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Minimal Controller Structure for Generic Pole Placement

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***Abstract*— In this paper we address the generic pole placement problem for a system represented by differential algebraic equations. The genericity aspect is relevant when dealing with large dynamical systems where the plant equations are sparse. We capture the sparsity structure of the plant equations into an edge weighted and undirected bipartite graph. We propose an algorithm that furnishes a 'minimal' controller structure for achieving generic arbitrary pole placement: minimality in the sense of the sparsity within controller equations. More precisely, we introduce a procedure to come up with a set of controller equations such that, in addition to generically achieving arbitrary pole placement, the bipartite graph constructed for this controller has the minimum number of edges amongst all controllers that generically achieve arbitrary pole placement. The algorithm we propose involves finding a minimum number of paths that cover a given set of vertices corresponding to plant equations. We introduce an integer that captures the extent of MIMO features inside the plant equations, since this turns out to crucially decide the minimum number of required edges.**

This paper's minimal controller structure problem and the proposed solution turn out to also solve the problem of generically completing a given rectangular polynomial matrix into a *unimodular* matrix using the minimum number of nonzero entries.

***Index Terms*— structural controllability, genericity, unimodular completion, bipartite graphs, maximum matching, minimum cover.**

I. INTRODUCTION

Often when dealing with large scale dynamical systems, it is infeasible to perform numerical computation to determine the system's controllability properties and to compute a controller numerically. The system equations often have a sparsity structure that allows employing graph techniques to study properties in a 'structural'/generic sense. Notable amongst work in this area is that by [Lin74] in the context of state-space systems and later by many others for both regular and singular descriptor state-space systems. Results for generic solvability of various control problems are formulated based on an associated graph. We refer to the survey paper by [DCvdW03] for this.

The behavioral approach to modeling of dynamical systems has been adopted for structural studies in [vdW95]: there the generic *dimension* of a minimal state space realization is found by constructing a graph from the polynomial matrix. In [vdWM95] the pole placement problem with disturbance decoupling has been done for a system in *descriptor* form. They use a simplified version of the Dulmage-Mendelsohn (DM) decomposition on a bipartite graph associated with the plant to obtain conditions for generic solvability of the pole placement problem by computing the maximum and minimum weights for the components of the graph obtained from the DM decomposition.

In this paper, we deal with systems described by algebraic/differential equations of any order, and not just first order differential equations. For such systems, we pursue the problem of proposing a controller that generically achieves arbitrary pole placement and further has 'highest sparsity' structure in the controller equations. More precisely, amongst all controllers that generically achieve pole placement, we seek to construct a controller which has the least number of variables for each equation, totalled across equations. This problem is motivated by a minimality during design of a sensor-actuator network. We use the behavioral approach and associate the system equations to a polynomial matrix. An undirected and bipartite graph is constructed for the plant based on this associated polynomial matrix. Notion of minimality here refers to the graph of the controller having minimum number of edges.

The paper is organized as follows. Section II has problem formulation and statement of our main result under certain simplifying assumptions. Section III explains the preliminaries on behavioral theory, bipartite graphs and genericity. In Section IV some preliminary results are stated. This main result is generalised in Section V and Section VI. Section VII infers the graph theoretic results based on the MIMO characteristics of the plant. Section VIII contains concluding remarks.

II. PROBLEM FORMULATION & MAIN RESULT

We consider systems which are described by a set of ordinary linear differential equations with constant coefficients. The system behavior $\mathfrak{B}$ is defined to be the subspace of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ consisting of the solutions to the system equations: let $P(s) \in \mathbb{R}^{n\times m}[s]$.

$$\mathfrak{B} := \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m) \mid P\left(\frac{d}{dt}\right) w = 0 \right\}. \qquad (1)$$

This representation is called a *kernel representation* of $\mathfrak{B}$.

Since we seek only 'generic' results, we consider just the structural aspects of the system. In this context, we associate a weighted bipartite graph to the given system. A graph $G = (V, E)$ with vertex set V and edge set E is said to be bipartite if V can be partitioned into two subsets $\mathcal{R}$ and $\mathcal{C}$ such that no two vertices from the same subset are adjacent. We associate an edge weighted bipartite graph $G(\mathcal{R}, \mathcal{C}; E)$ to a polynomial matrix $P(s) \in \mathbb{R}^{n\times m}[s]$ as follows. The sets $\mathcal{R}$ and $\mathcal{C}$ denote the rows and columns of the polynomial matrix and are the two disjoint vertex sets of the bipartite graph G, i.e. $|\mathcal{R}| = n$, $|\mathcal{C}| = m$. By definition of G, an edge exists in the bipartite graph between vertex $u_i \in \mathcal{R}$ and $v_j \in \mathcal{C}$ if the $(i, j)^{\text{th}}$ entry of the matrix P is nonzero. It turns out that

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all nonconstant polynomial entries in $P(s)$ contribute to the results in the same manner irrespective of the degree. The constant polynomial entries are different since their roots are an empty set, and these entries have fewer conditions to satisfy. Hence we distinguish only between constant and nonconstant entries of the matrix. In this regard the edge set is classified into two types.

