Decentralized controllability of multi-inverter microgrids

Mahendra Singh, Rachel Kalpana Kalaimani, Madhu N. Belur and Shivkumar V. Iyer
Department of Electrical Engineering, Indian Institute of Technology Bombay,
Mumbai, India 400 076. Email: belur@ee.iitb.ac.in, Fax: +91 22 2572 3707

***Abstract*—Network of Voltage Source Inverters (VSI) to form a microgrid has played a central role in providing reliable power supply. It is very useful to control this network of VSI using decentralized controllers rather than centralized controllers due to less dependence of decentralized controllers on communication with other systems. A difficulty frequently encountered when studying and designing decentralized controllers is the controllability of the overall network using controllers with the constraint of being decentralized, i.e. the constraint that each controller can access only local sensors and only local actuators. This paper proves that the droop controller and its class of decentralized controllers indeed allows controllability of the system for *generically all* system parameter values. While this has been shown for *specific parameter values* using numerical methods, the graph theoretic approach followed in this paper helps in proving this generically for all system parameter values. We apply a recent result on arbitrary pole-placement by a controller with structural constraints to the case of a VSI connected to the grid, and to the case of two VSIs to show that indeed pole-placement can be achieved using decentralized controllers.**
Keywords: voltage source inverters, droop law, decentralized control, bipartite graphs, behavioral theory

I. Introduction

In this paper we prove that the single VSI connected to the grid with the VSI controlled by a controller imposing a dynamic equation between only the P-ω variables is able to achieve arbitrary pole-placement.

We next prove that when two VSIs are connected and the two inverters are controlled using local controllers of the kind where a dynamic equation is imposed only on the two local variables: active power P_i and the frequency ω_i is also able to achieve arbitrary pole-placement. Of course, this has been observed both experimentally and using simulations for only specific parameter values of the system. However, an important question is whether it is a structural property that such an interconnection procedure is able to achieve arbitrary pole-placement for (almost) all parameter values or if this is possible for just the specific values. We prove this property for 'generically' any value. More precisely, the set of values for which controllability is lost forms a 'thin set', i.e. a set of measure zero. In other words, unless parameter values are extremely carefully chosen and implemented with very high precision, the droop controller type of decentralized controller will always achieve pole placement. This is made precise within Section III.

The paper is organized as follows. The next section (Section II) contains the main results of this paper: that the decentralized droop controller is able to achieve arbitrary pole placement for the case of single VSI connected to the grid (Theorem 2) and for the case of a network of two VSIs (Theorem 3). Section III contains background about bipartite graphs and its links with structural controllability. Recent results (Proposition 7) about arbitrary pole placement with controller structural constraints are also reviewed in III: this result plays a key role in the proofs. Section IV explains about the inverter characteristics and the droop control law. Section V gives a mathematical model of the VSI. This section utilizes results developed in [6] in the context of the interconnection of a droop controller with a voltage source inverter. The proof of Theorem 2 for single VSI connected to the grid is in this section. Section VI contains the proof for the case of two VSIs. Conclusive remarks follow in Section VII. The rest of this section is devoted to the notation used in this paper.

Notation 1: $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ the set of complex numbers. $\mathbb{R}[s]$ is the set of polynomials in one indeterminate s over the field of real numbers. $\mathbb{R}^{m\times n}[s]$ represents the set of matrices of dimension $m \times n$ with polynomial entries. The determinant of a matrix R is represented as $\det(R)$. $\mathfrak{C}^{\infty}$ refers to the set of all infinitely differentiable functions.

II. Main results: generic decentralized controllability

This paper addresses the question about arbitrary pole placement using a decentralized droop controller when dealing with a network of VSIs: for the case of one and two VSIs. More precisely, consider a VSI connected to the grid, i.e. a system whose frequency is constant, and we intend to control the system variables in general, and frequency of the interconnected system in particular using a 'decentralized' controller: a controller that has access to just local variables. The well-known P-ω droop law is an example of such a decentralized controller. For example, can the decentralized controller achieve settling of the frequency of the various VSIs to the steady state frequency at a pre-specified rate of convergence? With just this intuitive idea of decentralized control and arbitrary pole-placement, we formulate the following results, the main results of this paper. The next section contains a precise formulation of the model of the VSI and a precise definition of what it means for a network of VSIs to be controllable by the decentralized droop controller. Theorem 2 states that a single VSI connected to the grid and controlled using a decentralized droop controller allows arbitrary pole placement, while Theorem 3 states the same

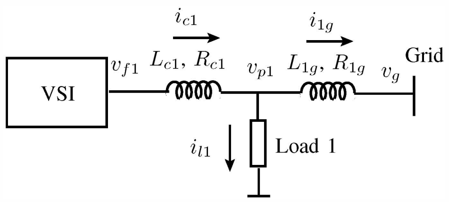

Fig. 1. Single VSI connected to the grid

for two VSIs. Both theorems are proved in the later sections after the required preliminaries. The variables ω, V_d, V_q, I_d, I_q stand respectively for the frequency and d,q components of the voltage v_f, current i_f at the output of an inverter: see Figures 1 and 2. Subscript 1 and 2 for these variables make them specific to Inverter 1 or Inverter 2; this is for the two VSI case. These variables are all local variables as far as the droop controller for each VSI is concerned.

