Stability margin of the dissipativity property

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***Abstract*— This paper is about an algorithm to compute the maximum amount of 'perturbation', within a suitable class of the quadratic supply rates, such that a given system is dissipative with respect to this class of perturbed supply rates. This turns out to also be a method to verify strict dissipativity of a behavior with respect to a given rate of supply of energy.**

Keywords: Dissipative system, behavior, strict dissipativity, quadratic supply rates

I. Introduction

Dissipative systems are those that absorb net energy when they interact with the environment through their 'manifest' variables. (The variables through which this interaction takes place are termed *manifest variables*.) This interaction is often a quadratic form in the manifest variables. For example, the power flowing in through the ports of an electric network is V^TI, where V is the vector of voltages across the ports, and I is the vector of currents through the ports (with the suitable sign convention). So, here the power is quadratic in the manifest variables (V, I).

In this paper we study dissipativity for more general supply rates, and we use the behavioral approach for this purpose. In this context, strict dissipativity is defined and we present an algorithm to compute the 'extent' (say, ϵ) to which the dissipativity is strict. This ϵ can be interpreted as the stability margin of the property of dissipativity for a given behavior. The computation of the stability margin turns out to open up a novel algorithm to compute the $\mathcal{H}_\infty$ norm of a transfer matrix; and this algorithm turns out to be computationally more efficient than the present method ([2]) used in Matlab.

This paper is structured as follows. This section contains briefly the notation that we use, and then basic definitions of behavioral theory. Preliminaries about dissipativity is covered in the next section. Section II also contains definitions and the relation between 'stability margin' of the dissipativity, strict dissipativity and the $\mathcal{H}_\infty$-norm of a transfer function. Section III contains an algorithm to compute this stability margin. A few examples are provided in section IV to illustrate the method. A few conclusive remarks form section V.

$\mathbb{R}$ and $\mathbb{C}$ denote the fields of real numbers and complex numbers respectively, and $\mathbb{R}[\xi]$ is the ring of polynomials in one indeterminate ξ. $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ is the set of infinitely often differentiable functions (i.e., smooth functions) that have domain $\mathbb{R}$ and co-domain $\mathbb{R}^{\mathtt{n}}$, the $\mathtt{n}$-dimensional real vector space. $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ is a subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ and contains all functions in $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ that are compactly supported. In order to systematically denote the number of components in a vector valued function w, we write $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, i.e. $\mathtt{w}$ denotes the number of components in w. Similarly, we write $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\ell)$.

A controllable behavior $\mathfrak{B}$ is a set of functions $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ that are in the image of some operator $M(\frac{\mathrm{d}}{\mathrm{d}t})$, where $M(\xi)$ is a polynomial matrix. More precisely,

$$\mathfrak{B} \;:= \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{ there exists } \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\ell) \text{ such that } w = M(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell\}.$$

We write $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. ($\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is the set of controllable behaviors.) In the above definition, $M \in \mathbb{R}^{\mathtt{w}\times\ell}[\xi]$, i.e. M is a polynomial matrix with $\mathtt{w}$ rows and ℓ columns. In general, M is not unique and $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ is called *an image representation* of $\mathfrak{B}$. Further, if $w_1 = w_2$ implies $\ell_1 = \ell_2$, we call this image representation *observable*. It turns out that a given image representation $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ is observable if and only if the complex matrix $M(\lambda)$ has constant rank for all $\lambda \in \mathbb{C}$. In light of the fact that controllable behaviors are precisely those that admit an observable image representation, we use such representations without loss of generality in this paper.

This paper is restricted to just controllable behaviors since we deal with only controllable behaviors here. A good exposition on the behavioral approach to systems and control can be found in [3], [4].

II. Dissipative systems

Dissipative systems are those for which the the net energy that has flown into the system is always non-negative. A big class of physical systems exhibit this property and hence the sustained interest in them. A formal treatment of dissipative systems theory in the behavioral approach can be found in [5]. In this section the essential definitions are covered.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and let $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be symmetric and nonsingular. $\mathfrak{B}$ is said to be *dissipative* with respect to Σ (or Σ-dissipative) if

$$\int_{\mathbb{R}} w^T \Sigma w \,\mathrm{d}t \geqslant 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}).$$

In this context Σ is called the *rate of supply of energy*, or simply, *supply rate*. The following theorem from [5] relates dissipativity property with a more concrete condition on Σ and $M(\xi)$, where $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ is an image representation of the behavior.

