2016 European Control Conference
June 29 - July 1, 2016. Aalborg, Denmark

New results and algorithms for computing storage functions: the lossless/all-pass cases

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Abstract— **Storage functions, which are also the Algebraic Riccati Inequality (ARI) solutions, play an important role in many optimal control/estimation problems. Extreme storage functions are solutions to the corresponding Algebraic Riccati Equation (ARE). While storage functions exist under assumption of dissipativity, a key assumption in formulation of the ARE/ARI is certain 'regularity conditions' on the feedthrough term in the input/state/output representation of the system. For example, lossless and all-pass systems do not meet such regularity conditions (nonsingularity of $D+D^T$ and $I-D^TD$ respectively). And hence the ARE does not exist for such systems. Consequently, computation of storage functions for lossless/all-pass systems is not possible by conventional ARE based methods, and therefore, for such systems, different techniques are needed. In this paper we present three new algorithms for computation of storage functions for lossless/all-pass systems and compare them for numerical accuracy and computational efficiency. Each of the proposed methods presented in this paper comes from different viewpoints. One is linked to the notion of Bezoutian of two polynomials, while another is motivated by Foster realization of LC circuits and the third method is linked to the notion of trajectories of minimal dissipation in the behavioral approach. A comparative study among the three methods shows that the method based on the Bezoutian is the best from both perspectives: computational time and numerical accuracy.**

Keywords: **Algebraic Riccati Equation (ARE), Bezoutian, optimal control, Hamiltonian systems.**

1. Introduction

Algebraic Riccati Equation/Inequality (ARE/ARI) plays a key role in many optimal and suboptimal control/estimation problems like Kalman filtering, LQ control, H_∞ and H_2 optimal control and model order reduction, etc [1]. Many conceptual and numerical methods have been developed for solving the ARE/ARI [3],[5],[9]. Different forms of the ARE/ARI arise in problems of optimal control and study of dissipative systems. For a system with input/state/output (i/s/o) representation, $\dot{x} = Ax + Bu$, and $y = Cx + Du$, the ARI pertaining to passivity of the system turns out to be of the form $A^TK + KA + (KB - C^T)(D + D^T)^{-1}(B^TK - C) \leqslant 0$. Note that existence of the ARE/ARI crucially depends on the nonsingularity of the matrix $D + D^T$. Similarly, for the case of small gain (bounded real systems), the ARI takes the form $A^TK + KA + C^TC + (KB + C^TD)(I - D^TD)^{-1}(B^TK + D^TC) \leqslant 0$. Again, its existence depends on the nonsingularity of $I - D^TD$. Such a condition on the feedthrough term D is called the "regularity condition". In other words, the ARE/ARI exists if and only if D satisfies the corresponding regularity condition. Thus, in any study involving ARE/ARI, it is essential to assume that the feedthrough matrix D satisfies the regularity condition.

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AREs/ARIs arise naturally in the study of dissipative systems; such systems are characterized by existence of *storage functions*. Interestingly, for a dissipative system, solutions of the ARE/ARI are interpreted as storage functions. Note that conservative systems[1] are a special type of dissipative systems and hence admit storage functions. Interestingly, a characteristic property of conservative systems is that, in any i/s/o representation of the system, the feedthrough matrix D does not satisfy the regularity condition. Therefore, conservative systems do not admit an ARE/ARI. Hence, computation of storage functions for conservative systems is not possible by conventional ARE/ARI based methods. Clearly, for such systems different techniques are needed. In this paper, we present alternate methods for such computations. We have mentioned earlier that when ARE/ARI exists, its solutions are interpreted as storage functions of a dissipative system. Under this interpretation, our proposed methods for computing storage functions of conservative systems, provide an alternative way of solving control problems, where ARE/ARI does not exist owing to failure of regularity condition on D.

The ARE/ARI associated to a system arises from a Linear Matrix Equality/Linear Matrix Inequality (LME/LMI). Existence of this LME/LMI, however, does not have to satisfy any regularity condition on D. Hence for conservative systems, although the ARE does not exist, the LME does exist. Note that one way of computing solutions of an ARE/ARI is to solve for the LME/LMI linked with the ARE/ARI [14]. There are various methods for solving such LMEs/LMIs in the literature: see [6] for different methods. Software packages like SLICOT are also available for computing solution to LME/LMI [4]. However, solution to the LME of a conservative system is not possible using standard methods of solving LMEs like interior point methods due to the absence of interior points to work with[2]. It can be verified through [1, Remark 5.8.1] that the LME admitted by a conservative system is of the Sylvester form $PX + X^TQ = R$ where P, Q, R, X are matrices of proper dimension. Numerous methods to solve the Sylvester equation are known in the literature [11]. However, most methods require matrices P and Q to be square, while the matrix equations encountered in the conservative case have P and Q nonsquare. Hence the known methods to solve LME do not work for the LME of conservative systems. The new methods developed in this paper to compute storage functions of conservative systems

[1] Lossless systems, with u input and y output, are conservative with respect to the "passivity supply rate" u^Ty and have $D + D^T = 0$. Similarly, all-pass systems are conservative with respect to the "bounded real supply rate" $u^Tu - y^Ty$. For all-pass systems $I - D^TD = 0$.

