Behaviors: more kinds of representations, controllability and elimination theorem

Notes for lecture 10 (May 12th, 2014)

In this lecture we will see some more kinds of representations: latent variable representations, image representations, state space systems and we will see the definition of controllability of a behavior.

Recall that

$\mathtt{m}$: number of inputs,	$\mathtt{p}$: number of outputs
$\mathtt{w}$: number of 'manifest' variables:	typically $\mathtt{m} + \mathtt{p}$
$\mathtt{n}$: (minimum) number of states	(McMillan degree)

1 More on polynomial matrices

In the previous lecture, we studied the Smith canonical form

$$U(\xi)R(\xi)V(\xi) = \begin{bmatrix} D(\xi) & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

with $D(\xi) \in \mathbb{R}^{\mathtt{r}\times\mathtt{r}}[\xi]$ being diagonal with all nonzero and monic polynomials: $\mathtt{r}$ is the (normal) rank of the polynomial matrix of $R(\xi)$. We motivated U as elementary *row* operations that do not change the set of solutions to $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. We will not delve on significance of V: we just note here that V is a 'coordinate transformation' that involves not just linear combinations of various components of variable w, but also *derivatives* of these components. Unimodularity of V ensures this transformation is one-to-one and onto, and hence is a coordinate transformation.

Please verify following facts (no need to submit these).

Fact 1.1 Most of the following can be solved by partitioning U and V of equation (1) conforming to that of the RHS. Also consider partitioning U^{-1} and V^{-1}.

- For any polynomial matrix R, there exists a unimodular U such that $U(\xi)R(\xi) = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$, with R_1 having full row rank.
- Suppose R is full row rank. Then R can be factored into $R(\xi) = F(\xi)R_c(\xi)$ with F square and nonsingular and $R_c(\xi)$ being left-prime[1]. Then,

[1] A full row rank polynomial matrix $R_c(\xi)$ is called left-prime if its Smith form equals $[I \quad 0]$, for an identity matrix I of suitable size.

  - $\det F$ is equal to a constant multiple of the diagonal matrix D that arises in the Smith form of R.
  - F and R_c are, in general, not unique.
  - Any polynomial vector $p \in \mathbb{R}^{\mathtt{w}}[\xi]$ such that $R(\xi)p(\xi) = 0$ satisfies $R_c(\xi)p(\xi) = 0$

- If $R_c(\xi)$ is left-prime, then $R_c(\xi)$ is full row rank.

- Suppose R_c is full row rank. Then, the following are equivalent.
  - $R_c(\xi)$ is left-prime,
  - $R_c(\lambda)$ is full row rank for every complex number $\lambda \in \mathbb{C}$,
  - Whenever[2] $R_c(\xi)$ can be factored into $R_c(\xi) = F(\xi)R_2(\xi)$ with F nonsingular, then F is unimodular,
  - There exists a *polynomial* right inverse $Q \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, i.e. $R_c(\xi)Q(\xi) = I$.

In view of the first fact above, we might as well assume any kernel representation we begin with $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ has $R(\xi)$ of full row rank. We call such a representation *minimal kernel representation*: this is without loss of generality.

The following exercise relates Jordan canonical form of A, its algebraic/geometric eigenvalues to the Smith canonical form of $\xi I - A$. Of course, determinant of D in the Smith form is the characteristic polynomial of A. Further, the sizes are same and the degrees of the polynomials in D have to add up to size of A.

Exercise 1.2 Suppose $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$. Let $n_a(\lambda_i)$ and $n_g(\lambda_i)$ denote the algebraic and geometric multiplicity of an eigenvalue λ_i.

- Suppose every eigenvalue has geometric multiplicity one, then show that $d_{\mathtt{n}}$ in the Smith form of $\xi I - A$ equals the characteristic polynomial of A.

- Consider the Smith canonical form of $\xi I - A$. Show that the number of 'ones' along the diagonal in the Smith canonical form equals $\mathtt{n} - \max_{\lambda_i} n_g(\lambda_i)$.

- Show that the number of polynomials d_i that have $(\xi - \lambda_i)$ as a factor is the geometric multiplicity of λ_i.

- Find[3] the Jordan canonical form of A, where A is such that $\xi I - A$ has the two polynomials 1 and $(\xi - 2)^2$ along the diagonal.

[2]This statement motivates the use of 'left-prime'.

[3]More generally, the Smith form of $\xi I - A$ contains all the information about the Jordan canonical form of A, and conversely, given the Jordan canonical form of A, the Smith form of $\xi I - A$ can be found.

- Use PBH test to show that (A, B) is controllable if and only if $[\xi I - A \;\; B]$ is left-prime.
- Show that the roots of the polynomials in the Smith form of $[\xi I - A \;\; B]$ are the uncontrollable eigenvalues of A.

2 Controllability

The set of LTI behaviors described by differential equations in $\mathtt{w}$ number of variables is denoted by $\mathfrak{L}^{\mathtt{w}}$. Equivalently, $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ if $\mathfrak{B}$ is the set of solutions to $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, for a polynomial matrix $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$.

