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Singular LQR Control, Impulse-Free Interconnection and Optimal PD Controller Design

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Abstract— **In this paper we consider the LQR control problem with no penalty on the input; this is addressed in the literature as the singular LQR control problem. We show that here the optimal controller is no longer a static controller but a PD controller. We also show that the closed loop system, i.e. the controlled system is a singular state space system. Singular system brings in the concern of existence of *inadmissible* initial conditions, i.e. initial conditions for which the solution is impulsive. Our main result is that there are no inadmissible initial conditions in the controlled system if and only if states which have relative degree one with respect to the input are penalised. Though the Algebraic Riccati equation is not defined for the singular case, we use the notion of storage function in dissipative systems theory to obtain the optimal cost function explicitly in terms of the initial conditions. We use this to prove that the initial conditions for which states of the autonomous (i.e. closed loop), singular system immediately jump to 0 have optimal cost 0.**

Our result that the optimal controller is a PD controller underlines a key intuitive statement for the dual problem, namely the Kalman-Bucy filter when measurements are noiseless: the minimum variance estimator *differentiates* the noiseless measurements. The MIMO case is not dealt in this paper due to space constraints and since it involves controllability indices and Forney indices in the result statements and proofs.

Index Terms— **Singular LQR problem, cheap control, PD control, impulses.**

I. Introduction

In control theory one of the important problems is the LQR control problem. In the regular case there is a penalty on the input. In this case the optimal control feedback law is a static feedback law which involves a solution of an Algebraic Riccati Equation. An important assumption here is that the matrix weighing the input in the cost function, usually denoted as R, is positive definite. When this matrix R is singular, the control law is different. This problem is called singular LQR problem. The need to consider distributional space for input, instead of smooth functions, has been recognised in the literature. The singular LQR problem has been approached by using several techniques like limiting case of a singular perturbation problem (see Clements and Anderson [1]). In Hautus and Silverman [2], the inputs are allowed to be distributional, but the inputs are restricted to those that ensure that the performance index is well-defined. Here the problem is converted into a nonsingular problem and structural properties are used to solve the problem. In Willems et al. [3] it is suggested that by a suitable input, the states are taken to a proper subspace of $\mathbb{R}^n$ where it is required to solve only a classical LQ problem. In Prattichzzo et al. [4] the problem is recast as an output nulling problem for Hamiltonian systems. A characterisation of admissible initial conditions is done to ensure that the optimal feedback is a static feedback and thus distributions in states and input are avoided.

In the case of Hautus and Silverman [2] and Willems et al. [3] only the regular part of the input is said to be implemented by a feedback. Since the optimal input involves distributions, it is expected that the ensuing state trajectories also belong to the class of distributions. In Hautus and Silverman [2] and Willems et al. [3] it is discussed that the state trajectories belong to distributions. In Willems [5] the use of a feedforward/PID control to go from the existing state space to a desired subspace is discussed. In our paper the optimal control law is a dynamic feedback law from the states and it is obtained explicitly as a PD controller. We also state a condition which ensures that the states do not have impulses for all initial conditions even if control input has impulses. The singular LQR problem is analysed from a different perspective using the concepts of behavioural theory, as a result of which the complexity of the analysis is reduced and also the controller design is done in a comparatively simple manner.

Notation 1.1: $\mathbb{R}$ denotes the set of real numbers. $\mathbb{R}[s]$ is the ring of polynomials in one indeterminate s over the field of real numbers. $\mathbb{R}^{m\times n}[s]$ represents the set of matrices of dimension $m \times n$ with polynomial entries. $\mathfrak{C}^\infty$ refers to the set of all infinitely differentiable functions. $\mathfrak{D}'$ represents the distributional space. I_n refers to identity matrix of dimension $n \times n$. The right kernel of a matrix A is denoted by $\ker A$. The image of a matrix A is denoted as $\text{im } A$.

