Hamiltonian matrix, orthogonal complements

Notes for lecture 12 (May 19th, 2014)

This lecture contains link between Hamiltonian matrix, ARE and (so-called) stationary trajectories. We will also define the 'orthogonal complement' of a behavior.

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1 Lossless systems, orthogonal complement

In the case of dissipativity, the storage function is not unique, in general. (Note that non-uniqueness of the storage function is *not* due to our choice of expressing the storage function in terms of different variables.) There is a special case[1] of dissipativity when the storage function is unique: lossless.

Consider $\Sigma = \Sigma^T \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. A system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is called Σ-lossless if

$$\int_{-\infty}^{\infty} w^T \Sigma w \,\mathrm{d}t = 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

***Theorem* 1.1** *Let $\mathfrak{B} \in \mathfrak{L}^{w}_{\text{cont}}$ and suppose $\Sigma = \Sigma^T \in \mathbb{R}^{w\times w}$. Suppose $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ is an image representation for $\mathfrak{B}$. Then, the following are equivalent.*

- *$\mathfrak{B}$ is Σ-lossless.*
- *$M(-\xi)^T \Sigma M(\xi)$ satisfies $M(-\xi)^T \Sigma M(\xi) = 0$.*
- *There exists a storage function $Q_\Psi(w)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) = w^T \Sigma w \textit{ for all } w \in \mathfrak{B}. \tag{1}$$

[1]Note that there are dissipative and non-lossless systems which can have a unique storage function: uniqueness of storage function only ensures non-strictness of the dissipativity, it does not ensure losslessness.

- $\int_{t_1}^{t_2} w^T \Sigma w \,\mathrm{d}t$ *is 'path-independent': i.e. the value of the integral depends only on values of* w *(and its derivatives) at* t_1 *and* t_2*, and does not depend on which trajectory in* $\mathfrak{B}$ w *assumes between* t_1 *and* t_2*.*

Closely related to lossless is the notion of an orthogonal complement of a controllable behavior. Given a controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and a symmetric, nonsingular matrix $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, *the* Σ*-orthogonal complement of* $\mathfrak{B}$, (denoted by $\mathfrak{B}^{\perp_\Sigma}$), is the set of all the trajectories $v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ such that $\int_{-\infty}^{\infty} v^T \Sigma w \,\mathrm{d}t = 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$.
When $\Sigma = I$, the Σ-orthogonal complement $\mathfrak{B}^{\perp_\Sigma}$ is written as just $\mathfrak{B}^\perp$.

2 Euler Lagrange equation

We briefly link the differential equations $\partial\Phi'(\frac{\mathrm{d}}{\mathrm{d}t})\ell = 0$ with the EL equation (for the simplified case). Consider minimizing or maximizing a performance functional $\int V(\ell, \dot{\ell}) \,\mathrm{d}t$, with y unconstrained, and $\mathfrak{C}^\infty$. Then the optimum trajectories y^* satisfy

$$\frac{\partial V}{\partial \ell} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V}{\partial \dot{\ell}} = 0.$$

Of course, we are dealing with a situation simplified in many ways.

3 Hamiltonian matrix

A matrix $H \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is called a Hamiltonian matrix if H is similar to $-H^T$. Hamiltonian matrices arise in different contexts; in our context, they are closely related to kernel of $\partial\Phi'(\frac{\mathrm{d}}{\mathrm{d}t})$. For example, roots of $\partial\Phi'(\xi)$ and eigenvalues of H are the same (counted with multiplicity).

Recall that for lossless systems, $\partial\Phi'(\xi)$ was identically zero. (Vaguely) motivated by this, think of kernel of $\partial\Phi'(\frac{\mathrm{d}}{\mathrm{d}t})$ as 'lossless trajectories'.

For finite dimensional vector spaces, if $\mathfrak{V} \subseteq \mathbb{R}^{\mathtt{n}}$ is Σ-non-negative, then $\mathfrak{V} \cap \mathfrak{V}^{\perp_\Sigma}$ is a subspace of $\mathbb{R}^{\mathtt{n}}$ which is Σ-neutral. (A subspace $\mathfrak{V}$ is called Σ-neutral if $v^T\Sigma v = 0$ for all $v \in \mathfrak{V}$; neutral is same as lossless. See [GLR05] for an elaborate treatment on indefinite linear algebra.)

We will see that $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$, when autonomous, has a state representation that is a Hamiltonian matrix (as the state transition matrix). The link between ARE and 'the corresponding' Hamiltonian matrix is due to the following fact (that is best verified oneself).

Fact 3.1 *([TSH02, Section 13.4]) Let* $F, S, T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ *with* S *and* T *are symmetric. Due to*

$$\begin{bmatrix} X & -I \end{bmatrix} \begin{bmatrix} F & T \\ -S & -F^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = F^TX + XF + XTX + S \quad \textit{and} \quad \begin{bmatrix} X & -I \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = 0,$$

we note that solutions to the ARE $F^TX + XF + XTX + S = 0$ *are linked*[2] *to* $\mathtt{n}$*-dimensional*

[2] We used that X is symmetric. Also, an $\mathtt{n}$-dimensional invariant subspace (say image of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$) of H requires to have its top $\mathtt{n} \times \mathtt{n}$ block X_1 invertible, for this subspace to yield an ARE solution. In most ARE studies, this invertibility is a key step.

invariant subspaces of the $2\mathtt{n} \times 2\mathtt{n}$ matrix (say H) in the above equation. H is defined as the Hamiltonian matrix corresponding to this ARE.

