Passivity/dissipativity-preserving model reduction Algorithm (Lecture 14)

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Outline

- **Remaining behavioral-notation and**
- **(Half-line) dissipativity definition and results**
- **Dissipativity-preserving model-order reduction**

Half-line dissipativity: $\Sigma = \Sigma^T \in \mathbb{R}^{w \times w}$

- **Recall a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}_{\mathrm{cont}}$ was called Σ-dissipative if**

$$\int_{\mathbb{R}} w^T \Sigma w dt \geqslant 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

- **Call $\mathfrak{B}$ dissipative on $\mathbb{R}_-$ if for all $w \in \mathfrak{B} \cap \mathfrak{D}$ and for all T**

$$\int_{-\infty}^{T} w^T \Sigma w dt \geqslant 0.$$

('bounded from below')
(like physical storage)

- **and on $\mathbb{R}_+$ if $\int_T^{\infty} w^T \Sigma w dt \geqslant 0$.**
- **dissipative $\Leftrightarrow \exists$ storage function $Q_\Psi(w)$**
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When supply rate Σ equals $\gamma^2 u^T u - y^T y$ and for system with input u and output y (Case of maximal input cardinality: $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$)

- **dissipativity on $\mathbb{R}_-$ $\Leftrightarrow$ transfer matrix is stable (no poles in CRHP)**
- $\mathfrak{B}$ is Σ-dissipative on $\mathbb{R}_-$ $\Leftrightarrow$ $\mathfrak{B}^{\perp_\Sigma}$ is $-\Sigma$-dissipative on $\mathbb{R}_+$

Dissipativity on $\mathbb{R}_-$ $\Leftrightarrow$ maximum storage function $Q_{\Psi_{\max}}(w) \geqslant 0$ (i.e. $K_{\max} \geqslant 0$) ($Q_{\Psi_{\max}}(w)$: 'required supply')

Dissipativity on $\mathbb{R}_+$ $\Leftrightarrow$ minimum storage function $Q_{\Psi_{\min}}(w) \leqslant 0$ (i.e. $K_{\min} \leqslant 0$) ($Q_{\Psi_{\min}}(w)$: 'available storage')

Stability and half-line dissipativity

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Kernel/image 'dimensions'

Assume $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ has minimal kernel representation $R(\frac{d}{dt})w = 0$
(Full row rank $R(\xi)$) and image representation $w = M(\frac{d}{dt})\ell$

- **WLOG, choose $M(\xi)$ such that $M(\lambda)$ is full column rank for all $\lambda \in \mathbb{C}$.**
  (ℓ is 'observable' from w, also M is called 'right-prime')
- **$R(\xi) = [P(\xi) \;\; Q(\xi)]$ with $\det(P(\xi)) \neq 0$ and $w = (y, u)$, then transfer matrix from u to y is $G(s) = -P(s)^{-1}Q(s)$.**
- **Number of rows in R = number of outputs**
- **Number of columns in M = number of inputs**

McMillan degree

Again assume controllable $\mathfrak{B}$, and R is left-prime and M is right-prime

- Corresponding to $w = (y, u)$, also partition $M(\xi) = \begin{bmatrix} Y(\xi) \\ U(\xi) \end{bmatrix}$, $G(s) = -P(s)^{-1}Q(s) = Y(s)U(s)^{-1}$ (left/right (polynomial) coprime factorization of $G(s)$)
- Amongst all maximal nonsingular minors P in $R(\xi) = [P(\xi) \;\; Q(\xi)]$, find one with maximum determinantal degree: $\mathtt{n}(\mathfrak{B})$: McMillan degree
- Ensures $G(s)$ is proper: $\det U(s)$ has same degree, and is also maximum
- $\mathtt{n}(\mathfrak{B})$: least number of 'states' (defined using a 'concatenability' axiom)

Interested in w-behavior: manifest behavior $\mathfrak{B} \subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$: kernel representations
but conveniently

- **image representations: 'free' ℓ generates all trajectories**
- **state representations: state x 'being equal' allows concatenation of trajectories**
- **powerful/efficient/accurate manipulation of constant matrices: (E, A, B, C, D)**

$E\frac{d}{dt}x + Fx + Gw = 0$ **and**

$$\left(\begin{array}{l} \frac{d}{dt}x = Ax + Bw_1, \\ w_2 = Cx + Dw_1 \end{array} \right) \Leftrightarrow$$ transfer matrix $w_1 \to w_2$ is proper

(last one: i/s/o representation)

Few more 'convenient' representations

Interested in w-behavior: manifest behavior $\mathfrak{B} \subseteq \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$: kernel representations
but conveniently

- image representations: 'free' ℓ generates all trajectories
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- powerful/efficient/accurate manipulation of constant matrices: (E, A, B, C, D)

$E\frac{d}{dt}x + Fx + Gw = 0$ and

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(last one: i/s/o representation)

Output nulling and driving variable (state) representations

Interested in w only.
Dummy variables: x and d (with additional properties)

$$\dot{x} = Ax + Bd \text{ and } w = Cx + Dd \quad \text{(driving variable (d.v.))},$$
$$\dot{x} = Ax + Bw \text{ and } 0 = Cx + Dw \quad \text{(output nulling (o.n.))},$$

Just like $\mathfrak{B} = \ker R(\frac{d}{dt}) \Leftrightarrow \mathfrak{B}^{\perp} = \text{image } R(-\frac{d}{dt})^T$,

Can jump between i/s/o representations of $\mathfrak{B}$ and $\mathfrak{B}^{\perp}$

and d.v. representation of $\mathfrak{B}$ and o.n. of $\mathfrak{B}^{\perp}$ (and $\mathfrak{B}^{\perp_\Sigma}$)
We seek least number of variables in d, and in x:
'observability', 'trimness'