- Constant edge: if the entry in $P(s)$ corresponding to this edge is a *nonzero constant*.
- Nonconstant edge: if the entry in $P(s)$ corresponding to this edge is a polynomial of degree one or more.

Note that there is no edge between two vertices v_i and v_j if and only if the $(i,j)^{\text{th}}$ entry of $P(s)$ is 0. The degree of a *vertex* v in the graph is the number of edges incident on v, equivalently, the number of 'neighbors' of v.

For a bipartite graph $G(\mathcal{R},\mathcal{C};E)$, we need the subgraph *induced* due to a subset of edges $E_1 \subseteq E$: the bipartite graph $G[E_1]$ denotes the subgraph consisting of edges from E_1 and their endpoint vertices. A set of edges M in a graph G is called a *matching* if every vertex of the subgraph of G induced by M has degree at most 1. A maximum matching is a matching with maximum number of edges. A graph can have more than one maximum matching. An edge e in G is called *inadmissible* if e does not occur in any of the maximum matchings of G. See [ADH98, Section 10.3]. The cardinality of a matching M, denoted by $|M|$, is defined as the number of edges in M. In this paper, we consider polynomial matrices $P(s) \in \mathbb{R}^{n\times m}[s]$ with $n \leqslant m$, i.e. the graph G satisfies $|\mathcal{R}| \leqslant |\mathcal{C}|$. A matching M is said to be $\mathcal{R}$-*saturating* if $|M| = |\mathcal{R}|$. A $\mathcal{C}$-saturating matching is defined similarly. The special case when G satisfies $|\mathcal{R}| = |\mathcal{C}|$, an $\mathcal{R}$-saturating matching is also $\mathcal{C}$-saturating matching: we call such a matching a *perfect matching*. A detailed exposition of matching theory can be found in [LP86]. With the above brief preliminaries we come to the formulation of the problem. Let $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, with $P \in \mathbb{R}^{n\times m}[s]$ being full row rank, denote a kernel representation of the plant. We construct the bipartite graph for the plant, $G^p(\mathcal{R}_P,\mathcal{C};E_P)$ corresponding to the polynomial matrix $P(s)$. The graph of the controller is denoted by $G^k(\mathcal{R}_K,\mathcal{C};E_K)$. The vertex set $\mathcal{R}_P$ in G^p and $\mathcal{R}_K$ in G^k correspond to plant and controller *equations* respectively. The second set $\mathcal{C}$ corresponds to the *variables* that the equations involve. An edge $e \in E_P$ is called plant edge and an edge $e \in E_K$ is called controller edge.

The problem we address is to propose a controller structure which generically achieves arbitrary pole placement and such that the graph of the controller has minimum number of edges. This is motivated by a minimum sensor network design issue, for example. We state this as two problems: one from a control viewpoint and another, graph theoretic.

Problem 2.1: Given a plant $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, find a regular[1] controller structure, i.e. a graph $G^k(\mathcal{R}_K,\mathcal{C};E_K)$ corresponding to $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, which satisfies the following properties.

- Arbitrary pole placement is generically achievable with this controller structure.
- The total number of non-zero entries in K is minimum.

Proposition 4.1 and some more preliminaries help in understanding the equivalence of Problem 2.1 with the following problem. Condition 1 below ensures a regular controller, while condition 2 ensures arbitrary pole placement.

Problem 2.2: Given a graph $G^p(\mathcal{R}_P,\mathcal{C};E_P)$. Find a graph $G^k(\mathcal{R}_K,\mathcal{C};E_K)$ such that the following are satisfied.

1) In $G(\mathcal{R}_P \cup \mathcal{R}_K,\mathcal{C};E_P \cup E_K)$ there exists a perfect matching.
2) Every edge $e \in E_P$ that is admissible in $G(\mathcal{R}_P \cup \mathcal{R}_K,\mathcal{C};E_P \cup E_K)$ is in some cycle involving an edge e_K from E_K such that e_K is admissible in $G(\mathcal{R}_P \cup \mathcal{R}_K,\mathcal{C};E_P \cup E_K)$.
3) $G^k(\mathcal{R}_K,\mathcal{C};E_K)$ has the minimum number of edges amongst all graphs that satisfy condition 1 and 2.

