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Graph theoretic methods in the study of structural issues in control

Madhu N. Belur[1] and Debraj Chakraborty[1]

[1]Department of Electrical Engineering, Indian Institute of Technology Bombay
Powai, Mumbai 400 076, India
(Fax: +91-22-2572-3707; E-mail: {belur,dc}@ee.iitb.ac.in)

Abstract: This paper addresses questions regarding controllability and observability for 'generic parameter' dynamical systems. We use graph theoretic methods and hence these questions are answered in a 'structural sense'. We formulate necessary and sufficient conditions for controllability in terms of suitable maximum matchings in the bipartite graph constructed from the constraints and variables. The conditions for observability turn out to be similar. We relate these tests to Gilbert's controllability and observability tests.

Keywords: maximum matching, perfect matching, controllability, observability, Gilbert's tests

1. INTRODUCTION AND PRELIMINARIES

When dealing with very large dynamical systems, numerical computation is often not feasible. Under the assumption of genericity of parameters, one can answer questions about controllability, observability and arbitrary pole placement using graph theoretic tools. These issues are typically dealt as 'structural' issues in control in [3, 4], for example. While existing techniques to address structural aspects of control start from a (possibly singular) state space representation of the system, the results in this paper apply to more general models of dynamical systems: linear differential-algebraic equations of possibly higher order. The behavioral theory of systems allows this general approach; the required preliminaries for this paper is in Section 3and more details can be found in [5, 6]. The paper is organized as follows. Some definitions and graph theoretic preliminaries are covered in the following section. Section 3.reviews some concepts within the behavioral approach needed for this paper. Section 4relates polynomial matrix rank properties to those of its bipartite graph. Some important results of this paper are in this section. Section 5contains main results about controllability, stabilizability, observability and detectability. Section 6contains a state space example and shows how the classical Gilbert's tests for controllability and observability are a special case of the main results of this paper. Section 7has some conclusive remarks.

2. MATCHINGS IN A GRAPH

A graph in which the vertices can be partitioned into two sets so that each edge in the graph is a correspondence between a node in one set with a vertex in the other set is called a bipartite graph. For this paper, one vertex set denotes the constraints, C, and the other denotes variables, V. In a bipartite graph, a vertex in one set could be connected to two or more vertices in the other set of vertices. A subset of the edges such that there is a one-to-one correspondence between the vertices in a bipartite graph is called a matching, i.e. each vertex has at most one edge from this subset incident on it. The number of edges in a matching M is denoted by $|M|$. For a bipartite graph with vertex sets C and V, a matching M is said to be a perfect matching if $|M| = \min(|C|, |V|)$. A detailed exposition on these notions can be found in [2].

We introduce some notation regarding the vertices and edges occurring in a perfect matching in a bipartite graph. A perfect matching P corresponds to a set of constraints, variables and the entries corresponding to the edges in this perfect matching. We use $V(P)$ to denote the variables in the perfect matching, $C(P)$ denotes the set of constraints and $E(P)$ denotes the set of entries that occur in the perfect matching P.

As an example, consider the matrix $R \in \mathbb{R}^{p\times w}[\xi]$ with its nonzero entries: e_{ij}, marked by its row and column indices i and j respectively:

$$R := \begin{bmatrix} 0 & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \end{bmatrix}.$$

Keeping in mind that the rows of the matrix R correspond to constraints (equations), denote the rows by c_1 and c_2. Similarly, the columns correspond to variables, hence denote columns of R by v_1, v_2 and v_3. The bipartite graph is shown in Figure 1.

