On the Stability and Formations in Ad Hoc Multilane Vehicular Traffic

Rakesh U. Chavan, Madhu Belur, Debraj Chakraborty, and D. Manjunath
Dept. of Elecl. Engg., IIT Bombay

Abstract— **We develop a new model for traffic with on roads with multiple lanes but where the vehicles do not quite adhere to a lane discipline. To study the system the dynamics is split along two independent directions—the *y*-axis representing the direction of motion and and the *x*-axis representhe the lateral or the direction perpendicular to the direction of motion. Different influence graphs are used to model the interaction between the vehicles in these two directions. The model allows more than one 'layer' ahead to influence the dynamics of a car thus allowing for mutliple cars in its visible range to affect its motion. The stability of the dynamical system models are analyzed. Conditions are provided under which all cars can converge to constant spacing. The spacing will refer to distance among cars in two levels along the direction of motion and between the cars in the same level.**
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I. Introduction

Those of us that drive in Indian cities, we participate in what we may call 'functioning anarchy' (apologies to J K Galbraith), and willy nilly even precipitate some of the chaos that we see. Every one has favourite jibe and diatribe about our traffic. That long term and short term mechanisms to reduce travel times and associated uncertainties has been recognised for a while now and several attempts are indeed being made. Another important need is to understand the causes of the traffic behavior that we see and to possibly understand the consequent economics—rationale for such behavior, societal costs, etc. This paper is a step in that direction. In this paper, we develop a stylised model to represent the interaction among the vehicles on the road and use this to characterize the emerging macroscopic behaviour.

The basis for our model comes from the single-lane car-following models that have been studied for a while now, e.g., [1], [2], [3]. The basic premise of these models is that the cars follow each other in a single file and the acceleration strives to maintain a constant spacing between the cars. A dynamical system model for such a system is then analysed for stability. Multilane models are essentially extensions of the these single lane models with different extensions modeling the various interactions that are now possible. For example, in the multi-laned model of [11], multiple vehicular interactions in a single lane where used. In a completely approach, in [7] information from assistance systems are used and the model is tested with real driver behaviours. In this paper we also consider a multilane system except that we assume that there is no strict demarcation of lanes, i.e., wide roads on which multiple cars can drive abreast.

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Our multilane model is adapted from the single lane model of [4]. A directed graph models the influence. We too seek an equilibrium analysis of a dynamical system model that we develop. Our notion of stability refers to the condition that all cars attain the same velocity as the leader. Since this a car following model, we also have the notion of levels and the system will also seek the condition that cars in one 'level' maintain a fixed spacing from cars in the next 'level.' Our analysis will primarily use the Laplacian of the directed graph that models the influences which in turn will allow us to dissolve the lanes, so to say. The influence graph can also be a weighted graph to enable us to model the relative degrees of the influences.

The rest of the paper is organized as follows. We begin by introducing some premliminaries related to graph Laplacians. We then cover the assumptions in Section III. In Section IV we explain the bifurcation of the system along the two possible directions of motion. We discuss the control laws and dynamics along the direction of travel in Section IV-A. We then analyze the system along horizontal direction in Section IV-B. In Section V we provide a numerical results for our model and the behaviour of the system. Finally, we discuss some of the possible extensions of the model and its analysis.

II. Graph related preliminaries

Consider a graph $\mathscr{G} = (\mathrm{V}, \mathscr{E})$ with node set V and edge set $\mathscr{E}$. Each node in the graph represents a vehicle. The directed edges are introduced as follows: If vehicle j can be sensed by vehicle i ($i \neq j$), then we introduce an edge from j to i and denote it by $j \to i$. We define the Laplacian (L) for a directed graph with unit weights as follows: $\ell_{ij} := -1 \quad \text{if} \quad j \to i \quad \text{and} \quad \ell_{ii} := \text{indegree}$. As an example consider the Laplacian of the graph in Figure 1.

$$L := \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

For weighted directed graphs $\mathscr{G} = (\mathrm{V}, \mathscr{E}, w)$ is a directed graph $\mathscr{G} = (\mathrm{V}, \mathscr{E})$ along with a function $w : \mathscr{E} \to \mathbb{R}^+$ where $\mathbb{R}^+$ denotes the set of positive real numbers. Assigning weights w_{01}, w_{02}, w_{13}, and w_{23} to edges $0 \to 1$, $0 \to 2$, $1 \to 3$, and $2 \to 3$ in Figure 1. The Laplacian for the weighted