Of course, it is possible that $G(\mathcal{R}_P \cup \mathcal{R}_K,\mathcal{C};E_P \cup E_K)$ is disconnected, or its subgraph of *admissible* edges is disconnected: this suggests, loosely speaking, decoupled subsystems whose poles are assigned arbitrarily. This does not affect our results nor the proofs. The following theorem is one of the main results of this paper. We solve the problem for the special case when the graph of the plant has no cycles and no constant edges. The more general results (i.e with cycles and both constant and non-constant edges) follow later: Theorem 5.4 and Theorem 6.2. We assume that the graph of the plant after removing inadmissible edges is connected. In case there are many components, then it can be shown that the minimum number of edges is just the sum of edges required for each component. Therefore, without loss of generality, most of the results in this paper address the case of the graph being connected. We define a parameter e_{mimo}, which can be interpreted as the extent of MIMO characteristics of the plant. This is elaborated in Section VII and Remark 8.1.

Theorem 2.3: Let $G^p(\mathcal{R}_P,\mathcal{C};E_P)$ denote the graph of a controllable plant after removing the inadmissible edges. Assume $G^p(\mathcal{R}_P,\mathcal{C};E_P)$ is connected and has no cycles and no constant edges. Suppose at least N_p paths are required to cover the $\mathcal{R}_P$ vertices. Let $\mathcal{C}_p \subseteq \mathcal{C}$ denote the vertices that are covered by the N_p paths and define $e_{\text{mimo}} := |\mathcal{C}_p| - |\mathcal{R}_P|$. Suppose n_t denote the set of degree one vertices in $\mathcal{C}_p$. Define e_p by

(i) $e_p := |n_t| - e_{\text{mimo}}$, if $e_{\text{mimo}} < N_p$ and
(ii) $e_p := e_{\text{mimo}}$, if $e_{\text{mimo}} \geqslant N_p$.

Then there exists a controller that generically achieves arbitrary pole placement and whose bipartite graph $G^k(\mathcal{R}_K,\mathcal{C};E_K)$, with $|\mathcal{R}_K| = |\mathcal{C}| - |\mathcal{R}_P|$, has number of edges $|E_K| = e_p + |\mathcal{R}_K|$. Moreover, any controller that generically achieves arbitrary pole placement has at least $e_p + |\mathcal{R}_K|$ edges in its bipartite graph.

The proof of this theorem is skipped due to space constraints. Section IV has some preliminary results that will provide an insight to the proof.

III. Preliminaries

Subsections III-A and III-B respectively elaborate about behavioral theory and generic aspects in polynomial matri-

[1]The interconnection is said to be regular if rank of $\begin{bmatrix} P(s) \\ K(s) \end{bmatrix}$ is the sum of ranks of $P(s)$ and $K(s)$. Regularity of interconnection is closely related to implementation of controller in the feedback configuration. See [Wil97]. It's key role in the equivalence of controllability and arbitrary pole placement is brought out in Proposition 3.2.

ces.

A. Behavioral approach

In the previous section on problem formulation, the concept of behavior of a system was introduced. A behavior $\mathfrak{B}$ is controllable it is possible to patch from any past trajectory to any other desired trajectory using a another trajectory that satisfies the system laws, perhaps with some finite delay. A behavior $\mathfrak{B}$ is called *autonomous* if $w_1 = w_2$ whenever $w_1, w_2 \in \mathfrak{B}$ satisfy $w_1(t) = w_2(t)$ for all $t \leqslant 0$. We state the required results from behavioral literature in the following proposition for easy reference: see [PW98].

Proposition 3.1: Consider $P \in \mathbb{R}^{n\times m}[s]$ and let behavior $\mathfrak{B}$ have the kernel representation $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. Then,

1) $\mathfrak{B}$ is autonomous if and only if P has full column rank.
2) $\mathfrak{B}$ is controllable if and only if $P(\lambda)$ has constant row rank for every complex number $\lambda \in \mathbb{C}$.

The above kernel representation is called minimal if P has full row rank: this can be assumed without loss of generality.

1) Pole placement: Let $A(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ be a kernel representation of an autonomous behavior $\mathfrak{B}$. The determinant of A, called the characteristic polynomial (assumed monic, without loss of generality) of the system, is denoted by $\chi(\mathfrak{B})$. The roots of χ counted with multiplicities are called the poles of the behavior. The following proposition gives a necessary and sufficient condition for pole placement using the behavioral approach.

Proposition 3.2: [Wil97] Let $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, $P(s) \in \mathbb{R}^{n\times m}[s]$ denote a minimal kernel representation of the plant. Then, the plant is controllable if and only if, for any monic $d(s) \in \mathbb{R}[s]$, there exists a regular[2] controller $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ such that the corresponding closed system has characteristic polynomial $d(s)$, i.e. $\det \begin{bmatrix} P(s) \\ K(s) \end{bmatrix} = d(s)$.

