Tutorial Sheet EE 636 Matrix Computations: 5^{th} Jan 2023.

Q-1: For vectors $x, y \in \mathbb{R}^n$ show that the "dot-product" $x^T y$ is unchanged if Q acts on each of x, y (i.e. Qx, Qy dot product is same as $x^T y$) when Q is orthogonal

Q-2 Noting that $\|x\|_2^2 = x^T x$, show that $\|Qx\|_2 = \|x\|_2$ for any orthogonal Q & any $x \in \mathbb{R}^n$.

Q-3: Work/formulate 2 converses of statement 2 and show atleast one of them is wrong..

Q-4 Show that $\|A\|_F = \|QA\|_F$ for any $A \in \mathbb{R}^{n \times m}$ & $Q \in \mathbb{R}^{n \times n}$
$= \|A^T\|_F$

(Use Q2 above and $QA = \begin{bmatrix} | & | & & | \\ Qa_1 & Qa_2 & \cdots & Qa_m \\ | & | & & | \end{bmatrix}$, $A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_m \\ | & & | \end{bmatrix}$)

Q-5 Use defn of $\|A\|_2$ to prove $\|A\|_2 = \|UAV^T\|_2$ for any A & any orthogonal U, V.

Q-6 Thus show $\|A\|_2 = \sigma_{max}$ & $\|A\|_F^2 = \Sigma \sigma_i^2$ (σ_i: singular values of A)

Q-7 Show that ~~A~~ if A is rank 1, then $\|A\|_F = \|A\|_2$ and what about converse?

Q-8 In an SVD, $A = U \Sigma V^T$, check that each $u_i v_i^T$ has 2-norm $=1$ & Frobenius norm $=1$ and further.
$(u_i v_i^T)^T (u_j v_j^T) = 0$ for $i \neq j$. Some "orthonormal" set.

Q-9 Suppose A has rank $r < n$ & $A \in \mathbb{R}^{n \times n}$.
Consider "thin SVD" $A = \hat{U} \Sigma_r \hat{V}^T$ ($\hat{U}$ is first r columns of U & $\hat{V}$ is first r columns of V).

Find orthonormal basis for Image of A & Kernel of ~~A~~ A using $\hat{U}$ & $\hat{V}$.

Q-10: Check that transmitting entries of $\hat{U}$ & $\hat{V}$ & Σ_r is much cheaper than entries of $A \in \mathbb{R}^{n \times n}$ for $r << n$.

Find what rank 'r' is "break-even"?
(image compression: lossy ;)

Q-11:

Check that $K_2(A) = 1$ when A is an orthogonal matrix.

Q-12:

Suppose A is diagonal, find conditions on A for $K_2(A) = 1$. Conclude that "ill-conditioned" does not mean $\det(A)$ is too large or too small.

Q-13: For any induced p-norm, show

$$K_p(A)^{-1} \leq \min_{\substack{A+\Delta A \\ \text{singular}}} \frac{\|\Delta A\|_p}{\|A\|_p}.$$

Q-14: Show for induced 2-norm:

$$\left[K_2(A)\right]^{-1} = \min_{\substack{A+\Delta A \\ \text{singular}}} \frac{\|\Delta A\|_2}{\|A\|_2}$$

Q-15: Use $\text{rank}(AB) \leq \min(\text{rank } A, \text{rank } B)$ to conclude sum of r rank 1 matrices is atmost r.

Q-16: Find number of additions & multiplications for $A \cdot B$ for $A, B \in \mathbb{R}^{n \times n}$ using
- inner product procedure
- outer product procedure.

Q-17: $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal: check.

$\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$ is orthogonal: check.