Theorem 1: **:** Consider $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, and let $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be symmetric and nonsingular. Let $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ be an image representation of $\mathfrak{B}$. Then $\mathfrak{B}$ is Σ dissipative if and only if

$$M^T(-i\omega)\Sigma M(i\omega) \geqslant 0 \text{ for all } \omega \in \mathbb{R} .$$

Thus for each $\omega \in \mathbb{R}$, non-negativity of the Hermitian matrix $M^T(-i\omega)\Sigma M(i\omega)$ is necessary and sufficient for Σ dissipativity of $\mathfrak{B}$.

For dissipative systems, there could be nonzero trajectories in the behavior such that the net energy absorbed along these trajectories is zero. This is ruled out in the case of strict dissipativity which we define now. $\mathfrak{B}$ is called strictly dissipative with respect to Σ if there exists $\epsilon > 0$ such that

$$\int_{\mathbb{R}} w^T \Sigma w \, \mathrm{d}t \geqslant \epsilon \int_{\mathbb{R}} |w|^2 \, \mathrm{d}t \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}).$$

($|w|^2$ denotes $w^T w$.) In other words, $\mathfrak{B}$ is called strictly Σ-dissipative if $\mathfrak{B}$ is dissipative with respect to $\Sigma - \epsilon I$ for some $\epsilon > 0$, where I is the identity matrix of suitable size.

Notice that if $\mathfrak{B}$ is dissipative with respect to Σ, then $\mathfrak{B}$ is also dissipative with respect to $\Sigma - \epsilon I$ for $\epsilon \leqslant 0$. The situation is different for the case $\epsilon > 0$. This forms the central issue of this paper:

> Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, and $\Sigma \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ which is symmetric and nonsingular, find the maximum $\epsilon \in \mathbb{R}$ such that $\mathfrak{B}$ is $\Sigma - \epsilon I$ dissipative.

Denote this maximum ϵ by $\epsilon_{\max}$. The algorithm described in this paper computes $\epsilon_{\max}$, and clearly, $\epsilon_{\max} \geqslant 0$ is equivalent to dissipativity. Moreover, $\epsilon_{\max} > 0$ is equivalent to strict dissipativity.

Notice that $\epsilon_{\max}$ can also be interpreted as the 'stability margin' of the dissipativity property. Suppose $\mathfrak{B}$ is Σ-dissipative. Consider perturbations of Σ of the kind $\Sigma - \epsilon I$. The question that arises now is whether dissipativity of $\mathfrak{B}$ is retained under this class of perturbations for ϵ sufficiently small, and what is the maximum allowed ϵ that does not make $\mathfrak{B}$ non-dissipative. This viewpoint also leads to understanding $\epsilon_{\max}$ as the amount of robustness of the dissipativity of $\mathfrak{B}$ (under the specified class of perturbations of Σ).

We now relate this issue to the $\mathcal{H}_\infty$ norm of a transfer function. Let $g(s) = \frac{q(s)}{p(s)}$ be a transfer function, with p and q coprime polynomials. Assume g is proper and has no poles in the closed right half complex plane. Suppose $\|g(s)\|_{\mathcal{H}_\infty} = \gamma_0 < \gamma_1$. ($\gamma_1$ is an upper bound on the $\mathcal{H}_\infty$ norm of g.) Then, $\mathfrak{B}_g \in \mathfrak{L}^2_{\text{cont}}$ defined by the image representation

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} p(\frac{\mathrm{d}}{\mathrm{d}t}) \\ q(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \ell \tag{1}$$

is dissipative with respect to

$$\begin{bmatrix} \gamma_1^2 & 0 \\ 0 & -1 \end{bmatrix}.$$

Moreover, it can be verified that $\mathfrak{B}$ is also strictly Σ-dissipative with the corresponding $\epsilon_{\max}$ as below

$$\epsilon_{\max} = \frac{\gamma_1^2 - \gamma_0^2}{\gamma_0^2 + 1}.$$

In this fashion, we can start with an upper bound on the $\mathcal{H}_\infty$ norm of a transfer function, and use the method described in the next section to compute $\epsilon_{\max}$ and then calculate the $\mathcal{H}_\infty$ norm of g.

III. Algorithm

In this section we present an algorithm to compute $\epsilon_{\max}$, i.e. the maximum $\epsilon \in \mathbb{R}$ such that $\mathfrak{B}$ is dissipative with respect to $\Sigma - \epsilon I$.