II. Problem formulation and Main Results

Consider the following continuous, linear time invariant, single input system

$$\dot{x}(t) \quad = \quad Ax(t) + bu(t) \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$. The regular LQR problem defines the performance index as follows

$$J(x_0; u) = \int_0^\infty (x^TQx + u^TRu)dt. \tag{2}$$

where the matrix $Q \in \mathbb{R}^{n\times n}$ and Q is positive semidefinite i.e. $Q \geq 0$. Since we deal with singular LQR problem $R \in \mathbb{R}$ is singular i.e. $R = 0$ and hence the second term in the

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integral is 0. We make the following assumptions for this problem:

Assumptions 2.1: A1) The system (A, b) is controllable.
A2) (Q, A) is observable.
So for the singular case, the performance index J becomes

$$J(x_0; u) = \int_0^\infty x^T Q x dt. \tag{3}$$

We define the following two input spaces:

1) Regular inputs
$\mathcal{U}_{reg} = \mathfrak{L}_2^{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R})$
$= \left\{ u : \mathbb{R}^+ \to \mathbb{R} \mid \int_0^T |u|^2 \, dt < \infty \text{ for all } T \in \mathbb{R}^+ \right\}.$
2) Distributional inputs
$\mathcal{U}_{dist} = \{u \in \mathfrak{D}' | u = u^{\mathrm{imp}} + u^{\mathrm{reg}}\}.$
$u^{\mathrm{reg}} \in \mathcal{U}_{reg}$ and u^{imp} belongs to impulsive distributions. i.e. distributions of the type $\sum_{i=0}^N a_i \delta^{(i)}$ where $a_i \in \mathbb{R}$ and δ is the Dirac delta.

Let

$$J^*(x_0) := \inf_{u \in \mathcal{U}_{dist}} J(x_0, u). \tag{4}$$

The problems that we address in this paper are as follows:

1) Find conditions on A, b, Q, R and the space of initial conditions $\mathcal{X}_0 \subseteq \mathbb{R}^n$ such that $J^*(x_0)$ exists for all $x_0 \in \mathcal{X}_0$, and find conditions for $\mathcal{X}_0 = \mathbb{R}^n$
2) If $J^*(x_0)$ exists, find conditions for the existence of an optimal input $u^* \in \mathcal{U}_{dist}$ such that $J(x_0, u^*) = J^*(x_0)$.
3) When can u^* be obtained by a *feedback* controller? Is the controller a static controller (i.e. there exists a matrix F such that $u = Fx$) or a more general controller?
4) For the closed loop system,(i.e.controlled system) find conditions on A, b, Q under which there exist inadmissible initial conditions.
5) Find the subspace of initial conditions for which $x(t) = 0$ for $t > 0$ and verify that for these initial conditions $J^*(x_0) = 0$.
6) Find the subspace of initial conditions $\mathcal{X}_{0r} \subset \mathcal{X}_0$ for which the impulsive part of u is 0. i.e. in $u = u^{\mathrm{imp}} + u^{\mathrm{reg}}, u^{\mathrm{imp}} = 0$

A. Outline

The paper is outlined as follows. The rest of this section discusses the main results of this paper: The optimal controller for the singular LQR control problem is a PD controller and a condition which ensures that there are no impulses in the states of the controlled system . Section III discusses some behavioural preliminaries that are used for system representation and lemmas which are useful for controller design. Section IV deals with the optimal controller design. The regular LQR control problem is discussed first and is later applied suitably to the design of the controller for the singular case. The main result of this paper, Theorem 2.2 is proved here. In section V we discuss behavioural preliminaries for the states of a system and also about state maps. We characterise the space of initial conditions which jump to 0 and also for the case when optimal input doesn't have an impulse.

B. Main Results

Problems 1 and 2 have been studied extensively in the literature; for example in Hautus and Silverman [2] and Willems et al. [3]. It is discussed there that $J^*(x_0)$ exists if u is allowed to belong to the distributional space. Uniqueness of the optimal input has been proved in Mehrmann [6]. When an optimal input u^* exists, our main result below addresses the existence of feedback controller as a *PD controller*,[1] and the issue of inadmissible initial conditions for the closed loop system. In this regard we state the following theorem.

Theorem 2.2: For the singular LQR control problem under Assumption 2.1 and assuming $J^*(x_0)$ exists, the following statements hold.

