band polynomial is given below.

$$P(z) = a_0 + a_2z^{-2} + ... + a_{2p-2}z^{-2p+2} + z^{-2p+1} \\ +a_{2p-2}z^{-2p+2} + ... + a_0z^{-4p+2}$$

$P(z)$ can be factorized to obtain $H_0(z)$and $H_1(-z)$ filters. Based on the order of $P(z)$ and the kind of factorization we can design filter banks with different lengths of support. Using a 14th order half band polynomial, we can extract eight zeros at $z = -1$. If these eight zeros are equally divided between the two analysis and synthesis low pass filters along with factors of the remainder polynomial, it gives the standard 9/7 filters which are used in the JPEG2000 standards [4],[6]. The designed standard 9/7 filters are,

$$H_0(z) = k_9(z^4 + 4z^3 + 6z^2 + 4z + 1) \\ .(z^4 + a_1z^3 + a_2z^2 + a_1z + 1)$$

$$H_1(-z) = k_7(z^4 + 4z^3 + 6z^2 + 4z + 1).(z^2 + a_3z + 1)$$

where, $a_1 = -4.63046$, $a_2 = 9.59748$, $a_3 = -3.36953$ and $k_9 = 0.03782$, $k_7 = -0.064538$
In this design both the low pass filters ($H_0(z)$ $H_1(-z)$) have four zeros at $z = -1$,along with the 4th and 2nd order factors of the remainder term respectively. After extracting eight zeros from the 14th order half band polynomial, the remainder term is a complectly constraint polynomial. So in the case of standard 9/7 filters, we do not have any freedom of the selective parameter.
In this paper we increase the order of the half band polynomial, which after extraction of eight zeros gives the remainder term of higher order in terms of the selective parameter. We use the selective parameter to improve the pass-band behavior of the filter.

III. Higher order of remainder polynomial

In this case we shall assume a product polynomial of order more than 14. For illustration we assume product polynomials of order 18 and 22.

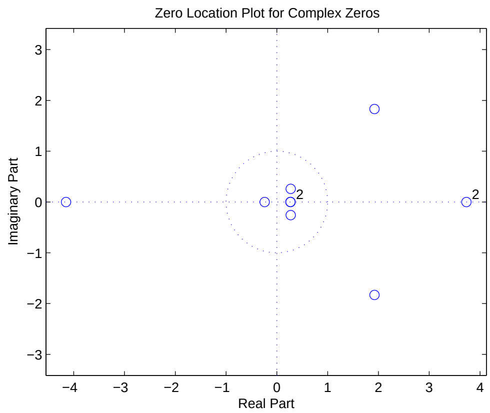

Fig. 2. Zero Location Plot

A. Product polynomial of 18th order

Here we use a half band product polynomial of 18-th order to design the filter bank with the same regularity as that of the standard 9/7 filter but better frequency response.

$$P(z) = a_0 + a_2z^{-2} + a_4z^{-4} + a_6z^{-6} + a_8z^{-8} + z^{-9} \\ +a_8z^{-10} + a_6z^{-12} + a_4z^{-14} + a_2z^{-16} + a_0z^{-18}$$

To design $H_0(z)$ and $H_1(-z)$, polynomial factorization can be used. We can easily extract the factor $(z+1)^8$ using synthetic division thus imposing eight zeros at $z = -1$ or $w = \pi$. The remainder term can be given as,

$$R(z) = a_0 - 8a_0z^{-1} + (\frac{-160}{65536} + 29a_0)z^{-2} + \\ (\frac{1280}{65536} - 64a_0)z^{-3} + (\frac{-4192}{65536} + 98a_0)z^{-4} + \\ (\frac{6656}{65536} - 122a_0)z^{-5} + (\frac{-4192}{65536} + 98a_0)z^{-6} + \\ (\frac{1280}{65536} - 64a_0)z^{-7} + (\frac{-160}{65536} + 29a_0)z^{-8} \\ -8a_0z^{-9} + a_0z^{-10}$$