Let $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ be an observable image representation of $\mathfrak{B}$. Define $P \in \mathbb{C}^{\ell \times \ell}[\omega, \epsilon]$ as follows:

$$P(\omega, \epsilon) := M^T(-i\omega)\Sigma M(i\omega) - \epsilon M^T(-i\omega)M(i\omega) \,. \tag{2}$$

$\mathbb{C}^{\ell \times \ell}[\omega, \epsilon]$ is a matrix of size $\ell \times \ell$ in which each entry is a polynomial in two indeterminates ω and ϵ, and with complex coefficients.

An underlying idea behind the algorithm is to find the maximum ϵ that makes $P(\omega, \epsilon)$ singular for some $\omega \in \mathbb{R}$. Assume that $P(\omega_0, \epsilon_4)$ is singular for some (finite) $\omega_0,\ \epsilon_4 \in \mathbb{R}$. To see that ϵ_4 is an upper bound for $\epsilon_{\max}$, proceed as follows.

Let $P(\omega_0, \epsilon_4)$ be singular for $\epsilon_4, \omega_0 \in \mathbb{R}$. Hence there exists $0 \neq v \in \mathbb{C}^\ell$ such that

$$(M(i\omega_0)v)^*(\Sigma - \epsilon_4 I)(M(i\omega_0)v) = 0.$$

Consider

$$\begin{aligned} &(M(i\omega_0)v)^*(\Sigma - \epsilon I)(M(i\omega_0)v) \\ &\quad = v^* P(\omega_0, \epsilon_4)v - (\epsilon - \epsilon_4)(M(i\omega_0)v)^*(M(i\omega_0)v) \\ &\quad = -(\epsilon - \epsilon_4)(M(i\omega_0)v)^*(M(i\omega_0)v) \,. \end{aligned}$$

Note that because $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, $M^T(-i\omega)M(i\omega)$ is nonsingular for all $\omega \in \mathbb{R}$. This results in the required conclusion: for $\epsilon > \epsilon_4$, $v^* P(\omega_0, \epsilon)v < 0$ and hence $P(\omega_0, \epsilon) \ngeqslant 0$ for $\epsilon > \epsilon_4$, This proves that ϵ_4 is an upper bound for $\epsilon_{\max}$.

A step-by-step procedure is given below. The explanations for the various steps are provided along with the algorithm. The formal proofs are skipped here, since they follow along similar lines as in [1] for an analogous problem.

Step 1) Define the two variable polynomial matrix $P(\omega, \epsilon)$ as in equation (2).

Step 2) Evaluate the polynomial $p_\epsilon(\omega) \in \mathbb{C}[\omega, \epsilon]$ defined as $p_\epsilon(\omega) := \det(P(\omega, \epsilon))$.

Step 3) For each $\omega, \epsilon \in \mathbb{R}$, the matrix $P(\omega, \epsilon)$ is a Hermitian matrix. We wish to find the maximum $\epsilon_{\max}$ such that for all $\epsilon \leqslant \epsilon_{\max}$,

$$P(\omega, \epsilon) \geqslant 0 \text{ for all } \omega \in \mathbb{R} .$$

Again, because of the nonsingularity of $M^T(-i\omega)M(i\omega)$ for all $\omega \in \mathbb{R}$, we have that for $\epsilon \ll 0$ (i.e. $\epsilon < 0$ and $|\epsilon|$ is very large) the above inequality is always satisfied as a strict inequality. In other words, for $\epsilon \ll 0$, $P(\omega, \epsilon)$ has ℓ real and positive eigenvalues for each $\omega \in \mathbb{R}$. This implies that for $\epsilon \ll 0$, $p_\epsilon(\omega)$ has no roots in $\mathbb{R}$. A graph of $p_\epsilon(\omega)$ versus ω is shown in figure 1.

Step 4) We now analyze the properties that the polynomial p_ϵ has, at this maximum value $\epsilon_{\max}$ we are seeking. As noted above, p_ϵ has no real roots for ϵ sufficiently negative and we look for the supremum $\epsilon_{\sup}$ such that p_ϵ has no real roots for all $\epsilon < \epsilon_{\sup}$. It can be shown that $\epsilon_{\max} = \epsilon_{\sup}$ (see the explanation just before beginning of this algorithm), and, further, that at $\epsilon = \epsilon_{\sup}$, p_ϵ has at least one *repeated* real root $\omega_0 \in \mathbb{R} \cup \infty$. We now distinguish the following two cases:

