2012 American Control Conference
Fairmont Queen Elizabeth, Montréal, Canada
June 27-June 29, 2012

Low Order Controller with Regional Pole Placement

Subashish Datta, Debraj Chakraborty and Madhu N. Belur

Abstract— **In this paper we propose a method to find a low order controller for a single input single output linear continuous time system which guarantees that the closed loop poles are placed within some pre-specified region in the complex plane. Additionally, the method can also ensure that any subset of the closed loop poles are placed at specific pre-designed locations. Further, it is possible to ensure that the resulting controller is proper. The problem is solved by formulating it as an LMI constrained optimization problem. The proposed method is demonstrated on a power system example.**

I. INTRODUCTION

The problem of finding low order controllers for continuous time linear systems for various control objectives has proved to be difficult due to the underlying non-convexity of the optimizations involved [1], [2], [3]. In the case of an n^{th} order linear time invariant (LTI) single-input single-output (SISO) system, if all the closed loop poles are specified, then it is well known that the minimum order of the controller which achieves these pole locations, is $(n-1)$ [4], [5]. However, if there are no precise requirements on the closed loop poles, but they are only required to belong to some pre-specified region in the complex plane, then these extra degrees of freedom can be used to reduce the controller order below $(n-1)$. We address this problem using a linear matrix inequality (LMI) based inner convex approximation for the closed loop characteristic polynomial stability region [6], [7]. It is shown that the low order proper controller, satisfying any regional pole placement requirements, can be found by solving at most $(n-1)$ semidefinite programs (SDPs).

In addition to regional pole placement requirements, the designer might want to specify some of the poles precisely [8], [9], [10]. Such a situation regularly occurs in the case of practical large order systems, where only few open loop poles are unstable or have undesired damping. The remaining open loop poles are, in general, stable and well damped. For example, in the case of interconnected power systems, the post-fault oscillatory response is specifically influenced by few poorly damped inter-area modes [11]. In such situations the designer wishes to place only those unstable poles (henceforth called fixed poles) at some desired locations to ensure desired performance following disturbances. There is no need to worry about the remaining poles (henceforth called free poles) as long as their damping/settling times do not exceed those in open loop.

To address such situations, here we present a method of obtaining output feedback low order controller which will ensure that (i) the fixed poles are placed at desired (precise) closed loop locations, and (ii) the free poles are assumed positions within some specified stable region in closed loop.

The requirements on the closed loop free poles are translated into constraints in the coefficient space of the characteristic polynomial through an inner convex approximation of the polynomial stability region [6], [7]. These constraints define an LMI on the coefficients of the polynomials associated with the output feedback controller. Thus the problem mentioned above is solved as a satisfiability problem with two types of constraints: (i) linear equality constraints arising out of the precise placement requirement of the closed loop fixed poles, and (ii) LMI constraint arising out of the regional placement requirement of the closed loop free poles.

The traditional approaches to find a low order controller for a linear system are the following: i) reduce the full order plant to a smaller order and then design a controller for the smaller order plant, ii) design a controller for the full order plant and then reduce this to a smaller order controller (see [12], [13] and the references therein). The reduction of the plant or the controller, which will satisfy the closed loop performance restrictions, is done by various model reduction approaches [12]. These methods provide no guarantees on the closed loop specifications and hence are necessarily iterative.

The low order controller design with pole placement requirements is addressed in [4], [5], where it is shown that the low order controller for an n^{th} order system is $(n-1)$. This however turns out to be disadvantageous for systems of large order, and hence obtaining a low order controller, with which the transient/time response characteristics of the closed loop system can be achieved, remains an important problem [1]. In [14], [15], the above problem is approached by posing it as a rank minimization problem (RMP). It is shown that if the associated feasible set is a hyper-lattice [14] then RMP problem can be posed as a trace minimization problem which in turns can be solved as an SDP. Similarly in [16], a convex suboptimal problem, associated with obtaining low order controller, is solved by using strictly positive realness condition. In these approaches convexification is achieved at the cost of optimality or some special system properties are assumed.

Many papers, like [7], [17] and [2], have focused on obtaining a fixed order controller for a plant with polytopic uncertainty. In [2], [3] it is shown that the set of all stabilizing controllers for a polytopic uncertain plant is non convex. Hence in [7], [17], [2], [3] a convex approximation, by fixing a central polynomial, is obtained by imposing strictly positive realness condition on the associated transfer function. A similar problem is treated in [3], where a trace minimization

S. Datta, D. Chakraborty and M. N. Belur are with the Department of Electrical Engineering, Indian Institute of Technology Bombay, India. subashish@iitb.ac.in, dc@ee.iitb.ac.in and belur@ee.iitb.ac.in

heuristic is proposed to minimize the rank of the Sylvester resultant matrix associated with plant coefficients.