It is to be noted that the remainder term is a function of parameter a_0. We now find out the roots of the remainder polynomial and divide ten roots in two groups, the first five roots form $R_1(z)$ and the remaining five roots give rise to $R_2(z)$. The final filters which we get are,

$$H_0(z) = (z+1)^4R_1(z) \qquad (5)$$

$$H_1(-z) = (z+1)^4R_2(z) \qquad (6)$$

We select the value of a_0 parameters such that the pass band ripples are minimized. We have used the discrete search range in the power of two form (2^{-1} to 2^{-20}

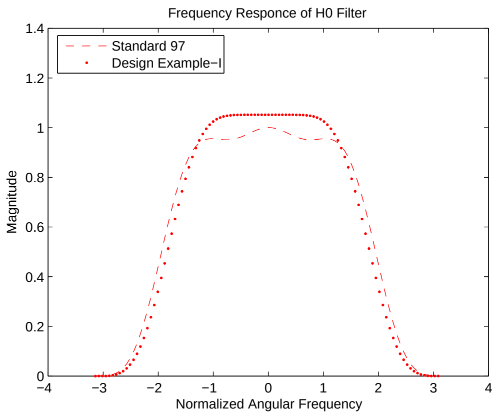

Fig. 3. Frequency response of H_0 Filters

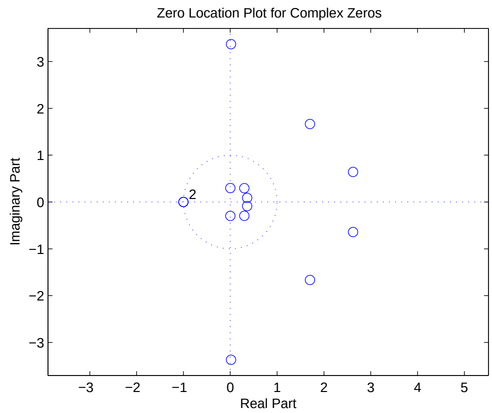

Fig. 4. Zero Location Plot

)for the a_0 value. Figure 2 shows the zeros location of the remainder polynomial for 18th order half band polynomial.As we extract eight zeros at $z=-1$ from the half band polynomial, the remaining ten zeros form the remainder polynomial. Figure 3 shows the comparison of the frequency responses of the standard 9/7 filter and the filters designed using the above method with $a_0 = 2^{-13}$.The impulse responses of the corresponding filters are,

$$\begin{aligned} h_0(n) = [&0.0010 \ -0.0032 \ -0.0199 \ 0.0720 \ 0.0465 \\ &-0.3585 \ -0.0227 \ 1.1740 \ 1.2867 \ 0.4073] \\ h_1(n) = [&1.0000 \ -3.1592 \ 2.8823 \ 0.0557 \ -0.8802 \\ &-0.1142 \ 0.1769 \ 0.0489 \ -0.0078 \ -0.0025] \end{aligned}$$

It can be observed that the designed filter has comparatively better pass band behavior than the standard 9/7 filter.

B. Product polynomial of 22th order

In this case we use half band product polynomial of 22-th order to design the filter bank with more selective parameters.

$$\begin{aligned} P(z) = a_0 + a_2z^{-2} + a_4z^{-4} + a_6z^{-6} + a_8z^{-8} + \\ a_{10}z^{-10} + z^{-11} + a_{10}z^{-12} + a_8z^{-14} + a_6z^{-16} \\ +a_4z^{-18} + a_2z^{-20} + a_0z^{-22} \end{aligned}$$

To design $H_0(z)$ and $H_1(-z)$ here we again use polynomial factorization. We can easily extract the factor $(z+1)^8$ thus imposing a eight zeros at $z=-1$ or $w=\Pi$ to keep same regularity as that of standard 9/7 filter. The remainder term can be given as,

$$\begin{aligned} R(z) = a_0 - 8a_0z^{-1} + (a_2 + 36a_0)z^{-2} + \\ (-8a_2 - 120a_0)z^{-3} + (\frac{-160}{65536} + 29a_2 + 302a_0)z^{-4} + \\ (\frac{1280}{65536} - 64a_2 - 122a_0)z^{-5} + (\frac{-4192}{65536} + 98a_2 + 813a_0)z^{-6} + \\ (\frac{6656}{65536} - 112a_2 - 912a_0)z^{-7} + (\frac{-4192}{65536} + 98a_2 + 813a_0)z^{-8} + \\ (\frac{1280}{65536} - 64a_2 - 122a_0)z^{-9} + (\frac{-160}{65536} + 29a_2 + 302a_0)z^{-10} + \\ (-8a_2 - 120a_0)z^{-11} + (a_2 + 36a_0)z^{-12} - 8a_0z^{-13} + a_0z^{-14} \end{aligned}$$