In the above bipartite graph, there are four perfect matchings, say, P_1, P_2, P_3 and P_4 corresponding to the edge pairs: $\{e_{12}, e_{21}\}$, $\{e_{12}, e_{23}\}$, $\{e_{13}, e_{22}\}$ and $\{e_{13}, e_{21}\}$, respectively. Notice that these perfect matchings are precisely the nonzero terms (with suitable signs) in the maximal minor determinants. More precisely, the maximal minor corresponding to the first two columns

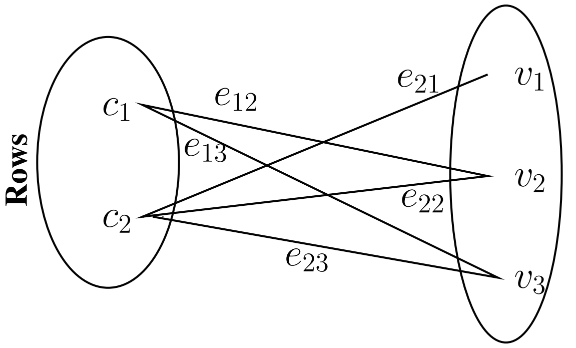

Fig. 1 Example of a bipartite graph

has only one nonzero term: product of the entries in the perfect matching P_1 defined above. The maximal minor due to the 2^{nd} and 3^{rd} columns has two nonzero terms, namely products of the entries in perfect matchings P_2 and P_3. Similarly, product of the entries in the perfect matching P_4 corresponds to the maximal minor: columns 1 and 3.

The above example brings out the following fact: for a square matrix R, the determinant of R comprises of the sum of products of the entries in the perfect matchings (with suitable signs). An upper triangular, or lower triangular, or diagonal square matrix has only one perfect matching, and hence the determinant comprises of just one nonzero term. When R is a non-square matrix, then the perfect matchings correspond to nonzero terms in the concerned maximal minors. This interpretation of perfect matchings plays a key role in analysis of system properties, as shown below in Section 4.

3. POLYNOMIAL MATRICES AND LTI BEHAVIORS

A linear time invariant dynamical system is described by differential equations $R(\frac{d}{dt})w = 0$, where $R \in \mathbb{R}^{p\times w}[\xi]$ is a polynomial matrix. The behavior $\mathfrak{B}$ of this system is defined as the set of solutions w that satisfy $R(\frac{d}{dt})w = 0$:

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{W}}) \mid \mathfrak{R}(\frac{\mathfrak{d}}{\mathfrak{dt}})\mathfrak{w} = \mathfrak{o}\}.$$

The superscript $\mathtt{w}$ in $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{W}})$ denotes the number of components in a typical element w in the behavior $\mathfrak{B}$. Moreover, we say the behavior $\mathfrak{B}$ is an element of $\mathfrak{L}^{\mathtt{w}}$, the set of such linear time invariant systems described by ordinary differential equations.

Let $R(\frac{d}{dt})w = 0$ be a kernel representation of $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$. A kernel representation is said to be minimal if the row dimension of R is the minimum of all kernel representations of $\mathcal{P}$: in this case R has full row rank. Minimality of a kernel representation can be assumed without loss of generality because, otherwise, one can always premultiply R by a unimodular matrix to obtain linearly independent laws.

The zeros of a polynomial matrix $R(\xi)$ play a central role while studying controllability/observability and other related properties of behaviors. This concept is similar to that of a nonsingular matrix. A square polynomial matrix $R \in \mathbb{R}^{w\times w}[\xi]$ is said to be nonsingular if $\det(R) \neq 0$. The roots of the polynomial $\det(R)$ are called the zeros of R. We define the zeros of a polynomial matrix, not necessarily square in a similar fashion. The zeros of $R(\xi) \in \mathbb{R}^{p\times w}[\xi]$ is defined to be the set of those complex numbers $\lambda \in \mathbb{C}$ where the rank of the polynomial matrix 'falls', more precisely:

$$\text{zeros}(R) := \{\lambda \in \mathbb{C} \mid \text{rank } (R(\lambda)) < \text{rank } (R(\xi))\}.$$

Unimodular matrices play an important role in the manipulation of equations describing a dynamical system. The polynomial matrix $U \in \mathbb{R}^{w\times w}[\xi]$ is called unimodular if $\det(U)$ is a nonzero constant. In other words, these are square nonsingular polynomial matrices whose zero set is empty.