Rest of the paper is organized as follows. The problem is formulated in Section II-A after introducing some preliminary notations. Following [6] and [7], a procedure to find a stable convex LMI region in the polynomial coefficients space is included in Section II-B. In Section III, a methodology for obtaining low order controller is presented. It is shown that the low order controller can be obtained by solving at most $(n-1)$ SDPs. A numerical example demonstrating the application of the proposed theory on a linearized model of a 4-machine, 2-area power system [18] is included in Section IV.

II. Notation and Problem Formulation

A. Problem Formulation

Consider a LTI single-input single-output (SISO) system represented by the following transfer function.

$$P(s) = \frac{b(s)}{a(s)} = \frac{b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \qquad (1)$$

where the polynomials $a(s)$ and $b(s)$ are co-prime. Assume that the plant $P(s)$ is strictly proper. Let us consider an output feedback controller of the following form:

$$C(s) = \frac{y(s)}{x(s)} = \frac{y_m s^m + y_{n-1}s^{n-1} + \ldots + y_1 s + y_0}{x_m s^m + x_{m-1}s^{m-1} + \ldots + x_1 s + x_0} \qquad (2)$$

with $x_m \neq 0$ and $m \leq (n-1)$. The closed loop, comprising of plant $P(s)$ and controller $C(s)$, is shown in Fig. 1. According to the inter-connection the closed loop characteristic polynomial would be

$$\sigma(s) = a(s)x(s) + b(s)y(s) \qquad (3)$$

with degree $(n+m)$.

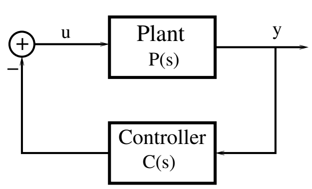

Fig. 1. Closed loop system comprising of plant $P(s)$ and controller $C(s)$

It is well known [4], [5] that if all the $(n+m)$ poles of the closed loop system are specified then the minimum order controller is $m = n-1$. However, as discussed in the introduction, we are considering the case where a subset of the closed loop poles are specified while remaining are free. Let us assume that out of $(n+m)$ closed loop poles, q poles are free and are not associated with any desired closed loop location, whereas the remaining $(n+m-q)$ poles are fixed poles and are required to be placed at $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n+m-q}\}$ in closed loop. Note that λ_i's for $i = 1, \ldots, (n+m-q)$ should be chosen from a set consisting of self conjugate complex numbers. Assume that the q free poles are required to be located inside the stable region $\mathbb{S}$ of the complex plane $\mathbb{C}$. Following [6], we will define $\mathbb{S}$ as follows:

$$\mathbb{S} = \left\{ s \in \mathbb{C} : \begin{bmatrix} 1 & s^* \end{bmatrix} \underbrace{\begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}}_{S} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0 \right\} \qquad (4)$$

where s^* denotes the complex conjugate of s and $S \in \mathbb{R}^{2\times 2}$. It has been shown that this region $\mathbb{S}$ can be used to represent some common stability regions in the complex plane (like arbitrary half planes and discs [6]). So the problem described in the introduction can simply be formulated as:

Problem 1: Find a low order $(m < (n-1))$ proper (i.e. biproper or strictly proper) controller $C_{mo}(s)$ such that the closed loop poles have the following properties:

1) $(n+m-q)$ out of the total $(n+m)$ poles are placed at $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n+m-q}\}$ and
2) remaining q poles are placed anywhere in $\mathbb{S}$.

