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# A new result on passivity preserving model reduction

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**Abstract**

The problem of model reduction with preservation of passivity is investigated. The approach is based on positive real interpolation, and is inspired by the similarity between Löwner and Pick matrices. The former are important in problems of general rational interpolation while the latter in problems of interpolation by positive real functions. It follows that interpolation of the original set of data together with an appropriately defined mirror-image set of data yields automatically positive real interpolants. Subsequently, we show how this result can be implemented using a Krylov projection procedure. The ensuing model reduction method preserves stability and passivity and can be implemented efficiently for the large-scale systems.


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## 1. Introduction

In VLSI design, the verification of chip performance involves modeling of various parts of the chip. Some of the resulting models are linear, and are defined in the usual state space form

$$\mathbf{\Sigma}: \ \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \tag{1.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state, $\mathbf{u}(t) \in \mathbb{R}^m$ is the input and $\mathbf{y}(t) \in \mathbb{R}^p$ is the output of the system at time $t$, and $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are constant matrices of appropriate dimension. The dimension $n$ of the state is defined as the *complexity* of $\mathbf{\Sigma}$. Depending on the accuracy requirements, this complexity can be very high $n \sim 10^5, 10^6$, and thus direct simulation may prove hard to impossible. Therefore, the need for simplification or *model reduction* of $\mathbf{\Sigma}$, arises. We are thus looking for a system of the form

$$\hat{\mathbf{\Sigma}}: \ \dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t), \quad \hat{\mathbf{y}}(t) = \hat{\mathbf{C}}\hat{\mathbf{x}}(t) + \hat{\mathbf{D}}\mathbf{u}(t), \tag{1.2}$$

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where $\hat{\mathbf{x}}(t) \in \mathbb{R}^k$, $k$ is the complexity of $\hat{\Sigma}$, with $k < n$, and $\hat{\mathbf{y}}(t) \in \mathbb{R}^m$ is the output produced by the input $\mathbf{u}$.

The original model $\mathbf{\Sigma}$ in this case is stable and often passive, and thus during this dimension reduction both *stability* and *passivity* have to be *preserved*. This motivates the problem of reduction of a passive system given in state space form with preservation of stability and passivity; and since the dimension of the $\mathbf{\Sigma}$ may be very high, *numerically efficient* methods are needed for this task.

This problem has been studied by several researchers; for an overview of existing approaches see, e.g. [7,12,17,18,22].

In this paper, we present a new approach to the problem of model reduction which preserves passivity. It combines two ingredients, namely, Krylov projection methods and a new result on positive real interpolation. The first aspect is important because Krylov methods are suitable for application to very high order systems. The second aspect uses rational interpolation as a way to achieve dimension reduction. In particular a choice of interpolation points is proposed which *guarantees* that the reduced model preserves the passivity property of the original high order system.

The paper is organized as follows. After a review of selected results on rational interpolation and the introduction of the Löwner matrix, Section 3 explores the similarity between the Löwner and the Pick matrices. This leads to a particular choice of the interpolation points so that the resulting interpolant is passive (and stable). As a byproduct we also present the solution of the positive real partial realization problem (with respect to the imaginary axis). Finally Section 4 makes use of the results developed in the preceding section together with Krylov projection to come up with the new method of model reduction which preserves passivity. The section concludes with some illustrative examples. There are also two appendices which discuss the derivation of the multiple point Pick and Löwner matrices.

## 2. Rational interpolation and the Löwner matrix

Consider the array of points

$$\mathscr{J} = \{(x_i, y_i) : x_i, y_i \in \mathbb{C}, i = 1, \ldots, N, \; x_i \neq x_j, i \neq j\}. \tag{2.3}$$

We are looking for all rational functions

$$\mathbf{y}(x) = \frac{\mathbf{n}(x)}{\mathbf{d}(x)}, \tag{2.4}$$

where the polynomials $\mathbf{n}$, $\mathbf{d}$ have no common factors, which *interpolate* the points of the array $\mathscr{J}$, i.e.

$$\mathbf{y}(x_i) = y_i, \quad i = 1, \ldots, N. \tag{2.5}$$

The *Lagrange interpolating polynomial* defined as $\mathbf{y}_0(x) = \sum_{j=1}^{N} y_j [\prod_{i\neq j}(x - x_i)][\prod_{i\neq j}(x_j - x_i)]^{-1}$, is the unique polynomial of degree less than $N$ which interpolates the points of the array $\mathscr{J}$. A parametrization of *all* solutions to (2.4), (2.5) can be given as $\mathbf{y}(x) = \mathbf{y}_0(x) + \mathbf{r}(x)\prod_{i=1}^{N}(x - x_i)$, where the parameter $\mathbf{r}(x)$ is an arbitrary rational function with no poles at the $x_i$'s.

The above formula allows one to say very little about the structure of the family of solutions of the interpolation problem (2.4), (2.5). In order to be able to investigate this solution set more closely, we introduce the following (scalar) *parameter*

$$\deg \mathbf{y} = \max\{\deg \mathbf{n}, \deg \mathbf{d}\},$$

which is sometimes referred to as the *McMillan degree* of the rational function $\mathbf{y}$. The following problems arise:

(a) Find the *admissible* degrees of complexity, i.e. those positive integers $\pi$ for which there exist solutions $\mathbf{y}(x)$ to the interpolation problem (2.4), (2.5), with $\deg \mathbf{y} = \pi$.

(b) Given an admissible degree $\pi$, construct *all* corresponding solutions.

**Remark 2.1.** (a) Usually one seeks interpolants with real coefficients. In this case if the complex pair $(x_i, y_i) \in \mathscr{J}$, the associated complex conjugate pair must belong to the array as well: $(x_i^*, y_i^*) \in \mathscr{J}$.

(b) In array (2.3) the points $x_i$ have been assumed *distinct*. In terms of the interpolation problem, this means that only the *value* of the underlying rational function is prescribed at each $x_i$. If the value of successive derivatives at the same points is also prescribed, we are dealing with the *multiple-point* or *confluent* interpolation problem. There is a vast amount of literature on the interpolation problem. The approach followed here has its origins in the papers of Antoulas and Anderson [2], Anderson and Antoulas [4], Antoulas et al. [6]. Some of the results discussed below can also be found in Belevitch [8].

### 2.1. A rational Lagrange-type formula and the Löwner matrix

The idea now is to come up with a Lagrange polynomial-type formula which would be valid for rational functions. Before introducing this formula, we partition the array $\mathscr{J}$ in two disjoint subarrays $\mathscr{J}_C$ and $\mathscr{J}_R$ as follows:

$$\mathscr{J}_C = \{(x_i, y_i) : i = 1, \ldots, r\},$$
$$\mathscr{J}_R = \{(\bar{x}_i, \bar{y}_i) : i = 1, \ldots, p\},$$

where for simplicity of notation some of the points have being redefined as follows: $\bar{x}_i = x_{r+i}$, $\bar{y}_i = y_{r+i}$, $i=1, \ldots, p$, $p+r=N$. Consider the function $\mathbf{y}$ defined through the following equation:

$$\sum_{i=1}^{r} c_i \frac{\mathbf{y}(x) - y_i}{x - x_i} = 0, \quad c_i \neq 0,$$
$$i = 1, \ldots, r, \;\; r \leqslant N.$$

Solving for $\mathbf{y}(x)$ we obtain

$$\mathbf{y}(x) = \frac{\sum_{j=1}^{r} y_j c_j \prod_{i \neq j} (x - x_i)}{\sum_{j=1}^{r} c_j \prod_{i \neq j} (x - x_i)}, \quad c_j \neq 0. \tag{2.6}$$

Clearly, the above formula, which can be regarded as the rational equivalent of the Lagrange interpolating polynomial, interpolates the first $r$ points of the array $\mathscr{J}$, i.e. the points of the array $\mathscr{J}_C$. In order for $\mathbf{y}(x)$ to interpolate the points of the array $\mathscr{J}_R$, the coefficients $c_i$ must satisfy the following equation:

$$\mathbf{L}\,\mathbf{c} = 0,$$

where

$$\mathbf{c} = [\, c_1^* \quad \cdots \quad c_r^* \,]^* \in \mathbb{C}^r$$

and

$$\mathbf{L} = \begin{bmatrix} \frac{\bar{y}_1 - y_1}{\bar{x}_1 - x_1} & \cdots & \frac{\bar{y}_1 - y_r}{\bar{x}_1 - x_r} \\ \vdots & & \vdots \\ \frac{\bar{y}_p - y_1}{\bar{x}_p - x_1} & \cdots & \frac{\bar{y}_p - y_r}{\bar{x}_p - x_r} \end{bmatrix} \in \mathbb{C}^{p \times r}. \tag{2.7}$$

The superscript $(\cdot)^*$ is used to denote complex conjugation followed by transposition (thus if the quantity in question is a scalar, superscript denotes complex conjugation). $\mathbf{L}$ is called the *Löwner matrix* defined by means of the *row array* $\mathscr{J}_R$ and the *column array* $\mathscr{J}_C$, and turns out to be an important tool for studying the rational interpolation problem.

The key result in connection with the Löwner matrix is the following.