We also need the notion of controllability of a system. The behavior $\mathfrak{B}$ is called controllable if for any $w_1, w_2 \in \mathfrak{B}$, there exist $w_3 \in \mathfrak{B}$ and $T \in \mathbb{R}$ such that $w_3(t) = w_1(t)$ for $t < 0$ and $w_3(t) = w_2(t)$ for $t > T$. If $R(\frac{d}{dt})w = 0$ is a kernel representation of $\mathfrak{B}$, then controllability of $\mathfrak{B}$ is equivalent to the zero set of R being empty.

The results of this paper also concern observability in dynamical systems. Consider $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ having two types of variables: to-be-deduced variable w and accessible variable c. The question of observability concerns the ability to deduce the to-be-deduced variable from the accessible variable. w is said to be observable from c in $\mathcal{P}_{\text{full}}$ if (w_1, c) and $(w_2, c) \in \mathcal{P}_{\text{full}}$ imply that $w_1 = w_2$. When $\mathcal{P}_{\text{full}}$ is described by a kernel representation $R_w(\frac{d}{dt})w + R_c(\frac{d}{dt})c = 0$, then observability of w from c in $\mathcal{P}_{\text{full}}$ is equivalent to $R_w(\xi)$ having full column rank for every complex number λ. This is equivalent to R_w being a full column rank polynomial matrix, and having its zero set empty. Another situation when full column rank comes into picture is when dealing with autonomous systems. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called autonomous if one can conclude that $w_1 = w_2$ whenever $w_1, w_2 \in \mathfrak{B}$ satisfy $w_1(t) = w_2(t)$ for all $t \leqslant 0$. If $R \in \mathbb{R}^{w\times w}[\xi]$ induces a minimal kernel representation of an autonomous $\mathfrak{B}$, then without loss of generality, one can assume that $\det R$ is monic: this polynomial is called the characteristic polynomial of $\mathfrak{B}$ and is denoted by $\chi_{\mathfrak{B}}$.

The roots of $\det R$ are called the poles of the autonomous behavior $\mathfrak{B}$. The notions of $\chi_{\mathfrak{B}}$ and poles are needed when studying the pole placement problem and its solution (see Results 3 and 8 below).

4. GENERICITY OF PARAMETERS

The notion of structural property makes the key assumption of genericity of parameters. All the nonzero entries are assumed to be independent, i.e. they do not satisfy any nontrivial algebraic relations (see [4]). The following result relates genericity of parameters, polynomial matrix rank properties and the concerned bipartite graph.

Theorem 1: Consider the bipartite graph constructed using the polynomial matrix $R \in \mathbb{R}^{w \times w}[\xi]$. The following are true.

1. R is generically nonsingular if and only if the corresponding bipartite graph has at least one perfect matching.
2. R is generically unimodular if and only if every perfect matching has only constant entries.

Since we noted in Section 2that a perfect matching corresponds to a nonzero term in the determinant expansion, the first statement in the above theorem follows by assumption of genericity: cancellation of terms is ruled out due to genericity. The second statement ensures that the nonzero terms in the determinant expansion are nonzero constants, causing the zero set to be empty, and hence unimodularity.

We now deal with the case that a polynomial matrix R is not square. Consider the bipartite graph associated to R. The representation is generically minimal if and only if there exists a perfect matching M and $|M| = |C|$. Of course, this requires $|C| \leqslant |V|$ to begin with. A system satisfying the last inequality strictly is also called 'open', while a system that satisfies $|C| = |V|$ and the condition for minimality: $|M| = |C|$ is, in fact, autonomous.

In the context of context of controllability and observability, we ask the question of zeros for a polynomial matrix R which is possibly nonsquare. The assumption of genericity of polynomials leads to the fact that two (or more) nonzero polynomials are generically coprime, i.e. if p and $q \in \mathbb{R}[\xi]$ are nonzero polynomials, then the polynomial matrix $R := [p \;\; q]$ generically has no zeros. Further, if degree of p is zero (i.e., if p is a nonzero constant), then R has no zeros even if q is the zero polynomial. (Note that the zero polynomial is defined to have degree $-\infty$.) The following theorem deals with this issue for general nonsquare polynomial matrices. We assume more number of columns than rows only for simplicity; the same result is true for the converse case also.