Denote the unspecified closed loop poles of the system as $\{-\mu_1, -\mu_2, \ldots, -\mu_q\}$. Hence the characteristic equation of the closed loop system will be

$$\sigma(s) = \underbrace{\left[\prod_{j=1}^{q}(s+\mu_j)\right]}_{\alpha(s)} \underbrace{\left[\prod_{i=1}^{n+m-q}(s+\lambda_i)\right]}_{\beta(s)} \qquad (5)$$

where $\alpha(s) := s^q + \alpha_{q-1}s^{q-1} + \ldots + \alpha_1 s + \alpha_0$ and $\beta(s) := s^{n+m-q} + \beta_{n+m-q-1}s^{n+m-q-1} + \ldots + \beta_1 s + \beta_0$. In (5), $\alpha(s)$ is a monic polynomial of unknown coefficients while $\beta(s)$ is a monic polynomial of known coefficients (completely defined from the problem specifications). The only requirement on $\alpha(s)$ is that the roots should be located in a pre-specified region $\mathbb{S} \subset \mathbb{C}$ defined in (4). Next, denote the set of all q^{th} degree monic polynomials with real coefficients as $\mathbb{R}[s]$ and define the set $C_s := \{\alpha(s) \in \mathbb{R}[s] :$ roots of $\alpha(s) \in \mathbb{S}\}$. Then, Problem 1 can be restated as follows:

Problem 2: Find a low order $(m < (n-1))$ proper controller $C_{mo}(s)$ such that the closed loop has the following properties:

1) $(n+m-q)$ out of the total $(n+m)$ poles are placed at $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n+m-q}\}$ and
2) the polynomial $\alpha(s) \in C_s$.

However, note that the set $C_s \subset \mathbb{R}[s]$ in Problem 2 is not a convex set for $q \geq 3$ (see [19], [6]) which leads to a non-convex satisfiability problem. Hence, to make it convex, we replace C_s with an inner convex approximation of C_s. For this purpose, we briefly discuss a result from [6], [7] in the next section.

B. LMI stability region in the polynomial coefficient space

Assume that $\widehat{\alpha}(s)$ be a polynomial in the stability region C_s. Define the coefficient vectors corresponding to $\widehat{\alpha}(s)$ and $\alpha(s)$ (defined in (5)) as follows $\widehat{\alpha} := \begin{bmatrix}\widehat{\alpha}_0 \ \widehat{\alpha}_1 \ldots \widehat{\alpha}_{q-1}\end{bmatrix}^T \in \mathbb{R}^q$ and $\alpha := \begin{bmatrix}\alpha_0 \ \alpha_1 \ldots \alpha_{q-1}\end{bmatrix}^T \in \mathbb{R}^q$ respectively. Further, let $\alpha_e := \begin{bmatrix}\alpha^T & 1\end{bmatrix}^T \in \mathbb{R}^{q+1}$ and $\widehat{\alpha}_e := \begin{bmatrix}\widehat{\alpha}^T & 1\end{bmatrix}^T \in \mathbb{R}^{q+1}$. For a given $\widehat{\alpha}(s) \in C_s$ define the following set:

$$\mathbb{S}_{LMI} := \left\{ \alpha(s) \in \mathbb{R}[s] : \alpha_e \widehat{\alpha}_e^T + \widehat{\alpha}_e \alpha_e^T - \Pi^T (S \otimes P) \Pi \geq 0 \right\} \qquad (6)$$

for some $P = P^T \in \mathbb{R}^{q\times q}$. In the above inequality $\otimes$ refers to the Kronecker product, ≥ 0 implies a positive semidefinite matrix, $S \in \mathbb{R}^{2\times 2}$ refers to a symmetric matrix introduced in the definition of $\mathbb{S}$ given in (4), and $\Pi \in \mathbb{R}^{2q\times(q+1)}$ denotes a projection matrix given by

$$\Pi = \begin{bmatrix} 1 & & 0 & \cdots & 0 \\ & \ddots & 1 & & \\ & & 1 & \ddots & \\ 0 & \cdots & 0 & & 1 \end{bmatrix}^T_{(q+1)\times 2q}$$

It was shown in [6, Theorem 1] that for any given stable polynomial $\widehat{\alpha}(s) \in C_s$, the polynomial $\alpha(s)$ is also in C_s provided there exists a symmetric matrix $P \in \mathbb{R}^{q\times q}$ satisfying the matrix inequality $\alpha_e \widehat{\alpha}_e^T + \widehat{\alpha}_e \alpha_e^T - \Pi^T(S\otimes P)\Pi \geq 0$. Hence corresponding to each $\widehat{\alpha}(s) \in C_s$ there exists a set $\mathbb{S}_{LMI} \subseteq C_s$.

Furthermore, the inequality $\alpha_e \widehat{\alpha}_e^T + \widehat{\alpha}_e \alpha_e^T - \Pi^T(S\otimes P)\Pi \geq 0$, introduced in (6), is linear in the unknowns α_e and P. This helps us to convexify Problem 2 by replacing C_s with $\mathbb{S}_{LMI}$. Now, since $\mathbb{S}_{LMI} \subseteq C_s$ we can pose the following problem.