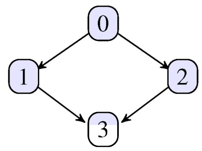


Fig. 1: Graph with 4 nodes

graph is as follows:

$$L := \begin{bmatrix} 0 & 0 & 0 & 0 \\ -w_{01} & w_{01} & 0 & 0 \\ -w_{02} & 0 & w_{02} & 0 \\ 0 & -w_{13} & -w_{23} & w_{13}+w_{23} \end{bmatrix}$$

In the context of directed graphs we will define the neighbour set ($\mathcal{N}$) as all cars directly visible to the car in consideration. For example, in Figure 1 $\mathcal{N}(3) = 1,2$.

III. Assumptions

We make some assumptions regarding the capabilities of the system. Some of the assumptions are made for simplicity of exposition.

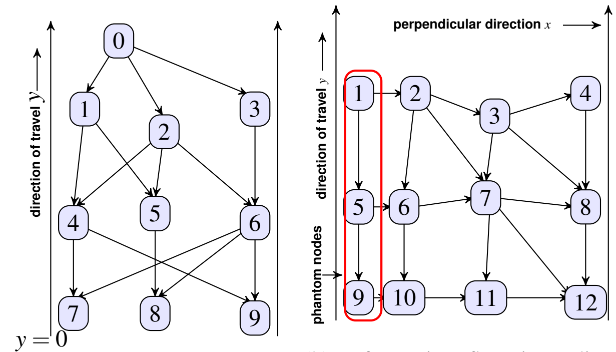
1) We assume that all cars (nodes) have the same dynamics and control laws.
2) The maximum number of neighbours will depend on the visibility of cars. We assume identical drivers, so the maximum number of visible neighbours is same for all cars. We are allowing only directly visible cars to influence the car in consideration: more precisely car 5 in Figure 2a can see cars 1, 2, 3 but not car 0.
3) We fix certain number of cars (cars $1,2$, and 3 in figure 2a) as leaders. These cars together are occupying the whole road width. Though they have been termed as leaders we assume these cars follow a 'phantom leader' as shown in Figure 2a for motion in the direction of travel.
4) We have two different connected graphs for determining the control laws governing motion along the x and y axis. We consider y axis as motion in the direction of travel and x axis as perpendicular to the direction of travel. We assume the laws to be independent of each other along the axes.

IV. Multilane traffic

The motion of cars will take place along two directions. There will be two separate graphs, one for control along the y axis and the other for motion along the x axis. We first analyze the system in the y direction and then we move on to analyze the x direction motion.

A. Y axis motion

Consider a stream of cars as shown in Figure 2a. Node zero is the phantom leader and is moving at a constant equilibrium speed v_0. All cars have positive y coordinates. The arrows indicate the flow of information amongst the cars. We refer to car 0 as level 0, cars $1,2$, and 3 as level one and so on.



(a) Information flow in y axis

(b) Information flow in x direction with 3 cars along the road

Fig. 2: Graphs for information flow

Let the acceleration, velocity and position for car i be represented as a_{yi}, v_{yi} and y_i. Note that car 9 will have the lowest y coordinate and car 0 will have the highest one.

We determine the acceleration of car $i+1$ as follows:

$$a_{y(i+1)} := \sum_{j=\mathcal{N}(i+1)} b(v_{yj} - v_{y(i+1)}) + k(y_j - y_{i+1} - g_y)$$

where $\mathcal{N}(i+1)$ represents the neighbour set of car $i+1$, b and k are constants (loosely speaking they play the role of damping and spring constant), and g_y is the parameter used to determine the required equilibrium distance.

These laws can be represented compactly for $n+1$ cars using the Laplacian of the graph as follows:

$$\begin{bmatrix} \dot{y} \\ \dot{v_y} \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{I} \\ -k\ L & -b\ L \end{bmatrix} \begin{bmatrix} y \\ v_y \end{bmatrix} - k g_y \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \qquad (1)$$

where $y, \dot{y}, v_y, \dot{v_y} \in \mathbb{R}^{n+1}$, $\mathbf{1} \in \mathbb{R}^{n}$, $\mathbf{0} \in \mathbb{R}^{n+5}$.

Below is our first main result. The above control law guarantees stability and also fetches the desired spacing between levels.

