Dissipative systems, storage functions

Notes for lecture 11 (May 14th, 2014)

This lecture contains more about dissipative systems, maximum/minimum storage functions and Algebraic Riccati Equations (ARE). In this lecture, we also took a simple LQ problem and obtained the optimum feedback law.

1 Dissipative systems

Consider $\Sigma = \Sigma^T \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. A system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is called Σ-dissipative if

$$\int_{-\infty}^{\infty} w^T \Sigma w \, \mathrm{d}t \geqslant 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

(Recall that $\mathfrak{D}$ is defined as the set of compactly supported $\mathfrak{C}^{\infty}$ trajectories.) Restricting to trajectories in $\mathfrak{B} \cap \mathfrak{D}$ ensures integral is well-defined. Secondly, it is like checking for all trajectories 'starting at rest' and 'ending at rest'. This removes 'initially stored energy' from the calculations. Finally, for *controllable* systems, the compactly supported trajectories in $\mathfrak{B}$ are 'dense' in $\mathfrak{B}$: see [PS98] and [WT02, Page 55].

For this course, we restrict ourselves to just *controllable* behaviors ($\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$). Work on uncontrollable dissipative systems can be found in the literature, and is outside the scope of this course.

***Theorem* 1.1** *Let $\mathfrak{B} \in \mathfrak{L}^{w}_{\text{cont}}$ and suppose $\Sigma = \Sigma^T \in \mathbb{R}^{w\times w}$. Suppose $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ is an image representation for $\mathfrak{B}$. Then, the following are equivalent.*

- *$\mathfrak{B}$ is Σ-dissipative.*
- *$M(-\xi)^T \Sigma M(\xi)$ satisfies $M(-j\omega)^T \Sigma M(j\omega) \geqslant 0$ for all $\omega \in \mathbb{R}$.*
- *There exists a storage function[1] $Q_\Psi(w)$ i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) \leqslant w^T \Sigma w \text{ for all } w \in \mathfrak{B}. \tag{1}$$

$Q_\Psi(w)$ is a *Quadratic Differential Form* (QDF) in w: this is nothing but a quadratic form in w and (a finite number of) its derivatives.

Amongst many representations (kernel, image, the more-general latent-variable) that we saw till now, the state space representation is special. There are three important state-space representations with some structure, and a fourth one: a more general state space

[1]Equation (1) can be taken as the definition of a storage function.

representation. In each case the manifest variable is w (partitioned sometimes into $w = (w_1, w_2)$), and the matrices A, B, C and D are possibly different.

$$\overset{\bullet}{x} = Ax + Bw_1 \quad \text{and } w_2 = Cx + Dw_1 \quad \text{(input/state/output)}, \tag{2}$$

$$\overset{\bullet}{x} = Ax + Bd \quad \text{and } w = Cx + Dd \quad \text{(driving variable)}, \tag{3}$$

$$\overset{\bullet}{x} = Ax + Bw \quad \text{and } 0 = Cx + Dw \quad \text{(output nulling)}, \tag{4}$$

$$E\overset{\bullet}{x} + Fx + Gw = 0 \quad \text{(the most general state-space)}. \tag{5}$$

(We will see these state representations in more detail in following lectures.)

The storage function might require *derivatives* of the manifest variables in order to express it, but when expressed in the 'state', every storage function can be expressed in terms of a *static* function of the state: $x^T K x$ for a constant matrix K.

Alternatively, the storage function can also be expressed in terms of the latent variable ℓ of the image representation $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$.

Depending on the requirement/convenience, we will express the storage function in terms of either w (the manifest variable), or ℓ (of the image representation) or x, the state variable (any one of the equations (2)-(4)). The ARI solutions are the ones when the storage function is expressed as a state function.

The storage functions are not unique, in general: for controllable systems, they are a bounded set, with a maximum $Q_{\Psi_{\max}}$ and minimum $Q_{\Psi_{\min}}$.

All storage functions Q_Ψ satisfy

$$Q_{\Psi_{\min}}(w) \leqslant Q_\Psi(w) \leqslant Q_{\Psi_{\max}}(w).$$

The maximum and minimum storage functions satisfy a neat interpretation. Consider expressing the stored energy in terms of the state variable x. Accordingly, $Q_\Psi(w) = x^T K x$, say. Let $\mathfrak{B}_a$ denote all trajectories in $\mathfrak{B}$ such that at $t = 0$, the trajectory w has state $x = a \in \mathbb{R}^{\mathtt{n}}$. Then,

$$Q_{\Psi_{\max}}(w)(0) = a^T K_{\max} a = \inf_{w \in \mathfrak{B}_a \cap \mathfrak{D}} \int_{-\infty}^{0} w^T \Sigma w \, \mathrm{d}t \tag{6}$$

and

$$Q_{\Psi_{\min}}(w)(0) = a^T K_{\min} a = \sup_{w \in \mathfrak{B}_a \cap \mathfrak{D}} \int_{0}^{\infty} -w^T \Sigma w \, \mathrm{d}t \tag{7}$$

When expressed in terms of ℓ, the storage functions are easily calculated using the spectral factorization result due to Ran and Rodman.

***Theorem* 1.2** *Let $P \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ satisfy*[2] $P(-\xi) = P(\xi)^T$. *Further, suppose* $\det P \neq 0$. *Then, the following are equivalent.*

[2]Such a P is said to be para-Hermitian.

- $P(j\omega) \geqslant 0$ *for each* $\omega \in \mathbb{R}$.
- *There exists an almost Hurwitz* $H(\xi) \in \mathbb{R}^{m \times m}[\xi]$ *such that* $P(\xi) = H(-\xi)^T H(\xi)$.
- *There exists an almost anti-Hurwitz* $A(\xi) \in \mathbb{R}^{m \times m}[\xi]$ *such that* $P(\xi) = A(-\xi)^T A(\xi)$.