Problem 3: Find a low order $(m < (n-1))$ proper controller $C_{mo}(s)$ such that the closed loop has the following properties:

1) $(n+m-q)$ out of the total $(n+m)$ poles are placed at $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n+m-q}\}$ and
2) the polynomial $\alpha(s) \in \mathbb{S}_{LMI}$.

However to compute $\mathbb{S}_{LMI}$ explicitly we still need *a priori* a polynomial $\widehat{\alpha}(s) \in C_s$. This is referred to as the "central polynomial" in [6] and [7] where various domain dependent heuristics are provided for design choices for $\widehat{\alpha}(s)$. In our case $\widehat{\alpha}(s)$ can be chosen to be any q^{th} degree polynomial with roots in the stability region $\mathbb{S}$. Note that the convex stability region $\mathbb{S}_{LMI}$ is sensitive to the choice of central polynomial $\widehat{\alpha}(s)$ (see [6], [7]) and hence some conservativeness is introduced in to the proposed methodology due to this dependence.

III. Main Results

In this section we will show that Problem 3 can be re-written as an SDP. Before that let us define the following Toeplitz matrix

$$T(a) := \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \\ 1 & a_{n-1} & \cdots & a_1 \\ 0 & 1 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{(n+m+1)\times(m+1)} \quad (7)$$

corresponding to the polynomials $a(s)$ in (1). Similarly, corresponding to the polynomials $b(s)$, define the following Toeplitz matrix.

$$T(b) := \begin{bmatrix} b_0 & 0 & \cdots & 0 \\ b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-2} & b_{n-3} & \cdots & b_0 \\ b_{n-1} & b_{n-2} & \cdots & b_1 \\ 0 & b_{n-1} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{n-1} \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{(n+m+1)\times(m+1)} \quad (8)$$

The Sylvester's resultant matrix associated with $T(a)$ and $T(b)$ can be defined as follows:

$$S_l(a,b,2(m+1)) := \begin{bmatrix} T(a) & T(b) \end{bmatrix}_{(n+m+1)\times 2(m+1)} \quad (9)$$

Let us define a vector

$$\sigma =: \begin{bmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_{n+m-1} & \sigma_{n+m} \end{bmatrix}^T \in \mathbb{R}^{n+m+1}$$

associated with the closed loop characteristic polynomial $\sigma(s) := \sigma_{n+m}s^{n+m} + \sigma_{n+m-1}s^{n+m-1} + \cdots + \sigma_1 s + \sigma_0$. Define following vectors

$$x := \begin{bmatrix} x_0 & x_1 & \cdots & x_m \end{bmatrix}^T \in \mathbb{R}^{(m+1)}$$
$$y := \begin{bmatrix} y_0 & y_1 & \cdots & y_m \end{bmatrix}^T \in \mathbb{R}^{(m+1)}$$

corresponding to the polynomial $x(s)$ and $y(s)$ as defined in (2). Further define the controller coefficient vector

$$\upsilon := \begin{bmatrix} x^T & y^T \end{bmatrix}^T \in \mathbb{R}^{2(m+1)}. \quad (10)$$

Now, according to [4], [5], arbitrary pole placement with the controller $C(s)$ can be achieved from the following relation:

$$[S_l(a,b,2(m+1))]\,\upsilon = \sigma \quad (11)$$

From (11) it can be verified that when $m = n-1$ the matrix $S_l(a,b,2(m+1))$ is square and also non-singular ($a(s)$ and $b(s)$ are co-prime). Hence there is a unique controller coefficient vector υ corresponding to the specified σ. However, we are interested in finding a low order biproper or strictly proper controller $C_{mo}(s)$ i.e. for $m < (n-1)$, which will assure the pole placement requirements. Such a situation can be addressed in following way.

Recalling the expression for the required closed loop characteristic polynomial (5), the coefficients could be written as follows:

$$\begin{aligned} \sigma_0 &= \beta_0\alpha_0 \\ \sigma_1 &= \beta_0\alpha_1 + \beta_1\alpha_0 \\ &\vdots \\ \sigma_{n+m-1} &= \beta_{n+m-q-1} + \alpha_{q-1} \\ \sigma_{n+m} &= 1 \end{aligned} \quad (12)$$

Since $(-\lambda_1, -\lambda_2, \ldots, -\lambda_{n+m-q})$ are specified by the designer, the coefficients $\beta_0, \beta_1, \ldots, \beta_{n+m-q-1}$ in (12) are known quantities. However, the free poles $-\mu_1, -\mu_2, \ldots, -\mu_q$ are unspecified, so that $\alpha_0, \alpha_1, \ldots, \alpha_{q-1}$ are unknown.

