Dual-systems, orthogonal complement, Hamiltonian matrices, dissipativity-preserving model reduction (Lecture 13)

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Outline

- More analogies with indefinite linear algebra
- Lossless systems, orthogonal complement, dual/adjoint systems
- Hamiltonian matrices, trajectories of minimal dissipation
- Remaining behavioral-notation
- Dissipativity-preserving model-order reduction

Subspaces of $\mathbb{R}^w$

Consider subspace $\mathbb{V} \subset \mathbb{R}^{\mathtt{w}}$ and symmetric and nonsingular $\boldsymbol{\Sigma} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ (Think initially $\boldsymbol{\Sigma} = \mathrm{diag}\ (1, 1, 1, \ldots, -1, -1, -1, \ldots)$.)

- $\mathbb{V}$ **is $\boldsymbol{\Sigma}$-non-negative** $\Rightarrow \dim(\mathbb{V}) \leqslant \boldsymbol{\sigma}_+(\boldsymbol{\Sigma})$
- $\mathbb{V}$ **is $\boldsymbol{\Sigma}$-neutral** $\Rightarrow$ $\boldsymbol{\Sigma}$ **is indefinite (unless** $\mathbb{V} = 0$**)**
- $\mathbb{V} \cap \mathbb{V}^\perp = 0$
- $\mathbb{V} \oplus \mathbb{V}^\perp = \mathbb{R}^{\mathtt{w}}$ **and** $\dim(\mathbb{V}) + \dim(\mathbb{V}^\perp) = \mathtt{w}$
- $\mathbb{V}^{\perp_{\boldsymbol{\Sigma}}} = (\boldsymbol{\Sigma}\mathbb{V})^\perp = \boldsymbol{\Sigma}^{-1}\mathbb{V}^\perp$
- $\mathbb{V}$ **is strictly $\boldsymbol{\Sigma}$-non-negative** $\Leftrightarrow$ $\mathbb{V}$ **is $\boldsymbol{\Sigma}$-positive**

Suppose $\dim(\mathbb{V}) = \boldsymbol{\sigma}_+(\boldsymbol{\Sigma})$ **(then** $\dim(\mathbb{V}^{\perp_{\boldsymbol{\Sigma}}}) = \boldsymbol{\sigma}_-(\boldsymbol{\Sigma})$**)**

$\mathbb{V}$ **is $\boldsymbol{\Sigma}$-positive** $\Leftrightarrow$ $\mathbb{V}^\perp$ **is -$\boldsymbol{\Sigma}^{-1}$-positive** $\Leftrightarrow$ $\mathbb{V}^{\perp_{\boldsymbol{\Sigma}}}$ **is -$\boldsymbol{\Sigma}$-positive**

$\mathbb{V}$ is Σ-neutral $\equiv$ $\mathfrak{B}$ is Σ-lossless
all-pass, $\partial\Phi'(\xi) = 0$

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- **$\mathbb{V} \oplus \mathbb{V}^{\perp} = \mathbb{R}^{\mathtt{w}}$ and $\dim(\mathbb{V}) + \dim(\mathbb{V}^{\perp}) = \mathtt{w}$**
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For behaviors $\mathfrak{B} \subset \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w)$

The orthogonal complement of $\mathfrak{B}$ (in $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$): $\mathfrak{B}^\perp$
Adjoint system, dual system, co-state dynamics
(Dual Riccati: for the dual system)
Suppose $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ is symmetric and nonsingular. $\mathfrak{B}^{\perp_\Sigma}$ just 'normalization' w.r.t. Σ.

- **Number of inputs of $\mathfrak{B}$ (i.e. $\mathtt{m}(\mathfrak{B})$): column rank of $M(\xi)$ (Image representation $w = M(\frac{d}{dt})\ell$)**
- **$\mathfrak{B}$ is Σ-dissipative $\Rightarrow \mathtt{m}(\mathfrak{B}) \leqslant \sigma_+(\Sigma)$**
- **$\mathfrak{B} + \mathfrak{B}^\perp = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, though not direct sum**
- **$\mathfrak{B} \cap \mathfrak{B}^\perp \neq \{0\}$, the intersection is 'thin'**
- **Intersection is autonomous, i.e. finite dimensional**
- **$\mathfrak{B} \cap \mathfrak{B}^\perp \cap \mathfrak{D} = \{0\}$**
- **Intersection has dynamics $\frac{d}{dt}x = Hx$, for a Hamiltonian matrix H.**
- **(H is called Hamiltonian if $H^T \sim -H$)**

Intersection: central role in model-order reduction (dissipativity preserving)

Lossless and orthogonality

$\mathfrak{B}_1$ and $\mathfrak{B}_2$ (both controllable) are Σ-orthogonal :⇔

$$\int_{\mathbb{R}} w_1^T \Sigma w_2 dt = 0 \text{ for all } w_1 \in \mathfrak{B}_1, w_2 \in \mathfrak{B}_2 \cap \mathfrak{D}.$$

$\mathfrak{B}_1 \perp_\Sigma \mathfrak{B}_2 \Leftrightarrow \mathfrak{B}_1 \times \mathfrak{B}_2$ is lossless with respect to $\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix}$

Note: the Cartesian product $\mathfrak{B}_1 \times \mathfrak{B}_2 \subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{2\mathtt{w}})$, $(w_1, w_2) \in \mathfrak{B}_1 \times \mathfrak{B}_2 \Leftrightarrow w_1 \in \mathfrak{B}_1$ and $w_2 \in \mathfrak{B}_2$.