Theorem 2: Let $R \in \mathbb{R}^{p \times w}[\xi]$ be a polynomial matrix. Consider the bipartite graph constructed from the rows and columns of R and let P_i be the perfect matchings in this graph with vertex sets: constraints C and variables V. Suppose $p < w$, i.e. $|C| < |V|$. Then $R(\lambda)$ has full row rank for every complex number $\lambda \in \mathbb{C}$ if and only if at least one of the following statements is true:

1. there exists a subset of variables say V_U such that $|V_U| = |C|$ and all perfect matchings between V_U and C have only constant entries, or
2. the following two conditions are satisfied:

A there exist perfect matchings with unequal variable sets, i.e. $|\cup_i V(P_i)| > |C|$, and

B for every entry $e \in \cap_i E(P_i)$, we have $\deg(e) = 0$, i.e. the entry is a nonzero constant.

Property 1 above says that there is a unimodular matrix as a maximal minor in R, and this guarantees that R has full row rank and no zeros. Alternatively (Property 2), determinants of two or more maximal minors in R are generically coprime if and only if any common entries (edges) are nonzero constants. The second statement's condition B rules out the possibility that any nonconstant entries are common factors in the determinants of all the maximal minors.

5. MAIN RESULTS

In the behavioral approach, control is viewed as restriction of the plant behavior to a suitable subset called the controlled behavior $\mathcal{K}$. This restriction is achieved by interconnection of the plant with another system called the controller; after interconnection the variables have to satisfy the laws of both the plant and the controller. We work in the generality that the plant has two kinds of variables: the to-be-controlled variables w, and the control variables c. The controller can impose restrictions only on c: thus the controller behavior $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$, while the *full* plant behavior $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. The interconnection results in the *full controlled* behavior $\mathcal{K}_{\text{full}} := \mathcal{P}_{\text{full}} \wedge \mathcal{C}$ defined as

$$\mathcal{K}_{\text{full}} := \{(,) \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}+\mathtt{c}}) \mid (\mathfrak{w}, \mathfrak{c}) \in \mathcal{P}_{\text{full}} \text{ and } \in \mathcal{C}\}.$$

If $R_w(\frac{d}{dt})w + R_c(\frac{d}{dt})c = 0$ is a kernel representation of $\mathcal{P}_{\text{full}}$ and $C(\frac{d}{dt})c = 0$ is a kernel representation of $\mathcal{C}$, then a kernel representation of $\mathcal{K}_{\text{full}}$ is seen to be

$$\begin{bmatrix} R_w(\frac{d}{dt}) & R_c(\frac{d}{dt}) \\ 0 & C(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0.$$

We often require that the interconnection of two systems should satisfy 'regularity': the interconnection of

$\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is said to be regular if rank $[R_w \quad R_c]$ + rank C = rank $\begin{bmatrix} R_w & R_c \\ 0 & C \end{bmatrix}$. A controller $\mathcal{C}$ satisfying this is called a regular controller. See [7] for a detailed exposition. In this situation where the controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ can influence the to-be-controlled variable w through c, the notion of hidden behavior comes into picture. The hidden behavior $\mathcal{N} \in \mathfrak{L}^{\mathtt{w}}$ is defined as

$$\mathcal{N} := \{ \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{W}}) \mid (\mathfrak{w}, \mathfrak{o}) \in \mathcal{P}_{\text{full}}\}.$$

Observability of w from c in $\mathcal{P}_{\text{full}}$ is equivalent to $\mathcal{N} =$.

We need the following result from [1] regarding pole placement.

Proposition 3: Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. The following are equivalent.

- For every monic $p \in \mathbb{R}[\xi]$, there exists a regular controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that $\mathcal{P}_{\text{full}} \wedge \mathcal{C}$ is autonomous and has characteristic polynomial p,
- the hidden behavior $\mathcal{N}$ satisfies $\mathcal{N} =$, and $\mathcal{P}_{\text{full}}$ is controllable.