In the state space case, the polynomial d is of the same degree as the size of A. Unlike the state space case, the above result allows d to be of higher/lower degree. Of course, d of a lower degree requires the feedback to allow differentiating of the output: just like a PD controller would. This option is practically relevant when some measurements are noiseless. Since we focus on generic arbitrary pole placement, which is nothing but assigning the roots of χ counted with multiplicity, we ignore the 'monic' aspect of χ for the rest of this paper.

B. Generic Properties of polynomial matrices

Definition 3.3: A property P in terms of variables $a_1, \ldots, a_n$ is said to be satisfied generically if the set of values $a_1, \ldots, a_n \in \mathbb{R}$ that do *not satisfy property* P *form a non-trivial algebraic variety in* $\mathbb{R}^n$.

Since a non-trivial algebraic variety in $\mathbb{R}^n$ forms a 'thin set', i.e. a set of measure zero, a property P is said to be true generically in $\mathbb{R}^n$ if P is satisfied for *almost all* values in $\mathbb{R}^n$. For example any two nonzero polynomials $a(s)$ and $b(s)$ are generically coprime. In this case, $n = \deg a(s) + \deg b(s) + 2$. Generic coprimeness follows since the set of coefficients have to satisfy a nontrivial algebraic relation for the two polynomials to be coprime: the algebraic relation is nothing but the resultant [Kai80, Section 2.4.4] of $a(s)$ and $b(s)$.

The notion of an inadmissible edge plays a central role in this paper. Recall from Section II that in a graph $G(\mathcal{R}, \mathcal{C}; E)$ constructed from $P \in \mathbb{R}^{n\times n}[s]$ an edge e which does not occur in any maximal matching is called an inadmissible[3] edge of G. Consequently, the entry in P corresponding to this edge e does not play a role in the determinant expansion of any maximal minor of P. After removing the inadmissible edges from G the resulting subgraph is denoted as G_a. Clearly, G has an $\mathcal{R}$-saturating matching if and only if G_a has one. Due to the genericity assumption on P, and since the non-zero entries in P_a corresponding to G_a are also in P, we have the genericity property for P_a also.

IV. Preliminary results

This section gives some preliminary results that help in proving the main result of the paper about constructing a minimal controller structure which achieves arbitrary pole placement for the situation when the graph of the plant has no cycles as well as no constant plant edges.

We begin with some standard definitions. A path $\mathscr{P}$ in a graph G is a finite sequence of distinct vertices and edges, $\mathscr{P} : v_0 e_1 v_1 \ldots e_n v_n$, where edge e_i connects vertices v_{i-1} and v_i, for each $1 \leqslant i \leqslant n$. Further, $\mathscr{P}$ is not properly contained in another path. The initial vertex, v_0 and final vertex, v_n are called the *terminals* of the path. The degree of incidence of a vertex refers to the number of edges incident on that vertex. We shall henceforth refer vertices with degree of incidence equal to one as *degree-one* vertices. Since all paths are in some sense maximal, the terminals of every path are degree-one vertices. Conversely every degree-one vertex is a terminal of some path. A cycle $\mathscr{C}$ in a graph G is a finite sequence of vertices and edges, $\mathscr{C} : v_0 e_1 v_1 \ldots e_n v_n$, where edge e_i connects vertices v_{i-1} and v_i, for each $1 \leqslant i \leqslant n$, with $v_0 = v_n$ and all other vertices and edges distinct.

In order to achieve arbitrary pole placement for a given plant it is not always required that each controller equation involves every variable. In graph theoretic terms this means that there need not be an edge from $\mathcal{R}_K$ to every vertex in $\mathcal{C}$. Hence for a given controller structure, i.e. the graph of the controller, the following proposition from [KB11] states a necessary and sufficient condition for feasibility of arbitrary pole placement.

Proposition 4.1: Let $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, with $P \in \mathbb{R}^{n\times m}[s]$ being full row rank, denote a plant. Let G^k, the graph of a controller $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ with $K \in \mathbb{R}^{(m-n)\times m}[s]$ be given. Consider the bipartite graph $G^{\mathrm{aut}}(\mathcal{R}, \mathcal{C}; E)$ of the controlled system, constructed from $\begin{bmatrix} P^T & K^T \end{bmatrix}^T$. Let G_a^{aut} represent the graph obtained after removing the inadmissible edges. Then the following are equivalent.

1) Every non-constant plant edge in G_a^{aut} is in a cycle containing an admissible edge from G^k.