Before we go further into calculation procedures, we note two key behavioral notation aspects.

Notation: Suppose $\Sigma \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ and $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$, then we study $\Phi'(\zeta, \eta) := M(\zeta)^T \Sigma M(\eta)$. If $M(\xi)$ has $\mathtt{w}$ rows and $\mathtt{m}$ columns, then Φ' has an $\mathtt{m} \times \mathtt{m}$ <u>*two*</u>-variable polynomial matrix. The prime (') indicates action on ℓ instead of w. The ζ (zeta) indicates differentiation of the ℓ *the left*, while η (eta) indicates differentiation of ℓ *on the right.* They are merely separate place-holders, since we are dealing with *quadratic* forms in ℓ and its derivatives.
QDF $\equiv$ Quadratic Differential Form.

Another notation: ∂: acts on two-variable polynomial matrices and gives a one-variable polynomial matrix of the same size: $\zeta \to -\xi$, and $\eta \to \xi$. More precisely, for $\Phi(\zeta, \eta) \in \mathbb{R}^{\mathtt{m} \times \mathtt{m}}[\boldsymbol{z}]$, $\partial\Phi(\xi) := \Phi(-\xi, \xi)$. Loosely speaking, the minus sign on the *left* indicates the negative sign when one switches the derivative from one dependent variable to another, when integrating by parts. We will see this with an example in the next lecture.

The factorizations A and H above are, in general, not unique. Using this result and the second condition in Theorem 1.1, we obtain (almost) Hurwitz and anti-Hurwitz factorizations of $\partial\Phi'(\xi) := M(-\xi)^T \Sigma M(\xi)$ and use this to obtain the maximum and minimum storage functions are follows: for proof, please refer [WT98].

$$\Psi'_{\min}(\zeta, \eta) = \frac{\Phi'(\zeta, \eta) - H(\zeta)^T H(\eta)}{\zeta + \eta} \qquad \text{and} \qquad \Psi'_{\max}(\zeta, \eta) = \frac{\Phi'(\zeta, \eta) - A(\zeta)^T A(\eta)}{\zeta + \eta}$$

(Recall that $\Phi'(\zeta, \eta) := M(\zeta)^T \Sigma M(\eta)$: the prime gives the QDF expressed in ℓ instead of w.)

The roots of det $\partial\Phi(\xi)$ are called the *spectral zeros.* They play a key role in the dissipativity (or passivity) preserving model order reduction. The LHP ones of these spectral zeros come into $H(\xi)$ and the RHP ones into $A(\xi)$.

2 LQ problem

Using the fact that the storage function is a state function $x^T K x$ gives the ARE (through the LMI and the Schur complement). For $w = (x, u)$ and $w^T \Sigma w = x^T Q x + u^T R u$, with Q and R symmetric and $Q \geqslant 0$ and $R > 0$, we get the ARI

$$A^T K + KA - Q + K^T B R^{-1} B^T \leqslant 0. \tag{8}$$

For the LQ problem, once $K_{\min}$ is calculated, optimum cost is $-x_0 K_{\min} x_0$. Compare with other Riccati equation and the solutions (www.egr.msu.edu/classes/me851/jchoi: Lecture 14) (Available from: http://www.egr.msu.edu/classes/me851/jchoi/lecture/Lect_14.pdf) The Riccati equation in the LQ problem in the literature

$$A^T K + KA + Q - KBR^{-1}B^T K = 0$$

requires the *largest* solution to be taken: here the largest is the one that is stabilizing. Note the switch in the signs: see also [TSH02, Chapter 10].

Verify that this K of equation (8) gives the closed loop A_F as $A - BR^{-1}B^T K$ (from the Hamiltonian matrix and invariant subspace argument): see [TSH02, Section 13.4]. (This reference is available online in H.L. Trentelman's homepage: http://www.math.rug.nl/~trentelman.)

Exercise 2.1 Consider the system $\frac{\mathrm{d}}{\mathrm{d}t}x = 3x+2u$ and the performance cost $\int_0^\infty (4x^2+u^2)\,\mathrm{d}t$. Let the initial condition be $x(0) = 4$.

- Find a Hurwitz factorization of $\partial\Phi'(\xi)$ obtained from[3] $\Phi'(\zeta,\eta) := M^T(\zeta)\Sigma M(\eta)$, with $M(\xi)$ carefully chosen so that the state x equals ℓ.
- Obtain the optimum cost for this initial condition.
- Check that the LQ optimal feedback law from the literature gives closed loop x satisfying the same differential equation as $H(\frac{\mathrm{d}}{\mathrm{d}t})\ell = 0$, with H obtained from the above Hurwitz factorization.

(We will continue with this same exercise further in the next lecture.)

References

[PS98] H. K. Pillai and S. Shankar. A behavioral approach to control of distributed systems. *SIAM Journal on Control and Optimization*, 37:388–408, 1998.

[TSH02] H.L. Trentelman, A. A. Stoorvogel, and M. Hautus. Control theory for linear systems. In *Communications and Control Engineering Series*. Springer, 2002.

[WT98] J.C. Willems and H.L. Trentelman. On quadratic differential forms. *SIAM Journal on Control and Optimization*, 36:1703–1749, 1998.

[3] The prime used to go from Φ (supply rate in terms of w) to Φ' (the supply rate expressed in terms of ℓ) is same notation as in [WT98, page 1723].

[WT02] J.C. Willems and H.L. Trentelman. Synthesis of dissipative systems using quadratic differential forms: Part I. *IEEE Transactions on Automatic Control*, 47:53–69, 2002.