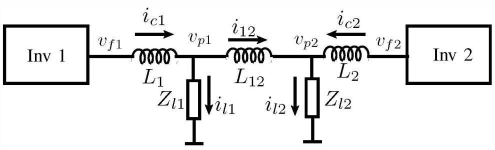

Fig. 2. Two-inverter-connected microgrid

Theorem 2: Consider the single Voltage Source Inverter (VSI) containing local load and connected to the grid. Suppose this VSI is controlled through a decentralized controller C that can access only the local variables: V_d, V_q, I_d, I_q and ω. Then arbitrary pole placement is achievable for the interconnected system, i.e. the VSI connected to the grid, using the decentralized controller C.

Theorem 3: Consider two Voltage Source Inverters (VSI) each containing local loads and connected to each other. Suppose each VSI is controlled through a decentralized controller that can access only the local variables: $V_{d,i}, V_{q,i}, I_{d,i}, I_{q,i}$ and ω_i for $i = 1, 2$. Then arbitrary pole placement is achievable for the network of two VSIs using this decentralized controller.

The above two results use the notion of arbitrary pole placement being achievable through a specific class of controllers, in particular, the decentralized controller. See Proposition 6 for the relation of arbitrary pole placement with controllability (using a specific class of controllers, in this case). We include a precise definition for easier reference.

Definition 4: Suppose variable w describes the equations of a to-be-controlled system and w is partitioned into (c, v) with c the variables accessible to any controller within a class C of controllers. We say arbitrary pole placement is achievable using a controller of class C if a set of controller equations described using just variables c is able to achieve arbitrary pole placement for the controlled system.

In view of the close relation between arbitrary pole placement and controllability, in particular, see Proposition 6, we use the two phrases interchangeably: 'arbitrary pole placement with a controller with specified structure' and 'controllable by a controller with specified structure'.

Of course, in this paper, we are concerned with the controller structure as 'local controller'. We deal with the overall system as comprising of subsystems with each subsystem having a pre-specified set of the local control variables. A decentralized controller is one that accesses just the local variables. Further, we ask this question in a 'generic sense'. This notion of genericity is made precise in in Definition 5.

III. Preliminaries

We briefly review preliminaries required for this paper. The next subsection reviews the behavioral approach to modeling and control of dynamical systems.

A. Behavioral approach

A linear behavior $\mathfrak{B}$ is defined as the subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ consisting of the solutions to a set of ordinary linear differential equations with constant coefficients; i.e.,

$$\mathfrak{B} := \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid P\left(\frac{d}{dt}\right) w = 0 \right\},$$

where $P(s)$ is a polynomial matrix having $\mathtt{w}$ number of columns: $P \in \mathbb{R}^{\bullet \times \mathtt{w}}[s]$. This representation is called a *kernel representation* of $\mathfrak{B}$. We call w as the *manifest variable*; these are the variables of interest. The behavior $\mathfrak{B}$ is called *controllable* if for any two trajectories $w_1, w_2 \in \mathfrak{B}$ there exists a $t_1 \geqslant 0$ and a trajectory $w \in \mathfrak{B}$ with the property

$$w(t) = \begin{cases} w_1(t) & t \leqslant 0 \\ w_2(t) & t \geqslant t_1 \end{cases} \tag{1}$$

$P(\frac{d}{dt})w = 0, P(s), \in \mathbb{R}^{\bullet \times \mathtt{w}}[s]$ is a kernel representation of a controllable behavior $\mathfrak{B}$ if and only if the rank of the matrix $P(\lambda)$ remains the same for all $\lambda \in \mathbb{C}$. We shall assume that the matrix $P(s)$ is of full row rank without loss of generality.

A behavior $\mathfrak{B}$ is called autonomous if one can conclude that $w_1 = w_2$ whenever $w_1, w_2 \in \mathfrak{B}$ satisfy $w_1(t) = w_2(t)$ for all $t \leqslant 0$. $P(\frac{d}{dt})w = 0, P(s) \in \mathbb{R}^{\bullet \times q}[s]$ is a kernel representation of an autonomous behavior $\mathfrak{B}$ if and only if P has full column rank, i.e. rank$(P) = q$. A detailed exposition of the behavioral approach can be found in [11].