[2] The interconnection is said to be regular if rank of $\begin{bmatrix} P(s) \\ K(s) \end{bmatrix}$ is the sum of ranks of $P(s)$ and $K(s)$. See Willems [Wil97]. In this paper, all interconnections are regular. Given a plant system, a controller is called regular if the interconnection between the plant and that controller is regular.

[3] Admissible/inadmissible edge have also been referred to respectively as allowed/forbidden in the literature: see [LP86], for example.

2) Arbitrary pole placement is possible generically with the given G^k.

In the problem considered in this paper we do not have any controller structure to begin with, rather we propose a controller structure which is minimal and at the same time ensures that each edge of the plant satisfies the conditions stated in the above proposition. In this regard a relevant question is about the structural controllability of the plant. The following lemma gives a necessary and sufficient condition for the plant to be controllable under the assumption that the graph of the plant has no cycles.

Lemma 4.2: Assume a plant $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. Let $G^p(\mathcal{R}_P, \mathcal{C}; E_P)$ be the bipartite graph associated to $P(s)$ with all inadmissible edges removed. Assume G^p has no cycles. Then plant is structurally controllable if and only if every path whose terminal is in $\mathcal{R}_P$ has length one and is a constant edge.

From the above lemma it follows that if the plant is controllable, then in the graph of the plant G^p all paths containing at least one non-constant plant edge has both its terminals in $\mathcal{C}$. The next major step is to complete all these paths to cycles using controller edges.

Lemma 4.3: Assume the graph of a controllable plant $G^p(\mathcal{R}_P, \mathcal{C}; E_P)$ is connected, has no cycles and has no constant plant edges. Let $d := |\mathcal{C}| - |\mathcal{R}_P|$. Suppose at least N_p paths are required to cover the $\mathcal{R}_P$ vertices in G^p. Let $\mathcal{C}_p \subseteq \mathcal{C}$ denote the vertices that are covered by the N_p paths and define $e_{\text{mimo}} := |\mathcal{C}_p| - |\mathcal{R}_P|$. Then $1 \leqslant e_{\text{mimo}} \leqslant \min\ (d, 2p-1)$.

The proofs of both the above lemmas are skipped due to space constraints.

V. Bipartite graph with cycles

In this section we deal with the case when there are cycles in the graph of the plant. We first find the *merged-cycles* graph defined as follows.

Definition 5.1: Let $G_a(\mathcal{R}, \mathcal{C}; E)$ denote the graph obtained from $G(\mathcal{R}, \mathcal{C}; E)$ by removing all inadmissible edges. The *merged-cycles* graph G_{mc} is defined by the bipartite graph obtained from G_a by repeating the following steps till there are no cycles in the graph. Initialize $G_{mc} := G_a$.

While there exists a cycle in G_{mc}, repeat:

- Consider a cycle in G_{mc} formed by a set of edges, $e_i \subset E$ which connect vertices in the set $r_i \subset \mathcal{R}$ and $c_i \subset \mathcal{C}$ and $|e_i| = |r_i| + |c_i|$ and $|r_i| = |c_i|$.
- Merge and replace all vertices in r_i into one single vertex r_{m_i} and vertices in c_i to one vertex c_{m_i}.
- The edge e_{m_i} between r_{m_i} and c_{m_i} is representative of all the edges in the e_i.
- If at least one of the edges in the cycle which was merged is a non-constant plant edge, then the edge e_{m_i} is also labelled a non-constant plant edge.

The resulting graph is called the *merged-cycles* graph G_{mc}.

Since a cycle is 'merged out' in each run of the above algorithm, we end in a 'merged-cycles' graph G_{mc} with no cycles in a finite number of operations, and moreover, for the new bipartite graph $G_{mc}(R, C; E)$, the difference $|C| - |R|$ is same as that in G_a. We illustrate this with the following example as shown in Figure 1. One can check that the final

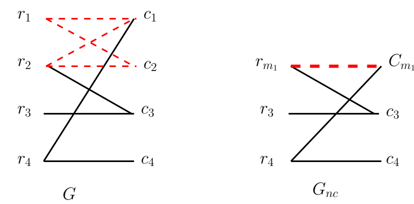

Fig. 1. Graph with no cycle

graph G_{mc} is independent of the sequence of merging the cycles. For arbitrary pole placement, all non-constant plant edges in G_a^p should form a cycle with controller edges or be inadmissible in G^{aut}. It is enough to perform this check on the simplified graph G_{mc}, due to the following result from matroid theory (in the context of 'circuits'): see [Nar97].