1) The optimal control u^* that achieves $J^*(x_0) = J(x_0, u^*)$ can be implemented by a feedback controller.
2) This optimal controller is a PD controller. i.e there exists $F_P, F_D \in \mathbb{R}^{1\times n}$ such that $u = F_P x + F_D \dot{x}$.
3) The closed loop system is a singular system of the form $E\dot{x} = A_F x$ with rank $E = n - 1$.
4) There are no inadmissible initial conditions in the closed loop system if and only if $Qb \neq 0$.

We prove these statements in the later sections since more preliminaries are required. The optimal cost involved with the LQR problem is given by an *extremum storage* function (see Willems and Trentelman [7]). Hence it is expected that the space of initial conditions for which the states jump to 0 should be contained in the kernel of the matrix in the storage function as the cost involved for these initial conditions is 0. We shall prove this later after making these notions precise. Since we have a PD controller, the optimal input u^*, could have impulses when the states have a jump. Hence we also find some initial conditions for which the input doesn't have an impulse.

Remark 2.3: We briefly bring out the close relation between the PD controller we have obtained for the LQR control problem (the cheap control case) and the Kalman-Bucy filter when measurements are noiseless. It is quite expected that when measurements are noiseless, then differentiating the measurement will help in getting a 'better' (i.e. a lower variance error) state estimate. The Deyst filter (see Deyst [8]) and (Maybeck [9]) indeed studies the singular Kalman-Bucy filter and obtains a differentiator as optimal estimator than just a static error feedback within the observer. In Schumacher [10], an optimal estimator constructed for the singular case is discussed to have an integrating part and a differentiating part. The duality between the regular LQR control problem and the minimum variance estimator has been extensively addressed in the literature over the last five decades. However from Theorem 2.2 our result that the optimal LQR controller is a PD controller provides the duality-link as to why the corresponding Kalman-Bucy filter (called the Deyst filter in this case) turns out to be a differentiator.

[1]For a given open loop state space system $\dot{x} = Ax + bu$, a static state feedback controller cannot make the closed loop system a singular system, but a PD controller indeed can result in the closed loop system being a singular system; this is the focus of this paper.

III. Polynomial Matrices and Behavioural Approach

In this section we start with some behavioural preliminaries that are required for this paper.

A. Behavioural Approach

A linear differential behaviour $\mathfrak{B}$ is defined to be the subspace of $\mathfrak{L}_1^{\text{loc}}$ consisting of the solutions to a set of ordinary linear differential equations with constant coefficients; i.e.,

$$\mathfrak{B} := \left\{ w \in \mathfrak{L}_1^{\text{loc}} \mid P\left(\frac{d}{dt}\right) w = 0 \right\},$$

where $P \in \mathbb{R}^{\bullet \times \mathtt{w}}[s]$. The differential equations are assumed to be satisfied in the *distributional sense*, i.e. weak sense. $P(\frac{d}{dt}) = 0$ is called a *kernel representation* of $\mathfrak{B}$. We call w as the *manifest variable*. For definition of controllablity of behaviours see Polderman and Willems [11]. One of the important properties of a controllable behaviour is that $P(\frac{d}{dt}) \in \mathbb{R}^{\bullet \times q}[s]$ is a kernel representation of a controllable behaviour $\mathfrak{B}$ if and only if the rank of the matrix $P(\lambda)$ remains the same for all $\lambda \in \mathbb{C}$. We shall assume that the matrix $P(s)$ is of full row rank without loss of generality i.e. the rows of $P(s)$ are linearly independent over $\mathbb{R}[s]$. Another important property of controllable behaviours is the existence of an *image representation,* i.e. there exists a matrix $M(s) \in \mathbb{R}^{q \times \bullet}[s]$ such that

$$\mathfrak{B} = \left\{ w \;\middle|\; w = M(\frac{d}{dt})\ell \text{ for some } \ell \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\bullet}) \right\},$$

where ℓ is called the *latent variable*. If the latent variables are observable from the manifest variables, then the image representation is called *an observable image representation.* An image representation can be assumed to be observable without loss of generality.

