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Synthesis of Dissipative Behaviors with Dynamics in the Weighting Matrices

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1 Introduction and preliminaries

$\mathcal{H}_\infty$-control has been a subject of intensive research for almost three decades now for two main reasons: proven performance in practical applications and the elegance of the theory. Weighted $\mathcal{H}_\infty$-control has been studied in several contexts because of its equally wide range of applications. In this paper we consider a more general formulation of this problem and show how the behavioral approach allows us to solve, as a special case, the weighted $\mathcal{H}_\infty$-control problem in a more straightforward fashion.

In order to make this paper self-contained, we cover some essential definitions, which we do in the rest of this section. The following section contains the formulation of the dissipativity synthesis problem (DSP) and how certain standard control problems can be formulated into a DSP. Section 3 contains the main results of this paper, which are some necessary conditions and certain sufficient conditions for the solvability of the DSP. We conclude this paper with a few remarks on the main results in section 4.

A linear differential controllable behavior $\mathfrak{B}$ is the set of those trajectories $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ that are in the image of the matrix differential operator $M(\frac{\mathrm{d}}{\mathrm{d}t})$, where $M(\xi)$ is some polynomial matrix with $\mathtt{w}$ rows. More precisely,

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{ there exists } \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\ell}) \text{ such that } w = M(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell \ \}.$$

The set of such controllable behaviors with $\mathtt{w}$ components is denoted by $\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. For the purpose of this paper, the easiest way to define the input cardinality of a behavior $\mathfrak{B}$ is the rank of the polynomial matrix $M(\xi)$ in the above equation. We denote the input cardinality of $\mathfrak{B}$ by $\mathtt{m}(\mathfrak{B})$. See [4] for a good exposition on the behavioral approach to systems and control.

The set of polynomial matrices (in one indeterminate ξ) with $\mathtt{w}$ rows and ℓ columns is denoted by $\mathbb{R}^{\mathtt{w}\times\ell}[\xi]$. We shall also use matrices in which each entry is a polynomial in *two* indeterminates ζ and η. $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is the set of such polynomial matrices with $\mathtt{w}$ rows and columns. Induced by $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, we have the *bilinear differential form* $L_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ defined as follows. Let $\Phi(\zeta,\eta)$ be written as a

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(finite) sum $\Phi(\zeta,\eta)=\Sigma_{k,\ell\in\mathbb{Z}_+}\Phi_{k\ell}\zeta^k\eta^\ell$ with $\Phi_{k\ell}\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Define $L_\Phi(w,v)$ by

$$L_\Phi(w,v):=\sum_{k,\ell\in\mathbb{Z}_+}(\tfrac{\mathrm{d}^k}{\mathrm{d}t^k}w)^T\Phi_{k\ell}(\tfrac{\mathrm{d}^\ell}{\mathrm{d}t^\ell}v).$$

Using the bilinear differential form L_Φ induced by $\Phi\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, we define the quadratic differential form $Q_\Phi:\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})\to\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ by

$$Q_\Phi(w):=L_\Phi(w,w)=\sum_{k,\ell\in\mathbb{Z}_+}(\tfrac{\mathrm{d}^k}{\mathrm{d}t^k}w)^T\Phi_{k\ell}(\tfrac{\mathrm{d}^\ell}{\mathrm{d}t^\ell}w).$$

In this paper we consider integrals of quadratic differential forms. Given $\Phi\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, we call $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ Φ-*dissipative* if $\int_{\mathbb{R}}Q_\Phi(w)\,\mathrm{d}t\geqslant 0$ for all $w\in\mathfrak{B}\cap\mathfrak{D}$. ($\mathfrak{B}\cap\mathfrak{D}$ is the subspace of those trajectories in $\mathfrak{B}$ which are compactly supported.) In this context, Φ is said to induce the *supply rate* Q_Φ.

$\Phi\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is called symmetric if $\Phi(\zeta,\eta)=\Phi^T(\eta,\zeta)$. Notice that when considering the quadratic differential form induced by Φ, we can assume that Φ is symmetric without loss of generality.