First note that $\sigma_0, \sigma_1, ..., \sigma_{n+m}$ can be eliminated from equations (11) and (12) to get $(n+m+1)$ number of linear equations:

$$\begin{aligned} a_0x_0 + b_0y_0 &= \beta_0\alpha_0 \\ a_1x_0 + a_0x_1 + b_1y_0 + b_0y_1 &= \beta_0\alpha_1 + \beta_1\alpha_0 \\ &\vdots \\ x_{m-1} + a_{n-1}x_m + b_{n-1}y_m &= \beta_{n+m-q-1} + \alpha_{q-1} \\ x_m &= 1 \end{aligned} \tag{13}$$

From (13), it is possible to express α_j $(j = 0, 1, \ldots, q-1)$ in terms of variables x_i's and y_i's $(i = 0, 1, \ldots, m)$. Compactly this can be written as $\alpha = F\upsilon + g$ where $F \in \mathbb{R}^{q\times(2m+2)}$ and $g \in \mathbb{R}^q$.

Now, excluding $x_m = 1$, the coefficients $\alpha_0, \ldots, \alpha_{q-1}$ can be back-substituted in the set of $(n+m)$ equations (13) to get $(n+m-q)$ linear equations in x_i's and y_i's (for $i = 0, 1, \ldots, m$). These equations can be written in the form: $E\upsilon + h = \mathbf{0}$ where $E \in \mathbb{R}^{(n+m-q)\times(2m+2)}$, $h \in \mathbb{R}^{(n+m-q)}$ and $\mathbf{0}$ is a zero vector of appropriate dimension. Including the equation $x_m - 1 = 0$ to the above set of equations, $E\upsilon + h = \mathbf{0}$ can be written as $\tilde{E}\upsilon + \tilde{h} = \mathbf{0}$. Hence, we get the following set of equations:

$$\alpha = F\upsilon + g \quad \text{and} \quad \tilde{E}\upsilon + \tilde{h} = \mathbf{0} \tag{14}$$

from the set of $(n+m)$ equations in (13). Corresponding to the relation $\alpha = F\upsilon + g$, define α_e as

$$\alpha_e = \tilde{F}\upsilon + \tilde{g} \quad \text{where } \tilde{F} = \begin{bmatrix} F_{q\times(2m+2)} \\ \mathbf{0}_{1\times(2m+2)} \end{bmatrix} \text{ and } \tilde{g} = \begin{bmatrix} g \\ 1 \end{bmatrix} \tag{15}$$

By using (15), the LMI introduced in the definition of $\mathbb{S}_{LMI}$ (see (6)) will be

$$\tilde{F}\upsilon\widehat{\alpha}_e^T + \widehat{\alpha}_e\upsilon^T\tilde{F}^T + \tilde{g}\widehat{\alpha}_e^T + \widehat{\alpha}_e\tilde{g}^T - \Pi^T(S\otimes P)\Pi \geq 0 \tag{16}$$

Then the following result holds:

Theorem 1: For any fixed $\widehat{\alpha}(s) \in C_s$, if for some $\upsilon \in \mathbb{R}^{2m+2}$ and for some $P = P^T \in \mathbb{R}^{q\times q}$, the relations (16) and $\tilde{E}\upsilon + \tilde{h} = \mathbf{0}$ hold, then the closed loop poles (roots of the polynomial defined in (3)) satisfy the following properties:

1) $(n+m-q)$ out of the total $(n+m)$ poles are $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n+m-q}\}$.
2) the remaining q poles $-\mu_i \in \mathbb{S}$ for $i = 1, \ldots, q$.

Furthermore, the resulting controller will be a m^{th} order biproper or strictly proper controller.