The theorem below is one of the main results: a necessary and sufficient condition for generic observability of w from c in $\mathcal{P}_{\text{full}}$.

Theorem 4: Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. Let $R_w(\frac{d}{dt})w + R_c(\frac{d}{dt})c = 0$ be a kernel representation of $\mathcal{P}_{\text{full}}$. The following are equivalent.

- $\mathcal{N} =$, i.e. w is observable from c in $\mathcal{P}_{\text{full}}$.
- At least one of conditions A and B below is true for the bipartite graph constructed from R_w:

A there exists a subset of constraints say C_U such that $|C_U| = |V|$ and all perfect matchings between C_U and V have only constant entries, or

B Conditions B1 and B2 are true:

B1 there exist perfect matchings with unequal variable sets, i.e. $|\cup_i V(P_i)| > |C|$, and

B2 for every entry $e \in \cap_i E(P_i)$, we have $\deg(e) = 0$, i.e. the entry is a nonzero constant.

The proof is similar to that outlined after Theorem 2. Condition A guarantees existence of a unimodular submatrix as a maximal minor, while condition B guarantees coprimeness of nonsingular maximal minors. The situation for controllability is quite similar.

Theorem 5: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$. Let $R(\frac{d}{dt})w = 0$ be a minimal kernel representation of $\mathcal{P}$. The following are equivalent.

- rank $R(\lambda)$ is constant for every $\lambda \in \mathbb{C}$.
- At least one of conditions A and B below is true for the bipartite graph constructed from R:

A there exists a subset of variables say V_U such that $|V_U| = |C|$ and all perfect matchings between V_U and C have only constant entries, or

B Conditions B1 and B2 are true:

B1 there exist perfect matchings with unequal variable sets, i.e. $|\cup_i V(P_i)| > |C|$, and

B2 for every entry $e \in \cap_i E(P_i)$, we have $\deg(e) = 0$, i.e. the entry is a nonzero constant.

In the absence of controllability, one often seeks its weaker notion, called stabilizability. A behavior $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ is said to be stabilizable if for every $w_1 \in \mathcal{P}$, there exists a $w \in \mathcal{P}$ such that $w(t) = w_1(t)$ for all $t \leqslant 0$ and $w(t) \to 0$ as $t \to \infty$. Thus a behavior is called stabilizable if every trajectory can be steered to zero asymptotically. A necessary and sufficient condition to check stabilizability of $\mathcal{P}$ using a kernel representation $R(\frac{d}{dt})w = 0$ is that rank $(R(\lambda))$ is constant for all $\lambda \in \mathbb{C}_+$, the closed right half complex plane. The result below gives a structural necessary condition for stabilizability.

Theorem 6: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$. Let $R(\frac{d}{dt})w = 0$ be a minimal kernel representation of $\mathcal{P}$. Suppose $\mathcal{P}$ is stabilizable, i.e. the rank $R(\lambda)$ is constant for every $\lambda \in \mathbb{C}_+$. Then, the bipartite graph of constraints-variables constructed using R satisfies the condition that for every entry e such that $e \in \cap_i E(P_i)$, the polynomial e is Hurwitz.

The above theorem signifies that any common factors in all the maximal minors have to be Hurwitz to ensure that $\mathcal{P}$ is stabilizable. An analogous result is straightforward for detectability. Detectability plays a role when we do not have property of observability of a to-be-deduced variable w from the accessible variable c. For $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, w is said to be detectable from c if we have $w_1(t) - w_2(t) \to 0$ as $t \to \infty$ whenever (w_1, c) and $(w_2, c) \in \mathcal{P}_{\text{full}}$. If $R_w(\frac{d}{dt})w + R_c(\frac{d}{dt})c = 0$ is a kernel representation of $\mathcal{P}_{\text{full}}$, then detectability of w from c is equivalent to $R(\lambda)$ being full column rank for every $\lambda \in \mathbb{C}$.