Definition 5: **Generic property:** A property P in terms of variables $a_1, \ldots, a_N$ is said to be satisfied generically if the set of values $a_1, \ldots, a_N \in \mathbb{R}$ that do *not satisfy property P are a subset of a proper algebraic variety in $\mathbb{R}^N$.*

Hence a property which is true for almost all values is said to be true *generically*. For example any two non-zero polynomials $a(s)$ and $b(s)$ are generically coprime.

B. Pole placement

Let $R(\frac{d}{dt})w = 0$ be a minimal kernel representation of an autonomous behavior. The determinant of R, denoted by $\mathcal{X}$ is called the characteristic polynomial. The roots of $\mathcal{X}$ together with its multiplicities are called the poles of the behavior.

We define the pole placement problem as follows: let $P(\frac{d}{dt})w = 0$, $P[s] \in \mathbb{R}^{n \times m}$ denote the kernel representation of the plant. We have to find conditions under which there exists, for every monic $d \in \mathbb{R}[s]$, a controller behavior given by $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ such that the controlled behavior given by

$$\begin{bmatrix} P(\frac{\mathrm{d}}{\mathrm{d}t}) \\ K(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} w = 0$$

has the characteristic polynomial, $\chi = d$ and the interconnection is regular[1]. The solution to this problem is given in the following proposition by [13].

Proposition 6: Let the plant be given by $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. Given any monic $d \in \mathbb{R}[s]$, there exists a regular controller, $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ such that χ of the closed loop system is d if and only if the plant is controllable.

C. Bipartite graphs

A graph with vertex set V and edge set E is denoted as $G = (V, E)$. This G is bipartite if V can be partitioned into two sets $\mathcal{R}$ and $\mathcal{C}$ such that no two vertices from the same set are adjacent. Given any polynomial matrix $P[s] \in \mathbb{R}^{m\times n}[s]$, we can associate with it an edge weighted bipartite graph $G = (\mathcal{R}, \mathcal{C}, E)$ explained as follows. The set $\mathcal{R}$ and $\mathcal{C}$ denote the rows and columns of $P(s)$. An edge in the bipartite graph between vertex $v_i \in \mathcal{R}$ and $v_j \in \mathcal{C}$ exists if the $(i, j)th$ entry of the matrix is non-zero. The degree of the polynomial in the $(i, j)th$ entry is assigned to be the weight of the edge in the bipartite graph. Hence if the entry is a constant, then the weight of the edge is 0 but this does not mean that there is no edge. There is no edge only if the entry is 0.

A set of edges in a graph $G = (\mathcal{R}, \mathcal{C}; E)$ is called a matching if no two edges have a common end vertex. The graph G has a perfect matching if $|\mathcal{R}| = |\mathcal{C}|$ and the number of edges in the matching is $|\mathcal{R}|$. We discuss the relation between the determinants of a square polynomial matrix P and perfect matchings (as we have a square matrix) of the bipartite graph G associated to P. Let M be a perfect matching in G. Then M corresponds to a non-zero term in the determinant expansion of M. The determinant expansion of P is the sum over all perfect matchings in G (with suitable signs).

D. Structural controllability

Let $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ with $P \in \mathbb{R}^{n\times m}[s]$ being full row rank, denote the kernel representation of the plant. We construct the bipartite graph $G^P = (\mathcal{R}, \mathcal{C} : E)$ for the polynomial matrix $P(s)$. Since the vertex set $\mathcal{R}$ in the graph G^P corresponds to the plant we shall denote it by $\mathcal{R}_P$. Assume that the plant is controllable. Hence from Proposition 6 there exists a regularly implementable controller $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0, K \in \mathbb{R}^{(m-n)\times m}[s]$ which ensures that the controlled system has poles at the desired locations. The graph of the controller is denoted by G^K. The vertex set corresponding to the rows in G^K is represented as $\mathcal{R}_K$. The kernel representation of the closed loop, autonomous system is

$$A(\frac{\mathrm{d}}{\mathrm{d}t})w = 0, \text{where } A = \begin{bmatrix} P \\ K \end{bmatrix} \in \mathbb{R}^{m\times m}[s] \qquad (2)$$

[1]The interconnection is said to be regular ([13]) if rank of $\begin{bmatrix} P(s) \\ K(s) \end{bmatrix}$ is the sum of ranks of $P(s)$ and $K(s)$. Regularity of interconnection is equivalent to implementability of the controller in the feedback configuration.

Let G^A denote the bipartite graph constructed from the rows and columns of the polynomial matrix A. The vertex set $\mathcal{R}$ is divided into two sets $\mathcal{R}_P$ and $\mathcal{R}_K$ corresponding to the plant equations and controller equations respectively.

