Analytic Hierarchy Process using Scilab

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Outline

AHP: what and why

- Vested interests make it difficult to analyze a case holistically
- Also, difficulty comparing technologies when many qualitative attributes involved.
- Development perspectives would rather be explicit than 'intuitive'.
- More reasons and theory for today:
  - The influence of development perspectives on the choice of technology, U. Subba Raju, N. Rangaraj and A.W. Date, Technology, Forcasting and Social Change, 1995
  - The Analytic Hierarchy Process, Thomas L. Saaty, Mc Graw Hill (1980)

Scilab/Matlab/Java

- Matlab program was developed by Subba Raju 10 years ago (multi-level).
- Bapuji Kanaparthy (T&D, 2009) upgraded to current Matlab versions (multi-level, with some limitations, bugs)
- Ganesh Ramkrishnan in Java (remote server)
- Bapuji : developed in Scilab (single level): today demo and some theory
- Multi-level in Scilab: volunteers?

Overall procedure

- Fix a development perspective (just one perspective: fine) (Example: businessman, government)
- Decide very very clearly on attributes that matter (for this perspective)
- Attributes are either qualitative or quantitative.
  Qualitative: technologies will be compared pairwise
  Quantitative: technologies have numbers w.r.t. this attribute
- Important: rename qualitative attributes as beneficial attributes
- For example: pollution (if qualitative) $\rightarrow$ eco-friendliness
- Quantitative attributes: can classify them as cost or benefit
- Cost : lower the better, benefit: higher the better

- Every pairwise comparison of n items: $n \times n$ matrix: gives one priority vector of n components.
- Compare attributes to obtain relative importances of the attributes
- Relative importances contained in a vector (priority vector)
- For each qualitative (beneficial) attribute, decide on which technology is more preferable, relatively to other technologies
- How do we compare pairwise and obtain priority vector?
- For quantitative attributes, 'priority vector' obtained directly from the numbers

Pairwise comparison

Suppose we want to compare 3 items it1, it2, it3 between each other.
Construct a matrix $X \in \mathbb{R}^{3\times 3}$
(Matrix with entries as real numbers and 3 rows and 3 columns)
"3 choose 2" $({}^3C_2)$ comparisons: 3 comparisons

$$X = \begin{bmatrix} 1 & a_{12} & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & a_{32} & 1 \end{bmatrix}$$

How to fill entries in X

Convention: $a_{12} := \frac{\text{Importance of it1}}{\text{Importance of it2}}$
$a_{12} > 1$: Item 1 is more important than item 2
$a_{12} = \frac{1}{a_{21}}$ and both are positive
Only strictly upper triangular elements of X to be chosen
When comparing n items pairwise, how many entries to be chosen?
How many strictly upper triangular elements in a square matrix?
Equal?
$\frac{n(n-1)}{2}$

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Consistency

Consider again $X \in \mathbb{R}^{3\times 3}$ with

$$X = \begin{bmatrix} 1 & a_{12} & a_{13} \\ \star & 1 & ? \\ \star & \star & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 2 & 6 \\ \star & 1 & 3 \\ \star & \star & 1 \end{bmatrix}$$

If item 1 is 2 times more important than item 2, and
if item 1 is 6 times more important than item 3, then
item 2 and item 3 compare in what manner ?
Item 2 is 3 times more important than item 3.
Or else, 'not consistent'

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Consistent positive reciprocal matrices

- Positive entries in a square matrix X
- Reciprocal: a_{ij} and a_{ji} are inverses
- Consistent : $a_{ij} = a_{ik} a_{kj}$
- First row decides whole matrix
- Not $\frac{n(n-1)}{2}$ but just $n-1$ entries decide X

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Consistent positive reciprocal matrices

- $a_{ij} = a_{ik}a_{kj}$
- A is a rank one matrix (exactly one eigenvalue is nonzero)
- All diagonal entries are 1.
- Sum of eigenvalues = trace (sum of diagonal elements)
- One eigenvalue = n and rest are all zero

Eigenvalue/eigenvector

Let A be a matrix in $\mathbb{C}^{n \times n}$. A complex number λ is said to be an eigenvalue if there exists some nonzero vector $v \in \mathbb{C}^n$ such that $Av = \lambda v$.

Eigenvector v gets just scaled when operated by A.

(Generally, matrix A both rotates and scales.)