Proof: Let us first fix $\widehat{\alpha}(s) \in C_s$. Assume that some $\upsilon \in \mathbb{R}^{(2m+2)}$ and $P = P^T \in \mathbb{R}^{q\times q}$ satisfy (16). Then

$$\alpha_e\widehat{\alpha}_e^T + \widehat{\alpha}_e\alpha_e^T - \Pi^T(S\otimes P)\Pi \geq 0.$$

Hence the polynomial $\alpha(s) \in \mathbb{S}_{LMI}$. But $\mathbb{S}_{LMI} \subseteq C_s$, so the roots of $\alpha(s)$ lie in $\mathbb{S}$. The $(n+m-q)$ equations $\tilde{E}\upsilon + \tilde{h} = \mathbf{0}$ imply that the $(n+m-q)$ roots of polynomial $\beta(s)$ (see (5)) are placed at $\{-\lambda_1, \ldots, -\lambda_{n+m-q}\}$.

Since $x_m = 1$ (see (13)), the corresponding coefficient vector associated with polynomial $x(s)$ would be $x = \begin{bmatrix} x_0 & x_1 & \cdots & x_{m-1} & 1 \end{bmatrix}^T$. Hence the denominator polynomial $x(s)$ of the controller $C_{mo}(s)$ is a monic polynomial of degree m. The polynomial $y(s)$, on the other hand is of degree not more than m, since there are only $m+1$ entries in vector υ corresponding to the polynomial $y(s)$. Hence the resulting controller will either be a biproper or strictly proper controller. ■

Note that, for $m < (n-1)$ the corresponding Sylvester resultant matrix $S_l(a, b, 2(m+1))$ is a tall matrix and hence (11) might not have a solution for specified σ. However, the problem we are interested in, does not have fixed σ (because of the unknown coefficients polynomial $\alpha(s)$) and hence there is a possibility that for some vector υ, (11) will be satisfied. According to Theorem 1, the controller vector υ satisfying the relations (16) and $\tilde{E}\upsilon + \tilde{h} = \mathbf{0}$, will guarantee that the pole placement requirements are achieved as well as (11) is satisfied. The conditions of Theorem 1 can be checked by solving the following SDP for increasing values of m.

Problem 4: Find $\max_{P,\upsilon,\gamma} \gamma$ subject to

(i) $\tilde{E}\upsilon + \tilde{h} = \mathbf{0}$

(ii) $\Pi^T(S\otimes P)\Pi - \tilde{F}\upsilon\widehat{\alpha}_e^T - \widehat{\alpha}_e\upsilon^T\tilde{F}^T - \tilde{g}\widehat{\alpha}_e^T - \widehat{\alpha}_e\tilde{g}^T + \gamma I_{q+1} \leq 0$

To obtain a low order controller we have to start with a first order controller $(m = 1)$ and check whether the solution γ to Problem 4 satisfies $\gamma > 0$. If this condition is not satisfied then we should increase the order of the controller by one and recheck the satisfiability condition. At the stage of $m = (n-1)$ it is guaranteed that the above problem has a feasible solution and hence to obtain the lowest order controller achievable through this method, we need to solve at most $(n-1)$ SDPs.

The above problem is an LMI constrained optimization with variables γ, υ and P and can be solved by using solvers like *SeDuMi* in MATLAB environment [20], [21]. It should be noted, however, that this formulation is sensitive to the choice of the stability region $\mathbb{S}_{LMI}$, which in turns depend on the selection of the central polynomial $\widehat{\alpha}(s)$. This observation is verified through numerical examples. At the current state of research, the central polynomial needs to be chosen heuristically. Hence it is possible for the proposed algorithm to produce a slightly higher order controller than the actual minimum possible, because of a bad choice of the central polynomial.

The controller design procedure described above is summarized in the following four steps:

Design Steps

1) Start with a first order controller $m = 1$.
2) Form the Toeplitz matrices according to (7) and (8) of appropriate dimension and the corresponding Sylvester resultant matrix $S_l(a, b, 2(m+1))$.
3) Define a stability region $\mathbb{S}$ in the complex plane for the free poles according to the requirement. Choose poles (equal to the number of free poles) from $\mathbb{S}$ to form a central polynomial $\widehat{\alpha}(s)$. Some trial and error adjustment may be required here for the choice of $\widehat{\alpha}(s)$. Solve Problem 4. If there is no feasible solution then go to Step-4.
4) Increase the order of the controller by one and go to Step-2. The increment of the controller should be

followed until $m \leq (n-1)$ condition is satisfied.

Note: We have assumed that the plant is strictly proper. Also the resulting controller is biproper or strictly proper. Now, according to [5, Chapter 3, Theorem 3.26], since all the closed loop poles are in the stable region of complex plane, the inter-connection shown in Fig. 1 is internally stable.