Let $\Phi\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Define $\partial\Phi\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ by $\partial\Phi(\xi)=\Phi(-\xi,\xi)$. It is easy to see that if Φ is symmetric then the the complex matrix $\partial\Phi(i\omega)$ is Hermitian for all $\omega\in\mathbb{R}$.

2 Dissipativity synthesis and its applications

In this paper we consider the problem of synthesis of dissipative behaviors. In this context we require the relation between the input cardinality of the behavior and the *signature* of the polynomial matrix that induces the supply rate. (The signature of a nonsingular symmetric constant matrix M, denoted by $\operatorname{sign}(M)$, is defined as $(\sigma_-(M),\sigma_+(M))$, where $\sigma_-(M)$ is the number of negative eigenvalues of M, and $\sigma_+(M)$ is the number of positive eigenvalues of M.) In order to make an analogous definition for the signature of a two variable polynomial matrix $\Phi(\zeta,\eta)$, we make certain assumptions on Φ, and these assumptions remain for the rest of this paper.

Assumption 1 : *Let $\Phi\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be symmetric. Assume that $\partial\Phi$ is nonsingular and that $\partial\Phi$ admits a J-spectral factorization: $\partial\Phi(\xi)=F^T(-\xi)JF(\xi)$ for some $F\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ and $J\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ of the form*

$$J=\begin{bmatrix}I_+ & 0\\ 0 & -I_-\end{bmatrix}.$$

Under these assumptions, we define $(\sigma_-(\partial\Phi),\sigma_+(\partial\Phi))=\operatorname{sign}(\partial\Phi):=\operatorname{sign}(J)$.

It is well known that J-spectral factorizability of $\partial\Phi$ is equivalent to $\partial\Phi(i\omega)$ having constant signature for almost all $\omega\in\mathbb{R}$ (see [5]). We are now ready to state the dissipativity synthesis problem (DSP).

Dissipativity synthesis problem (DSP): Suppose $\Phi\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ satisfies assumption 1 and let $\mathcal{N},\mathcal{P}\in\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with $\mathcal{N}\subseteq\mathcal{P}$. Find conditions under which there exists a behavior $\mathcal{K}\in\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ satisfying

1. $\mathcal{N}\subseteq\mathcal{K}\subseteq\mathcal{P}$,
2. $\mathcal{K}$ is Φ-dissipative, and
3. $\mathtt{m}(\mathcal{K})=\sigma_+(\partial\Phi)$.

$\mathcal{K}$ is called the controlled behavior, while $\mathcal{N}$ and $\mathcal{P}$ are called the hidden and the plant behaviors respectively. Each of the three conditions above have important implications in systems theory, and more on this can be found in [9, 6]. In [9, 6], however, the problem was solved only for the case that Φ is a constant. (This means that the supply rate does not depend on derivatives of the concerned variables.) Another important difference between the above problem and the one studied in [6, 9] is that the dissipativity there was required to hold on the *half-line*. Half-line dissipativity is a concept stronger than just dissipativity as we have defined above. The relation between half-line dissipativity and internal stability of the controlled behavior $\mathcal{K}$ has been brought out in [6]. In this paper we do not require $\mathcal{K}$ to be half-line dissipative and hence we do not go more into this concept. We only note that, as far as the $\mathcal{H}_\infty$-control problem is concerned, the above DSP implies that the closed loop behavior need not have to be internally stable, i.e. the corresponding transfer function is allowed to have poles in the closed right half plane also. Thus, strictly speaking, this problem is better termed the $\mathcal{L}_\infty$-control problem.

An example where we encounter dissipativity with respect to a supply rate induced by a nonconstant Φ is in mechanical systems. Here, power can be expressed as force F times the derivative of the position x, i.e. power$= F\frac{\mathrm{d}}{\mathrm{d}t}x$. Thus, for $w = (F, x)$, the supply rate $Q_\Phi(w) = F\frac{\mathrm{d}}{\mathrm{d}t}x$ is induced by $\Phi \in \mathbb{R}^{2\times 2}[\zeta, \eta]$ defined below

$$\Phi(\zeta, \eta) := \frac{1}{2}\begin{bmatrix} 0 & \eta \\ \zeta & 0 \end{bmatrix}.$$

Hence the problem of synthesis of a passive mechanical system (i.e. the controlled mechanical system can only *absorb* energy) can be formulated as a DSP, with the above Φ.