For an eigenvalue λ, if v is an eigenvector then so is $5v$, $-100v$ and $0.5v$, but not $0v$. (v is required to be a nonzero vector.)

Upto scaling, eigenvectors are unique.
(unless we have repeated eigenvalues.)
Eigenvalues can be found as those specific complex numbers λ such that

$$(A - \lambda I)v = 0$$

admits a nonzero solution v.
These λ's must make $A - \lambda I$ singular: solve for roots of $\det(sI - A)$.
For an $n \times n$ matrix A, at most n distinct eigenvalues.
Farthest amongst these from the origin: easier to compute: spectral radius.
(The farthest distance is called spectral radius.)

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Perron Frobenius theorem

Suppose A has all entries positive.

- Exactly one eigenvalue is farthest from origin, and this eigenvalue is positive
- Eigenvector corresponding to this eigenvalue has all entries nonzero and same sign
- (We can ensure all entries in eigenvector v are positive)
- Perron root, Perron Frobenius eigenvalue
- The principal eigenvector contains valuable information
- Graph theory (google/page-rank algorithm), Markov chains, demography

Consistency index (C.I.)

- Principal eigenvector information is less relevant if not consistent
- Consistency index: some careful measure about lack of consistency
- Formula accepted after wide scholarly debate $C.I. = \frac{\lambda_{max} - n}{n-1}$
- 0.1 is 'bad' : thumbrule

Pairwise comparison

$$X = \begin{bmatrix} 1 & a_{12} & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & a_{32} & 1 \end{bmatrix}$$

a_{ij} (entry in i-th row and j-th column)
Number of times i-th property is more important than j-th property.
a_{ij} can be chosen less than one if j-th property should be more important
Priority vector (principal eigenvector) shows relative importances : for qualitative attributes
(Ensure components add up to one.)

Priority vector and importances

Why should priority vector contain importances?
If A is consistent, positive reciprocal matrix, then

$A = vw^T$ (column vector v times row vector w^T, both nonzero)

(Any rank one matrix is like this.)
Here, w is the principal right eigenvector, and v is the principal left eigenvector.
Check this using scilab commands 'spec' to get eigenvalues and (right) eigenvectors, and use `spec(A')` to get left eigenvectors of A. (' means transpose.)
Given that A has *relative* importances, the right eigenvector will have *absolute* importances.

Quantitative attributes

- Machine cost, material cost, waste generated : cost
- Number of employees : benefit/cost (depends on perspective)
- Calculate priority vector directly (rather than pairwise comparison)

Priority vector

Priority vector gets built for

1. all attributes: priority vector of all attributes w
2. for each qualitative attribute, all technologies : columns of Normalized attributes matrix P

Scilab workspace contains

- Number of components in w is number of attributes: column vector
- Number of rows in P matrix: number of technologies, and
- ranking= 'score-vector' $= Pw$ (Matrix times vector) :
- Higher number in ranking vector: that technology is more preferable.
- For each priority vector, check that highest priority value is as expected.
- Consistency index less than 0.1 is a thumbrule.

Example: Gas depot owner's perspective

Suppose we have two technologies for transport of cylinders to individual homes: 1: petrol-powered tempo, 2: bicycle
Attributes (rename to have qualitative attributes as benefits):

Attribute name	Rename (if required)
Dependence-on-fuel, (qualitative)	1. Independence from fuel
Quickness-of-delivery, (qualitative)	2. Quickness already a benefit
Requirement of good-quality-roads for delivery, (qualitative)	non-requirement of good-quality roads or 3: multi-terrain-capability
Pollution (quantitative)	4. Emmissions per delivery: cost
Fuel cost (quantitative)	5. Fuel cost per delivery
labour cost (quantitative)	6. Labour cost per delivery

Sensitivity analysis

How sensitive is ranking to a particular attribute:
Final answer's sensitivity to

1. removal of an attribute
2. 'small' variations in importance of that attribute

$P(i, j)$ means element in the P matrix in i-th row and j-th column.
$P(:, [1\ 2\ 4\ 5])$ means all rows, and columns coming from [1 2 4 5] only
$P(:, :)$ means all rows and all columns (same as P.)
$w([1 : 3\ 5 : 9])$ means all entries from 1 to 9 except 4. (Check yourself first that there are 9 entries.)
First entry is indexed by 1 (and not zero).