A system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called controllable if for any w_1 and $w_2 \in \mathfrak{B}$, there exist $w_3 \in \mathfrak{B}$ and $T \geqslant 0$ such that

$$w_3(t) = \begin{cases} w_1(t) \text{ for } t \leqslant 0 \;, \\ w_2(t) \text{ for } t \geqslant T. \end{cases}$$

***Theorem* 2.1** *Let $\mathfrak{B} \in \mathfrak{L}^{w}$ and suppose $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is a minimal kernel representation. Then, the following are equivalent.*

1. *$\mathfrak{B}$ is controllable,*
2. *$R(\xi)$ is left-prime, i.e. $R(\lambda)$ is full row rank for every complex number $\lambda \in \mathbb{C}$.*
3. *$\mathfrak{B}$ has an image representation: there exists $M(\xi) \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ such that*

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{w}) \mid \text{ there exists } \ell \text{ such that } w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell\}.$$

3 Elimination of 'latent variables'

Consider again:

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{ there exists } \ell \text{ such that } w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell\}.$$

Often, additional variables (like ℓ) than the ones of interest (here: w).
Call all auxiliary variables: latent variables.

Sometimes latent variables inevitable when modeling systems from first principles.

In general,

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{ there exists } \ell \text{ such that } R(\tfrac{\mathrm{d}}{\mathrm{d}t})w + M(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0\}.$$

Project (w, ℓ) behavior to just w-variables.

Does $\mathfrak{B}$ have a *kernel representation*?

Always possible for $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, but not for all function-spaces.

***Theorem* 3.1** *Consider $\mathfrak{B}_{\text{full}}$ described by $R(\frac{\mathrm{d}}{\mathrm{d}t})w + M(\frac{\mathrm{d}}{\mathrm{d}t})\ell = 0$ (for polynomial matrices R and M).*
Then, there exists a kernel representation ($R_2(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$) for $\mathfrak{B}$ defined by

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{w}) \mid \text{ there exists } \ell \quad \text{such that} \quad R(\tfrac{\mathrm{d}}{\mathrm{d}t})w + M(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell = 0\}.$$

Obtain R_2 as follows. Find a unimodular U such that $U(\xi)M(\xi) = \begin{bmatrix} M_1 \\ 0 \end{bmatrix}$. Partition U conformably into $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$. Define $R_2(\xi) := U_2(\xi)R(\xi)$. Then, $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R_2(\frac{\mathrm{d}}{\mathrm{d}t})w = 0\}$. In general ker $R(\frac{\mathrm{d}}{\mathrm{d}t}) \supseteq \mathfrak{B}$: equality for $\mathfrak{C}^\infty$.

Note that the equality is not true for $\mathfrak{D}$: the set of compactly supported $\mathfrak{C}^\infty$ functions.

Note: U_2 used above is a so-called Maximal Left Annihilator of M. For a polynomial matrix $M(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$, define a Maximal Left Annihilator (MLA) $P(\xi) \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ if the following is satisfied:

- $P(\xi)M(\xi) = 0$
- $P(\xi)$ is full row rank
- If any polynomial matrix $\hat{P}(\xi)$ satisfies $\hat{P}(\xi)M(\xi) = 0$, then there exists a polynomial matrix $F(\xi)$ such that[4] $\hat{P}(\xi) = F(\xi)P(\xi)$.

Following facts can be verified easily.

Fact 3.2 Let $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$ be full column rank.

- In general, an MLA P of M is not unique. (Non-uniqueness can be characterized using unimodular matrices.)
- Any MLA P is left-prime and P has $\mathtt{w} - \mathtt{m}$ rows.

[4]This third property motivates the use of the word 'maximal'.

- Conversely, if $P \in \mathbb{R}^{(\mathtt{w}-\mathtt{m})\times\mathtt{w}}[\xi]$ is left-prime and satisfies $P(\xi)M(\xi) = 0$, then P is an MLA of $M(\xi)$.
- If $M(\xi)$ is left-invertible, then any MLA P can be used to obtain *all* left-inverses of M.
- For any nonsingular polynomial matrix $F(\xi) \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\xi]$, both $M(\xi)$ and $M(\xi)F(\xi)$ have the same set of MLAs.

4 Dissipative systems

Consider $\Sigma = \Sigma^T \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. A system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is called dissipative if

$$\int_{-\infty}^{\infty} w^T \Sigma w \, \mathrm{d}t \geqslant 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

For this course, we restrict ourselves to just *controllable* behaviors. Work on uncontrollable dissipative systems can be found in the literature, and outside the scope of this course.

Variety of Algebraic Riccati Equations: just different supply rates.

For example:

LQ control: $A^T P + PA - Q + PB^T R^{-1} BP = 0$

$\mathcal{H}_\infty$ norm (strictly proper): $A^T PPA + C^T C + PB^T BP = 0$

Passivity: $A^T P + PA + (KB - C^T)(D + D^T)^{-1}(B^T K - C) = 0$

Dissipativity, storage functions will unify these. Of course, well-known that there is a link with Linear Matrix Inequality (LMI). More precisely,

$$L(P) := \begin{bmatrix} A^T + PA - Q & PB^T \\ PB^T & R \end{bmatrix} \leqslant 0$$

Then, Schur complement with respect to R is exactly the ARE for the LQ control problem.

References

[PW98] J.W. Polderman and J.C. Willems. *Introduction to Mathematical Systems Theory: a Behavioral Approach*. Springer-Verlag, New York, 1998.

[WT98] J.C. Willems and H.L. Trentelman. On quadratic differential forms. *SIAM Journal on Control and Optimization*, 36:1703–1749, 1998.