**Lemma 2.1.** *Consider the array of points* $\mathscr{J}$ *defined by* (2.3), *consisting of samples taken from a given rational function* $\mathbf{y}(x)$. *Let* $\mathbf{L}$ *be any* $p \times r$ *Löwner matrix with* $p, r \geqslant \deg \mathbf{y}$. *It follows that* $\operatorname{rank} \mathbf{L} = \deg \mathbf{y}$. *Consequently*, *any square sub-Löwner matrix of* $\mathbf{L}$ *of size* $\deg \mathbf{y}$ *is non-singular*.

Given the array of points $\mathscr{J}$ defined by (2.3), we are now ready to tackle the interpolation problem (2.4), (2.5), and in particular, solve the two problems (a) and (b), posed earlier. The following definitions are needed first. (i) The *rank* of the array $\mathscr{J}$ is

$$\operatorname{rank} \mathscr{J} = \max_{\mathbf{L}} \{\operatorname{rank} \mathbf{L}\} =: q,$$

where the maximum is taken over all possible Löwner matrices which can be formed from $\mathscr{J}$. (ii) We will call a Löwner matrix *almost square*, if it has at most one more row than column or vice versa, the sum of the number of rows and columns being equal to $N$. A consequence of Lemma 2.1 is the following.

**Corollary 2.1.** *The rank of all Löwner matrices having at least q rows and q columns is equal to q. Consequently almost square Löwner matrices with at least q rows or columns have rank q.*

Assume that $2q < N$. For any Löwner matrix with $\operatorname{rank} \mathbf{L} = q$, there exists a column vector $\mathbf{c} \neq 0$ of appropriate dimension, satisfying

$$\mathbf{L}\,\mathbf{c} = 0 \quad \text{or } \mathbf{c}^* \mathbf{L} = 0. \tag{2.8}$$

In this case we can attach to $\mathbf{L}$ a rational function denoted by

$$\mathbf{y}_{\mathbf{L}}(x) = \frac{\mathbf{n}_{\mathbf{L}}(x)}{\mathbf{d}_{\mathbf{L}}(x)}, \tag{2.9}$$

using formula (2.6), i.e.

$$\mathbf{n_L}(x) = \sum_{j=1}^{r+1} c_j y_j \prod_{i \neq j} (x - x_i),$$
$$\mathbf{d_L}(x) = \sum_{j=1}^{r+1} c_j \prod_{i \neq j} (x - x_i). \tag{2.10}$$

The rational function $\mathbf{y_L}(x)$ just defined, has the following properties.

**Proposition 2.1.** (a) deg $\mathbf{y_L} \leqslant r \leqslant q < N$. (b) *There is a unique* $\mathbf{y_L}$ *attached to all* $\mathbf{L}$ *and* $\mathbf{c}$ *satisfying* (2.8), *as long as* rank $\mathbf{L} = q$. (c) *The numerator, denominator polynomials* $\mathbf{n_L}$, $\mathbf{d_L}$ *have* $q -$ deg $\mathbf{y_L}$ *common factors of the form* $(x - x_i)$. (d) $\mathbf{y_L}$ *interpolates exactly* $N - q +$ deg $\mathbf{y_L}$ *points of the array* $\mathscr{J}$.

As a consequence of this and Lemma 2.1, we obtain

**Corollary 2.2.** $\mathbf{y_L}$ *interpolates all given points if, and only if,* deg $\mathbf{y_L} = q$ *if, and only if, all* $q \times q$ *Löwner matrices which can be formed from the data array* $\mathscr{J}$ *are non-singular*.

We are now ready to state, from [2], the following result.

**Lemma 2.2.** *Given the array of* $N$ *points* $\mathscr{J}$, *let* rank $\mathscr{J} = q$. (a) *If* $2q < N$, *and all square Löwner matrices of size q which can be formed from* $\mathscr{J}$ *are non-singular, there is a unique interpolating function of minimal degree denoted by* $\mathbf{y}^{\min}(x)$ *and* deg $\mathbf{y}^{\min} = q$. (b) *Otherwise*, $\mathbf{y}^{\min}(x)$ *is not unique and* deg $\mathbf{y}^{\min} = N - q$.

The first part of the theorem follows from the previous corollary. The second part can be justified as follows. Part (b) of the proposition above says that as long as $\mathbf{L}$ has rank $q$ there is a unique rational function $\mathbf{y_L}$ attached to it. Consequently in order for $\mathbf{L}$ to yield a different rational function $\mathbf{y_L}$ defined by (2.9), (2.10), it will have to *lose* rank. This occurs when $\mathbf{L}$ has at most $q-1$ rows. In this case its rank is $q-1$ and there exists a column vector $\mathbf{c}$ such that $\mathbf{Lc} = 0$. Since $\mathbf{L}$ has $N - q + 1$ columns, the degree of the attached $\mathbf{y_L}$ will generically (i.e. for almost all $\mathbf{c}$) be $N - q$. It readily follows that for almost all $\mathbf{c}$, $\mathbf{y_L}$ will interpolate all the points of the array $\mathscr{J}$. This argument shows that there can *never* exist interpolating functions of degree between $q$ and $N - q$. The admissible degree problem can now be solved in terms of the rank of the array $\mathscr{J}$.

**Corollary 2.3.** *Under the assumptions of the lemma, if* deg $\mathbf{y}^{\min} = q$, *the admissible degrees consist of q together with all integers which are greater than or equal to* $N - q$; *if* deg $\mathbf{y}^{\min} = N - q$, *the admissible degrees consist of all integers which are greater than or equal to* $N - q$.

**Remark 2.2.** (i) If $2q = N$, the only solution $\mathbf{c}$ of (2.8) is $\mathbf{c} = \mathbf{0}$. Hence, $\mathbf{y_L}$, defined by (2.9), (2.10), does not exist, and part (b) of Lemma 2.2 applies.

(ii) In order to distinguish between case (a) and case (b) of Lemma 2.2, we *only* need to check the non-singularity of $2q + 1$ Löwner matrices. Construct from $\mathscr{J}$ *any* Löwner matrix of size $q \times (q + 1)$, with row, column sets denoted by $\mathscr{J}_{R_q}$, $\mathscr{J}_{C_q}$, and call it $\mathbf{L}_q$. The Löwner matrix $\mathbf{L}_q^*$ of size $(q + 1) \times q$ is now constructed; its row set $\mathscr{J}_{R_q^*}$ contains the points of the row set $\mathscr{J}_{R_q}$ together with the last point of the column set $\mathscr{J}_{C_q}$; moreover, its column set $\mathscr{J}_{C_q^*}$ contains the points of the column set $\mathscr{J}_{C_q}$ with the exception of the last one. The $2q + 1$ Löwner matrices which need to be checked are the $q \times q$ submatrices of $\mathbf{L}_q$ and $\mathbf{L}_q^*$.

## 3. Positive real interpolation and the Pick matrix

In several applications the rational interpolation problem with a positive realness constraint is relevant. A rational function $\mathbf{y}(x)$ with real coefficients, is called *positive real* if it is stable and it maps the open right-half plane into the closed right-half plane. These conditions can be expressed as follows:

(i) $\mathbf{y}(x)$ is analytic in the open right-hand-plane $\mathscr{R}e(x) > 0$,
(ii) $\mathbf{y}(x) + [\mathbf{y}(x)]^* \geqslant 0$ for $\mathscr{R}e(x) > 0$.

The following is a well-known consequence of this definition.

**Proposition 3.1.** (a) *If* $\mathbf{y}(x)$ *is analytic in* $\mathscr{R}e(x) \geqslant 0$, *and* $\mathscr{R}e(\mathbf{y}(x)) \geqslant 0$ *on the imaginary axis* $\mathscr{R}e(x) = 0$, *then* $\mathbf{y}(x)$ *is positive real.*

(b) *If* $\mathbf{y}(x)$ *is analytic in* $\mathscr{Re}(x) > 0$, *and* $\mathscr{Re}(\mathbf{y}(x)) \geqslant 0$ *on the imaginary axis* $\mathscr{Re}(x) = 0$, *then* $\mathbf{y}(x)$ *is positive real if, and only if, it has simple poles on the imaginary axis with positive residues.*

In this section, we will discuss aspects of the positive real interpolation problem. Given an array of points (2.3), we thus seek rational interpolants which are *positive real*. It is clear that this interpolation problem need not have a solution in general; the array consisting of the two pairs: (1, 1) and (2, −1) for instance, cannot be interpolated by means of a positive real function. According to [23] the necessary and sufficient condition for the existence of a positive real interpolant is that the following Hermitian *Pick matrix*:

$$\mathbf{\Pi} = \begin{bmatrix} \frac{y_1+y_1^*}{x_1+x_1^*} & \cdots & \frac{y_1+y_N^*}{x_1+x_N^*} \\ \vdots & & \vdots \\ \frac{y_N+y_1^*}{x_N+x_1^*} & \cdots & \frac{y_N+y_N^*}{x_N+x_N^*} \end{bmatrix} \in \mathbb{R}^{N\times N}, \qquad (3.11)$$

be positive definite or semi-definite: $\mathbf{\Pi} \geqslant 0$ (recall that for any scalar $z$, $z^*$ denotes its complex conjugate). Furthermore, if rank $\mathbf{\Pi} = k < N$, a unique positive real interpolating function of minimal degree $k$ exists. Otherwise interpolating functions of degree at least $N-1$, always exist.