B. Quadratic Differential Forms

Quadratic Differential forms (QDF's) are quadratic functionals of the manifest variables and a finite number of their derivatives. The QDF Q_Φ induced by $\Phi(\zeta, \eta) \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ is a map $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by

$$Q_\Phi(w) := \sum_{i,k} \left(\frac{d^i w}{dt^i}\right)^T \Phi_{ik} \left(\frac{d^k w}{dt^k}\right).$$

We can also assume that Φ is symmetric i.e. $\Phi(\zeta, \eta) = \Phi^T(\eta, \zeta)$. We often require the one variable polynomial matrix $\Phi(-s, s)$: we shall denote this by $\partial\Phi(s)$. Due to the symmetry of $\Phi(\zeta, \eta)$, the one variable polynomial matrix $\partial\Phi(s)$ is *para-Hermitian*, i.e., $\partial\Phi(-s) = \partial\Phi^T(s)$. See Willems and Trentelman [7] for a detailed study on QDFs.

C. State space systems

The above preliminaries are used to represent the system in our problem. Representing (1) using a kernel representation, with x and u as manifest variables we get

$$P(\frac{d}{dt}) \begin{bmatrix} x \\ u \end{bmatrix} = 0 \text{ where } P(s) = \begin{bmatrix} sI - A & -b \end{bmatrix}. \quad (5)$$

The system (1) is assumed to be controllable. Hence the kernel representation in (5) is also controllable. So an observable image representation exists. We discuss the construction of an observable image representation. Let determinant of $(sI - A)$ be $\chi(s)$. We define the Adjugate of $(sI - A)$ as adj(s):= $(sI - A)^{-1}\chi(s)$. Then an image representation of (5) is given by

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \text{adj}(s)b \\ \chi(s) \end{bmatrix} \ell \quad (6)$$

where ℓ is the latent variable. The image representation matrix $M(s)$ is $\begin{bmatrix} \text{adj}(s)b \\ \chi(s) \end{bmatrix}$. (This satisfies $PM = 0$.) We next show that this representation is observable. The McMillan [2] degree of a controllable kernel representation matrix and observable image representation matrix are the same.(See Polderman and Willems [11]) The McMillan degree of $P(s)$ is n. Hence the McMillan degree of $M(s)$ also should be n. In the $M(s)$ constructed, $\chi(s)$ is monic and has the highest degree n among the entries of $M(s)$. Hence the constructed $M(s)$ is observable.

Lemma 3.1: Assume $\begin{bmatrix} x \\ u \end{bmatrix} = M(\frac{d}{dt})\ell$ to be an observable image representation (constructed as mentioned above) for the system (1). Let $M(s) \in \mathbb{R}^{(n+1) \times 1}[s]$ be partitioned as $\begin{bmatrix} M_x \\ M_u \end{bmatrix}$, where $M_x \in \mathbb{R}^{n \times 1}[s]$ and $M_u \in \mathbb{R}[s]$. Here $M_x =$ adj$(s)b$ and $M_u = \chi(s)$. Then the following statements hold:

1) In M_x, the coefficients of s^{n-1} are precisely the entries of b.
2) M_x is observable.

Remark 3.2: Since we have an observable image representation there also exists a matrix $F(s)$, such that $F(s)M(s) = I$. This matrix $F(s)$ is called a *left inverse* of the matrix $M(s)$. In $M(s)$, M_x is observable as stated in Lemma 3.1. Hence a left inverse exists for M_x. Let this be F_x. Therefore one of the left inverse of the matrix $M(s)$ is given by $[F_x \;\; 0]$, where $F_x(s) \in \mathbb{R}^{1 \times n}[s]$.

Lemma 3.3: Let $P(s) \in \mathbb{R}^{n \times (m+n)}[s]$. Let $M(s) \in \mathbb{R}^{(m+n) \times m}[s]$ be any maximal right annihilator[3] of $P(s)$ and $F \in \mathbb{R}^{m \times (m+n)}[s]$ be any left inverse of $M(s)$. Then the matrix $\begin{bmatrix} P(s) \\ F(s) \end{bmatrix}$ is unimodular.