Theorem 4.1: Consider a directed graph with unit weight for all communication links. Suppose each node (except for the leader) has the same indegree. Then the control law (1) achieves a stable equilibrium point with any positive desired constant spacing.

Theorem 4.2: Consider a weighted directed graph such that the total weight across all incoming edges is the same for each node. Then the control law (1) achieves a stable equilibrium point with any positive desired constant spacing.

Corollary 4.3: The Laplacian of the weighted directed graph has the following properties:

1) The row corresponding to the phantom leader has all zeros.
2) Each car senses only those cars in the level ahead of it in y direction. This ensures the absence of cycles in the graph and the matrix has a lower triangular structure.
3) L has non-negative eigenvalues and eigenvalue at 0 is simple.

4) Under assumptions of Theorem 4.2 and 4.1, the diagonal entries will be same for all rows excluding rows 0 to 3.
5) The left eigenvector corresponding to eigenvalue 0 will be of the form $\begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}$ where 1 corresponds to car 0 (leader car).
6) The Laplacian nullspace consists of the vector $[1\ 1\ \dots\ 1]$.

The proof of these properties are straightforward and hence skipped.

Remark 4.4: The computations require only information of the graph. The weights can then be choosen by the cars individualy, thus making the computation local to each car. In this sense the control law is distributed.

With these developements we now proceed to prove our main results.

Proof of Theorem 4.1: Representing the states of the control law (1) as $\tilde{y}_i = y_i - g_y$. The stability part of the theorem directly follows from [9, Lemma 4.1 and Theorem 4.1]. From Corollary 4.3.5 velocities of all cars converge to leader's velocity.

The spacing can be obtained by observing the difference system at equilibrium. Note that the indegree of all cars is the same. For ease of exposition we will provide the proof for an indegree of 2 using Figure 2a. At equilibrium, the terms corresponding to spacing in the control law can be represented as follows:

$$(-1)\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \end{bmatrix} = \begin{bmatrix} g_y - y_1 - y_2 \\ g_y - y_1 - y_2 \\ g_y - y_2 - y_3 \\ g_y \\ g_y \\ g_y \end{bmatrix}$$

Equations corresponding to nodes $7,8,9$ are obtained from the graph structure. Cars $1,2,3$ are assumed to be leaders and having the same y coordinate. From the above equations it can be seen that $y_i - y_j = g_y/2$ where i,j belong to consecutive levels with $i > j$. The 2 in the denominator correponds to the indegree of the graph.

Here the indegree of the node is more important rather than the actual connections. These conditions can be easily extended to nodes having an indegree of M and the spacing corresponding to such graphs would be g_y/M between consecutive levels.

Also, all shifted solutions will satisfy these conditions from Corollary 4.3.6.

□

However this theorem will only work when the indegree of each node is same. We relax this condition by setting weights along the communication links. This is achieved in Theorem 4.2.

Proof of Theorem 4.2: Stability part of this theorem is same as Theorem 4.1. We proceed to show the spacing details for cars in different levels. Though the indegree could be different now, the net weight being same plays the required role here: we analyze the terms corresponding to spacing in the control law at equilibrium as before.

Consider Figure 2a. In order to have different indegree's: add an extra edge connecting cars 6 to 1 with weight w_{61}. The Laplacian for this modified weighted graph is as follows:

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -w_1 & w_1 & 0 & 0 & 0 & 0 & 0 \\ -w_1 & 0 & w_1 & 0 & 0 & 0 & 0 \\ -w_1 & 0 & 0 & w_1 & 0 & 0 & 0 \\ 0 & -w_{41} & -w_{42} & 0 & w_{44} & 0 & 0 \\ 0 & -w_{51} & -w_{52} & 0 & 0 & w_{55} & 0 \\ 0 & -w_{61} & -w_{62} & -w_{63} & 0 & 0 & w_{66} \end{bmatrix}$$

where w_{ij} represents the weight along link $j \to i\ \forall\ i \neq j$, $w_{ii} = \sum_{j=1}^{n} w_{ij}$. Writing the equations for car 5 and car 6 we get:

$$y_5 = y_1 - g_y\left(\frac{1}{w_{51}+w_{52}}\right),$$
$$y_6 = y_1 - g_y\left(\frac{1}{w_{61}+w_{62}+w_{63}}\right).$$

However, $w_{55} = w_{66}$ by assumption. Hence the spacing for car 5 and car 6 from the cars in the above level will remain same. This same argument can be extended for any distribution of weights along the edges as long as the net weight remains the same for all nodes. This completes the proof of the theorem. □

We now turn our attention to the motion along x axis or the horizontal movement of cars.