Further, verify that

$$H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} (F + TX).$$

Thus, if H has no eigenvalues on the imaginary axis, choosing 'the' $\mathtt{n}$-dimensional invariant subspace corresponding to all OLHP eigenvalues gives an X (assuming invertibility of X_1 as mentioned in Footnote 2) that is stabilizing*: this is due to $F + TX$ being Hurwitz.*

4 Minimal dissipation trajectories

One of the passivity preserving model order reduction methods proposed in [Ant05, Sor05] turns out to 'retain' (a lower dimensional subspace of) the set of trajectories of minimal dissipation ([MTR09]).

Consider a nonsingular $\Sigma = \Sigma^T \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ and suppose $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is Σ-dissipative. As proposed in [MTR09], for a $w \in \mathfrak{B}$, consider the *change* $J_w(\delta)$ in dissipation[3] (about w) if w is changed to $w + \delta$, for $\delta \in \mathfrak{B} \cap \mathfrak{D}$:

$$J_w(\delta) := \int_{-\infty}^{\infty} (Q_\Delta(w + \delta) - Q_\Delta(w))\, \mathrm{d}t.$$

A trajectory $w \in \mathfrak{B}$ is said to be a *trajectory of minimal dissipation* if $J_w(\delta) \geqslant 0$ for all $\delta \in \mathfrak{B} \cap \mathfrak{D}$. Any small change in w causes *increase* of net dissipated energy: in that sense, these are local minima (see [MTR09, page 177].)

The link between the set of trajectories (in a Σ-dissipative behavior $\mathfrak{B}$) of minimal dissipation (denoted by $\mathfrak{B}^*$) and $\mathfrak{B}^{\perp_\Sigma}$ is [MTR09, Theorem 3.4], which states $\mathfrak{B}^* = \mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$. Notice that $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ is just the set of those trajectories $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$, where ℓ, is no longer free/generic, but in fact, satisfies $\partial\Phi'(\frac{\mathrm{d}}{\mathrm{d}t})\ell = 0$.

5 Strict dissipativity

Quite unfortunately, the lossless case is not handled by the ARE/ARI. The ARE and the ARI are best suited for 'strict dissipativity', which is a kind of opposite to losslessness. Consider $\Sigma = \Sigma^T \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$. A system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is called *strictly Σ-dissipative* if there exists an $\epsilon > 0$ such that

$$\int_{-\infty}^{\infty} w^T \Sigma w \, \mathrm{d}t \geqslant \epsilon \int_{-\infty}^{\infty} w^T w \, \mathrm{d}t \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

[3] A dissipation function $Q_\Delta(w)$ (a function of time, that depends on the trajectory w) is defined as the amount of supplied power that didn't go into storing energy, i.e. $Q_\Delta(w) := w^T \Sigma w - \frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w)$. Since storage functions are not unique, we speak of a dissipation function Q_Δ *corresponding* to a storage function Q_Ψ. Inspite of this dependence on Q_Ψ, along *compactly supported* trajectories, the 'net power' dissipated depends only on w: for more details, see [WT98].

General concerns: lossless part in $\mathfrak{B}$: moreover, lossless part is non-autonomous. In the LMI (corresponding to dissipativity), one needs to take the Schur complement with respect to the 'lower right' block: which ought to be sign-*definite*: only then the LMI yields the ARI/ARE. This in turn allows defining the Hamiltonian matrix.

Strict dissipativity lays to rest any concerns about singularity of the above lower-right block. (Of course, strict dissipativity at just 'the ∞ frequency' is good enough: [KBAR14].)

***Theorem* 5.1** *Assume $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ is symmetric and nonsingular. Suppose $\mathfrak{B}$ is strictly Σ-dissipative. Then,*

1. *$\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ is autonomous. $\boldsymbol{n}(\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}) = 2\boldsymbol{n}(\mathfrak{B})$.*

2. *The ARE exists.*

3. *The Hamiltonian matrix exists.*

Dissipation at the ∞ frequency is denoted by the matrix P in the matrices in the last section. In order to obtain the Hamiltonian matrix as indicated there, some more development relating $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$ is required. Key property is that $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ has a state transition matrix exactly the Hamiltonian matrix H: this will be elaborated in this pdf-file by 20th May, 2pm.

The two statements within 1 above can be viewed as regular interconnection and regular feedback interconnection (see [Wil97, JPKB13]) between the 'plant' $\mathfrak{B}$ and the 'controller' $\mathfrak{B}^{\perp_\Sigma}$ (defined next). Study of these interconnections seems inessential to pursue model-order reduction, and hence we do not pursue further here.

(In Lecture 11, we began with an exercise, which we continue with now.)