[2] Methods to solve LMIs usually use interior point methods that require 'strict feasibility' and these methods are not suited for the equality case.

do not use the existing LME/LMI solving techniques. Each method utilizes a significantly different approach.

The rest of the paper is organized as follows: Section 2 summarizes the preliminaries required in the paper. In Section 3, we formulate and prove new results that help computation of storage function K: one based on Bezoutian of two polynomials, second one based on partial fraction expansion and the third one linked to the notion of trajectories of minimal dissipation in behavioral approach [13]. In Section 4, we use these main results and propose three numerical algorithms to compute storage function of conservative systems. Section 5 contains a comparison of the algorithms based on their computational time and numerical accuracy. Some concluding remarks are presented in Section 6.

2. Notation and Preliminaries

We follow standard notation in this paper: $\mathbb{R}$ and $\mathbb{C}$ denote fields of real and complex numbers respectively. The ring of polynomials in ξ with real coefficients is denoted by $\mathbb{R}[\xi]$. The set $\mathbb{R}^{\mathtt{w}\times\mathtt{p}}[\xi]$ denotes all $\mathtt{w}\times\mathtt{p}$ matrices with entries from $\mathbb{R}[\xi]$. We use $\bullet$ when a dimension need not be specified: for example, $\mathbb{R}^{\mathtt{w}\times\bullet}$ denotes the set of real constant matrices having $\mathtt{w}$ rows. $\mathbb{R}[\zeta,\eta]$ denotes the set of real polynomials in two indeterminates: ζ and η. The set of $\mathtt{w}\times\mathtt{w}$ matrices with entries in $\mathbb{R}[\zeta,\eta]$ is denoted by $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. The set $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ denotes the space of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$, and $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ denotes the subspace of all compactly supported trajectories in $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. We represent a block diagonal matrix $A=\operatorname{diag}\,(A_1,A_2,\ldots,A_{\mathtt{m}})$ where $A_1,\ldots,A_{\mathtt{m}}$ are square matrices of possibly different sizes.

A. Behavior

This section contains the essential preliminaries of the behavioral approach: an elaborate exposition can be found in [10]. A linear differential behavior, denoted by $\mathfrak{B}$, is defined as the set of all infinitely often differentiable trajectories that satisfy a system of ordinary linear differential equations with constant coefficients, i.e.,

$$\mathfrak{B}:=\left\{w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})\mid R\left(\frac{d}{dt}\right)w=0\right\},\text{where } R(\xi)\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi].$$

We denote the linear differential behaviors with $\mathtt{w}$ number of variables by $\mathfrak{L}^{\mathtt{w}}$. The behavior $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}$ can be represented as $\mathfrak{B}=\ker R(\frac{d}{dt})$, called a *kernel representation* of $\mathfrak{B}$. We will restrict to SISO systems: here $R(\xi)\in\mathbb{R}^{1\times 2}[\xi]$ and $R(\xi)$ is nonzero.

There can be various ways in which a behavior $\mathfrak{B}$ can be represented. One of the ways of representing a linear differential behavior $\mathfrak{B}$ is the *latent variable representation*.

$$\mathfrak{B}:=\{w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})\mid \text{ there exists } \ell\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{m}})\text{such that}$$
$$R(\frac{d}{dt})w=M(\frac{d}{dt})\ell\},\ \text{where } M(\xi)\in\mathbb{R}^{\bullet\times\mathtt{m}}[\xi]. \tag{1}$$

Here ℓ is called a latent variable of the behavior. For the SISO case, the general trajectory-level patchability definition of controllability of a system specializes to coprimeness of numerator $n(s)$ and denominator $d(s)$ of $G(s)$ (see [15]). We represent the set of all controllable behaviors with $\mathtt{w}$ variables as $\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. It is known that a controllable behavior $\mathfrak{B}$ admits an *image representation* defined as

$$\mathfrak{B}:=\left\{w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})\mid\exists\,\ell\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{m}})\text{ such that }w=M\left(\frac{d}{dt}\right)\ell\right\}$$

where $M(\xi)\in\mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$. If $M(\xi)$ is such that $M(\lambda)$ has full column rank for all $\lambda\in\mathbb{C}$, then it is called an *observable* image representation (see [15, Section 2]).