IV. LQR Control

In this section we design the optimal controller and discuss the proof of our main result, Theorem 2.2

A. Outline of the proof of Theorem 2.2

We first design an optimal controller using standard LQR control method. We then discuss that this optimal controller can be implemented as a feedback from states of the system. We append it to the plant equations and do elementary row

[2]For the purpose of this paper, the McMillan degree is defined as the minimum number of states required in a state space realisation.

[3]$M(s) \in \mathbb{R}^{(m+n) \times m}[s]$ is said to be a maximal right annihilator of $P(s) \in \mathbb{R}^{n \times (m+n)}[s]$ if $M(s)$ has full column rank for all $\lambda \in \mathbb{C}$ and satisfies $PM = 0$.

operations so as to reduce the degree of the terms in the controller equation. We show that the optimal controller is a PD controller. Using this fact we show that the closed loop system is a singular system. We then show that this closed loop system will not have zeros at infinity (i.e. no impulses) for all $x_0 \in \mathbb{R}^n$ if and only if $Qb \neq 0$.

B. Regular LQR problem

We first review the regular LQR problem. The performance index (2) is written as a QDF, with x and u as the manifest variables.

$$J(x_0, u) = \int_0^\infty Q_\Phi \begin{pmatrix} x \\ u \end{pmatrix} dt \quad \text{where } \Phi = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$$

This QDF induced by Φ is written in terms of latent variable using the image representation matrix.

$$J(x_0, u) = \int_0^\infty Q_{\Phi'}(\ell)\, dt$$
$$\text{where } \Phi'(\zeta,\eta) = M^T(\zeta)\Phi M(\eta) \quad (7)$$

Remark 4.1: If $\Phi \in \mathbb{R}^{n\times n}[\zeta,\eta]$ and $\partial\Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$ then there exists a polynomial matrix $H \in \mathbb{R}^{n\times n}[s]$, with H *Hurwitz* i.e. the determinant of H has all its roots in the open left half of the complex plane and

$$\partial\Phi(s) = H^T(-s)H(s). \quad (8)$$

$H(s)$ is said to be a Hurwitz factor of $\partial\Phi(s)$.
The following proposition from Willems [13], reworded to the notation of our paper, gives the optimal trajectories for the LQR control problem, using characterisation of the stationary and stable trajectories of the performance functional in (2).

Proposition 4.2: (Willems [13, Proposition 1]) Let $\mathfrak{B}^*$ denote the set of all optimal trajectories $w^* : \mathbb{R} \longrightarrow \mathbb{R}^q$. Let $\partial\Phi(i\omega) > 0$ for all $\omega \in \mathbb{R}$. Then $w^* \in \mathfrak{B}^*$ if and only if

$$H(\frac{d}{dt})w^* = 0 \quad (9)$$

where H is the Hurwitz factor of $\partial\Phi$.

The following steps explain the design of the controller for the LQR problem

Algorithm 4.3:
1) The performance index is written as a QDF in terms of latent variable ℓ.
2) From Proposition 4.2 the optimal controller is given by
$$H(\frac{d}{dt})\ell = 0.$$
3) Let $F(s)$ be a left inverse of $M(s)$. The controller equation is written in terms of manifest variables by substituting $\ell = F(\frac{d}{dt})w$.
$$H(\frac{d}{dt})F(\frac{d}{dt})w = 0. \quad (10)$$
This optimal controller can be implemented as a feedback controller.
4) A kernel representation of the controlled system with the controller equation appended is given by
$$\begin{bmatrix} P(\frac{d}{dt}) \\ H(\frac{d}{dt})F(\frac{d}{dt}) \end{bmatrix} w = 0 \quad (11)$$

Remark 4.4: The question arises whether the controlled system defined in equation (11) is autonomous, i.e. whether $\begin{bmatrix} P(s) \\ H(s)F(s) \end{bmatrix}$ has full rank. Consider the following two equations which follow from the definitions of $P(s), M(s)$ and $F(s)$

$$P(s)M(s) = 0$$
$$F(s)M(s) = I.$$

We can see that the rows of $P(s)$ and $F(s)$ are independent. $H(s)$ is nonsingular and hence the plant equation $P(\frac{d}{dt})w = 0$ and controller equation $H(\frac{d}{dt})F(\frac{d}{dt})w = 0$ are independent.