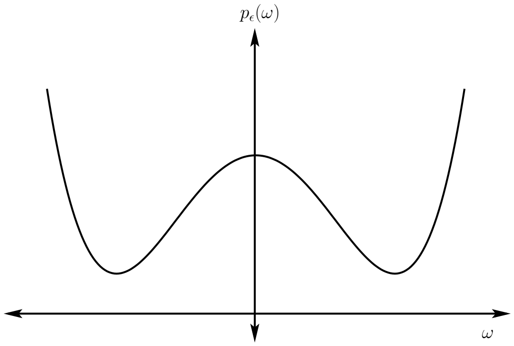

Fig. 1. for $\epsilon \ll 0$ (or equivalently, for $\epsilon < \epsilon_{\text{sup}}$)

Case 1: At $\epsilon = \epsilon_{\text{sup}}$, the repeated root of p_ϵ is finite. This case is dealt in the next step.
Case 2: The other case is that, as $\epsilon \to \epsilon_{\text{sup}}$, at least two roots of p_ϵ become unbounded. In this case, the degree of p_ϵ has fallen (by at least two), and, loosely speaking, we say $\omega_0 = \infty$ is the multiple root of p_ϵ at $\epsilon = \epsilon_{\text{sup}}$. This case is analyzed in **Step 10** below.

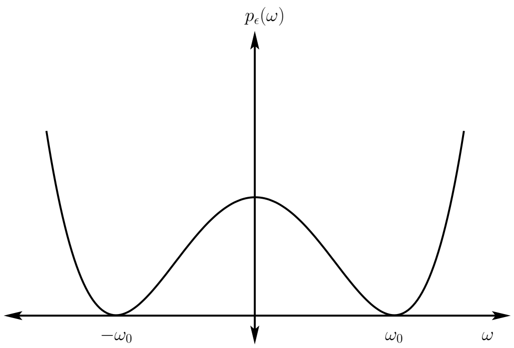

Fig. 2. for $\epsilon = \epsilon_{\max}$

Step 5) Case 1. At $\epsilon = \epsilon_{\max}$, $p_\epsilon(\omega)$ has a root ω_0 in $\mathbb{R}$. Further, by the sign definiteness of $P(\omega, \epsilon)$ for values lower than $\epsilon_{\max}$, we infer that this real root ω_0 should have an even multiplicity. This implies that $\frac{\mathrm{d}p_\epsilon}{\mathrm{d}\omega}(\omega)$ also has the same root. See figure 2. We compute $q_\epsilon(\omega) := \frac{\mathrm{d}p_\epsilon}{\mathrm{d}\omega}(\omega)$. We now determine all values of ϵ for which p_ϵ and q_ϵ are not coprime.

Step 6) We write the polynomials p_ϵ and q_ϵ as below. Notice that p_ϵ is an even polynomial in ω and hence its derivative q_ϵ is an odd polynomial in ω.

$$\begin{aligned} p_\epsilon(\omega) &= a_0(\epsilon) + \omega^2 a_2(\epsilon) + \cdots + \omega^n a_n(\epsilon) \\ q_\epsilon(\omega) &= \omega b_1(\epsilon) + \cdots + \omega^{n-1} b_{n-1}(\epsilon) \end{aligned} \tag{3}$$

for $a_i, b_i \in \mathbb{C}[\epsilon]$.

Step 7) We now use the result that two polynomials are coprime if and only if their Sylvester resultant is nonsingular. We form the Sylvester resultant $S \in \mathbb{C}^{(2n-1)\times(2n-1)}[\epsilon]$.

$$S = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-2} & 0 & a_n & 0 & \cdots & 0 \\ 0 & a_0 & \cdots & a_{n-3} & a_{n-2} & 0 & a_n & 0 & \cdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_0 & a_1 & a_2 & \cdots & 0 & a_n \\ 0 & b_1 & \cdots & 0 & b_{n-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & b_{n-3} & 0 & b_{n-1} & 0 & 0 & \cdots \\ \vdots & \ddots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & b_1 & \cdots & \cdots & b_{n-1} & 0 \\ 0 & \cdots & 0 & 0 & 0 & b_1 & \cdots & 0 & b_{n-1} \end{bmatrix} \tag{4}$$

with a_i, b_i polynomials from p_ϵ and q_ϵ. The a_i's constitute the first $n-1$ rows and the b_i's constitute the remaining n rows of S.

Step 8) We compute $\det(S)$. Assume that $\det(S) \neq 0$ The case that $\det(S) = 0$ is analyzed in in **Step 11** (Case 3) below.