B. Quadratic Differential Forms and Dissipativity

In this subsection we provide basic details about quadratic differential forms (QDF): for a detailed study, see [15]. Consider a two-variable polynomial matrix

$$\phi(\zeta,\eta):=\sum_{j,k}\phi_{jk}\zeta^j\eta^k\ \in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta],\ \text{where }\phi_{jk}\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}.$$

The QDF Q_ϕ induced by $\phi(\zeta,\eta)$ is a map $Q_\phi:\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})\to\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ defined as

$$Q_\phi(w):=\sum_{j,k}(\frac{d^jw}{dt^j})^T\ \phi_{jk}\ (\frac{d^kw}{dt^k}).$$

A quadratic form induced by a real symmetric *constant* matrix is a special case, often needed in this paper: and we shall denote it by $Q_\Sigma(w)=w^T\Sigma w$ where $\Sigma\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}$.

Following [15], we call a controllable behavior $\mathfrak{B}$ *dissipative* with respect to a symmetric nonsingular matrix Σ, or simply Σ-*dissipative*, if

$$\int_{\mathbb{R}}w^T\Sigma w\,dt\geqslant 0\text{ for every }w\in\mathfrak{B}\cap\mathfrak{D}. \tag{2}$$

Q_Σ is often called the *supply rate*. For simplicity, we will also call Σ the supply rate. For a constant nonsingular symmetric matrix Σ, the number of positive eigenvalues of Σ is called its *positive signature* and is denoted by $\sigma_+(\Sigma)$.

For a controllable behavior $\mathfrak{B}$, with supply rate Σ, the two variable polynomial matrix $\psi\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is said to induce a *storage function* Q_ψ for $\mathfrak{B}$ with respect to Q_Σ if

$$\frac{d}{dt}Q_\psi(w)\leqslant Q_\Sigma(w)\text{ for all }w\in\mathfrak{B}. \tag{3}$$

Storage function captures the intuition that the rate of increase of stored energy in a dissipative system is at most the power supplied. The storage function with respect to a supply rate is not unique in general for a given system, however, there is uniqueness for the lossless case (see [15]).

We define $\overset{\bullet}{\Psi}(\zeta,\eta):=(\zeta+\eta)\Psi(\zeta,\eta)$. Note that $\frac{d}{dt}Q_\psi(w)=Q_{\overset{\bullet}{\psi}}$ for all $w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. Given $\Sigma\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and a system described by the observable image representation $w=M(\frac{d}{dt})\ell$, it helps to define the QDF $Q_\Phi(\ell)$, in the *latent variables*, induced by $\Phi(\zeta,\eta)\in\mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\zeta,\eta]$ with

$$\Phi(\zeta,\eta):=M(\zeta)^T\,\Sigma M(\eta). \tag{4}$$

For a Σ-dissipative system, with $\Sigma\in\mathbb{R}^{2\times 2}$ symmetric and nonsingular, as in the SISO case, dissipativity implies $\sigma_+(\Sigma)\geqslant 1$: see [15, Remark 5.11].

C. Conservative systems

For $\Sigma\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, a controllable behavior $\mathfrak{B}$ is said to be Σ-*conservative* if inequality (2) is satisfied with *equality*, i.e.

$$\int_{\mathbb{R}}Q_\Sigma(w)\,dt=0\text{ for all }w\in\mathfrak{B}\cap\mathfrak{D}.$$

It follows that conservative systems satisfy equation (3), too, with equality. Further, the storage function $\Psi \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\zeta,\eta]$ associated with a Σ-conservative system satisfies

$$\dot{\Psi}(\zeta,\eta) = \Phi(\zeta,\eta) \text{ i.e. } \Psi(\zeta,\eta) = \frac{M(\zeta)^T \Sigma M(\eta)}{(\zeta+\eta)}. \quad (5)$$

Equation (5) gives us the unique storage function of the conservative system in the latent variables.

In this paper, we elaborate on the methods for SISO systems with power = 2 input $\times$ output as the supply rate. We refer to this as the *passivity supply rate* i.e.

$$Q_\Sigma = \begin{bmatrix} u \\ y \end{bmatrix}^T \Sigma \begin{bmatrix} u \\ y \end{bmatrix} \text{ induced by } \Sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (6)$$

where u, y are the input and output of the system, respectively. In this paper, as in the literature, we will call conservative systems with respect to the passivity supply rate *lossless systems*: see Footnote 1. Electrical circuits consisting of ideal inductors and/or capacitors have lossless behaviors. In order to simplify the exposition in this paper, we shall be using the passivity supply rate and deal with lossless systems only. However, all the methods reported in this paper can be applied to systems conservative with respect to other supply rates too.