Proposition 5.2: Consider cycles $\mathscr{C}_1$, $\mathscr{C}_2$ in a bipartite graph $G(\mathcal{R}, \mathcal{C}; E)$. Let $e(\mathscr{C})$ denote the set of edges in $\mathscr{C}$. Then the set $e(\mathscr{C}_1) \cup e(\mathscr{C}_2) - e(\mathscr{C}_1) \cap e(\mathscr{C}_2)$ is also a cycle.

In the context of maximizing the usage of merged vertices as terminals of paths which are to be completed to cycles we require the following definition of distance between two vertices.

Definition 5.3: In a graph G, the distance between two vertices v_1 and v_2 denoted as $\text{dist}(v_1, v_2)$ is defined as the minimum number of edges between v_1 and v_2.

We state our next main result about the minimal controller structure for pole placement when the graph of the plant is not acyclic. In this case the minimum number of controller edges required is possibly more because a merged plant edge ought not be made inadmissible.

Theorem 5.4: Let $G_{mc}(\mathcal{R}_P, \mathcal{C}, E_P)$ denote the graph of a controllable plant obtained after merging cycles. Suppose G_{mc} is connected and has no constant plant edges. Assume at least N_p paths are required to cover the $\mathcal{R}_P$ vertices. Let $\mathcal{C}_p \subseteq \mathcal{C}$ denote the vertices that are covered by the N_p paths and define $e_{\text{mimo}} := |\mathcal{C}_p| - |\mathcal{R}_P|$. Suppose n_t denotes the set of degree-one vertices in $\mathcal{C}_p$. Define e_p by

(i) $e_p := |n_t| - e_{\text{mimo}}$ if $e_{\text{mimo}} < N_p$.
(ii) $e_p := e_{\text{mimo}}$ if $e_{\text{mimo}} \geqslant N_p$.

Define γ through the sets $\mathcal{C}_m$ and A as follows:

$$\begin{aligned} \mathcal{C}_m &:= \{v \in \mathcal{C} \backslash \mathcal{C}_p \mid v \text{ is a merged vertex in } G_{mc}\}. \\ A &:= \{v \in n_t \mid v \text{ is not a merged vertex and} \\ &\qquad\qquad \text{dist}(v, v_1) = 2 \text{ for some } v_1 \in \mathcal{C}_m\}. \\ \gamma &:= |\mathcal{C}_m| - |A|. \end{aligned}$$

Then there exists a controller that generically achieves arbitrary pole placement and whose bipartite graph $G^k(\mathcal{R}_K, \mathcal{C}; E_K)$, with $|\mathcal{R}_K| = |\mathcal{C}| - |\mathcal{R}_P|$, has number of edges $|E_K| = e_p + \gamma + |\mathcal{R}_K|$, and moreover, bipartite graph of any controller that achieves arbitrary pole placement has at least $e_p + \gamma + |\mathcal{R}_K|$ edges.

VI. Bipartite graph with constant plant edges

In this section we consider the case when the graph of the plant has constant plant edges. As explained in the previous section we first find the merged-cycles graph and then proceed to find the $\overline{\mathcal{R}_P^c}$ vertices defined below.

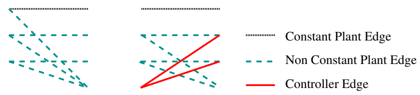

Fig. 2. G^p with constant edge

Definition 6.1: Let $G^p(\mathcal{R}_P, \mathcal{C}; E_P)$ denote the graph of a controllable plant. A maximal constant vertex set, denoted by $\mathcal{R}_P^c \subset \mathcal{R}_P$, is such that it satisfies the following:

1) There is at least one non-constant edge incident on each of the vertices in $\mathcal{R}_P^c$.
2) The vertex in $\mathcal{C}$ corresponding to each of the above non-constant edges, denoted as $\mathcal{C}^c$, are distinct.
3) The set $\mathcal{R}_P^c$ is not a proper subset of any other set satisfying the above two properties.

Define corresponding to the maximal constant vertex set a minimal non-constant vertex set $\overline{\mathcal{R}_P^c} := \mathcal{R} \backslash \mathcal{R}_P^c$.

Since our problem is to minimize the number of controller edges the above classification of constant vertices is helpful: It is no longer required to cover vertices in $\mathcal{R}_P^c$ by paths thus resulting in a possible reduction of controller edges. This is illustrated in the following figure. Here if all $\mathcal{R}_P$ vertices are to be covered, then $N_p = 2$ and hence $k_{\min} = 3$. Since there is a constant plant edge we cover only $\overline{\mathcal{R}_P^c}$ vertices and hence $N_p = 1$ and consequently $k_{\min} = 2$. The following theorem is the main result for the general case when there are cycles as well as constant plant edges.