In general, $\mathtt{m}(\mathfrak{B}_1) + \mathtt{m}(\mathfrak{B}_2) \leqslant \mathtt{w}$

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More generally, $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma} =: \mathfrak{B}^*$

- $\mathfrak{B}^*$: trajectories in $\mathfrak{B}$ of 'minimal dissipation' (Minh, Trentelman & Rapisarda: MCSS 2009)
- Retain a lower dimension of $\mathfrak{B}^*$ into the reduced order model
- Restriction on $\mathfrak{B}$ (to $\mathfrak{B}^*$) can also be achieved by forcing ℓ to satisfy equations (instead of free/generic):

$$w = M(\frac{d}{dt})\ell \quad \text{and} \quad \partial\Phi'(\frac{d}{dt})\ell = 0.$$

- (In general, view additional laws as a controller: feedback? controller)
- Is intersection autonomous? Is $\det \partial\Phi'(\xi) \equiv 0$?

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At maximality

Suppose $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$ (then $\mathtt{m}(\mathfrak{B}^{\perp_\Sigma}) = \sigma_-(\Sigma)$)

- $\mathfrak{B}$ is Σ-dissipative $\Leftrightarrow$ $\mathfrak{B}^{\perp_\Sigma}$ is -Σ-dissipative
- $\mathfrak{B}$ is strictly Σ-dissipative $\Leftrightarrow$ $\mathfrak{B}^{\perp_\Sigma}$ is strictly -Σ-dissipative
- $\mathfrak{B}$ is strictly Σ-dissipative $\Rightarrow$ ARE has a solution K
- $\mathfrak{B}^{\perp_\Sigma}$ is strictly -Σ-dissipative $\Rightarrow$ 'Dual ARE' has a solution P (say)
- With careful state-space basis choice, ARE solutions $-K^{-1}$ = Dual ARE solutions P

References:

- Section 10 of QDF paper by Willems & Trentelman, 1998, SIAM Journal on Control & Optimization
- Proposition 12 of Willems & Trentelman (Part I), IEEE-TAC, 2002 (Synthesis of dissipative systems)

Non-strict dissipativity

- **$\boldsymbol{\Sigma}$-lossless $\Leftrightarrow \mathfrak{B} \subseteq \mathfrak{B}^{\perp_{\Sigma}}$**
- **Consider system with proper SISO transfer function $G(s)$ with no poles in CRHP.**
- **$\|G(s)\|_{\mathcal{H}_\infty} \geqslant \gamma \Leftrightarrow$ dissipative with respect to $\gamma^2 u^2 - y^2$**
- **$(\cdots > \cdots \Leftrightarrow$ strict $\cdots$, etc.)**
- **$\|G(s)\|_{\mathcal{H}_\infty} = \gamma \Leftrightarrow$ dissipative with respect to $\gamma^2 u^2 - y^2$, but**
- **Either $\gamma^2 I - D^T D$ is singular (lack of strictness at infinity frequency) or there exists $\omega_0 \in \mathbb{R}$ such that $\mathcal{H}_\infty$ norm is attained at ω_0.**
- **The former case: the Hamiltonian matrix H does not exist (singular descriptor system)**
- **The latter case: the Hamiltonian matrix H (exists and) has eigenvalue at $\pm j\omega_0$.**

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Strictness

Compared to just dissipativity, strict dissipativity helps
Suppose $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and $\mathfrak{B}$ is strictly Σ-dissipativity
$\Rightarrow \partial\Phi'(j\omega) > 0$ for each $\omega \in \mathbb{R}$.

- The P in $u^T(P)u$ within the LMI is invertible. (Dissipativity at ∞ frequency is strict.)
- The strict LMI has a solution: feasibility
- The ARE and the Hamiltonian matrix H exist
- The strict ARI has a solution and $K_{\max} > K_{\min}$ (Scherer)
- In fact, H has no eigenvalues on the imaginary axis
- $\det \partial\Phi'(\xi)$ has no roots on $j\mathbb{R}$
- (Note that $\det \partial\Phi'(\xi) = \det(\xi I - H)$: spectral zeros)

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Passivity preserving Model order reduction

- **Many papers in the literature: Feldman, Freund (1995, 1999 IEEE-TAC), Ober (1998, SIAM Con & Opt), PRIMA**
- **Based on positive real balancing**
- **'Simultaneous diagonalization': similarity transformation or congruence transformation?**
  $A \to S^{-1}AS$ **or** $P \to S^T PS$
- **Note that for storage** $x^T Kx$**, state space coordinate transformation due to** S **means storage** $z^T(S^T KS)z$**.**
- **Find coordinate transformation such that max/min of ARE/Dual-ARE solutions are 'balanced' (Simultaneously diagonalized: Antoulas, SIAM 2004 book)**
- **In this course, passivity preserving model reduction by**
  - **'interpolation at spectral zeros' (Antoulas/Sorensen: SCL 2005)**
  - **preserving trajectories of minimal dissipation (Minh, Trentelman, Rapisarda: MCSS 2009)**

Problem formulation

Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and symmetric nonsingular $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$
Suppose $\mathfrak{B}$ is strictly Σ-dissipative and suppose $\mathtt{n}$ is the McMillan degree of $\mathfrak{B}$
(McMillan degree: model order: minimum number of states)
Choose $k < \mathtt{n}$. Find $\hat{\mathfrak{B}} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ such that

1. **$\hat{\mathfrak{B}}$ has McMillan degree at most k**
2. **$\mathtt{m}(\hat{\mathfrak{B}}) = \mathtt{m}(\mathfrak{B})$**
3. **$\hat{\mathfrak{B}}$ is also strictly Σ-dissipative**
4. **$\hat{\mathfrak{B}}$ satisfies $\hat{\mathfrak{B}}^* \subset \mathfrak{B}^*$**

(Fourth point: trajectories in $\mathfrak{B}$ of minimal dissipation are retained into $\hat{\mathfrak{B}}$)
(Problem formulation correct except for stability aspect)