The positive real interpolation problem and the associated Pick matrix have been extensively studied in the literature; for system theoretic treatments of these issues we refer to the work of Genin and coworkers [10,11], as well as Georgiou and co-workers, see e.g. [13,14], and references therein.

### *3.1. Löwner and Pick matrices: distinct points*

We now compare (2.7) with (3.11). It follows that if the row array for the former is taken as $(x_i, y_i)$, $i = 1, \ldots, N$ and the column array as $(-x_i^*, -y_i^*)$, $i = 1, \ldots, N$, then

$$\mathbf{\Pi} = \mathbf{L}.$$

Given the array $\mathscr{J}$ defined by (2.3), this motivates the definition of the *mirror-image array*:

$$\mathscr{J}^* = \{(-x_i^*, -y_i^*), i = 1, \ldots, N\}. \qquad (3.12)$$

Thus the Löwner matrix constructed with *row array* $\mathscr{J}$ and *column array* $\mathscr{J}^*$ is the same as the Pick matrix. The following is the main result of this section:

**Lemma 3.1.** *All minimal-degree interpolants of the augmented array* $\mathscr{J} \cup \mathscr{J}^*$ *are positive real.*

We will mention three ways of proving this result. The first is an ab initio proof of the fact that interpolation of the original array *together* with the mirror image array, yields positive realness *automatically*; this can be found in [20] and [19]. The second is by means of the *positive real lemma* and can be found in [21].

Finally, the third is by transforming the positive real interpolation problem to a bounded real interpolation problem and then applying the result of [3]; this latter result provides an *algebraization of the bounded real interpolation problem*. Here are a few details. It is well known that the scalar function $\mathbf{y}(x)$ of the complex variable $x$, is positive real if, and only if, the function $\mathbf{w}(x) = [1-\mathbf{y}(x)/1+\mathbf{y}(x)]$ is *bounded real*, that is, it is stable and its magnitude on the imaginary axis does not exceed one. Furthermore the function $\mathbf{w}((1-z)/(1+z))$, where $z=(1-x)/(1+x)$, is bounded real with respect to the unit circle, that is has all its poles inside the unit disc and its magnitude on the unit circle does not exceed one. A moment's reflection shows that the positive real interpolation problem can be transformed into a bounded real interpolation problem either with respect to the imaginary axis or the unit disc. If the original pairs of points for the PR problem are $(x_i, y_i)$, $i = 1, \ldots, N$, the interpolation points of the BR problem with respect to the imaginary axis are $(w_i, x_i)$, where $w_i = (1-y_i)/(1+y_i)$, $i = 1, \ldots, N$, and those of the BR problem with respect to the unit circle are $(w_i, z_i)$, where $z_i = (1-x_i)/(1+x_i)$, $i = 1, \ldots, N$. The necessary and sufficient condition for the solvability of each of these three problems, which will be denoted by "PR", "BR$_1$", "BR$_2$", is the positive definiteness of each of the following Pick matrices, respectively:

$$\mathbf{\Pi}_{\mathrm{PR}} = \left[\frac{y_i + y_j^*}{x_i + x_j^*}\right]_{1\leqslant i,j\leqslant N},$$

$$\mathbf{\Pi}_{\mathrm{BR}_1} = \left[\frac{1 - w_i w_j^*}{x_i + x_j^*}\right]_{1\leqslant i,j\leqslant N},$$

$$\mathbf{\Pi}_{\mathrm{BR}_2} = \left[\frac{1 - w_i w_j^*}{1 - z_i z_j^*}\right]_{1\leqslant i,j\leqslant N}.$$

**Remark 3.1.** (a) The above result provides an algebraization of the positive real interpolation problem. It says, namely, that if $\mathbf{\Pi} \geqslant \mathbf{0}$, the minimal-degree rational functions which interpolate *simultaneously* the original array *and* its mirror image array, are automatically *positive real* and hence *stable* as well.

(b) It readily follows that interpolants of the augmented array constructed by means of the Löwner matrix, satisfy

$$\mathbf{y}(x) + \mathbf{y}(-x)|_{x=x_i} = 0.$$

In general the roots of $\mathbf{y}(x) + \mathbf{y}(-x)$ are called the *spectral zeros* of the underlying linear system (see also (4.16)). Thus the construction of positive real interpolants using the Löwner matrix, forces them to have the *given* interpolation points as *spectral zeros*.

### *3.2. Löwner and Pick matrices: multiple points*

In this section, we will draw attention to the fact that the Pick matrix and the Löwner matrix for multiple interpolation points, although not equal, are *congruent*. As already stated Löwner matrices have the property that rational interpolants can be constructed by computing their nullspace. Consequently these properties are inherited by Pick matrices as well. These considerations lead to Theorem 3.1, which is the analogue of Lemma 3.1, and provides an *algebraization* of the multiple-point positive real interpolation problem.

**Lemma 3.2** (*Positive real Pick matrix for a single repeated point*). *Given the interpolation point* $\lambda \in \mathbb{R}_+$, *with corresponding interpolation values* $(y^0, y^1, \ldots, y^{N-1})$, *let*

$$\psi_k = \frac{(2\lambda)^k}{k!}\, y^k, \quad k = 0, 1, \ldots, N-1.$$

*There exists a positive real function interpolating the above multiple point iff the* symmetric Toeplitz matrix $\mathbf{\Pi} \in \mathbb{R}^{N\times N}$ *is positive semi-definite*

$$\mathbf{\Pi} = \mathbf{\Xi} + \mathbf{\Xi}^* \geqslant 0, \tag{3.13}$$

*where* $\mathbf{\Xi}$ *is upper triangular, Toeplitz, with first row equal to*

$$\mathbf{\Xi}_{1,:} = [\psi_0 \quad \psi_1 \quad \psi_2 + \psi_1 \quad \psi_3 + 2\psi_2 + \psi_1 \\ \times \psi_4 + 3\psi_3 + 3\psi_2 + \psi_1 \\ \times \psi_5 + 4\psi_4 + 6\psi_3 + 4\psi_2 + \psi_1 \quad \cdots]. \tag{3.14}$$

Thus the $(1, \ell)$ entry of $\mathbf{\Xi}$, $\ell \geqslant 2$, is a linear combination of $\psi_i$, $i = 1, \ldots, \ell-1$, whose coefficients are equal to the coefficients of the binomial expansion of $(x+y)^{\ell-2}$.

#### *3.2.1. Connection with the Löwner matrix*

We will now show that $\mathbf{\Pi}$ defined by (3.13) is a special case of the Löwner matrix. Towards this goal, consider the Löwner matrix, which results by having as row array $\{\sigma; y_s^0, y_s^1, \ldots\}$ and as column array $\{\tau; y_t^0, y_t^1, \ldots\}$. The $(k, \ell)$th entry is

$$\mathbf{L}_{k,\ell} = \frac{\mathrm{d}^{k-1}}{\mathrm{d}s^{k-1}} \frac{\mathrm{d}^{\ell-1}}{\mathrm{d}t^{\ell-1}} \left[\frac{\mathbf{y}(s) - \mathbf{y}(t)}{s-t}\right]_{s=\sigma, t=\tau}, \\ k, \ell = 1, 2, \ldots, n.$$

Recall definition (A.1) of $\mathbf{W}$; we also define

$$\mathbf{D} = \mathrm{diag}\left[1, \mu, \frac{\mu^2}{2!}, \frac{\mu^3}{3!}, \cdots\right], \\ \mathbf{J} = \mathrm{diag}\,[1, -1, 1, -1, \cdots], \\ \mathbf{V} = \mathbf{WJ} \quad \text{and} \quad \mu = s - t.$$

It is rather straightforward but tedious to show that a Pick-like matrix which we shall denote by $\mathbf{P}$, is obtained from the Löwner matrix $\mathbf{L}$, by means of the following transformations:

$$\mathbf{P} = \mu \mathbf{V}^{-*}\mathbf{DJLDV}^{-1}, \quad w = y_s^0 - y_t^0. \tag{3.15}$$

As an example, for the case of 4 points, i.e., $(s; y_s^0, y_s^1, y_s^2, y_s^3)$ and $(t; y_t^0, y_t^1, y_t^2, y_t^3)$, the Löwner matrix is shown in Appendix A. The Pick matrix for this case by means of (3.15) on the other hand, is

$$\mathbf{P} = \begin{bmatrix} \phi_0 - \chi_0 & \chi_1 & \chi_1 - \chi_2 & \chi_3 - 2\chi_2 + \chi_1 \\ \phi_1 & \phi_0 - \chi_0 & \chi_1 & \chi_1 - \chi_2 \\ \phi_2 + \phi_1 & \phi_1 & \phi_0 - \chi_0 & \chi_1 \\ \phi_3 + 2\phi_2 + \phi_1 & \phi_2 + \phi_1 & \phi_1 & \phi_0 - \chi_0 \end{bmatrix}, \\ \phi_i = \frac{(s-t)^i}{i!}\, y_s^i, \quad \chi_i = \frac{(s-t)^i}{i!}\, y_t^i.$$

Comparing $\mathbf{P}$ with the Pick matrix obtained in (3.14) we conclude that they match, provided the interpolation points satisfy the following *mirror-image constraints*:

$$\mathbf{\Pi} = \mathbf{P} \text{ for } t = -s \text{ and } \chi_0 = -\phi_0,$$
$$\chi_1 = \phi_1, \chi_2 = -\phi_2, \ldots, \chi_k = (-1)^{k+1}\phi_k, \ldots .$$

An important consequence of the correspondence between Löwner and Pick matrices for multiple points is summarized in the following theorem. It is the counterpart of Lemma 3.1.