In the bipartite graph an edge which is incident on $\mathcal{R}_P$ is called a *plant edge* and an edge which is incident on $\mathcal{R}_K$ is called a *controller edge.* A plant edge with non-zero weight will be called as non-constant plant edge. In our problem we are given with a controller which has a specific structure. We use the following necessary and sufficient condition from [8] for arbitrary pole placement with the given controller structure.

Proposition 7: Let $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ with $P \in \mathbb{R}^{n\times m}[s]$ being full row rank, denote the plant. Let G^K, the graph of a controller K be given. Let $\mathfrak{L}$ denote the set of controllers with graph G^K. Consider the bipartite graph $G^A = (\mathcal{R}, \mathcal{C}; E)$ of the controlled system, constructed from the rows and columns of $\begin{bmatrix} P^T & K^T \end{bmatrix}^T$. Then the following are equivalent.

1) In G^A every non-constant plant edge is either part of some cycle involving controller edges or does not occur in any perfect matching.
2) Arbitrary pole placement is possible generically using the given controller with graph G^K.

IV. Inverter model and droop control law

This section reviews the model of an inverter that we use in this paper. We then briefly describe the droop controller.

A. Inverter model

Figure 3, has the topology of a standard Uninterrupted Power Supply (UPS). An inverter converts the dc output voltage Vdc across the dc capacitor to a three phase ac output voltage. Insulated Gate Bipolar Transistors (IGBTs) are used as a switching device. Switches S_1 to S_6 are IGBTs. The switches operate at a frequency of approximately 5 kHz. The inductor L_f and the capacitor C_f form a low pass $L - C$ filter that removes the high frequency switching harmonics generated by the inverter. The voltages v_{fa}, v_{fb}, v_{fc} across the filter capacitor C_f bank are the output voltages of the inverter while the currents i_{ca}, i_{cb}, i_{cc} are the output currents of the UPS.

In the circuit in Figure 3, the ac part is a three phase circuit. The three phase variables are output voltage v_f, output current i_c and the current through the inductor L_f which is i_f. Each variable consists of three components a, b and c. Example for v_f the three components are v_{fa}, v_{fb} and v_{fc}. We use Clarke's transformation to transform the three phase variables so as to obtain the complex variables. For example, the transformed output current is $i_c = i_{cd} + ji_{cq}$. Subscripts 'd', 'q' refer to the real and imaginary components respectively.

B. Droop control law

In distributed electric power systems, a droop control strategy is popular. In the droop control strategy [7], [2]–[5], [10], [12],the frequency ω and the magnitude $\mathbf{V}$ of the UPS output voltages are varied as follows. The frequency ω is varied with respect to the active power p supplied by the UPS while the magnitude $\mathbf{V}$ is varied with respect to the reactive power q

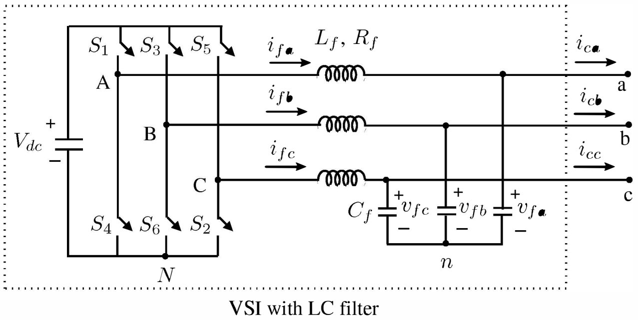

Fig. 3. Topology of the Uninterrupted Power Supply (UPS)

supplied. Mathematically, the droop controller equations for inverter m are written as follows [7], [2]–[5], [10], [12]:

$$\omega_m = s\delta_m = \omega_0 - k_{pm}p_m \tag{3}$$

$$\mathbf{V}_m = \mathbf{V}_0 - k_{qm}q_m \tag{4}$$

where ω_0, $\mathbf{V}_0$ are the nominal values of angular frequency and voltage magnitude, respectively. p_m, q_m are the active and reactive power supplied by inverter, respectively, while k_{pm}, k_{qm} are the droop coefficients. The first equation is called the p-ω droop control law and the second equation called q-V droop control law.

V. Proof of Controllability for Grid-VSI Network

In this section we review the mathematical model of a single UPS and construct the bipartite graph corresponding to the system of equations of a VSI connected to the grid. The equations we use are those that have been derived in [6] for a linearized model: they were validated against experimental and simulation results. We then apply Proposition 7 above to infer that the decentralized droop controller indeed achieves arbitrary pole placement.

A. Mathematical model: UPS

This section consists of mathematical model of a line interactive UPS (in Figure 3) connected to a three phase ac grid, system equations of controlled system, kernel representation of the system as shown in Figure 1. In the previous section, the topology of the UPS has been described. In the Figure 1, a line interactive UPS is connected to a three phase ac grid. The UPS has a local load that is connected to a ac grid through a cable which has inductance L_1 and resistance R_{1g}. R_{1g} is the parasitic resistance of this cable. The load of the UPS is normally supplied by ac grid but if ac grid fails, then load is supplied by UPS. The UPS is interfaced to the grid and load by a inductance L_{c1}. v_g is the three phase ac grid voltage.