Theorem 6.2: Let $G_{mc}(\mathcal{R}_P, \mathcal{C}, E_P)$ denote the graph of a controllable plant obtained after merging cycles. Suppose G_{mc} is connected and at least N_p paths are required to cover the $\overline{\mathcal{R}_P^c}$ vertices in G_{mc}. Let $\mathcal{C}_p \subseteq \mathcal{C}$ and $\mathcal{R}(p) \subseteq \mathcal{R}_P$ denote the set of vertices that are covered by the N_p paths and $e_{\text{mimo}} = |\mathcal{C}_p| - |\mathcal{R}(p)|$. Suppose n_t denotes the set of degree-one vertices in $\mathcal{C}_p$. Define e_p by

(i) $e_p := |n_t| - e_{\text{mimo}}$, if $e_{\text{mimo}} < N_p$ and
(ii) $e_p := e_{\text{mimo}}$, if $e_{\text{mimo}} \geqslant N_p$

Define γ through the sets $\mathcal{C}_m$ and A as follows:

$$\begin{aligned} \mathcal{C}_m &:= \{v \in \mathcal{C} \backslash \mathcal{C}_p \mid v \text{ is a merged vertex in } G_{mc}\}. \\ A &:= \{v \in n_t \mid v \text{ is not a merged vertex and} \\ &\qquad \text{dist}(v, v_1) = 2 \text{ for some } v_1 \in \mathcal{C}_m\}. \\ \gamma &:= |\mathcal{C}_m| - |A|. \end{aligned}$$

Then there exists a controller that generically achieves arbitrary pole placement and whose bipartite graph $G^k(\mathcal{R}_K, \mathcal{C}; E_K)$, with $|\mathcal{R}_K| = |\mathcal{C}| - |\mathcal{R}_P|$, has number of edges $|E_K| = e_p + |\gamma| + |\mathcal{R}_K|$, and moreover, bipartite graph of any controller that generically achieves arbitrary pole placement has at least $e_p + |\gamma| + |\mathcal{R}_K|$ edges.

Hence a minimal controller structure is proposed which achieves arbitrary pole placement in a generic sense.

VII. Extent of MIMO, e_{mimo}: Special Cases: Series Cascade (SISO), MISO, SIMO

In the previous section we stated the result for the minimum number of controller edges required for pole placement for a given plant structure. It is clear from Theorem 2.3 that for the minimum number, in addition to the number of paths that are to be completed to cycles, the index e_{mimo} also played an important role. The result was solely graph theoretical and provided less insight about the system. In this section we illustrate how the index e_{mimo} is suggestive of the extent of MIMO characteristics of the plant. We also bring out the significance of the number of paths.

A. Significance of e_{mimo} and N_p

We assume that the graph G_{mc} constructed for plant is connected. From Definition 5.1 it follows that this graph has only paths and no cycles. Let $d(v)$ denote the degree of incidence of a vertex v. There could be just one path or if there are more than one path, then the paths have common vertices and edges as the graph is assumed to be connected. The presence of more than one path in a component of G_{mc} implies that there are vertices in G_{mc} with $d(v) > 2$. We restrict our analysis to the following three cases.

1) $d(v) \leqslant 2$ for all $v \in \mathcal{R}_P \cup \mathcal{C}$.
2) $d(v) \leqslant 2$ for all $v \in \mathcal{C}$.
3) $d(v) \leqslant 2$ for all $v \in \mathcal{R}_P$.

Hence in the first case there is only one path in a component. It turns out that the above cases translate to the following conventional types of input-output structure.

1) Series cascade: SISO
2) MISO
3) SIMO

We explain this translation by considering three subsystems of the plant that are connected in each of the above cases as shown in Figures 3(a), 4(a) and 5(a). Assume each

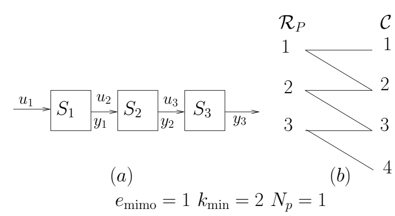

Fig. 3. Series cascade: SISO

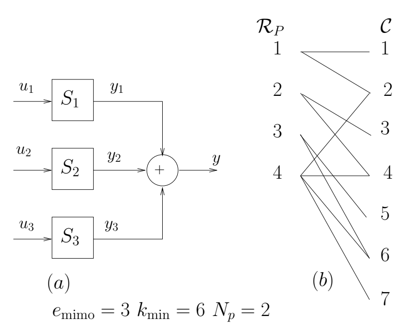

Fig. 4. MISO

subsystem S_i has transfer function $\frac{n_i(s)}{d_i(s)}$. Hence the differential equation for each S_i is $d(\frac{\mathrm{d}}{\mathrm{d}t})y_i = n(\frac{\mathrm{d}}{\mathrm{d}t})u_i$. Suppose $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is the kernel representation for the plant then the matrix P in each of the above cases is given below. The non-zero entries are denoted as $*$.