**Theorem 3.1.** *Given an array of N interpolating points with multiplicities*

$$\mathscr{J} = \{(x_i;\, y_{i,0}, y_{i,1}, \ldots, y_{i,k_i}), i = 1, \ldots, n;$$
$$x_i \neq x_j, i \neq j, k_1 + \cdots + k_n = N\},$$

*its mirror image array is defined as*

$$\mathscr{J}^* = \{(-x_i^*; -y_{i,0}^*, y_{i,1}^*, \ldots, (-1)^{k_i+1} y_{i,k_i}^*),$$
$$i = 1, \ldots, n; k_1 + \cdots + k_n = N\}.$$

*Let* $\mathbf{\Pi}$ *be the Löwner matrix constructed with* $\mathscr{J}$ *as the row and* $\mathscr{J}^*$ *as the column array. If* $\mathbf{\Pi} \geqslant 0$, *any minimal interpolant of the original array* $\mathscr{J}$ together *with the mirror image array* $\mathscr{J}^*$, *is* positive real.

**Remark 3.2.** Given $\mathbf{c}$ such that $\mathbf{\Pi}\,\mathbf{c} = 0$, the vector $\mathbf{f}$ such that $\mathbf{L}\,\mathbf{f} = \mathbf{0}$, is given by $\mathbf{f} = \mathbf{D}\mathbf{V}^{-1}\mathbf{c}$. It follows that one can construct positive real interpolants by *computing the null space of a Pick matrix*, and by using the formulae applicable to Löwner matrices. For details on this construction using the Löwner matrix we refer to the references cited earlier, namely [2,5,6]. These results are summarized in Appendices A and B at the end of the paper.

### *3.3. Positive real partial realization*

If we wish to investigate the existence of positive real interpolants for a given set of interpolation points, we have to check positive definiteness of the Pick matrix $\mathbf{\Pi}$ defined by (3.11); likewise for the case of repeated interpolation points, one needs to check the Pick matrix $\mathbf{\Pi}$ defined by (3.13); thereby, the $\psi_i$ depend on the interpolation point as well as the value of the corresponding derivative of the interpolating function.

We now turn our attention to the positive real partial realization problem. First we notice that if $\mathbf{y}(s)$ is such an interpolating function for the given data, the function $\mathbf{G}(s) = \mathbf{y}(s^{-1} + \lambda)$ has Markov parameters $\psi_i$. Therefore, we conclude that if the data consists of Markov parameters $\psi_k$, $k = 0, 1, 2, \ldots$, the condition $\mathbf{\Xi} + \mathbf{\Xi}^* \geqslant \mathbf{0}$, is sufficient for the existence of a positive real realization. If however, $\psi_0 > 0$, this matrix is always positive definite for sufficiently small $\lambda$. Therefore if $\psi_0 > 0$, there always exists a *positive real* rational function of degree at most *N*, which matches *any* arbitrary set of Markov parameters $\psi_k \in \mathbb{R}$, $k = 1, \ldots, N$. This leads to the following result which provides the solution of the positive real (pr) realization problem.

**Lemma 3.3.** *Given is a finite sequence of real numbers* $\psi_0, \psi_1, \ldots, \psi_N$. *If* $\psi_0 \neq 0$, *there exists a positive real rational function having the* $\psi_i$, $i = 0, 1, \ldots, N$ *as Markov parameters if, and only if,* $\psi_0 > 0$. *In this case there exist positive real realizations of degree N.*

**Remark 3.3.** (a) A more general statement than the one given in the above lemma is as follows. Consider again the sequence $\psi_i$, $i = 0, 1, \ldots, N$. There exists a positive real realization of this sequence, if either $\psi_0 > 0$, or otherwise the sequence is of the form:

$$(0, \psi_1, 0, \psi_3, \ldots, 0, \psi_{2k-1}, \psi_{2k}, \ldots, \psi_N),$$

where $\psi_{2i-2} = 0$, $\psi_{2i-1} > 0$, for $i = 1, 2, \ldots, k$, while $\psi_{2k}$ which is assumed without loss of generality to be different from zero is in fact negative $\psi_{2k} < 0$; the rest of the sequence $\psi_\ell$, $\ell = 2k+1, \ldots, N$, can be arbitrary.

The proof of this result follows by considering the Markov parameters of the inverse of the rational function whose Markov parameters are $\psi_i$. Thus a sequence with two consecutive zeros has no positive real realization. Finally, it should be mentioned that insight into the problem of positive realness of a rational function given all its Markov parameters is provided in [16].

(b) The positive real (pr) partial realization problem for the discrete time case (i.e. for functions which are stable with respect to the unit disc) was first studied by Carathéodory more than 80 years ago. More recently this problem has been studied by Georgiou and co-workers; for an overview see [14]. The main difference between the discrete- and continuous-time versions of the pr partial problem is that in the former case solvability requires the positive definiteness of an associated Pick matrix (which has Toeplitz structure, and is formed using the Markov parameters) while in contrast, in Lemma 3.3 only the leading Markov parameter $\psi_0$ needs to be positive. This difference is due to the fact that positive real partial realization with respect to the imaginary axis corresponds to *boundary interpolation*.

We conclude this section with an example illustrating the construction of positive real interpolants from the Löwner matrix, in the case of the *positive real partial realization* problem.

**Example 3.1** (*Positive real partial realization*). A positive real rational function with Markov parameters $1, -1, 1, -1,$ is sought. We transform this realization problem to an interpolation problem at the point $t = \frac{1}{2}$; the corresponding interpolating function values are $y_t^0 = 1$, $y_t^1 = -1$, $y_t^2 = 2$, $y_t^3 = -6$, while the mirror image points at $s = -\frac{1}{2}$ are $y_s^0 = -1$, $y_s^1 = -1$, $y_s^2 = -2$, $y_s^3 = -6$.

First, we check whether a solution exists. The Pick matrix (3.13)

$$\mathbf{\Pi} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} > 0,$$

is positive definite; a corresponding solution is computed by means of the Löwner matrix (see Appendix B):

$$\mathbf{L} = \begin{bmatrix} 2 & -3 & 8 & -30 \\ 3 & -6 & 20 & -90 \\ 8 & -20 & 80 & -420 \end{bmatrix};$$

its null space is spanned by $\mathbf{c} = [60\ 120\ 45\ 4]^*$, (that is $\mathbf{Lc} = 0$). We are now ready to construct a minimal interpolant. Again from Appendix B, we need the following expressions:

$$\mathbf{Y}_0 = \frac{\mathbf{y} - 1}{x - \frac{1}{2}}, \quad \mathbf{Y}_1 = \frac{1}{x - \frac{1}{2}} + \frac{\mathbf{y} - 1}{(x - \frac{1}{2})^2},$$
$$\mathbf{Y}_2 = \frac{-2}{x - \frac{1}{2}} + \frac{2}{(x - \frac{1}{2})^2} + \frac{2(\mathbf{y} - 1)}{(x - \frac{1}{2})^3},$$
$$\mathbf{Y}_3 = \frac{6}{x - \frac{1}{2}} - \frac{6}{(x - \frac{1}{2})^2} + \frac{6}{(x - \frac{1}{2})^3} + \frac{6(\mathbf{y} - 1)}{(x - \frac{1}{2})^4}.$$

Solving the equation $60\,\mathbf{Y}_0 + 120\,\mathbf{Y}_1 + 45\,\mathbf{Y}_2 + 4\,\mathbf{Y}_3 = 0,$ for $\mathbf{y}$, yields the following interpolant:

$$\mathbf{y}(x) = \frac{1}{2} \cdot \frac{8x^3 + 60x^2 + 22x + 5}{40x^3 + 20x^2 + 10x + 1}.$$

It is readily checked that at $x = \frac{1}{2}$ the value of $\mathbf{y}$ and of its first three derivatives is $1, -1, 2, -6$, while at $x = -\frac{1}{2}$ the corresponding values are $-1, -1, -2, -36$ (the third derivative in this case does not match, as expected). Furthermore the function is positive real since it is stable and positive on the imaginary axis. Finally it is easily verified that the function

$$\mathbf{G}(x) = \mathbf{y}\left(\frac{1}{x} + \frac{1}{2}\right) = \frac{4x^3 + 11x^2 + 9x + 1}{4x^3 + 15x^2 + 20x + 10},$$

solves the positive real partial realization problem since it is positive real and has Markov parameters $1, -1, 1, -1$.