IV. Numerical Examples

Example 1: The transfer function associated with a linearized model of 4-machine, 2-area power system [18] is given below.

$$P(s) = \frac{-0.03s^4 - 13.02s^3 - 28.74s^2 - 1323.29s + 9.83}{s^5 + 34.44s^4 + 1529.92s^3 + 825.27s^2 + 23419.86s + 2350.48}$$

The open loop poles of the plant $P(s)$ are given in Table I.

Let us find out a low order controller which will guarantee that in the closed loop, two poles are placed at $-0.4000 \pm 3.9352i$ and the settling time of the remaining poles (free poles) is not more than 8 second. Such requirements on the free poles can be achieved by choosing the stability region as closed left half of a vertical line at -0.5 in the complex plane. Hence (4) will take the following form:

$$\mathbb{S} = \left\{ s \in \mathbb{C} : \begin{bmatrix} 1 & s^* \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0 \right\} \tag{17}$$

According to the design steps, we first try with 1^{st} order controller to achieve our objective. However, there does not exist a feasible solution to the Problem 4 at this stage. Hence the next step is to try with 2^{nd} order controller and it is observed that the Problem 4 has a feasible solution. The results are discussed in the next section.

Second order controller : The order of the plant $P(s)$ is $n = 5$. The order of the controller $C_{mo}(s)$ is $m = 2$ and hence the corresponding Sylvester matrix would be

$$[S_l(a,b,6)]_{(8\times 6)} = \begin{bmatrix} 2350.48 & 0 & 0 & 9.83 & 0 & 0 \\ 23419.86 & 2350.48 & 0 & -1323.29 & 9.83 & 0 \\ 825.27 & 23419.86 & 2350.48 & -28.74 & -1323.29 & 9.83 \\ 1529.92 & 825.27 & 23419.86 & -13.02 & -28.74 & -1323.29 \\ 34.44 & 1529.92 & 825.27 & -0.03 & -13.02 & -28.74 \\ 1 & 34.44 & 1529.92 & 0 & -0.03 & -13.02 \\ 0 & 1 & 34.44 & 0 & 0 & -0.03 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The number of closed loop poles is seven. Among them two poles are already specified. The remaining five poles can take any positions in the stability region defined in (17).

To form the central polynomial $\widehat{\alpha}(s)$, -2.5, $-3.5 \pm 1i$ and $-5 \pm 2i$ are chosen inside the stability region $\mathbb{S}$. Corresponding to these poles the central polynomial would be

$$\widehat{\alpha}(s) = s^5 + 19.50s^4 + 154.75s^3 + 616.12s^2 + 1223s + 960.62$$

Following the discussion in Section III, (14) will take the following form:

$$\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix}^T = \begin{bmatrix} 150.23 & 0 & 0 & 0.62 & 0 & 0 \\ 1489.19 & 150.23 & 0 & -84.61 & 0.62 & 0 \\ -33.00 & 1489.19 & 150.23 & 2.44 & -84.61 & 0.62 \\ 4.29 & -33.00 & 1489.19 & 4.44 & 2.44 & -84.61 \\ 4.09 & 4.29 & -33.00 & -0.38 & 4.44 & 2.44 \end{bmatrix} \upsilon + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

TABLE I

Pole Locations Table

Open loop poles	Closed loop poles
$-17.1230 \pm 35.8508i$	$-0.4 \pm 3.9352i$
$-0.0468 \pm 3.9352i$	$-7.0538 \pm 6.6492i$
-0.1007	$-2.0457 \pm 2.6052i$
	-13.8114

and

$$\begin{bmatrix} -6.56 & 64 & 67.12 & -4.14 & -6.04 & 69.6 \\ -4.09 & -3.29 & 67.44 & 0.38 & -4.44 & -2.4 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \upsilon + \begin{bmatrix} -15.6 \\ -0.8 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\upsilon = \begin{bmatrix} x_0 & x_1 & x_2 & y_0 & y_1 & y_2 \end{bmatrix}^T$. Solving Problem 4, the resulting controller coefficient vector

$$\upsilon = \begin{bmatrix} 88.6643 & 1.6827 & 1 & 1463.3813 & 6.9415 & 93.7103 \end{bmatrix}^T$$

and hence the corresponding 2^{nd} order controller would be:

$$C_{mo}(s) = \frac{93.71s^2 + 6.94s + 1463.38}{s^2 + 1.68s + 88.66}$$

The closed loop poles are given in Table I. Notice that two poles are placed at $-0.4000 \pm 3.9352i$ and free poles have assumed positions in $\mathbb{S}$ as defined in (17). Hence all the requirements on closed loop poles are achieved with a 2^{nd} order controller.