$$\underset{SISO}{\begin{bmatrix} * & * & & \\ & * & * & \\ & & * & * \end{bmatrix}} \quad \underset{MISO}{\begin{bmatrix} * & * & & & & & \\ & & * & * & & & \\ & & & & * & * & \\ & * & & * & & * & * \end{bmatrix}} \quad \underset{SIMO}{\begin{bmatrix} * & & & * \\ & * & & * \\ & & * & * \end{bmatrix}}$$

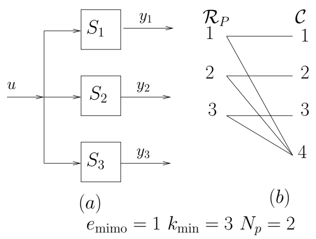

Fig. 5. SIMO

The bipartite graph constructed for each of these cases are given in Figures 3(b), 4(b) and 5(b) along with the values of e_{mimo}, N_p and minimum number of controller edges required, $k_{\min}$. Lemma 4.3 provides an upper and lower bounds for the index e_{mimo}. In the series cascade: SISO and SIMO case the lower bound of e_{mimo} is achieved. In the MISO case, the upper bound of e_{mimo}, i.e. min $(d, 2p-1) =$ min $(3, 2 \times 2 - 1) = 3$, is achieved. Hence we conclude that when a fixed number of subsystems are interconnected, the MISO input-output structure requires more controller edges for pole placement. In general several subsystems are interconnected in various combinations in a plant. A higher value of e_{mimo} suggests that in the given plant, majority of the interconnections have MISO type input-output structure. Similarly a lower value of e_{mimo} indicates the prominence of series cascade: SISO input-output structure in plant. In our example with three subsystems $k_{\min}$ for series cascade: SISO case is 2 and for MISO case is 6.

In the SIMO case $e_{\text{mimo}} = 1$; this is same as series cascade: SISO case. However, the number of controller edges required is 3 which is more than the SISO case. Here the role played by the number of paths N_p is evident: more controller edges are required for pole placement. This is summarized in the following theorem and elaborated in Remark 8.1.

Theorem 7.1: Assume a controllable plant with n subsystems interconnected with the input-output structure as series cascade:SISO, MISO and SIMO. Then in the bipartite graph associated with the plant, the value of e_{mimo} (extent of MIMO), the number of paths N_p and the minimum number of controller edges required for pole placement $k_{\min}$ for each case are given in the following table.

Table I: SISO, MISO, SIMO: key parameters

Type	e_{mimo}	N_p		$k_{\min}$
SISO	1	1		2
MISO	n	n even: $n/2$	n odd: $(n+1)/2$	$2n$
SIMO	1	n even: $n/2$	n odd: $(n+1)/2$	n

The proof of this is skipped as it can be verified directly from the graph associated with the plant. The series cascade: SISO case covers the series cascading of many subsystems. In case of parallel interconnection the merged-cycles graph, G_{mc} results in the SISO type again.

VIII. Concluding remarks

In this paper we considered the generic pole placement problem. The structural aspects of a given plant were incorporated in a bipartite graph which was used for all analysis of the problem. The main results of this paper was about proposing a minimal controller structure for a given plant such that arbitrary pole placement is achieved: Theorems 2.3, 5.4 and 6.2. An explicit expression was given for the minimum number of controller edges in the graph of the controller in terms of the number of paths and the index e_{mimo} of the plant graph. Achieving arbitrary pole placement is same as ensuring the polynomial matrix corresponding to the closed loop is square, nonsingular, and, in fact, unimodular. Thus we addressed the question of unimodular completion using the least number of nonzero entries in the completion.

Remark 8.1: For a given MIMO system, our main results crucially used two parameters: e_{mimo}, the 'extent of MIMO characteristics', and N_p the number of paths required to cover the vertices corresponding to plant equations. In this remark, we explain the significance of these parameters. Table I suggests that more paths N_p cause more number of nonzero entries $k_{\min}$ in the controller equations due to the requirement to 'feed back' more number of plant outputs or assign larger number of plant inputs. Of course, by Proposition 4.1, since every inadmissible plant edge is required to be in a cycle containing controller edges, more N_p clearly causes more $k_{\min}$. The role played by e_{mimo} is less obvious. As depicted for the special cases in Table I, the index e_{mimo} is higher if the plant is more under-determined, i.e. more number of controller equations are required in order to make the closed loop system autonomous. Of course, these arguments are applicable after the merging of cycles. In this sense, e_{mimo} gives an idea of the extent of Multi-Input-Multi-Output structure within a system.

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