We now briefly describe how the weighted $\mathcal{H}_\infty$-control problem (or rather the weighted $\mathcal{L}_\infty$-control problem) can be viewed as a special case of the DSP. Consider the $\mathcal{H}_\infty$-disturbance attenuation control problem, in which the problem is to design a controller that ensures that the effect of exogenous disturbance d on the endogenous to-be-regulated output z is sufficiently small. Interpret $G(s)$ as the to-be-shaped transfer function from d to z. For several applications, it is useful to shape the frequency response of not $G(s)$ but of $W(s)G(s)$, with $W(s)$ a weighting function. (Both $G(s)$ and $W(s)$ are assumed to be real rational transfer matrices.) Well-studied applications include high frequency roll-off, partial pole placement, etc.. More on this can be found in [3, 7].

This problem can be rephrased as finding a controller such that $\mathcal{L}_\infty$-norm of $W(s)G(s)$ is less than or equal to a given positive γ. Assume for simplicity that both $G(s)$ and $W(s)$ are scalar transfer functions. This problem can be reformulated into a DSP as explained below.

Define $w := (d, z)$ and choose $\Phi \in \mathbb{R}^{2\times 2}[\zeta, \eta]$ as

$$\Phi(\zeta, \eta) := \begin{bmatrix} \gamma^2 d(\zeta)d(\eta) & 0 \\ 0 & -n(\zeta)n(\eta) \end{bmatrix}$$

where $W(s) = \frac{n(s)}{d(s)}$ is a factorization of $W(s)$. Once the dissipativity synthesis problem is solved, we obtain a $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ that satisfies the requirements. Corresponding to this $\mathcal{K}$, we obtain the transfer function $G(s)$ from d to z. One can verify that requiring $\mathcal{K}$ to be Φ-dissipative is equivalent to requiring the $\mathcal{L}_\infty$-norm of $W(s)G(s)$ to be less than or equal to γ. Notice that both $G(s)$ and $W(s)$ are allowed to have poles in the open right half plane also. This illustrates how the weighted $\mathcal{L}_\infty$-control problem can be posed as a special case of the DSP.

3 Main results

In this paper, we provide necessary conditions for the existence of a $\mathcal{K}$ satisfying the requirements of the DSP, and we prove sufficiency of these necessary conditions together with certain additional conditions.

To be able to state these conditions, we need some more definitions relating to behaviors and their orthogonal complement. Consider a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, the orthogonal complement $\mathfrak{B}^\perp$ of $\mathfrak{B}$ is defined as

$$\mathfrak{B}^\perp := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \int_{\mathbb{R}} w^T v \,\mathrm{d}t = 0 \text{ for all } v \in \mathfrak{B} \cap \mathfrak{D}\}.$$

The orthogonal complement $\mathfrak{B}^\perp$ of a controllable behavior $\mathfrak{B}$ turns out to be a controllable behavior too. These facts, together with other relations about orthogonality and their proofs, can be found in [1]. We also require the notion of orthogonal complement with respect to a specific $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. The Φ-orthogonal complement $\mathfrak{B}^{\perp_\Phi}$ of $\mathfrak{B}$ is defined as $(\partial\Phi(\frac{\mathrm{d}}{\mathrm{d}t})\mathfrak{B})^\perp$. One can show that $\mathfrak{B}^{\perp_\Phi}$ is the largest controllable behavior such that

$$\int_{\mathbb{R}} L_\Phi(w, v)\,\mathrm{d}t = 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D} \text{ and } v \in \mathfrak{B}^{\perp_\Phi} \cap \mathfrak{D}.$$

Using this notion of the Φ-orthogonal complement of a behavior we are ready to state the necessary conditions for the solvability of the DSP.