## 4. Model reduction of passive systems in state space form

*Passive systems*. Consider a linear system $\mathbf{\Sigma}$ described as in (1.1)

$$\mathbf{\Sigma} : \dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{Bu}(t), \quad \mathbf{y}(t) = \mathbf{Cx}(t) + \mathbf{Du}(t).$$

We will assume that $(\mathbf{A}, \mathbf{B})$ is reachable and $(\mathbf{C}, \mathbf{A})$ observable. This system is *passive* if it is stable (i.e. the eigenvalues of $\mathbf{A}$ have non-positive real part), $m = p$, the real part of $\int_{-\infty}^{t} \mathbf{u}^*(\tau)\mathbf{y}(\tau)\,\mathrm{d}\tau$ is non-negative for all time $t$, and for all input–output pairs of functions $(\mathbf{u}, \mathbf{y})$ which satisfy the system equations; this definition can be found in many papers and texts, see e.g. [1]. In the sequel only the case $m = p = 1$ will be considered.

A classical result asserts that passivity of $\mathbf{\Sigma}$ is equivalent to the *positive realness* of the associated transfer function

$$\mathbf{G}(s) = \mathbf{D} + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}.$$

In other words $\mathbf{G}$ must satisfy the conditions listed at the beginning of Section 3. In the sequel we will make use of the *spectral zeros* of passive systems. They are defined as the zeros of the quantity $\mathbf{G}(s)+\mathbf{G}^*(-s)$. In the scalar case if $\mathbf{G}(s)=\frac{\mathbf{n}(s)}{\mathbf{d}(s)}$, $\mathbf{G}^*(-s)=\mathbf{G}(-s)$ and

$$\begin{aligned}\mathbf{G}(s)+\mathbf{G}(-s) &= \frac{\mathbf{n}(s)}{\mathbf{d}(s)}+\frac{\mathbf{n}(-s)}{\mathbf{d}(-s)} \\ &= \frac{\mathbf{n}(s)\mathbf{d}(-s)+\mathbf{d}(s)\mathbf{n}(-s)}{\mathbf{d}(s)\mathbf{d}(-s)} \\ &= \frac{\mathbf{r}(s)\mathbf{r}(-s)}{\mathbf{d}(s)\mathbf{d}(-s)}. \end{aligned} \tag{4.16}$$

The polynomial $\mathbf{r}$ has roots in the (closed) left-half plane and due to the positive realness of $\mathbf{G}$, its coefficients are real; this means that the spectral zeros cannot be purely imaginary. The *stable spectral zeros* are defined as the roots of $\mathbf{r}(s)$. In terms of the state space realization of the transfer function the spectral zeros are all $\lambda$ such that

$$\begin{pmatrix} \mathbf{A} & \mathbf{0} & \mathbf{B} \\ \mathbf{0} & -\mathbf{A}^* & -\mathbf{C}^* \\ \mathbf{C} & \mathbf{B}^* & \mathbf{D}+\mathbf{D}^* \end{pmatrix} - \lambda \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{4.17}$$

has rank less than $2n+1$. If $\mathbf{D}+\mathbf{D}^*$ is invertible, these numbers are the eigenvalues of the following *Hamiltonian* matrix:

$$\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}^* \end{pmatrix} - \begin{pmatrix} \mathbf{B} \\ -\mathbf{C}^* \end{pmatrix} (\mathbf{D}+\mathbf{D}^*)^{-1} (\mathbf{C} \quad \mathbf{B}^*).$$

*Model reduction by projection*. As already mentioned in the introduction, we seek reduced systems of form (1.2), i.e.

$$\hat{\mathbf{\Sigma}} : \dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t),$$
$$\hat{\mathbf{y}}(t) = \hat{\mathbf{C}}\hat{\mathbf{x}}(t) + \hat{\mathbf{D}}\mathbf{u}(t),$$

where the complexity $k$ of $\hat{\mathbf{\Sigma}}$ is (much) less than that of $\mathbf{\Sigma}$: $k<n$. This reduction must preserve both *stability* and *passivity* and it must be numerically efficient.

It is well known that model reduction by means of rational interpolation methods can be implemented efficiently (iteratively) using the *Lanczos* and/or *Arnoldi* procedures.

Suppose that we are given a system $\mathbf{\Sigma} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$, as above, and we wish to find a lower dimensional model $\hat{\mathbf{\Sigma}} = \begin{pmatrix} \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ \hat{\mathbf{C}} & \hat{\mathbf{D}} \end{pmatrix}$, where $\hat{\mathbf{A}} \in \mathbb{R}^{k\times k}$, $k<n$, such that $\hat{\mathbf{\Sigma}}$ preserves some properties of the original system, like stability and passivity. We will study this problem through appropriate projection methods. In other words, we will seek $\mathbf{V} \in \mathbb{R}^{n\times k}$ and $\mathbf{W} \in \mathbb{R}^{n\times k}$ such that $\mathbf{V}\mathbf{W}^*$ is a projection (i.e. $\mathbf{W}^*\mathbf{V}=\mathbf{I}_k$) and

$$\hat{\mathbf{A}} = \mathbf{W}^*\mathbf{A}\mathbf{V}, \quad \hat{\mathbf{B}} = \mathbf{W}^*\mathbf{B}, \quad \hat{\mathbf{C}} = \mathbf{C}\mathbf{V}. \tag{4.18}$$

Given $2k$ distinct points $s_1, \dots, s_{2k}$, let

$$\tilde{\mathbf{V}} = [(s_1\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B} \cdots (s_k\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}], \tag{4.19}$$

$$\tilde{\mathbf{W}} = [(s_{k+1}\mathbf{I}_n - \mathbf{A}^*)^{-1}\mathbf{C}^* \cdots (s_{2k}\mathbf{I}_n - \mathbf{A}^*)^{-1}\mathbf{C}^*]. \tag{4.20}$$

The initial idea of the result below is due to [9] and later developments are due to [15].

**Proposition 4.1.** *Assuming that* $\det \tilde{\mathbf{W}}^*\tilde{\mathbf{V}} \neq 0$, *the projected system* $\hat{\mathbf{\Sigma}}$ *defined by* (4.18) *where* $\mathbf{V}=\tilde{\mathbf{V}}$ *and* $\mathbf{W}=\tilde{\mathbf{W}}(\tilde{\mathbf{V}}^*\tilde{\mathbf{W}})^{-1}$ *interpolates the transfer function of* $\mathbf{\Sigma}$ *at the points* $s_i$:

$$\hat{\mathbf{G}}(s_i) = \mathbf{G}(s_i), \quad i=1,2,\dots,2k.$$

*Model reduction and preservation of passivity*. We will now combine this method of interpolation by projection with the positive real interpolation result stated in Lemma 3.1. If a reduced order model of degree $k$ is sought, the interpolation points $s_1,\dots,s_k$ and $s_{k+1},\dots,s_{2k}$ must be chosen. Because of (4.19) and (4.20) these points have to be samples of $\mathbf{G}(s)$. This leads to the choice of the interpolation points as *spectral zeros* of the original system $\mathbf{\Sigma}$. Here is the main result of this section.

**Lemma 4.1.** *If the interpolation points* $s_j$ *in* (4.19), (4.20), *are chosen as spectral zeros of the original passive system* $\mathbf{\Sigma}$ *defined by* (1.1), *the reduced system* $\hat{\mathbf{\Sigma}}$ *defined by* (4.18) *is both stable and passive.*

**Remark 4.1.** (a) The projector matrices $\tilde{\mathbf{V}}$, $\tilde{\mathbf{W}}$ can actually be obtained without the explicit computation of the spectral zeros. This is achieved through the computation of certain structured invariant subspaces of a generalized eigenvalue problem associated with the structured matrix (4.17). This idea is due to Sorensen and is developed in [21].

(b) If the transfer function of the original system $\mathbf{\Sigma}$ is strictly proper (i.e. $\mathbf{D}=\mathbf{0}$), the reduced order model $\hat{\mathbf{\Sigma}}$ obtained using the proposed method where the interpolation points are *finite* spectral zeros, is *lossless*. In other words, the transfer function is positive real with poles and zeros on the imaginary axis. For a proof of this fact see [19]; an illustration is provided in the examples that follow.

We conclude this section with some examples.

