Preliminaries about behaviors

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We will use behaviors to make ARE study easier. Behavioral techniques help in shifting from one 'representation' to another:

Representation \equiv system equations

Dissipativity (and storage functions) conveniently links: Algebraic Riccati Equations (ARE) (and ARI), Linear Matrix Inequalities (LMI), Hamiltonian matrix, Stationary trajectories, Euler Lagrange equations, and many more.

Some special cases of dissipativity: passivity, \mathcal{H}_{∞} -norm, sector nonlinearity

Different system representations: kernel, image, state-space, transfer function matrix

In transfer function matrix: right-coprime factorization (same as image representation) left-coprime factorization (same as kernel representation modulo controllable)

Throughout these lectures:

- m: number of inputs, p: number of outputs
- w: number of 'manifest' variables: typically m + p
- n: (minimum) number of states (McMillan degree)

$$G(s) \in \mathbb{R}^{\mathfrak{p} \times \mathfrak{m}}(s), \quad G(s) = P(s)^{-1}Q(s) = V(s)U(s)^{-1}$$

with $P, Q, U, V \in \mathbb{R}^{\bullet \times \bullet}[s]$. More precisely, $P, Q \in \mathbb{R}^{p \times \bullet}[s]$ and $U, V \in \mathbb{R}^{\bullet \times \mathfrak{m}}[s]$.

1 Behavior

A 'system' is nothing but the set of trajectories that the system allows. The system 'behavior' is the set of allowed trajectories, i.e. those that the system laws allow. Suppose the system variables are w.

 $\mathfrak{B} := \{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathsf{w}}) \mid w \text{ satisfies the system laws } \}.$

 \mathfrak{C}^{∞} : trajectory is infinitely often differentiable: primarily for convenience. Some notions do depend on the signal space used. \mathfrak{L}^{1}_{loc} is another frequently used space: this includes step, ramp and other such signals.

For model order reduction, \mathfrak{C}^{∞} is good enough.

Example:

Consider system with transfer function $G(s) = \frac{s-1}{s-8}$, with input u and output y. Then,

$$w = \begin{bmatrix} u \\ y \end{bmatrix} \text{ and } \mathfrak{B} = \{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^2) \mid \frac{\mathrm{d}}{\mathrm{d}t}u - u - \frac{\mathrm{d}}{\mathrm{d}t}y + 8y = 0 \}$$

Kernel representation: define $R(\xi) \in \mathbb{R}^{1 \times 2}[\xi]$ by $R(\xi) = [\xi - 1 \ 8 - \xi]$. Then, $\mathfrak{B} = \ker R(\frac{\mathrm{d}}{\mathrm{d}t})$, with signal space being \mathfrak{C}^{∞} .

All linear time-invariant 'lumped' (ordinary) differential equations can be written as such a set of differential equations $R(\frac{d}{dt})w = 0$, with $R(\xi) \in \mathbb{R}^{p \times w}[\xi]$. Note that number of equations is p (number of rows of $R(\xi)$) and number of variables is w: number of columns of $R(\xi)$.

Consider again a set of differential equations $R(\frac{d}{dt})w = 0$.

'Elementary row operations' on R do not change the set of solutions.

- Interchange two rows of R: premultiply R by (more generally) a permutation matrix¹.
- Multiply an equation by a nonzero *constant*: premultiply R by diagonal nonsingular constant matrix
- Differentiate an equation and add to another equation: premultiply by matrix P which has just one entry (say (i, j)-th entry) different from the identity matrix: $P(i, j) = p(\xi), inj.$

Any number and sequence of above three types of operations do not change the set of solutions/system-behavior. 'Sequence' here refers to product of the corresponding matrices. A matrix $U(\xi) \in \mathbb{R}^{p \times p}[\xi]$ is called unimodular if its determinant is a nonzero constant.

Theorem 1.1 Let $U \in \mathbb{R}^{p \times p}[\xi]$. Then the following are equivalent.

- U is unimodular.
- U can be written as a (non-unique) product of elementary matrices.
- The inverse of U exists and is (not just rational, but in fact) polynomial

In this lecture we also saw the Smith canonical form, (normal) rank of a polynomial matrix, unimodular completion and its link with the Bezout identity (as a special case).

Exercises will be in the lecture notes of lecture 10.

Suggested reading: [Kai80] for elaborate treatment on polynomial matrices and the Smith canonical form. Please see [PW98] for behavioral theory and a systematic algorithm to obtain the Smith canonical form. A relatively concise introduction to behaviors can be found in research papers, for example, [MTR09, WT98]

 $^{^{-1}}$ A square matrix P is called a permutation matrix if each row has exactly one entry equal to one, and rest all zero, and further, each column also satisfies this property.

References

- [Kai80] T. Kailath. *Linear Systems*. Englewood Cliffs, Prentice-Hall, 1980.
- [MTR09] H. B. Minh, H. L. Trentelman, and P. Rapisarda. Dissipativity preserving model reduction by retention of trajectories of minimal dissipation. *Mathematics of Control, Signals and Systems*, 21(3):171–201, 2009.
- [PW98] J.W. Polderman and J.C. Willems. Introduction to Mathematical Systems Theory: a Behavioral Approach. Springer-Verlag, New York, 1998.
- [WT98] J.C. Willems and H.L. Trentelman. On quadratic differential forms. SIAM Journal on Control and Optimization, 36:1703–1749, 1998.