Theorem 2 : *If $\mathcal{K}$ satisfying the requirements of the DSP above exists, then the following two conditions are satisfied:*

1. *$\mathcal{N}$ is Φ-dissipative, and*
2. *$\mathcal{P}^{\perp_\Phi}$ is $(-\Phi)$-dissipative.*

In addition to the above two conditions that are necessary for the existence of a $\mathcal{K}$, we shall assume certain 'regularity assumptions' on the dissipativities of $\mathcal{N}$ and $\mathcal{P}^{\perp_\Phi}$. These three conditions are shown to be sufficient for the existence of a suitable $\mathcal{K}$, in the following theorem.

Theorem 3 : *$\mathcal{K}$ satisfying the requirements in the DSP above exists if the following conditions are satisfied:*

1. *$\mathcal{N}$ is Φ-dissipative,*
2. *$\mathcal{P}^{\perp_\Phi}$ is $(-\Phi)$-dissipative, and*
3. $\mathtt{m}(\mathcal{N} \cap \mathcal{N}^{\perp_\Phi}) = 0 = \mathtt{m}(\mathcal{P} \cap \mathcal{P}^{\perp_\Phi})$.

The third condition in the above theorem is a kind of strictness on the dissipativity of the concerned behavior. More precisely, it can be shown (see [2]) that if $\mathcal{N}$ is Φ dissipative, then $\mathtt{m}(\mathcal{N} \cap \mathcal{N}^{\perp_\Phi}) = 0$ is equivalent to

$$\left.\begin{array}{r} w \in \mathcal{N} \cap \mathfrak{D} \\ \int_{\mathbb{R}} Q_\Phi(w)\,\mathrm{d}t = 0 \end{array}\right\} \Rightarrow w = 0.$$

In other words, the zero trajectory is the only compact support trajectory in $\mathcal{N}$ for which $\int_{\mathbb{R}} Q_\Phi(w)\,\mathrm{d}t = 0$. It is this kind of strictness of Φ-dissipativity of $\mathcal{N}$ that has been assumed in the above theorem. A similar remark holds for the condition $\mathtt{m}(\mathcal{P} \cap \mathcal{P}^{\perp_\Phi}) = 0$. The proof of the above theorems can be found in [2].

4 Remarks

We have provided some necessary conditions for the solvability of the dissipativity synthesis problem. We have also shown sufficience of these necessary conditions, together with additional conditions. The regularity assumptions make the situation similar to the 'regular case' in the proof of the corresponding dissipativity synthesis problem as solved in [9]. Though the regularity assumptions make the proof much simpler, one would expect that the two necessary conditions are also sufficient for the solvability of the DSP. This issue is explored in [2].

References

[1] M.N. Belur, *Control in a Behavioral Context*, Doctoral dissertation, University of Groningen, The Netherlands, 2003.

[2] M.N. Belur, H.K. Pillai and H.L. Trentelman, Weighted $\mathcal{H}_\infty$-control: a behavioral approach, *to-be-submitted*.

[3] H. Kwakernaak, A polynomial approach to minimax frequency domain optimization of multivariable systems, *International Journal of Control*, volume 44, pages 117-156, 1986.

[4] J.W. Polderman and J.C. Willems, *Introduction to Mathematical Systems Theory: a Behavioral Approach*, Springer Verlag, 1997.

[5] A. C. M. Ran and L. Rodman, Factorization of matrix polynomials with symmetries, *SIAM Journal on Matrix Analysis and Applications*, volume 15, 1994, pages 845-864.

[6] H.L. Trentelman and J.C. Willems, Synthesis of dissipative systems using quadratic differential forms: part II, *IEEE Transactions on Automatic Control*, volume 47, pages 70-86, 2002.

[7] M.C. Tsai, E.J.M. Geddes and I. Postlethwaite, Pole-zero cancellations and closed-loop properties of an $\mathcal{H}_\infty$-mixed sensitivity design problem, *Automatica*, volume 3, 1992, pages 519-530.

[8] J.C. Willems and H.L. Trentelman, On quadratic differential forms, *SIAM Journal on Control and Optimization*, volume 36, 1998, pages 1703-1749.

[9] J.C. Willems and H.L. Trentelman, Synthesis of dissipative systems using quadratic differential forms: part I, *IEEE Transactions on Automatic Control*, volume 47, pages 53-69, 2002.