We review the model of the to-be-controlled system, the plant. The complete derivation is given in the reference [6], [7]. The mathematical model of the microgrid is written using the system variable $\mathbf{w}$.

$$\mathbf{w} = \begin{bmatrix} v_{f1} & i_{c1} & v_{p1} & i_{l1} & u_1 & v_g & i_{1g} \end{bmatrix} \tag{5}$$

where v_{f1}, i_{c1}, v_{p1} and i_{l1} are the three phase variables in d-q complex form. We have $v_{f1} := v_{f1d} + v_{f1q}$, $i_{c1} := i_{c1d} + i_{c1q}, v_{p1} := v_{p1d} + v_{p1q}$, and $i_{l1} := i_{l1d} + i_{l1q}$. The variable $u_1 = \delta_1 - jV_1$ is also local to the UPS, where δ_1 is defined as the phase angle of complex variable v_{f1} while V_1 is defined as the magnitude [3], [7], *i.e.*

$$\delta_1 = \tan^{-1}(\frac{v_{f1q}}{v_{f1d}}) \quad \text{and} \quad V_1 = \sqrt{\frac{v_{f1d}^2 + v_{f1q}^2}{V_{f1d}^2 + V_{f1q}^2}} \tag{6}$$

where V_{f1d} and V_{f1q} are the values of v_{f1d} and v_{f1q} at the equilibrium point about which the equations are linearized. δ_1 and V_1 are the inputs to the system. The current flowing in the cable between UPS to grid is $i_{1g} := i_{1gd} + ji_{1gq}$ and the voltage of ac grid is $v_g := v_{gd} + jv_{gq}$. The grid voltage v_g is external input to the system because the grid and the UPS are two independent entities. The input-output partitioning of the manifest variable $\mathbf{w}$ makes this clear.

The variables in w defines the behavior of the system in (Figure 1). It is clear from (6) that the relations of δ_1 and V_1 are non-linear. We use the standard network laws such as Kirchoff's Voltage Law and Kirchoff's Current Law to obtain the other variables of $\mathbf{w}$ and hence are related by linear differential equations. Therefore we linearize the entire system about an equilibrium point so that we can define the behavior of the system as the solution set of equations and express all components of the vector $\mathbf{w}$ as deviations about the equilibrium point. The symbol Δ denotes the deviation of the variables from the equilibrium point. The system equations, after careful manipulations as elaborated in [6], are described in terms of the variables: $\Delta v_{f1}, \Delta i_{c1}, \Delta v_{p1}, \Delta i_{l1}$, $\Delta u_1, \Delta v_g$ and Δi_{1g}. These are the deviations of the concerned variables in Figure 1.

The behavior of the system can be written as

$$\mathfrak{B} = \left\{ \begin{bmatrix} \Delta\mathbf{w}_1 \\ \Delta\mathbf{w}_2 \end{bmatrix} \in C^\infty(\mathbb{R}, \mathbb{C}^n) | \mathbf{A}_{\textbf{total}} \left(\frac{d}{dt}\right) \begin{bmatrix} \Delta\mathbf{w}_1 \\ \Delta\mathbf{w}_2 \end{bmatrix} = 0 \right\} \tag{7}$$

The equations describing the controlled system after careful manipulation as elaborated in [7] turn out to be:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & a_1 & a_2 \\ 0 & -1 & 0 & 0 & a_2 & -a_1 \\ 0 & 0 & -1 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & -1 & b_2 & -b_1 \\ -m_1\frac{I_{c1d}}{s} & -m_1\frac{I_{c1q}}{s} & -m_1\frac{V_{f1d}}{s} & -m_1\frac{I_{f1q}}{s} & -1 & 0 \\ n_1 I_{c1q} & -n_1 I_{c1d} & -n_1 V_{f1q} & n_1 V_{f1d} & 0 & -1 \end{bmatrix} \begin{bmatrix} \Delta v_{f1d} \\ \Delta v_{f1q} \\ \Delta i_{c1d} \\ \Delta i_{c1q} \\ \Delta\delta_1 \\ \Delta V_1 \end{bmatrix} = 0 \tag{8}$$

where $\Delta\delta_1$ is a deviation in angle. Note that due to the P-ω droop law, $\Delta\delta_1$ is an integral function of the variables Δv_{f1d}, Δv_{f1q} and Δi_{c1d}, Δi_{c1d} while ΔV_1 depends linearly on these variables.