**Example 4.1.** We will now illustrate the above ideas by reducing three systems involving RLC ladder circuits.

(a) The first one has the transfer function

$$\mathbf{G}(s) = \frac{s^2+2s+3}{s(s^2+2s+5)} = \frac{1}{s+\frac{2}{s+2+\frac{3}{s}}},$$

which is positive real. A minimal realization is

$$\mathbf{A} = \begin{bmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & -2 & -\sqrt{3} \\ 0 & \sqrt{3} & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$\mathbf{C} = [1 \quad 0 \quad 0].$$

The spectral zeros of the system are the zeros of

$$\mathbf{G}(s) + \mathbf{G}(-s) = \frac{8}{(s^2+2s+5)(s^2-2s+5)};$$

therefore this system has 4 infinite and no finite spectral zeros. In this case according to [15], expressions (4.19) and (4.20) are modified as follows

$$\mathbf{V} = [\mathbf{B}\ \mathbf{AB}] = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix},$$

$$\tilde{\mathbf{W}}^* = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} \Rightarrow \mathbf{W}^* = (\tilde{\mathbf{W}}^*\mathbf{V})^{-1}\tilde{\mathbf{W}}^*$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

$$\hat{\mathbf{A}} = \mathbf{W}^*\mathbf{AV} = \begin{bmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix},$$
$$\hat{\mathbf{B}} = \mathbf{W}^*\mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{C}} = \mathbf{CV} = [1 \quad 0]$$
$$\Rightarrow \hat{\mathbf{G}}(s) = \hat{\mathbf{C}}(s\mathbf{I}-\hat{\mathbf{A}})^{-1}\hat{\mathbf{B}} = \frac{s+2}{s^2+2s+2}$$
$$\Rightarrow \hat{\mathbf{G}}(s) + \hat{\mathbf{G}}(-s) = \frac{8}{(s^2+2s+2)(s^2-2s+2)}.$$

(b) The second system is defined by the transfer function

$$\mathbf{G}(s) = \frac{s^2+s+3}{s^3+2s^2+6s+5} = \frac{1}{s+1+\frac{2}{s+\frac{3}{s+1}}};$$

a minimal realization of $\mathbf{G}$ is

$$\mathbf{A} = \begin{bmatrix} -1 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{3} \\ 0 & \sqrt{3} & -1 \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = [1\ 0\ 0].$$

The spectral zeros $\lambda_i$ are the zeros of the expression

$$\mathbf{G}(s) + \mathbf{G}(-s)$$
$$= \frac{2(s^4+5s^2+15)}{(s^3+2s^2+6s+5)(-s^3+2s^2-6s+5)},$$

that is

$$\lambda_1 = .8285 + 1.7851i$$
$$\lambda_2 = .8285 - 1.7851i$$
$$\lambda_3 = -.8285 + 1.7851i$$
$$\lambda_4 = -.8285 - 1.7851i$$
$$\lambda_{5,6} = \infty.$$

Thus, choosing the first four spectral zeros we obtain

$$\mathbf{V}_2 = [\,(\lambda_1\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} \quad (\lambda_2\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\,],$$
$$\tilde{\mathbf{W}}_2^* = \begin{bmatrix} \mathbf{C}(\lambda_3\mathbf{I}-\mathbf{A})^{-1} \\ \mathbf{C}(\lambda_4\mathbf{I}-\mathbf{A})^{-1} \end{bmatrix} \Rightarrow \mathbf{W}_2^* = (\tilde{\mathbf{W}}_2^*\mathbf{V}_2)^{-1}\tilde{\mathbf{W}}_2^*.$$
$$\Rightarrow \mathbf{V}_2 = \begin{bmatrix} .3017 & .1400 \\ .1402 & .1998 \\ -.0266 & .1633 \end{bmatrix},$$
$$\mathbf{W}_2^* = (\tilde{\mathbf{W}}_2^*\mathbf{V}_2)^{-1}\tilde{\mathbf{W}}_2^* = \begin{bmatrix} 1.6571 & 2.6758 & -4.6954 \\ 0 & .9429 & 4.9710 \end{bmatrix}.$$

The corresponding reduced order system is given by

$$\hat{\mathbf{A}}_2 = \mathbf{W}_2^*\mathbf{AV}_2 = \begin{bmatrix} -.8285 & -1.7851 \\ 1.7851 & .8285 \end{bmatrix},$$
$$\hat{\mathbf{B}}_2 = \mathbf{W}_2^*\mathbf{B} = \begin{bmatrix} 1.6571 \\ 0 \end{bmatrix},$$
$$\hat{\mathbf{C}}_2 = \mathbf{CV}_2 = [.3017 \quad .1400]$$
$$\Rightarrow \hat{\mathbf{G}}_2(s) = \frac{s}{2s^2+5}.$$

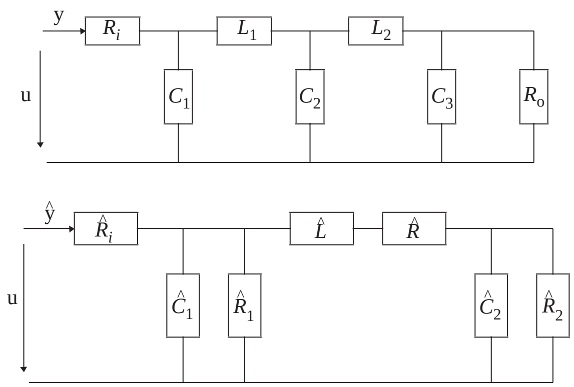


Fig. 1. Original RLC circuit $\mathbf{\Sigma}$ (above) and reduced-order RLC circuit $\mathbf{\Sigma}_2$ (below).

Furthermore to obtain a first-order model we need to use the two infinite spectral zeros. Thus $\mathbf{W}_1^* = \mathbf{C}$ and $\mathbf{V}_1 = \mathbf{B}$ (notice that $\mathbf{W}_1^*\mathbf{V}_1 = 1$). Thus $\hat{\mathbf{A}}_1 = \mathbf{W}_1^*\mathbf{A}\mathbf{V}_1 = -1$, $\hat{\mathbf{B}}_1 = \mathbf{W}_1^*\mathbf{B} = 1$, $\hat{\mathbf{C}}_1 = \mathbf{C}\mathbf{V}_1 = 1$; this implies that $\hat{\mathbf{G}}_1(s) = 1/(s+1)$.

(c) Finally, we consider the RLC ladder network shown in Fig. 1 (top). The state variables are: $x_1$, the voltage across $C_1$; $x_2$, the current through $L_1$; $x_3$, the voltage across $C_2$; $x_4$, the current through $L_2$; and $x_5$, the voltage across $C_3$. The input is the voltage $u$ and the output is the current $y$ as shown in the figure below. We assume that all the capacitors and inductors have the value $\frac{1}{10}$, while $R_i = \frac{1}{2}$, $R_o = 5$. The transfer function turns out to be $\mathbf{G}(s) = \frac{2s^5+4s^4+800s^3+1200s^2+60000s+40000}{s^5+22s^4+440s^3+6600s^2+38000s+220000}$, while a minimal realization is:

$$\mathbf{A} = \begin{bmatrix} -20 & -10 & 0 & 0 & 0 \\ 10 & 0 & -10 & 0 & 0 \\ 0 & 10 & 0 & -10 & 0 \\ 0 & 0 & 10 & 0 & -10 \\ 0 & 0 & 0 & 10 & -2 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 20 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = [-2 \quad 0 \quad 0 \quad 0 \quad 0], \quad \mathbf{D} = 2.$$

By earlier formulas the zeros of the stable spectral factor are: $\lambda_{1,2} = -0.536 \pm 17.367i$, $\lambda_{3,4} = -1.593 \pm 10.073i$, and $\lambda_5 = -2.113$; thus the zeros of the anti-stable spectral factor are $\lambda_{6,7} = -\lambda_{1,2}$, $\lambda_{8,9} = -\lambda_{3,4}$, $\lambda_{10} = -\lambda_5$.

We will construct approximants of dimension $k=3$. Notice that in order to end up with real reduced matrices $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$, whenever a complex spectral zero is selected, its complex conjugate has to be selected as well. Recall formulas (4.19), (4.20). There are two possible choices of spectral zeros, which give rise to distinct systems, namely (a) $s_1 = \lambda_1$, $s_2 = \lambda_2$, $s_3 = \lambda_5$, $s_4 = \lambda_6$, $s_5 = \lambda_7$, $s_6 = \lambda_{10}$ and (b) $s_1 = \lambda_3$, $s_2 = \lambda_4$, $s_3 = \lambda_5$, $s_4 = \lambda_7$, $s_5 = \lambda_8$, $s_6 = \lambda_{10}$. We will denote the resulting systems by $\mathbf{\Sigma}_1$, $\mathbf{\Sigma}_2$, respectively.

In addition, we will compute two further reduced order systems of the same complexity. The first denoted by $\mathbf{\Sigma}_{\text{bal}}$ is obtained by balanced truncation, while the second $\mathbf{\Sigma}_{\text{prbal}}$ is obtained by positive real balanced truncation. For details see chapter 7.5 of the book [1].

In addition the system $\mathbf{\Sigma}_2$ is as follows

$$\mathbf{A}_2 = \begin{bmatrix} -17.40 & -7.79 & 0 \\ 7.79 & -0.38 & -6.34 \\ 0 & 6.34 & -0.76 \end{bmatrix},$$

$$\mathbf{B}_2 = \begin{bmatrix} 5.82 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C}_2 = [-5.82 \quad 0 \quad 0],$$

$$\hat{\mathbf{D}} = 2.$$

The transfer function $\mathbf{G}_2(s) = \frac{2s^3+3.17s^2+203.38s+128.52}{s^3+18.54s^2+121.10s+751.30}$. Furthermore a realization in terms of RLC elements is shown in the lower part of Fig. 1; their values are: $\hat{R}_i = 0.5$, $\hat{C}_1 = 0.118$, $\hat{R}_1 = 19.432$, $\hat{L} = 0.140$, $\hat{R} = 0.053$, $\hat{C}_2 = 0.178$, $\hat{R}_2 = 7.360$.