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Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ and symmetric nonsingular $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$
Suppose $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and suppose $\mathtt{n}$ is the McMillan degree of $\mathfrak{B}$
Choose $k < \mathtt{n}$. Find $\hat{\mathfrak{B}} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ such that

1. **$\hat{\mathfrak{B}}$ has McMillan degree at most k**
2. **$\mathtt{m}(\hat{\mathfrak{B}}) = \mathtt{m}(\mathfrak{B})$**
3. **$\hat{\mathfrak{B}}$ is also strictly Σ-dissipative on $\mathbb{R}_-$**
4. **$\hat{\mathfrak{B}}$ satisfies $(\hat{\mathfrak{B}}^*)_{\mathrm{anti-stab}} \subset \mathfrak{B}^*$**

(Fourth point: trajectories in $\mathfrak{B}$ of minimal dissipation are retained into $\hat{\mathfrak{B}}$)
$\mathfrak{B}^* = M(\frac{d}{dt}) \ker \partial\Phi'(\frac{d}{dt})$ and
strict dissipativity $\Leftrightarrow$ no $j\mathbb{R}$ roots of $\det \partial\Phi'(\xi)$

Proposed by Sorensen, SCL 2005, and as interpreted in Minh, Trentelman & Rapisarda (MCSS, 2009)

$w^T \Sigma w = u^T y$, $w = (u, y)$

$\frac{d}{dt}x = Ax + Bu$, and $y = Cx + Du$ for $\mathfrak{B}$, and hence

$\mathfrak{B}^{\perp_\Sigma}$ represented by $\frac{d}{dt}z = -A^T z + C^T u$, $y = B^T z - D^T u$

(Try $\frac{d}{dt}x^T z \stackrel{?}{=} u^T y$)

Interconnecting (& assuming strict passivity $\Rightarrow D + D^T > 0$)

$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = H \begin{bmatrix} x \\ z \end{bmatrix}$ and $\begin{bmatrix} u \\ y \end{bmatrix} = L \begin{bmatrix} x \\ y \end{bmatrix}$ with H and L respectively as

$$\begin{bmatrix} A-B(D+D^T)^{-1}C & B(D+D^T)^{-1}B^T \\ -C^T(D+D^T)^{-1}C & -A^T+C^T(D+D^T)^{-1}B^T \end{bmatrix}, \begin{bmatrix} -(D+D^T)^{-1}C & (D+D^T)^{-1}B^T \\ C-D(D+D^T)^{-1}C & D(D+D^T)^{-1}B^T \end{bmatrix}$$

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Choose anti-Hurwitz $R \in \mathbb{R}^{k \times k}$ (from ORHP spectral zeros) and corresponding real X and Y such that

$$H \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} R.$$

Strict dissipativities $\Rightarrow$ X and Y are both full column rank. They are 'part' of maximal ARE solution (known to be symmetric), same argument helps $X^T Y \in \mathbb{R}^{k \times k}$ being symmetric and positive definite.

- Obtain $X^T Y = QS^2Q^T$ with $Q^T = Q^{-1}$, and S diagonal.
- Define $V := XQS^{-1}$ and $W := YQS^{-1}$,
- $\hat{A} := W^T AV$, $\hat{B} := W^T B$, $\hat{C} := CV$ and $\hat{D} := D$
- Define reduced order system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$.

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- **Define reduced order system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$.**

Some verifications

$W^T V$ is identity matrix and
WV^T satisfies $(WV^T)^2 = WV^T$
?? $X^T Y$ is the largest ARE solution of the reduced system??
Recall: we sought $\hat{B}$:

1. $\hat{\mathfrak{B}}$ has McMillan degree at most k
2. $\mathtt{m}(\hat{\mathfrak{B}}) = \mathtt{m}(\mathfrak{B})$
3. $\hat{\mathfrak{B}}$ is strictly Σ-dissipative on $\mathbb{R}_-$
4. $\hat{\mathfrak{B}}$ satisfies $(\hat{\mathfrak{B}}^*)_{\text{anti-stab}} \subset \mathfrak{B}^*$

With $\hat{X} := \hat{Y} := SQ^T$ (Sorensen, SCL-'05), Minh, et al gets

$$\hat{H} \begin{bmatrix} \hat{X} \\ \hat{Y} \end{bmatrix} = \begin{bmatrix} \hat{X} \\ \hat{Y} \end{bmatrix} R$$

Further, $\hat{L}\hat{X} = LX$ and $\hat{L}\hat{Y} = LY$ give $(\hat{\mathfrak{B}}^*)_{\text{anti-stab}} \subset \mathfrak{B}^*$

Pick and Löwner matrices: Antoulas, SCL, 2005

- **Lagrange interpolating polynomials**
- **Rational interpolant with degree constraint → 'Löwner' matrices**
- **Link with Nevanlinna Pick interpolation problem**
- **Given N pairs $(x_i, y_i) \in \mathbb{C}^2$, find p.r. interpolant $G(s)$**
- **Pick matrix Π with Π_{ij} defined as**

$$\frac{y_i + y_j^*}{x_i + x_j^*} \quad \text{and} \quad \frac{1 - w_i w_j^*}{x_i + x_j^*} \quad \text{and} \quad \frac{1 - w_i w_j^*}{1 - z_i z_j^*}$$

depending on P.R., B.R. (OLHP), B.R. ($|z| = 1$), with

$$w_i := \frac{1 - y_i}{1 + y_i} \quad \text{and} \quad z_i := \frac{1 - x_i}{1 + x_i}$$

"Model reduction by interpolating at (some) spectral zeros"
"Pick matrix ≡ minimum energy required across trajectories in ker $A(\frac{d}{dt})$" (QDF, Willems & Trentelman, SIAM 1998)