Step 9) We then compute the roots of the polynomial $\det(S)$. It is precisely at these roots that p_ϵ and q_ϵ have a common root ω_0. We are looking for only real ϵ, and hence we restrict ourselves to the real roots of $\det(S)$. Moreover, it is possible that at some ϵ which is a real root of $\det(S)$, the common root ω_0 of p_ϵ and q_ϵ is not real. This value of ϵ is also not a candidate for $\epsilon_{\max}$. Hence we consider only those real roots of $\det(S)$ that result in p_ϵ and q_ϵ having a common root in $\mathbb{R}$. The minimum of these real roots of $\det(S)$: ϵ_1 is a candidate for the $\epsilon_{\max}$ we are seeking.

Step 10) Case 2. This case happens when, as $\epsilon \to \epsilon_{\max}$, a root of $p(\omega, \epsilon)$ approaches ∞. (For a very simple example, we can see that the root ω_0 of the polynomial $p(\omega, \epsilon) = \epsilon\omega + 1$ approaches ∞ as $\epsilon \to 0$. An example of a transfer function $g(s)$ which results in a behavior of the kind as in equation (1) is covered in the next section.) This happens precisely when ϵ approaches some root ϵ' of $a_n(\epsilon)$, the leading coefficient of $p_\epsilon(\omega)$.
The minimum ϵ_2 of the real roots of $a_n(\epsilon)$ is another candidate for $\epsilon_{\max}$.

Step 11) Case 3. This case is when $\det(S) = 0$. This happens when p_ϵ and $\frac{\mathrm{d}p_\epsilon}{\mathrm{d}\omega}$ are not coprime for all values of ϵ, or in other words, not coprime as polynomial in two variables. This implies that p_ϵ and q_ϵ can be factored as

$$\begin{aligned} p_\epsilon(\omega) &= r_\epsilon(\omega)\ p'_\epsilon(\omega) \\ q_\epsilon(\omega) &= r_\epsilon(\omega)\ q'_\epsilon(\omega) \end{aligned}$$

for some suitable nontrivial factor $r_\epsilon(\omega)$. We continue with p'_ϵ and q'_ϵ and it turns out that after dividing out factors of the kind $r_\epsilon(\omega)$, the Sylvester resultant S' obtained by p'_ϵ and the corresponding q'_ϵ satisfies $\det(S') \neq 0$. We proceed through Steps 7 to 9 with p'_ϵ and q'_ϵ instead. We form the Sylvester resultant and then proceed as in **Step 9** and obtain ϵ_1.
It turns out that the factor $r_\epsilon(\omega)$ does not contain any additional information about $\epsilon_{\max}$.

Step 12) Define $\epsilon_{\max} := \min(\epsilon_1, \epsilon_2)$. This completes the algorithm.

IV. Examples

In this section, we provide some simple examples which are typical of the three cases described above. We consider transfer functions and the associated behaviors are obtained as in equation (1) above.

Consider $g(s) = \frac{1}{s+1}$, and

$$\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}. \qquad (5)$$

We see that this is an example of Case 1, i.e. p_ϵ and q_ϵ are not coprime for $\epsilon_{\max} = \frac{3}{2}$, with the common root being finite ($\omega_0 = 0$). $g(s) = \frac{1}{s^2+s+1}$ is an example of a similar nature. Notice that in both these examples, the $\mathcal{H}_\infty$ norm of $g(s)$ is *attained* at a finite frequency.

Now consider $g(s) = \frac{s+1}{s+2}$ and Σ as in equation (5). Here, one can verify that the situation in Case 2 (of the algorithm) is relevant. This is obviously because the $\mathcal{H}_\infty$ norm of $g(s)$ is not attained at any finite frequency, but

$$\|\frac{s+1}{s+2}\|_{\mathcal{H}_\infty} = \lim_{\omega\to\infty} |\frac{i\omega+1}{i\omega+2}| = 1.$$

For an example of Case 3, we need a matrix of transfer functions; amongst the simplest being

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}.$$

We proceed with Σ and M as follows

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} \xi+1 & 0 \\ 0 & \xi+1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Straightforward calculations show that $\det(S) = 0$ and applying the procedure, we get $\epsilon_{\max} = \frac{3}{2}$.

V. Remarks

We presented an algorithm to determine the maximum perturbation $\epsilon_{\max}$ on the supply rate Σ (within a certain class of perturbations), such that a given behavior $\mathfrak{B}$ retains dissipativity within this perturbed class of supply rates. As explained above, we call this $\epsilon_{\max}$ the stability margin of the dissipativity property.

A second remark is that the algorithm of section III requires only a slight modification before it can be used as an efficient method to compute the $\mathcal{H}_\infty$ norm of a transfer matrix. This issue has been addressed in [1]. The proofs of the various claims above can also be found in [1].

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