The frequency responses of the four approximants are plotted together with that of the original system in Fig. 2. Finally, the frequency responses of the four error systems are plotted in Fig. 3.

## 5. Conclusions

Inspired by the relationship between the Löwner matrix and the Pick matrix, a new method for positive real rational interpolation is proposed. This method yields rational functions, which interpolate the original set of points together with an associated *mirror image* set of points. If the associated Pick matrix is positive (semi) definite the minimal degree interpolants are positive real. This result is extended to the case

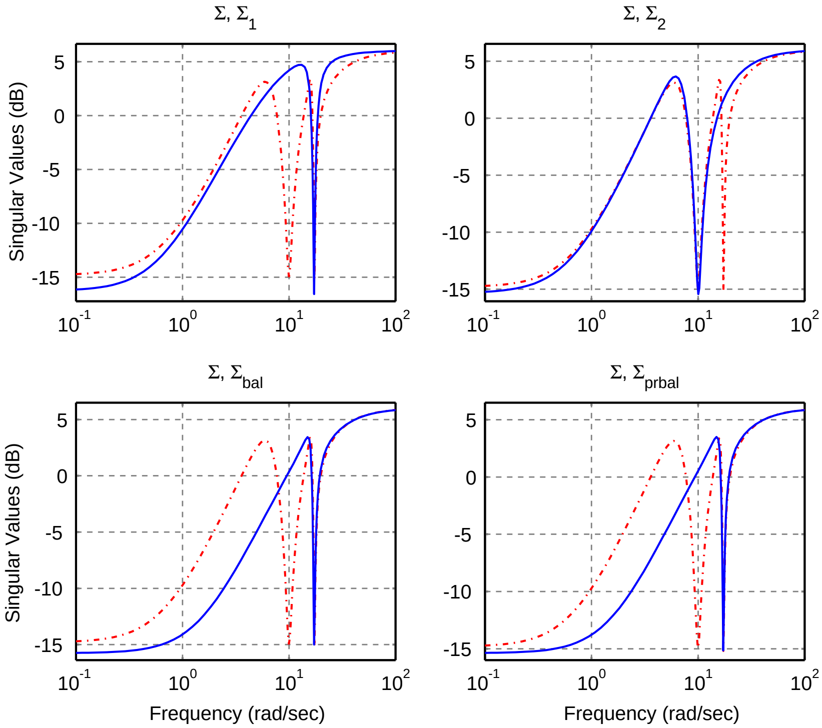


Fig. 2. Frequency response curves. In all four panes, the dash–dot curve is that of the original system, while the continuous curves are those of the system $\mathbf{\Sigma}_1$ (upper left-hand pane), $\mathbf{\Sigma}_2$ (upper right-hand pane), $\mathbf{\Sigma}_{\rm bal}$ (lower left-hand pane), and $\mathbf{\Sigma}_{\rm prbal}$ (lower right-hand pane), respectively.

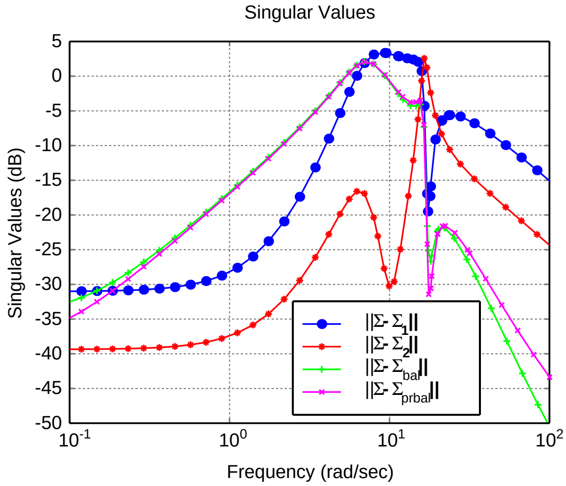


Fig. 3. Frequency response curves of the four error systems, namely $\mathbf{\Sigma}-\mathbf{\Sigma}_1$, $\mathbf{\Sigma}-\mathbf{\Sigma}_2$, $\mathbf{\Sigma}-\mathbf{\Sigma}_{\rm bal}$, and $\mathbf{\Sigma}-\mathbf{\Sigma}_{\rm prbal}$.

of interpolation with multiplicities (confluent case), namely to the case where besides the value a number of derivatives of the function at the given points are specified. In particular, the problem of positive real partial realization (all interpolation information consists of Markov parameters) is addressed and it is shown that in contrast to its discrete-time counterpart, if the leading Markov parameter is nonzero, its positivity is the only condition needed for solvability.

Subsequently, this result is combined with the rational Krylov projection procedure to obtain a model reduction method, which preserves stability and passivity and can be applied to large-scale systems.

## Appendix A

Consider the linear system described by the transfer function $\mathbf{G}(s)$. Then $(\mathbf{u}_k, \mathbf{y}_k)$ is an input–output pair for this system, where

$$\mathbf{u}_k(t) = \frac{t^k}{k!}\mathrm{e}^{\lambda t},$$
$$\mathbf{y}_k(t) = \sum_{\ell=0}^{k} \frac{\mathbf{G}^{(\ell)}}{\ell!}\,\mathbf{u}_{k-\ell}(t), \quad k = 0, 1, \ldots, n,$$
where $\mathbf{G}^{(\ell)} = \left.\dfrac{\mathrm{d}^\ell \mathbf{G}(s)}{\mathrm{d}s^\ell}\right|_{s=\lambda}$.

Now, if $\mathbf{u}(t) = \sum_{i=0}^{n} \alpha_i\, \mathbf{u}_i(t)$, $\alpha_i \in \mathbb{C}$, the output will be $\mathbf{y}(t) = \sum_{i=0}^{n} \alpha_i\, \mathbf{y}_i(t)$. The positive realness condition is obtained by requiring that $\mathscr{Re}\int_{-\infty}^{t} \mathbf{u}(\tau)^*\mathbf{y}(\tau)\,\mathrm{d}\tau \geqslant 0$, for all $t \in \mathbb{R}$ and all $\alpha_i \in \mathbb{C}$. Let $\mathbf{U}_n = [\mathbf{u}_0\ \mathbf{u}_1\ \cdots\ \mathbf{u}_n]$ and $\mathbf{Y}_n = [\mathbf{y}_0\ \mathbf{y}_1\ \cdots\ \mathbf{y}_n]$. Then $\mathbf{Y}_n = \mathbf{U}_n\mathbf{X}$, where

$$\mathbf{X} = \begin{bmatrix} \frac{\mathbf{G}^{(0)}}{0!} & \frac{\mathbf{G}^{(1)}}{1!} & \cdots & \frac{\mathbf{G}^{(n)}}{n!} \\ & \frac{\mathbf{G}^{0}}{0!} & \cdots & \frac{\mathbf{G}^{(n-1)}}{(n-1)!} \\ & & \ddots & \\ & & & \frac{\mathbf{G}^{(0)}}{0!} \end{bmatrix}.$$

Thus $\int \mathbf{u}^*\mathbf{y} = \int [\alpha_0^*\ \alpha_1^*\ \cdots\ \alpha_n^*]\,\mathbf{U}_n^*\,\mathbf{Y}_n[\alpha_0^*\ \alpha_1^*\ \cdots\ \alpha_n^*]^*$. Therefore, we need to compute the integral $\Phi = \int \mathbf{U}_n^*\mathbf{U}_n$, and consequently the following matrix must be positive semi-definite: $\mathbf{M} = \Phi\,\mathbf{X} + \mathbf{X}^*\,\Phi^* \geqslant 0$.

**Proposition A.1.** *The following holds* $\Phi = [\frac{1}{\sqrt{2\lambda}}\,\mathbf{T}^*\,\mathbf{\Delta}\mathbf{W}^*][\mathbf{W}\mathbf{\Delta}\mathbf{T}\,\frac{1}{\sqrt{2\lambda}}]$, *where*

$$\mathbf{T} = \begin{bmatrix} 1 & \frac{t}{1!} & \cdots & \frac{t^n}{n!} \\ & 1 & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \mathrm{e}^{\lambda t},$$

$$\mathbf{\Delta} = \mathrm{diag}\left[1,\ \frac{1}{2\lambda},\ \frac{1}{(2\lambda)^2}, \cdots \frac{1}{(2\lambda)^n}\right],$$

$$\mathbf{W} = \begin{bmatrix} 1 & -1 & 1 & -1 & \cdots \\ 0 & 1 & -2 & 3 & \\ 0 & 0 & 1 & -3 & \cdots \\ 0 & 0 & 0 & 1 & \\ \vdots & & \vdots & & \vdots \end{bmatrix}. \tag{A.1}$$