B. X axis motion

Our control law and the assumptions on information flow graph have to be modified slightly for representing this situation. The cars are assumed to acquire horizontal velocity either when they want to overtake a specific car or when they want to converge to the desired distance. Once the target distance is reached the cars will no longer move in the x direction.

Let the boundary of the road be represented by phantom cars. In Figure 2b, cars $1,5$ and 9 represent phantom cars. They will have the same x coordinates. They will not take any input from other cars but cars in immediate vicinity will try to maintain a constant distance from them. Furthermore, we need to impose constraints on cars $2,3,4$ in the first level to ensure that a specific spacing is maintained for the remaining cars.

Here, we consider only the case in which the weights are different for edges but the net weight for all the cars is same. Though the more general case can be handled too, we assume here that the information flow between cars in the same level takes place from left to right (away from the phantom cars). This assumption is similar to the assumption of information flow along the y axis.

Let the acceleration, velocity and position for car i be represented as a_{xi}, v_{xi} and x_i. Let the x coordinate vary from left to right. Car 1 will have the lowest x coordinate and car

4 will have the highest one in the same level. This pattern repeats for all levels.

The control law used here is:

$$\begin{bmatrix} \dot{x} \\ \dot{v}_x \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{I} \\ -k\ L & -b\ L \end{bmatrix} \begin{bmatrix} x \\ v_x \end{bmatrix} + k g_x \begin{bmatrix} \mathbf{0} \\ B \end{bmatrix} \tag{2}$$

where $x, v_x, \mathbf{1} \in \mathbb{R}^n$ and $\mathbf{0} \in \mathbb{R}^n$ and $0 \in \mathbb{R}^{n\times n}$. $B := \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & z_1 & z_2 & z_3 & 0 & \dots & z_n \end{bmatrix}$: with z_1, z_2, z_3 are constants decided by the cars 6, 7, 8 locally to ensure constant spacing. We can keep them to be same however this would give a spacing depending on the graph structure and weights assigned to specific edges.

The '0' entries in B correspond to the phantom cars. This corresponds to not giving horizontal velocity to the phantom cars.

Theorem 4.5: Consider a weighted directed graph such that the total incoming weights are same for each node. Then there exists a choice of B for control law (2) such that a desired constant spacing can be attained between cars in the same level.

Proof of Theorem 4.5: We shift the states to $\tilde{x}_i := x_i - g_{xi}$ where i is any non-phantom car, g_{xi} corresponds to the i^{th} component of g_x. We conclude about the stability of the system as before using [9, Lemma 4.1 and Theorem 4.1]. The velocities will converge to 0 which is the equilibrium state for the cars.

We now look at the difference equations for determining the horizontal spacing of cars in the same level. Notice that the horizontal spacing in two different levels can be same.

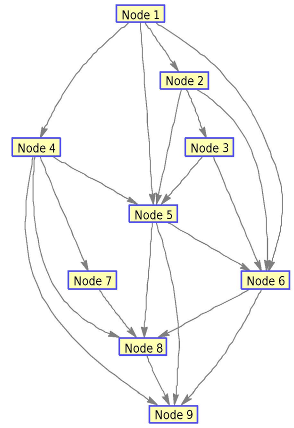
We look at two levels of cars in the graph. The same procedure can then be repeated for the remaining levels taken two at a time. The partial Laplacian for Figure 2b is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & m_{62} & 0 & 0 & m_{65} & -m_{66} & 0 & 0 \\ 0 & m_{72} & m_{73} & 0 & 0 & m_{76} & -m_{77} & 0 \\ 0 & 0 & m_{83} & m_{84} & 0 & 0 & m_{87} & -m_{88} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ g_x\ z_1 \\ g_x\ z_2 \\ g_x\ z_3 \end{bmatrix} \tag{3}$$

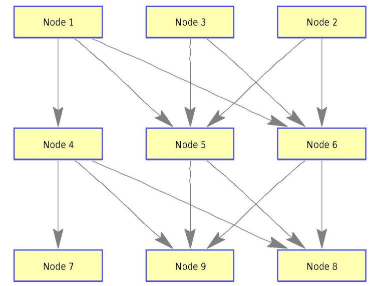
The first row in (3) signifies that the x coordinate of car 1 and car 5 is same. This condition will repeat for all phantom nodes as expected.