Exercise 5.2 Consider the system $\frac{\mathrm{d}}{\mathrm{d}t}x = 3x + 2u$ and the performance cost $\int_0^\infty (4x^2 + u^2)\,\mathrm{d}t$. Let the initial condition be $x(0) = 4$.

- Find all stationary trajectories $\mathfrak{B}^*$ in $\mathfrak{B}$ using the Euler-Lagrange equation.

- Find $\mathfrak{B}^*$ as the set of trajectories of 'minimal dissipation' (as $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$).

- Check if a first order representation of $\mathfrak{B}^*$ results in a Hamiltonian matrix H.

- Compare H with the one linked to the corresponding ARE, and use H to obtain the stabilizing ARE solution (K_{min}, in our case).

Exercise 5.3 Consider the following circuit.

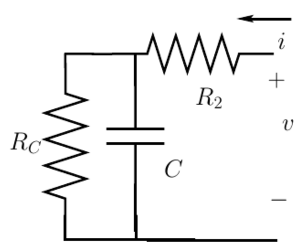

Figure 1: RC circuit

Let the capacitance C be 1 F and the resistances R_2 and R_C be equal to 3 Ω and 1 Ω respectively. Find the minimum energy required at the port to charge the capacitor to 4 V (from initially discharged state). Also find the maximum energy one can extract out from the port if the capacitor is initially charged to 4 V. Why is it reasonable that the *actual* energy stored is 'exactly in between' the maximum and the minimum storage functions?

6 Autonomous systems

This section contains briefly about autonomous systems to the extent we need for model order reduction. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called autonomous if

$$\text{whenever } w_1, w_2 \in \mathfrak{B} \text{ satisfy } w_1(t) = w_2(t) \Rightarrow w_1 = w_2.$$

***Theorem* 6.1** *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ have minimal kernel representation $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. Then, the following are equivalent.*

1. *$\mathfrak{B}$ is autonomous.*
2. *$R(\xi)$ is square and nonsingular.*
3. *$\mathfrak{B}$ is* finite dimensional *as a vector space over $\mathbb{R}$.*
4. *$\mathfrak{B}$ is a finite linear combination of only*[4] *exponentials, i.e., assuming for simplicity*[5] det (R) *has only real distinct roots $\lambda_1, \ldots, \lambda_N$, with $N =$* deg det *R:*

$$w \in \mathfrak{B} \Leftrightarrow w = \sum_{i=1}^{N} a_i v_i e^{\lambda_i} t \textit{ for } a_i \in \mathbb{R} \textit{ and } v_i \in \mathbb{R}^{\mathtt{w}} \backslash 0.$$

Excepting the (trivial) case when $\mathfrak{B} = \{0\}$, which is both controllable and autonomous, in general, autonomous means uncontrollable. In fact, autonomous means, not just uncontrollable, but in fact, controllable-part equal to zero. For this course, we stick to just controllable behaviors $\mathfrak{B}$ for the purpose of model-order reduction, but $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ will be autonomous (under strict dissipativity assumptions, etc.: as mentioned in Theorem 5.1).

7 LMI, ARE and Hamiltonian matrix H

For three key supply rates $x^T Q x + u^T R u$ (the LQ problem), $u^T y$ (passivity) and $\gamma^2 u^T u - y^T y$ ($\mathcal{L}_\infty$ norm at most γ), we list the LMI, ARE and Hamiltonian matrix. In each case, assume $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$ and $y = Cx + Du$ is a minimal state space realization (i.e. controllable and observable realization). The LMI can be obtained by $x^T K x$ as a storage function. The ARE is obtained by taking Schur complement w.r.t. the lower right block (say, P, the one corresponding to $u^T P u$) and the Hamiltonian matrix is constructed from the ARE as elaborated in [TSH02, Section 13.4].

[4]In 'exponential functions', we allow sinusoids and cosinusoids (due to complex exponents) and also polynomial combination of exponentials (when repeated roots).

[5]Real roots ensures no sinusoids/cosinusoids, and distinct ensures no polynomials required: see [PW98] for the general autonomous case.

(Some signs might not be correct. Please read critically. -Madhu)

Supply rate	LMI	P (the dissipation at ∞ frequency)	H
$\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$	$\begin{bmatrix} A^TK+KA-Q & KB \\ B^TK & -R \end{bmatrix}$	R	$\begin{bmatrix} A & Q \\ B^TR^{-1}B & -A^T \end{bmatrix}$
$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$	$\begin{bmatrix} A^TK+KA & KB-C^T \\ B^TK-C & -(D+D^T) \end{bmatrix}$	$(D+D^T)$	$\begin{bmatrix} A-BP^{-1}C & BP^{-1}B^T \\ -C^TP^{-1}C & -(A-BP^{-1}C)^T \end{bmatrix}$
$\begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix}$	$\begin{bmatrix} A^TK+KA+C^TC & KB+C^TD \\ D^TC+B^TK & D^TD-\gamma^2 I \end{bmatrix}$	$(\gamma^2 I - D^TD)$	$\begin{bmatrix} A+BP^{-1}D^TC & BP^{-1}B^T \\ -C^TP^{-1}C & -(A+BP^{-1}D^TC)^T \end{bmatrix}$

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