More representations, controllability
Lecture 10

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Outline

- **More on polynomial matrices**
- **Controllability**
- **Other representations**
- **Dissipative systems**

Minimal kernel representations

Recall $U(\xi)R(\xi)V(\xi) = \begin{bmatrix} D(\xi) & 0 \\ 0 & 0 \end{bmatrix}$.

There is some inconsistency across literature about use of the 'invariant polynomials/ and 'elementary divisors'.

In any case: d_1 is gcd of all 1×1 minors of R.

$d_1 \cdot d_2 \cdots d_k$ is gcd of all $k \times k$ minors of R.

Using just premultiplication by unimodular matrices,

- $UR = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$, with R_1 having full row rank.
- Ignore zero rows: trivially-satisfied equations
- Assume full row rank: without loss of generality.
- $R(\frac{d}{dt})w = 0$ is called minimal kernel representation if $R(\xi)$ is full row rank.

$\mathfrak{L}^{\mathtt{w}}$: the set of LTI behaviors described by differential equations in $\mathtt{w}$ number of variables.

Equivalently, $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ if $\mathfrak{B}$ is the set of solutions to $R(\frac{d}{dt})w = 0$, for a polynomial matrix $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$.

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$URV = [D \quad 0]$
Interpret $V(\frac{d}{dt})$ as a 'coordinate tranformation' from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ to $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$
since $V(\frac{d}{dt})$ is one-to-one and onto (and linear).

Recall:

m: number of inputs,	p: number of outputs
w: number of 'manifest' variables:	typically $\mathtt{m} + \mathtt{p}$
n: (minimum) number of states	(McMillan degree)

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We also saw

- **the Smith canonical form of a polynomial matrix**
- **(normal) rank of a polynomial matrix**
- **unimodular completion (of a wide, nonsquare matrix) and its special case: Bezout identity**
- **Input/output partitions**

For a square constant matrix A, the Smith form of $\xi I - A$ is closely related to Jordan canonical form of A.

A system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called controllable if for any w_1 and $w_2 \in \mathfrak{B}$, there exist $w_3 \in \mathfrak{B}$ and $T \geqslant 0$ such that

$$w_3(t) = \begin{cases} w_1(t) \text{ for } t \leqslant 0 \;, \\ w_2(t) \text{ for } t \geqslant T. \end{cases}$$

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and suppose $R(\frac{d}{dt})w = 0$ is a minimal kernel representation. Then, the following are equivalent.

1. $\mathfrak{B}$ is controllable,
2. $R(\xi)$ is left-prime, i.e. $R(\lambda)$ is full row rank for every complex number $\lambda \in \mathbb{C}$.
3. $\mathfrak{B}$ has an image representation: there exists $M(\xi) \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ such that

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{ there exists } \ell \text{ such that } w = M(\frac{d}{dt})\ell\}.$$

(Compare PBH rank test for state space systems.)

Consider again:

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{ there exists } \ell \text{ such that } w = M(\frac{d}{dt})\ell\}.$$

Often, additional variables (like ℓ) than the ones of interest (here: w).
Call all auxiliary variables: latent variables.
Sometimes latent variables inevitable when modeling systems from first principles.
In general,

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{ there exists } \ell \text{ such that } R(\tfrac{d}{dt})w + M(\tfrac{d}{dt})\ell = 0\}.$$

Project (w, ℓ) behavior to just w-variables.
Does $\mathfrak{B}$ have a kernel representation?
Always possible for $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, but not for all function-spaces.

Consider $\mathfrak{B}_{\text{full}}$ described by $R(\frac{d}{dt})w + M(\frac{d}{dt})\ell = 0$ (for polynomial matrices R and M).
Then, there exists a kernel representation ($R_2(\frac{d}{dt})w = 0$) for $\mathfrak{B}$ defined by

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{ there exists } \ell \text{ such that } R(\tfrac{d}{dt})w + M(\tfrac{d}{dt})\ell = 0\}.$$

Obtain R_2 as follows. Find a unimodular U such that $U(\xi)M(\xi) = \begin{bmatrix} M_1 \\ 0 \end{bmatrix}$. Partition U conformably into $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$.
Define $R_2(\xi) := U_2(\xi)R(\xi)$.
Then, $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R_2(\frac{d}{dt})w = 0\}$.
In general $\ker R_2(\frac{d}{dt}) \supseteq \mathfrak{B}$: equality for $\mathfrak{C}^\infty$.
Equality not true for $\mathfrak{D}$: the set of compactly supported $\mathfrak{C}^\infty$ functions.
Note: U_2 is a so-called Maximal Left Annihilator (please see notes of Lecture 10) of M.

For the rest of the course, assume the system is controllable.
$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$

Power$(w) := w^T \Sigma w$, **with** $\Sigma = \Sigma^T \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$**: supply rate**

$\mathfrak{B}$ **is called dissipative with respect to supply rate** $w^T \Sigma w$ **if**

$$\int_{-\infty}^{\infty} w^T \Sigma w dt \geqslant 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

Along any system trajectory (starting from rest and ending at rest), 'net energy' is absorbed.

Integral inequality insisted only on $\mathfrak{B} \cap \mathfrak{D}$**: denseness issues related to controllability.**

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Examples

Ignoring stability aspects (for this slide):

- $G(s)$ is positive real $\Leftrightarrow \Sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $w^T \Sigma w = 2uy$
- $G(s)$ has $\mathfrak{L}_\infty$ norm at most $\gamma \Leftrightarrow w^T \Sigma w = \gamma^2 u^2 - y^2$.
- In LQ control, $w^T \Sigma w = x^T Q x + u^T R u$
- $y = \phi(u)$, and ϕ is a 'sector' nonlinearity, $\phi \in$ sector (α, β):

$$(y - \alpha u)(u - \frac{y}{\beta}) = \begin{bmatrix} u \\ y \end{bmatrix} \begin{bmatrix} -\alpha & \frac{(\alpha+\beta)}{2\beta} \\ \frac{(\alpha+\beta)}{2\beta} & \frac{-1}{\beta} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \geqslant 0.$$

- Popov criteria, involving 'dynamic' notions of power

Interconnection of Σ-dissipative and $-\Sigma$-dissipative systems yields stability: Megretski & Rantzer: IQC paper

Next lecture: dissipativity of LTI systems linked to Algebraic Riccati Equation solutions

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