Remark 4.5: The degree of $H(s)$ as obtained in step 2 is n. The reason is as follows. Consider the matrix Φ' in (7). Substitute for M as $\begin{bmatrix} (\text{adj}(s)b) \\ \mathcal{X}(s) \end{bmatrix}$ and then from this get the matrix $\partial\Phi'(s)$, which after expanding is as follows

$$\partial\Phi'(s) = (\text{adj}(-s)b)^T Q(\text{adj}(s)b) + \mathcal{X}^T(-s)R\mathcal{X}(s) \quad (12)$$

The degree of $\mathcal{X}(s)$ is n, which is the only term with maximum degree in the matrix M. Hence if R is not 0, then the degree of $\partial\Phi'(s)$ is $2n$. After the Hurwitz factorisation, the obtained $H(s)$ is of degree n.

C. R singular

We now consider the case when R is singular (i.e. $R = 0$). The controller design is the same as for the regular case. The steps of Algorithm 4.3 are applicable here. Hence for the singular case also the optimal control input u^* that achieves $J^*(x_0) = J(x_0, u^*)$ can be implemented by a feedback controller. This proves the statement 1 of Theorem 2.2. But there is a fall in the degree of $H(s)$ in the singular case.
Fall in degree of $H(s)$: If R is singular then degree of $H(s)$ is less than $2n$, as the maximum degree term $\mathcal{X}(s)$ doesn't appear in the expression for $\partial\Phi'(s)$ and also there are possibilities of some cancellations in the remaining terms. Let the degree of $\partial\Phi'(s)$ in this case be $2k$. Hence degree of $H(s)$ is k. Here $k < n$.

D. PD Controller

We show that the optimal controller is a PD controller.

Lemma 4.6: The highest degree among all the terms of the controller equation is at least 1.
Next we prove the second part of Theorem 2.2 where we show that the above mentioned degree can be made to be precisely 1.
Proof of statement 2 of Theorem 2.2
The proof involves the following three steps:
1) We consider the matrix in (11) and show that by doing suitable elementary row operations we can make all the entries in controller equation have degree 0.
2) We then show that after step 1 the $(n+1, n+1)$ entry in the resulting matrix has to be 0.
3) The controller equation is modified by row operations in order to write the control law explicitly involving the

input. In this process we show that the optimal controller is a PD controller given by the following equation.

$$u = F_P x + F_D \dot{x}. \tag{13}$$

where $F_P, F_D \in \mathbb{R}^{1\times n}$.

E. Singular Closed Loop System

We have seen that the optimal controller used is a PD controller. This is substituted in the original system equation and simplified as follows

$$\left.\begin{array}{rcl} \dot{x} &=& Ax + b(F_p x + F_d \dot{x}) \\ \dot{x} &=& Ax + bF_p x + bF_d \dot{x} \\ (I - bF_d)\dot{x} &=& (A + bF_p)x \\ E\dot{x} &=& A_F x \end{array}\right\} \tag{14}$$

where E is $(I - bF_d)$ and A_F is $(A + bF_p)$.

Proof of statement 3 of Theorem 2.2

In the equations (14), the control variable u was eliminated. This can be achieved by a series of elementary row operations, whereby the entries in the column corresponding to the input in the plant equation are made 0. Therefore the matrix is as follows

$$\begin{bmatrix} [sE - A_F]_{n\times n} & 0_{n\times 1} \\ C(s) & -1 \end{bmatrix} \tag{15}$$

These elementary row operations do not affect the degree of determinant. Hence the degree of the determinant in both the matrices (11) and (15) will be the same. The determinant of the matrix in (15) is the determinant of the matrix $[sE - A_F]$. The degree of determinant of (11) is same as that of $H(s)$, i.e. k. Hence $\deg\,(\det\,(sE - A_F))$ is also k. Hence this fall in degree is possible only if E is singular. From (14) E is given by

$$E = I_n - bF_D \tag{16}$$

Hence E is a rank-one update of the identity matrix. The rank can fall at most by 1. It is known E is singular. Hence the rank of the matrix E is $n - 1$. ■

The system with the PD controller is given by

$$E\dot{x}(t) = A_F x(t) \tag{17}$$

It is known that E is singular, so we are to deal with a singular system. We now prove the last part of Theorem 2.2. We make use of the following proposition.