We use the above system of equations (8) to construct a bipartite graph as shown Figure 4. The matrix $\mathbf{K}$ in the kernel representation for the controller has ten non-zero elements. Hence there will be ten edges from R_K to $\mathcal{C}$. In the plant equations, the matrix $\mathbf{P}$ has twelve non-zero elements among which four elements have degree zero. The bipartite graph for the plant and the controller is given in Figure 4.

From Proposition 7, we have that the controllability of the plant with respect to the above controller is ensured if the following conditions are satisfied. First there should exist a

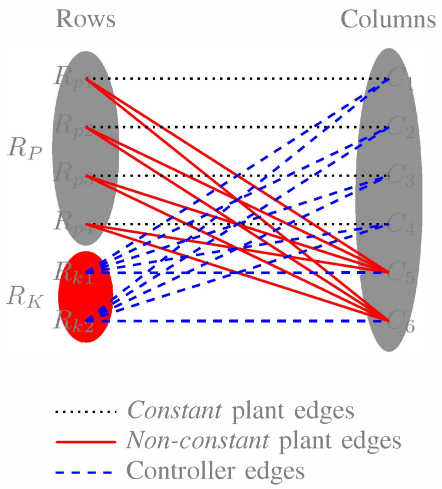

Fig. 4. Bipartite graph for single VSI connected to the grid

perfect matching, i.e. a one-to-one correspondence between vertices on the left and right. Then check for the following condition on all plant edges that correspond to non-constants (the thick, red edges in Figure 4). Amongst these plant edges, every edge either occurs in a cycle involving the controller edges (thick, blue, dashed in the figure) or does not occur in any perfect matching. We proceed to check this condition: this would complete the proof of Theorem 2.

In Figure 4 there are eight non-constant plant edges shown by red dashed lines which are emerging from plant row vertices R_{p1}, R_{p2}, R_{p3}, R_{p4} to column vertices C_5 and C_6. These plant edges are non-constant edges and are part of cycle with controller edges. Therefore from Proposition 7, we have arbitrary pole placement. Here the eight non-constant plant edges are in one or more of eight cycles involving two or more of the ten controller edges. These cycles are as follows: Cycles starting with controller vertex R_{k1} are as follows:

1) $R_{k1} - C_5 - R_{p1} - C_1 - R_{k1}$.
2) $R_{k1} - C_5 - R_{p2} - C_2 - R_{k1}$.
3) $R_{k1} - C_5 - R_{p3} - C_3 - R_{k1}$.
4) $R_{k1} - C_5 - R_{p4} - C_4 - R_{k1}$.

and cycles starting with controller vertex R_{k2} are as follows:

1) $R_{k2} - C_6 - R_{p1} - C_1 - R_{k2}$.
2) $R_{k2} - C_6 - R_{p2} - C_2 - R_{k2}$.
3) $R_{k2} - C_6 - R_{p3} - C_3 - R_{k2}$.
4) $R_{k2} - C_6 - R_{p4} - C_4 - R_{k2}$.

In all eight cycles, at least one controller edge is a part of cycle with one non-zero weighted plant edge. Therefore the condition in Proposition 7 is satisfied. Hence the system has arbitrary pole placement.

VI. Proof of Controllability for Two VSI Network

In [6], from the topology of three-inverter-ring-connected microgrid the derivation for the mathematical model of that system has been explained. Here we do the derivation of two-inverter-connected system in a similar manner. In Figure 2, inverters are denoted by “Inv 1”, “Inv 2” and they have local loads Z_{l1}, Z_{l2}. These loads are linear passive loads consisting of resistances R_{l1}, R_{l2} and inductances L_{l1}, L_{l2} and the cable connecting the two inverters has inductance L_{12}.

A. Mathematical Model of the system

Inorder to reduce the complexity we consider the system as two subsystems. Similar to the single inverter case, we use Clarke's transformation on the three phase variables to transform them into complex variables.

Now we explain how the complex variables of the micro grid are grouped. The vector containing all the variables of micro grid is

$$\mathbf{x} = \begin{bmatrix}\mathbf{u} & \mathbf{x}_v & \mathbf{x}_c & \mathbf{x}_l & \mathbf{x}_{int}\end{bmatrix} \tag{9}$$

where the variables are further defined as: $\mathbf{u} := (u_1, u_2)$, $\mathbf{x}_v := (v_{f1}, v_{f2})$, $\mathbf{x}_c := (i_{c1}, i_{c2})$, $\mathbf{x}_l := (i_{l1}, i_{l2})$ and $\mathbf{x}_{int} := i_{12}$. The variables in above equations are complex, *e.g.*, $v_{f1} = v_{f1d} + jv_{f1d}$. The complex variable $u_1 = \delta_1 - jV_1$. u_2 follows similarly.