It is to be noted that the remainder term is the function of two parameters a_0 and a_2.
We now find out the roots of the remainder polynomial and divide them in to two factors $R_1(z)$ and $R_2(z)$. We select the value of the a_0 and a_2 parameters such that the overall frequency response of the designed filter is improved as explained above. The final filters which we get are,

$$H_0(z) = (z+1)^4R_1(z) \tag{7}$$

$$H_1(-z) = (z+1)^4R_2(z) \tag{8}$$

We have divided the fourteen roots in two groups, first seven roots to form $R_1(z)$ and remaining seven roots to give rise to $R_2(z)$.

Figure 4 shows the locations of 14 zeros of the remainder polynomials and Figure 5 shows the comparison of frequency responses of the standard 9/7 filter and the filters designed using the above method with $a_0 = 2^{-13}$ and $a_2 = \frac{-2^{-14}}{4}$. The impulse response of the corresponding filters is,

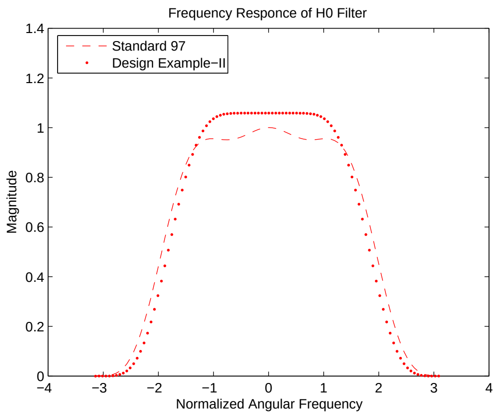

Fig. 5. Frequency response of H_0 Filters

$$
\begin{aligned}
h_0(n) = [&0.0010 \ -0.0036\ 0.0044\ -0.0018\ -0.0235 \\
&0.0855\ 0.0007\ -0.3346\ 0.1190\ 1.0221 \\
&0.9214\ 0.2555] \\
h_1(n) = [&1.0000\ -3.6063\ 4.0004\ -0.4658\ -1.3097 \\
&-0.0028\ 0.3345\ 0.0921\ -0.0072\ -0.0171 \\
&-0.0141\ -0.0039]
\end{aligned}
$$

In case of the standard 9/7 filter the remainder polynomial has no free parameters, but as we increase the order of the half band polynomial we get the remainder polynomial as a function of more variables. We can observe that the frequency response of the filter can be improved by increasing the order of the remainder polynomial. The frequency response of the designed filter is better in comparison with the standard 9/7 filter.It is to be noted that though $R(z)$ is a symmetric polynomial, the root distribution in terms of $R_1(z)$ and $R_2(z)$ cause the symmetry to be lost. This non symmetric nature is also reflected in the H_0 and G_0 filter impulse responses. Due to the requirement of linear phase it is necessary that the impulse response should have symmetry. In the next section we present an algorithm for the symmetric factorization of remainder polynomial $R(z)$ in symmetric factors $R_1(z)$ and $R_2(z)$ using the Chebyshev polynomial.

IV. Algorithm for Symmetric Factorization

To have a symmetric factorization of remainder terms we present the transformation of variable approach, which includes Chebyshev polynomial factorization.

For the sake of explanation, we assume a parametric form of the remainder term of fourth order.

$$R(z) = p_0 + p_1 z^{-1} + p_2 z^{-2} + p_1 z^{-3} + p_0 z^{-4} \quad (9)$$

We first represent $R(z)$ (remainder of half band product polynomial after extracting $(z+1)^8$ term) in zero phase form.