Furthermore, noticing that upper triangular Toeplitz matrices commute, that is $\mathbf{TX} = \mathbf{XT}$ we have

$$\begin{aligned} \mathbf{\Phi X} &= \left[\frac{1}{\sqrt{2\lambda}}\,\mathbf{T}^*\mathbf{\Delta W}^*\right]\left[\mathbf{W\Delta T}\,\frac{1}{\sqrt{2\lambda}}\right]\mathbf{X} \\ &= \mathbf{T}^*\left[\frac{1}{\sqrt{2\lambda}}\mathbf{\Delta W}^*\right]\left[\mathbf{W\Delta X}\frac{1}{\sqrt{2\lambda}}\right]\mathbf{T} \\ &= \mathbf{T}^*\mathbf{\Delta}\frac{1}{\sqrt{2\lambda}}\left[\mathbf{W}^*\mathbf{W}\bar{\mathbf{X}}\right]\frac{1}{\sqrt{2\lambda}}\mathbf{\Delta T}, \end{aligned}$$

where $\bar{\mathbf{X}}\mathbf{\Delta} = \mathbf{\Delta X}$. Thus $\mathbf{\Phi X} + \mathbf{X}^*\mathbf{\Phi}^* \geqslant \mathbf{0}$ iff $\mathbf{W}^*\mathbf{W}\bar{\mathbf{X}} + \bar{\mathbf{X}}^*\mathbf{W}^*\mathbf{W} \geqslant \mathbf{0}$ iff $\mathbf{W}\bar{\mathbf{X}}\mathbf{W}^{-1} + \mathbf{W}^{-*}\bar{\mathbf{X}}^*\mathbf{W}^* \geqslant \mathbf{0}$; this latter matrix $\mathbf{\Xi} = \mathbf{W}\bar{\mathbf{X}}\mathbf{W}^{-1}$, is a Toeplitz matrix which depends on the values of the function and the particular interpolation point. The main conclusion is stated in Lemma 3.2.

## Appendix B

Below is the $4 \times 3$ Löwner matrix constructed from the two multiple interpolation points: $(s;\, y_s^0,\, y_s^1,\, y_s^2)$ and $(t;\, y_t^0,\, y_t^1,\, y_t^2,\, y_t^3)$

$$\mathbf{L} = \begin{bmatrix} \frac{w}{\mu} & -\frac{y_t^1}{\mu} + \frac{w}{\mu^2} & -\frac{y_t^2}{\mu} - \frac{2y_t^1}{\mu^2} + \frac{2w}{\mu^3} & -\frac{y_t^3}{\mu} - \frac{3y_t^2}{\mu^2} - \frac{6y_t^1}{\mu^3} + \frac{6w}{\mu^4} \\ \frac{y_s^1}{\mu} - \frac{w}{\mu^2} & \frac{y_s^1 + y_t^1}{\mu^2} - \frac{2w}{\mu^3} & \frac{y_t^2}{\mu^2} + \frac{2y_s^1 + 4y_t^1}{\mu^3} - \frac{6w}{\mu^4} & \frac{y_t^3}{\mu^2} + \frac{6y_t^2}{\mu^3} + \frac{6y_s^1 + 18y_t^1}{\mu^4} - \frac{24w}{\mu^5} \\ \frac{y_s^2}{\mu} - \frac{2y_s^1}{\mu^2} + \frac{2w}{\mu^3} & \frac{y_s^2}{\mu^2} - \frac{2y_t^1 + 4y_s^1}{\mu^3} + \frac{6w}{\mu^4} & \frac{2y_s^2 - 2y_t^2}{\mu^3} - \frac{12y_s^1 + 12y_t^1}{\mu^4} + \frac{24w}{\mu^5} & -\frac{2y_t^3}{\mu^3} + \frac{6y_s^2 - 18y_t^2}{\mu^4} - \frac{48y_s^1 + 72y_t^1}{\mu^5} + \frac{120w}{\mu^6} \end{bmatrix},$$

where $\mu = s - t$ and $w = y_s^0 - y_t^0$. Furthermore, if $\mathbf{c} \in \mathbb{R}^4$ is such that $\mathbf{Lc} = 0$, following the theory outlined in Section 2, a minimal degree interpolating function is recovered by solving for $y(x)$ the equation $c_1\,\mathbf{Y}_0 + c_2\,\mathbf{Y}_1 + c_3\,\mathbf{Y}_2 + c_4\,\mathbf{Y}_3 = 0$, where

$$\mathbf{Y}_0 = \frac{\mathbf{y}(x) - y_t^0}{x - t}, \quad \mathbf{Y}_1 = -\frac{y_t^1}{x - t} + \frac{\mathbf{y}(x) - y_t^0}{(x - t)^2},$$
$$\mathbf{Y}_2 = -\frac{y_t^2}{x - t} - 2\frac{y_t^1}{(x - t)^2} + 2\frac{\mathbf{y}(x) - y_t^0}{(x - t)^3},$$

$$\mathbf{Y}_3 = -\frac{y_t^3}{x - t} - 3\frac{y_t^2}{(x-t)^2} - 6\frac{y_t^1}{(x-t)^3} + 6\frac{\mathbf{y}(x) - y_t^0}{(x-t)^4}.$$

## References

[1] A.C. Antoulas, Approximation of large-scale dynamical systems, Adv. Des. Control, SIAM, Philadelphia, 2004.

[2] A.C. Antoulas, B.D.O. Anderson, On the scalar rational interpolation problem, IMA J. Math. Control and Inform. (special issue on parametrization problems, D. Hinrichsen, J.C. Willems (Eds.)) 3 (1986) 61–88.

[3] A.C. Antoulas, B.D.O. Anderson, On the problem of stable rational interpolation, Linear Algebra Appl. 122–124 (1989) 301–329.

[4] B.D.O. Anderson, A.C. Antoulas, Rational interpolation and state variable realization, Linear Algebra Appl. (Special Issue on Matrix Problems) 137/138 (1990) 479–509.

[5] A.C. Antoulas, B.D.O. Anderson, State space and polynomial approaches to rational interpolation, in: M.A. Kaashoek, J.H. van Schuppen, A.C.M. Ran (Eds.), Progress in Systems and Control Theory III: Realization and Modeling in System Theory, Birkhäuser, Basel, 1990, pp. 73–82.

[6] A.C. Antoulas, J.A. Ball, J. Kang, J.C. Willems, On the solution of the minimal rational interpolation problem, Linear Algebra Appl. (Special Issue on Matrix Problems) 137/138 (1990) 511–573.

[7] Z. Bai, R. Freund, A partial Padé-via-Lanczos method for reduced order modeling, Linear Algebra Appl. 332–334 (2001) 139–164.

[8] V. Belevitch, Interpolation matrices, Philips Res. Rep. 25 (1970) 337–369.

[9] C. De Villemagne, R. Skelton, Model reduction using a projection formulation, Int. J. Control 40 (1987) 2142–2169.

[10] Ph. Delsarte, Y. Genin, Y. Kamp, The Nevanlinna Pick problem for matrix-valued functions, SIAM J. Appl. Math. 36 (1979) 47–61.

[11] Ph. Delsarte, Y. Genin, Y. Kamp, On the role of the Nevanlinna Pick problem in circuit and system theory, Circuit Theory Appl. 9 (1981) 177–187.

[12] R.W. Freund, Passive reduced-order modeling via Krylov subspace methods, Numerical Analysis Manuscripts No. 00-3-02, Bell Laboratories, 2000.

[13] T.T. Georgiou, The interpolation problem with a degree constraint, IEEE Trans. Automatic Control AC-44 (1999) 631–635.

[14] T.T. Georgiou, Analytic interpolation and the degree constraint, Int. J. Appl. Math. Comput. Sci. 11 (2001) 271–279.

[15] E.J. Grimme, Krylov projection methods for model reduction, Ph.D. Thesis, University of Illinois, Urbana-Champaign, 1997.

[16] N. Hamada, B.D.O. Anderson, Markov parameter characterization of the strict positive real problem, IEEE Trans. Circuits Systems CAS: 31 (1984) 814–819.

[17] S.-Y. Kim, N. Gopal, L.T. Pillage, Time-domain macromodels for VLSI interconnect analysis, IEEE Trans. Computer-Aided Design Integrated Circuits Systems 13 (1994) 1257–1270.

[18] L. Knockaert, D. De Zutter, Passive reduced order multiport modeling: the Padé-Laguerre, Krylov-Arnoldi-SVD connection, Int. J. Electron. Commun. AEÜ-53 (1999) 254–260.

[19] A.J. Mayo, The positive real interpolation problem in a behavioral setting, MS Thesis, Department of Electrical and Computer Engineering, Rice University, 2004.

[20] A.J. Mayo, A.C. Antoulas, A behavioral approach to positive real interpolation and realization, J. Math. Computer Modeling of Dyn. Systems (Special Issue on The Behavioral Approach to Systems and Modeling, H.L. Trentelman, P. Rapisarda) 8 (2002) 445–455.

[21] D.C. Sorensen, Passivity preserving model reduction via interpolation of spectral zeros, Systems Control Lett. 10.1016/j.sysconle.2004.07.006.

[22] J.M. Wang, C.-C. Chu, Q. Yu, E.S. Kuh, On projection-based algorithms for model-order reduction of interconnects, IEEE Trans. Circuits Systems—I: Fund. Theory Appl. CAS: 49 (2002) 1564–1585.

[23] D.C. Youla, M. Saito, Interpolation with positive real functions, J. Franklin Institute 284 (1967) 77–108.