For the Laplacian the first row corresponding to node 1 is all zeros. The rank of the connected Laplacian is $n-1$ where n is the number of nodes. This is guaranteed by the existence of a directed spanning tree in the graph [9, Lemma 4.3]. Thus the remaining rows have to be independent to ensure that the rank is $n-1$. This will ensure existence of at least one solution of (3). Adjusting the constant z_1, z_2 and z_3, the solution corresponding to equal spacing can also be determined. The cars in the second level have information of the target position of cars in the first level, the remaining equations can also be solved to obtained the final spacing as $x_6 = x_5 + \tilde{g}_x$ and so on. This computation is still distributed (refer Remark 4.4) as the constants are local to a particular car.

From Corollary 4.3.6 all solutions are are satisfy constant spacing property. This completes the proof. □



(a) Influence graph for motion along x direction



(b) Influence graph for motion along y direction

Fig. 3: Influence graphs

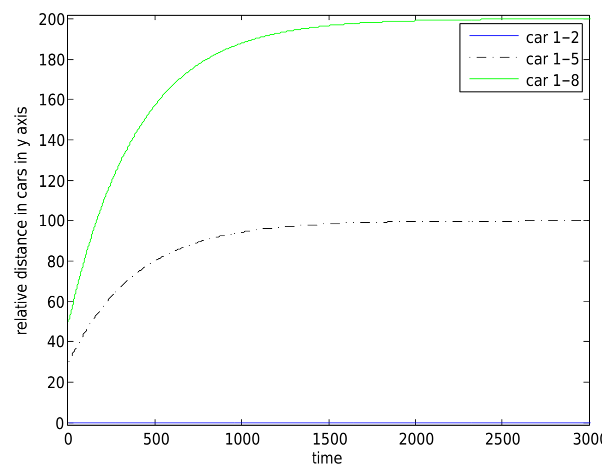


Fig. 4: X *vs* Y coordinate for motion of cars

We now show an example illustrating the convergence in the multilane case. We would like to obtain equal spacing along both horizontal and vertical directions independently. Different spacings can also be achieved by adjusting the constant g_y and g_x in (1) and (2) respectively.

V. Numerical Example

Consider 9 cars with a maximum of 3 cars fitting along the width of the road. The final spacing requirements are 100 and 70 along y and x direction respectively. The initial position coordinates along both directions are as follows: $y = [\ 60\ 60\ 60\ 30\ 30\ 30\ 10\ 10\ 10\ ]^T$ and $x = [\ 0\ 20\ 40\ 0\ 60\ 90\ 0\ 25\ 50\ ]^T$. The parameters used are $b = 0.4$ and $k = 0.001$. The graphs in both directions are allowed to change during the simulation. The final graphs for obtained for y direction is Figure 3b and for x direction is Figure 3a. Cars are considered to be in the same level based on tolerance band. The final values of convergence are shown in the Figure 5. It can be seen that cars 2, 5, 8

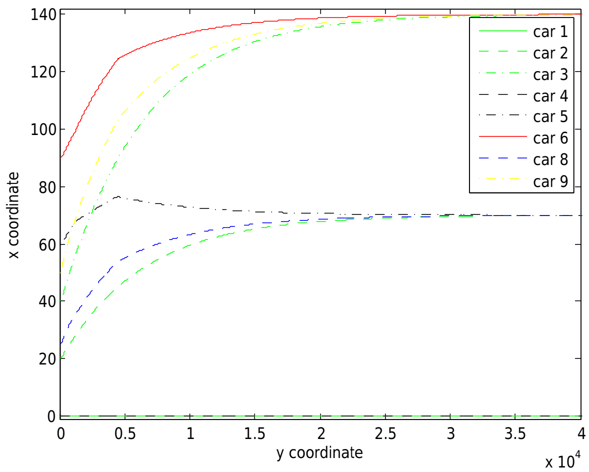


Fig. 5: X vs Y coordinate for motion of cars

have settled at an x coordinate of 70 and cars 3, 6, 9 have settled at a x coordinate of 140. Though the x coordinate for three cars is same their y coordinates are different as shown in Figure 4.

VI. Future work

In this paper we provided some insights on multilane formations of vehicles with a control law utilising the natural structure present in the system. Different formations can be obtained based on the spacing requirements using the control law. Optimality analysis of such a formation in terms of traffic throughput can reveal interesting characteristics of cars in the Indian scenario.

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