Theorem 7: Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}}$ and let $R_w(\frac{d}{dt})w + R_c(\frac{d}{dt})c = 0$ be a kernel representation. Suppose w is detectable from c in $\mathcal{P}_{\text{full}}$. Then, the bipartite graph of constraints-variables constructed using R_w satisfies has at least one perfect matching P such that $|P| = |V|$ and for every entry e such that $e \in \cap_i E(P_i)$, the polynomial e is Hurwitz.

Using Proposition 3, Theorems 4 and 5, we obtain the following result.

Corollary 8: Consider $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ described by the minimal kernel representation $R_w(\frac{d}{dt})w + R_c(\frac{d}{dt})c = 0$. Construct the bipartite graphs $G_{\mathcal{N}}$ from R_w and $G_{\mathcal{P}_{\text{full}}}$ from $[R_w \ R_c]$. Then arbitrary pole placement is possible using a regular controller generically if and only if the following conditions are satisfied:

1. $G_{\mathcal{N}}$ satisfies the necessary and sufficient conditions of

Theorem 4 for generic observability of w from c in $\mathcal{P}_{\text{full}}$,
2. $G_{\mathcal{P}_{\text{full}}}$ satisfies the necessary and sufficient conditions of Theorem 5 for generic controllability of $\mathcal{P}_{\text{full}}$.

6. GILBERT'S CONTROLLABILITY/OBSERVABILITY TESTS

In this section we apply the above results to a state space description of a system and we note that the results reduce to the familiar Gilbert's tests for controllability and observability.

Consider the example: $\frac{d}{dt}x = Ax + Bu$ and $y = Cx$, where

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}.$$

We construct the bipartite graphs from $R(\xi) = [\xi I - A \;\; B]$ and $R(\xi) = \begin{bmatrix} \xi I - A \\ C \end{bmatrix}$ to check controllability of (A, B) and observability of (C, A) as shown in Figures 2 and 3 respectively.

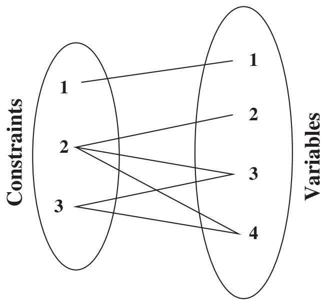

Fig. 2 Controllability bipartite graph

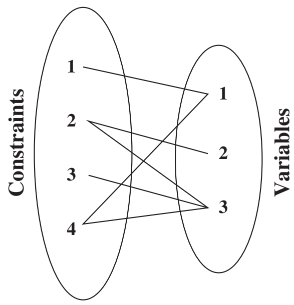

Fig. 3 Observability bipartite graph

Notice that the eigenvalue 3 of A is unobservable, and this would be observable if at least one of the entries $A(3, 2)$ or $C(1, 2)$ were nonzero. Similarly, (A, B) would be controllable if $B(1)$ were nonzero, thus ensuring that the eigenvalue 2 of A is controllable. Thus, Theorems 4 and 5 boil down to the familiar Gilbert conditions for controllability and observability. Further, (A, B) is not stabilizable since the necessary conditions for stabilizability (Theorem 6) is not satisfied: $(\xi - 2)$ is a common factor of all perfect matchings constructed from $[\xi I - A \;\; B]$, and this factor is not Hurwitz.

7. CONCLUSION

We formulated conditions for a system to be controllable and for arbitrary pole placement in terms of properties of a bipartite graph. The genericity of parameters helped to obtain necessary and sufficient conditions without requiring any numerical computation: an important issue when dealing with large dynamical systems. The results obtained, in this sense, are structural properties. It should be noted that, loosely speaking, in the absence of genericity, the conditions continue to be necessary, like they were in the case of stabilizability and detectability.

An important direction for future work is the case of other representations of system equations: latent variable representations and/or image representations (when the system is controllable). While premultiplication by a unimodular matrix to the set of equations is allowed in general, the structure of zero and nonzero entries is easily destroyed by such premultiplication, and moreover, the 'algebraically independent' assumption of genericity is lost too. Noting these points, a careful extension of these results to image/latent variable representations appears challenging.

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