Proposition 4.7: (Verghese et al. [14]) The free response of the system with the state space representation as in (17) has no impulsive solutions if and only if $\deg\,(\det\,(sE - A_F)) = \text{rank } E$.

Proof of statement 4 of Theorem 2.2

If Part: Here we show that if there are no impulses then $Qb \neq 0$. If there are no impulses then from Proposition 4.7 $\deg\,(\det\,(sE - A_F)) = \text{rank } E = n - 1$. The $\deg\,(\det\,(sE - A_F))$ is same as the $\deg\,(\det\,(H(s)))$ as discussed in the proof of statement 3 of Theorem 2.2. So the $\deg\,(\det\,(H(s)))$ has to be $n - 1$ and therefore the $\deg\,(\det\,(\partial\Phi'(s)))$ should be $2(n-1)$. In this regard consider the matrix $\partial\Phi'(s)$ expanded as in (12), which is shown below.

$$\partial\Phi'(s) = (\text{adj}(-s)b)^T Q(\text{adj}(s)b) + \mathcal{X}^T(-s)R\mathcal{X}(s)$$

From Lemma 3.1 we have

$$\text{adj}(s)b = bs^{n-1} + N(s)$$

where $N(s) \in \mathbb{R}^{n\times 1}[s]$, with entries having degree at most $n - 2$. Substituting this in the above equation we get

$$\begin{array}{rcl} \partial\Phi'(s) &=& \left[b(-s)^{n-1} + N(-s)\right]^T Q \left[bs^{n-1} + N(s)\right] \\ && +\ a(-s)Ra(s). \end{array}$$

Here $R = 0$, expanding the above equation

$$\begin{array}{rcl} \partial\Phi'(s) &=& (-1)^{n-1}b^T Qbs^{2n-2} + b^T QN(s)(-s)^{n-1} + \\ && N(-s)^T Qbs^{n-1} + N(-s)^T QN(s) \end{array} \tag{18}$$

So in order that $\deg\,(\det\,(\partial\Phi'(s)))$ be $2(n-1)$, b should not be in the kernel of Q i.e. $Qb \neq 0$.

Only If Part: Here we show that if $Qb \neq 0$ then there are no impulsive solutions. If $Qb \neq 0$ then from (18) the $\deg\,(\det\,(\partial\Phi'(s)))$ is $2(n-1)$. Therefore the $\deg\,(\det\,(H(s)))$ is $n-1$. Hence the $\deg\,(\det\,(sE - A_F))$ is $n - 1$, which is same as the rank of E. Hence there are no impulsive solutions for all initial conditions. ■

The condition that $Qb \neq 0$ is similar to the fact that states with relative degree one with respect to the input are penalised. This will be shown later after explaining about the controller canonical form.

V. Zero Optimal Cost, Optimal Input Without Impulse

In this section, we find the space of initial conditions for which the states jump to 0 for $t > 0$ and also for which the optimal control input doesn't have an impulse. It was shown in Rapisarda and Willems [15] that a set of state variables can be obtained through the manifest variables by a *state map*, $X \in \mathbb{R}^{\bullet\times \mathtt{w}}[s]$, which gives the state variables by $x = X(\frac{d}{dt})w$. Among all state maps, if X has the minimum number of rows, then it is called a minimal state map. The next section discuss about the state map for the system under consideration.