$$R(z) = z^{-2}(p_0 z^2 + p_1 z^1 + p_2 + p_1 z^{-1} + p_0 z^{-2})$$

After regrouping the terms with similar powers of z we can rewrite the term as,

$$R(z) = z^{-2}(p_0(z^2 + z^{-2}) + p_1(z^1 + z^{-1}) + p_2)$$

Now we use the transformation of variables,
We replace $\frac{(z^1+z^{-1})}{2} = Z$, and the term $z^n + z^{-n}$ can be written as $z^n + z^{-n} = 2T_n(\frac{(z^1+z^{-1})}{2})$ where, T_n is n-th order Chebyshev polynomial. The generalized form of the Chebyshev polynomial is given as follow,

$$
\begin{aligned}
T_0(v) &= 1 \\
T_1(v) &= v \\
&\mathit{and} \\
T_n(v) &= 2vT_{n-1}(v) - T_{n-2}(v)
\end{aligned}
$$

which gives,

$$R(z) = z^{-2}(2p_0(2Z^2 - 1) + 2p_1 Z + p_2) \quad (10)$$

Now the polynomial will be in terms of Z variable. We factorize the polynomial in the Chebyshev domain.Let us assume it has roots M_1 and M_2,

$$R(z) = z^{-2}((Z - M_1)(Z - M_2)) \quad (11)$$

Then we divide the roots into two factors, say $R_1(Z)$ and $R_2(Z)$. It is to be noted that these factors are in non symmetric form. Here we re-substitute $Z = \frac{(z^1+z^{-1})}{2}$. Due to the nature of the transformation variable the new $R_1(z)$ and $R_2(z)$ are in symmetric form [5].

$$R(z) = z^{-2}(((\frac{z+z^{-1}}{2}) - M_1)((\frac{z+z^{-1}}{2}) - M_2)) \quad (12)$$

simplification gives,

$$
\begin{aligned}
R_1(z) &= \frac{z^{-1} - 2M_1 z^{-2} + z^{-3}}{2} \\
R_2(z) &= \frac{z^{-1} - 2M_2 z^{-2} + z^{-3}}{2}
\end{aligned}
$$

This symmetric nature of $R_1(Z)$ and $R_2(Z)$ give rise symmetric filters using equation (7),(8).

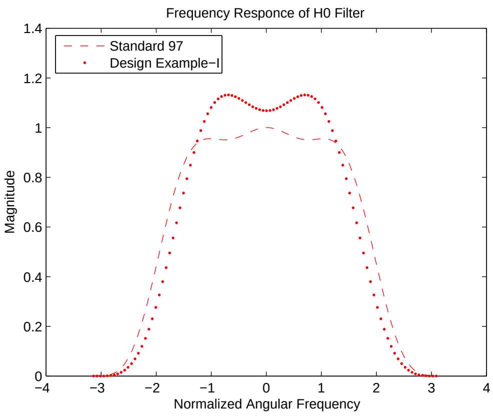

Fig. 6. Frequency response of H_0 Filters

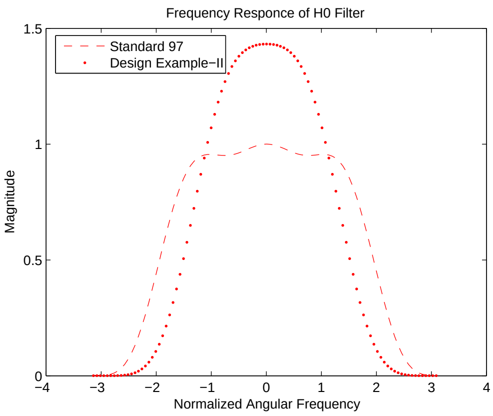

Fig. 7. Frequency response of H_0 Filters

Figure 6 shows the comparison between the frequency response of the modified symmetric filter (derived from the 18th order half band polynomial) and the standard 9/7 filter. A further improvement in the frequency response can be achieved by tuning the a_0 parameter in case of 18th order half band product polynomial. For the 22th order half band product polynomial a_0,a_2 can be tuned for improving the frequency response. Figure 7 shows the comparison between the frequency response of the modified symmetric filter (derived from the 22th order half band polynomial) and the standard 9/7 filter.

V. Summary and Conclusion

In this paper we have presented an algorithm for the design of 9/7 like filters with improved frequency response. It is shown that, using a higher order half band product polynomial, we can improve the frequency response of the filters keeping the same regularity. It is observed that with the higher order of half band product polynomial, the number of independent parameters also increases. This can be used to tune the frequency response further. We also present an algorithm to factor the remainder polynomial into symmetric factors, which yield symmetric filters with an improved frequency response.

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