The next set of equations is formed by the load laws at the two inverters. For example, load law at Inverter 1 is written in the small signal sense as follows:

$$\Delta v_{f1} - Z_{l1}\Delta i_{l1} = 0.$$

where $Z_{l1} = R_{l1} + j\omega L_{l1}$. The load laws at each of the two inverters, the KCL and KVL at the nodes and around loops together give the following system of equations for the controlled system, i.e. the plant and controller equations:

$$\begin{bmatrix} -\mathbf{A}_u^r & \mathbf{A}_u^i & \mathbf{I} & 0 & 0 & 0 \\ -\mathbf{A}_u^i & -\mathbf{A}_u^r & 0 & \mathbf{I} & 0 & 0 \\ -\mathbf{A}_{imp}^r & \mathbf{A}_{imp}^i & 0 & 0 & \mathbf{I} & 0 \\ -\mathbf{A}_{imp}^i & -\mathbf{A}_u^r & 0 & 0 & 0 & \mathbf{I} \\ s\mathbf{I} & 0 & \mathbf{C}_{vd} & \mathbf{C}_{vq} & \mathbf{C}_{id} & \mathbf{C}_{iq} \\ 0 & \mathbf{I} & \mathbf{D}_{vd} & \mathbf{D}_{vq} & \mathbf{D}_{id} & \mathbf{D}_{iq} \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta\mathbf{V} \\ \Delta\mathbf{x}_{vd} \\ \Delta\mathbf{x}_{vq} \\ \Delta\mathbf{x}_{cd} \\ \Delta\mathbf{x}_{cq} \end{bmatrix} = 0 \tag{10}$$

where $\mathbf{A}_{imp} = \mathbf{A}_l - \mathbf{A}_{int}\mathbf{A}_z$, matrices $\mathbf{A}_{imp}^r$ and $\mathbf{A}_{imp}^i$ are the real and imaginary components of $\mathbf{A}_{imp}$, $\mathbf{A}_u^r$ and $\mathbf{A}_u^i$ are the real and imaginary components of $\mathbf{A}_u$, and the vectors are defined as: $\delta := [\delta_1 \;\; \delta_2]$, $\mathbf{V} := -[\mathbf{V}_1 \;\; \mathbf{V}_2]$, $\mathbf{x}_{vd} := [v_{f1d} \;\; v_{f2d}]$, $\mathbf{x}_{vq} := [v_{f1q} \;\; v_{f2q}]$, $\mathbf{x}_{cd} := [i_{c1d} \;\; v_{c2d}]$ and $\mathbf{x}_{cq} := [i_{c1d} \;\; i_{c2q}]$. Further, the constant matrices $\mathbf{C}_{vd}, \mathbf{C}_{vq}, \mathbf{C}_{id}, \mathbf{C}_{iq}\mathbf{D}_{vd}, \mathbf{D}_{vq}, \mathbf{D}_{id}, \mathbf{D}_{iq}$ are each 2×2 diagonal matrices that contain the droop values and the voltage/current d-q values at the operating point. The above system (10) has twelve real variables, and the matrix in Equation (10) is of size 12×12. The first eight equations represent the plant equations and the last four represent the controller equations.

B. Bipartite graph

Here again we construct an edge weighted bipartite graph from the kernel representation of the system in (10). The polynomial matrix from which the bipartite graph is constructed

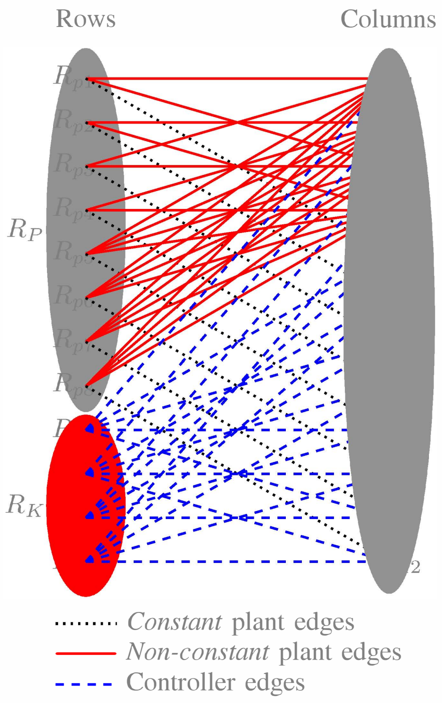

Fig. 5. Bipartite graph for network of two VSI system

for the plant and controller which are P and K respectively are given below.