A. State Map

The system is assumed to be controllable, hence there exists a controller canonical form as shown below. This is obtained by using a similarity transform $T \in \mathbb{R}^{n\times n}$ on the representation in (1).

$$\dot{x} = A_c x + b_c u \tag{19}$$

In the performance index, the Q becomes $\tilde{Q}$, where $\tilde{Q} = T^T QT$. The structure of A_c and b_c is given by

$$A_c = \begin{bmatrix} 0 & 1 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ & & & \vdots & & \\ 0 & 0 & . & . & . & 1 \\ * & * & . & . & . & * \end{bmatrix} \qquad b_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

where $*$ refers to non zero numbers. Let the observable image representation for this system be

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} M_x^{'} \\ M_u^{'} \end{bmatrix} \ell$$

Using A_c and b_c the steps explained in Lemma 3.1 are followed to get $M_x^{'}$ as follows.

$$M_x^{'} = \begin{bmatrix} 1 & s & s^2 & \dots & s^{n-1} \end{bmatrix}^T \tag{20}$$

This $M_x^{'}$ is considered as a state map.

Remark 5.1: Consider the expression for $M_x^{'}$. The state with relative degree one with respect to input is the nth state. The coefficient of s^{n-1} in $M_x^{'}$ is precisely b as explained in Lemma 3.1. Hence to say that this state is to be penalised in the cost function is equivalent to saying that $Qb \neq 0$.

B. Storage Function and Optimal Cost

It has been discussed in Trentelman and Willems [16] that every storage function is a state function. The storage function is given by

$$\psi(\zeta, \eta) = \frac{\phi(\zeta, \eta) - H^T(\zeta)H(\eta)}{\zeta + \eta}$$

where $H(s)$ is the Hurwitz factor as mentioned in Lemma 4.1. The $\psi(\zeta, \eta) \in \mathbb{R}(\zeta, \eta)$ can be written in the following form

$$\psi(\zeta, \eta) = \begin{bmatrix} 1 & \zeta & \dots & \zeta^{n-1} \end{bmatrix} K \begin{bmatrix} 1 & \eta & \dots & \eta^{n-1} \end{bmatrix}^T$$

where $K \in \mathbb{R}^{n \times n}$. So

$$\psi(\zeta, \eta) = X^T(\zeta)KX(\eta) \tag{21}$$

where $X(\zeta) = \begin{bmatrix} 1 & \zeta & \zeta^2 & \dots & \zeta^{n-1} \end{bmatrix}^T$. This X matrix acts as a state map, which is same as the state map given by matrix $M_x^{'}$ in (20). The optimum value for the performance index (2) is given to be $x_0^T K x_0$. (See Willems and Trentelman [7]). Hence if $x_0 \in \ker\, K$ then the cost involved is 0.

C. Zero Cost

Let $\mathfrak{J}$ denote the set of initial conditions for which $x(t) = 0$ for $t > 0$.

Theorem 5.2: For $\dot{x} = Ax + bu$, with the desired optimal control law $u = F_P x + F_D \dot{x}$, the space $\mathfrak{J}$, the initial conditions with 0 cost ($\ker K$) and $\operatorname{im} b$ are related as follows

$$\mathfrak{J} = \operatorname{im} b \;\text{ and }\; \mathfrak{J} \subseteq \ker K$$

Remark 5.3: $E\dot{x} = A_F x$ denotes the closed loop system for the singular LQR problem (from statement 3 of Theorem 2.2). Let $v \in \mathbb{R}^{1 \times n}$ such that $vE = 0$. Let $\mathcal{X}_{0r} \subset \mathcal{X}$ denote the space of initial conditions for which $u^{\text{imp}} = 0$. Then $\ker\, vA_F \subset \mathcal{X}_{0r}$.

VI. Conclusion

The singular LQR control problem was considered. An optimal controller was designed, which was shown to be a state feedback controller. The optimal controller was a PD controller. In this case the closed loop system turned out to be a singular system. Hence for the closed loop system to be free from impulses it was discussed that a necessary and sufficient condition is $Qb \neq 0$. This condition is equivalent to the fact that states which has relative degree one with respect to the input should be penalised. We also showed that the initial conditions for which states immediately jump to zero is contained in the space of initial conditions for which the optimal cost involved is 0.

VII. Acknowledgments

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