V. Conclusion

The problem of obtaining a low order biproper/strictly proper controller for a SISO LTI system is studied here. It is shown that a low order controller can be obtained by solving at most $(n-1)$ SDPs. The resulting controller ensures that the fixed poles are moved to the desired locations in the complex plane while the free poles are placed anywhere within the *pre-specified* stable region.

References

[1] D. Bernstein, "Some open problems in matrix theory arising in linear systems and control," *Linear Algebra and its Applications 162-164*, pp. 409–432, 1992.

[2] A. Karimi, H. Khatibi, and R. Longchamp, "Robust control of polytopic systems by convex optimization," *Automatica*, vol. 43, no. 8, pp. 1395–1402, 2007.

[3] H. Khatibi and A. Karimi, "A convex constraint for directly designing a low-order controller in polynomial approach," *International Journal of Robust and Nonlinear Control*. submitted.

[4] K. Wellstead, *Robust polynomial controller design*. PhD thesis, Brunel University, U.K., 1991.

[5] L.Qiu and K. Zhou, *Introduction to Feedback Control*. Upper Saddle River, New Jersy: Prentice Hall, 2009.

[6] D. Henrion, M. Sebek, and V. Kucera, "Positive polynomials and robust stabilization with fixed-order controllers," *IEEE Transactions on Automatic Control*, vol. 48, no. 7, pp. 1178–1186, 2003.

[7] F. Yang, M. Gani, and D. Henrion, "Fixed-order robust H_∞ controller design with regional pole assignment," *IEEE Transactions on Automatic Control*, vol. 52, no. 10, pp. 1959–1963, 2007.

[8] S. Datta, B. Chaudhuri, and D. Chakraborty, "Partial pole placement with minimum norm controller," in *Proceedings of the 49th IEEE Conference on Decision and Control*, pp. 5001–5006, 2010.

[9] S. Datta, D. Pal, and D. Chakraborty, "Partial pole placement and controller norm optimization over polynomial stability region," in *Proceedings of the 18th IFAC World Congress, 2011*, pp. 10129–10134, 2011.

[10] S. Datta, D. Chakraborty, and B. Chaudhuri, "Partial pole placement with controller optimization," *IEEE Transactions on Automatic Control*, 2011. accepted for publication.

[11] P. Kundur, *Power system stability and control*. McGraw-Hill : New York, London, The EPRI power system engineering series, 1994.
[12] G. Obinata and B. D. O. Anderson, *Model Reduction for Control System Design*. Springer-Verlag London, 2001.
[13] D. U. Campos-Delgado, R. Femat, and E. Ruiz-Velazquez, "Design of reduced order controllers via H_∞ and parametric optimization: comparison for an active suspension system," *European Journal of Control*, vol. 9, no. 1, 2003.
[14] M. Mesbahi and G. Papavassilopoulos, "On the rank minimization problem over a positive semi-definite linear matrix inequality," *IEEE Transactions on Automatic Control*, vol. 42, no. 2, pp. 239–243, 1997.
[15] M. Mesbahi, "On the rank minimization problem and its control applications," *Systems and Control Letters*, vol. 33, pp. 31–36, 1998.
[16] S. Wang and J. Chow, "Low-order controller design for SISO systems using coprime factors and LMI," *IEEE Transactions on Automatic Control*, vol. 45, no. 6, pp. 1166–1169, 2000.
[17] H. Khatibi, A. Karimi, and R. Longchamp, "Fixed-order controller design for polytopic systems using LMIs," *IEEE Transactions on Automatic Control*, vol. 53, no. 1, pp. 428–434, 2008.
[18] B. Pal and B. Chaudhuri, *Robust control in power systems*. Springer : New York, Power electronics and power systems, 2005.
[19] J. Ackermann, "Parameter space design of robust control systems," *IEEE Transactions on Automatic Control*, vol. AC-25, pp. 1058–1072, 1980.
[20] J. F. Sturm, "Using *SeDuMi* 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. cs.SC, pp. 625–653, 2005.
[21] *SeDuMi*, "Sedumi 1.3." http://sedumi.ie.lehigh.edu, 2010.