$$\mathbf{P} = \begin{bmatrix} * & 0 & * & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

$$\mathbf{K} = \begin{bmatrix} * & 0 & 0 & 0 & * & 0 & * & 0 & * & 0 & * & 0 \\ 0 & * & 0 & 0 & 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 1 & 0 & * & 0 & * & 0 & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & 0 & * & 0 & * & 0 & * & 0 & * \end{bmatrix} \tag{12}$$

where * denotes the entries that are non-constant. The matrix $\mathbf{K}$ has twenty non-zero elements. Hence there will be twenty edges from R_K to $\mathcal{C}$. $\mathbf{P}$ has thirty two non-zero elements among which eight elements have degree zero. The bipartite graph for the plant and the controller is given in Figure 5. We are to verify that in addition to existence of a perfect matching, each non-constant plant edge is either in a cycle involving controller edges or in none of the perfect matchings. This condition can be inspected visually for all the thick-red edges in Figure 5. For example: C_1-R_{p1}-C_3-R_{p3}-C_7-R_{k1}-C_1 is a cycle that contains a few plant thick-red edges and also contains controller edges (dashed blue). Similarly, every thick-red plant edge needs to be contained in some such cycle involving controller edges too, unless such a thick-red plant edge occurs in none of the perfect matchings. After the straightforward verification, one deduces that the two-VSI network is also controllable using the decentralized droop controller.

VII. Conclusion and Future work

In this paper we proved that decentralized controllers of the form of the droop controllers can indeed achieve pole-placement of the microgrid. Of course, droop controllers are special in the sense that they impose a *static* law between the variables, unlike a dynamic law that, depending on the location of the specified poles, might have to be imposed in order to achieve these poles. This procedure for proving that generically the droop controller can achieve arbitrary pole placement can be easily extended to a network with any number of inverters by systematically constructing the bipartite graph and verifying that every plant edge satisfies the conditions listed in Proposition 7. We emphasize that while controllability using the droop controller has been observed for *specific* system parameter values, we have *proved* this to be the case for almost all system parameter values, and almost all operating conditions. Moreover, this check was performed without recourse to extensive computation.

References

[1] M.N. Belur and S. Sivasubramanian, Structural controllability and generic Smith normal form, *http://arxiv.org/pdf/1006.5207*, 2010.

[2] M.C. Chandorkar, D.M. Divan, and R. Adapa, Control of parallel connected inverters in standalone AC supply systems, *IEEE Transactions on Industry Applications*, vol. 29, pp. 136-143, 1993.

[3] E.A.A. Coelho, P.C. Cortizo, and P.F.D. Garcia, Small-signal stability for parallel-connected inverters in stand-alone AC supply systems, *IEEE Transactions on Industry Applications*, vol. 38, pp. 533-542, 2002.

[4] J.M. Guerrero, J. Matas, L.G. de Vicuna, M. Castilla, and J. Miret, Wireless-control strategy for parallel operation of distributed-generation inverters, *IEEE Transactions on Industrial Electronics*, vol. 53, pp. 1461-1470, 2006.

[5] J.M. Guerrero, L. Hang, and J. Uceda, Control of distributed uninterruptible power supply systems, *IEEE Transactions on Industrial Electronics*, vol. 55, pp. 2845-2859, 2008.

[6] S.V. Iyer, M.N. Belur, and M.C. Chandorkar, A generalized computational method to determine stability of a multi-inverter microgrid, *IEEE Transactions on Power Electronics*, vol. 25, pp. 2420-2432, 2010.

[7] S.V. Iyer, M.N. Belur, and M.C. Chandorkar, Decentralized control of a line interactive uninterruptible power supply (UPS), *Proceedings of the IEEE American Control Conference (ACC)*, vol. 30, pp. 3293-3298, 2010.

[8] R.K. Kalaimani and M.N. Belur, Generic pole assignability of DAE systems with controller structural constraints, *Proceedings of the International Conference on Advances in Control and Optimization of Dynamical Systems (ACODS)*, Bangalore, India, 2011.

[9] C.-T. Lin, Structural controllability, *IEEE Transactions on Automatic Control*, vol. 19, pp. 201-208, 1974.

[10] N. Pogaku, M. Prodanovic, and T.C. Green, Modeling, analysis and testing of autonomous operation of an inverter-based microgrid, *IEEE Transactions on Power Electronics*, vol. 22, pp. 613-625, 2007.

[11] J.W. Polderman and J.C. Willems, *Introduction to Mathematical Systems Theory: a Behavioral Approach*, Springer-Verlag, New York, 1998.

[12] C.K. Sao and P.W. Lehn, Autonomous load sharing of voltage source converters, *IEEE Transactions on Power Delivery*, vol. 20, pp. 1009-1016, 2005.

[13] J.C. Willems, On interconnections, control and feedback, *IEEE Transactions on Automatic Control*, vol. 42, pp. 326-339, 1997.

[14] J. van der Woude and K. Murota, Disturbance decoupling with pole placement for structured systems: a graph theoretic approach, *SIAM Journal of Matrix Analysis & Applications*, vol. 16, pp. 922-942, 1995.