D. State representation and minimality

A latent variable x is said to be a state if, whenever (w_1,x_1) and (w_2,x_2) satisfies the linear differential equations $R(\frac{d}{dt})w = M(\frac{d}{dt})x$ describing $\mathfrak{B}$ and $x_1(0) = x_2(0)$ then, the concatenation $(w_1,x_1) \wedge (w_2,x_2)$ at $t = 0$ also satisfies (in a weak/distributional sense) $R(\frac{d}{dt})w = M(\frac{d}{dt})x$.

A state space description is said to be *minimal* if the number of components in the state x is the minimum amongst all possible state representations. The number of states corresponding to a minimal state representation of $\mathfrak{B}$ is called the *McMillan degree* of the behavior $\mathfrak{B}$. When the state x is not minimal but is observable from the system variable w, it is known that one or more components in x satisfy a static relation. In Section 3-C, we develop a method based on these static relations that are satisfied between the state x of the given lossless system and the 'dual state' z of the adjoint system: these static relations give us the unique storage function for the lossless case: see Theorem 3.5 below.

E. Controller canonical form

Given a transfer function $G(s)$, controller canonical i/s/o representation

$$\dot{x} = Ax + Bu,\ y = Cx + Du \quad (7)$$

is well known: see [7, Section 5.1] for an elaboration. The form we are using in this paper has the denominator coefficients of $G(s)$ as the last row of A and 1's along the superdiagonal of A. The coefficients of the numerator of $G(s)$ are the entries of C and $B = e_{\mathtt{n}}$. The states of the system corresponding to this controller canonical representation can also be written as $x = \begin{bmatrix} \ell & \dot{\ell} & \dots & \ell^{(\mathtt{n}-1)} \end{bmatrix}^T$ where $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} d(s) \\ n(s) \end{smallmatrix}\right] \ell$ and $x \in \mathbb{R}^{\mathtt{n}}$ is referred to as states in the canonical basis.

For a general state space SISO lossless system, $\dot{x} = Ax + Bu$, $y = Cx$, not necessarily in controller canonical form, assuming the storage function is a state function $x^T K x$ for a symmetric matrix K (see [15, Theorem 5.5]) and using equality in the inequality (3) results in an LME $\begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & 0 \end{bmatrix} = 0$, which further gives the matrix equations

$$A^T K + KA = 0 \qquad \text{and} \qquad KB - C^T = 0. \quad (8)$$

We use equation (8) crucially in comparison of the obtained K for error analysis.

F. Minimal polynomial basis (MPB)

This section contains a brief introduction to annihilators of polynomial matrices. Consider the polynomial matrix $R(s) \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}[s]$ of rank $\mathtt{n}$. Let the set $\{p_1(s), p_2(s), \dots, p_{\mathtt{m}-\mathtt{n}}(s)\}$ be a nullspace basis of $R(s)$ ordered with degrees $d_1 \leqslant d_2 \leqslant \dots \leqslant d_{\mathtt{m}-\mathtt{n}}$. The set $\{p_1(s), p_2(s), \dots, p_{\mathtt{m}-\mathtt{n}}(s)\}$ is said to be a *minimal polynomial basis* of $R(s)$ if every other nullspace basis $\{q_1(s), q_2(s), \dots, q_{\mathtt{m}-\mathtt{n}}(s)\}$ with degree $c_1 \leqslant c_2 \leqslant \dots \leqslant c_{\mathtt{m}-\mathtt{n}}$ is such that $d_i \leqslant c_i$, for $i = 1, 2, \dots, \mathtt{m}-\mathtt{n}$. The degrees of the vectors of minimal polynomial basis of $R(s)$ are called the *(Forney invariant) minimal indices* or *Kronecker indices* (more details in [7, Section 6.5.4]).

3. Main Results

In this section we present the main results: Theorems 3.1, 3.3 and 3.5 each of which gives a new algorithm to calculate the unique storage function for lossless systems. We present the new algorithms for *SISO lossless systems*.

A. Bezoutian based method

Consider a SISO system $G(s) = \frac{n(s)}{d(s)}$, that is lossless with respect to $\Sigma = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. The following is an observable image representation of the corresponding behavior: $w = M(\frac{d}{dt})\ell$ where $w = \begin{bmatrix} u \\ y \end{bmatrix}$, $M(\xi) := \begin{bmatrix} d(\xi) \\ n(\xi) \end{bmatrix}$. Define $\Phi(\zeta,\eta) := M(\zeta)^T \Sigma M(\eta)$. Using equation (5), we get, $\dot{\Psi} = \Phi$. Therefore, it follows for the passivity supply rate, that

$$\Psi(\zeta,\eta) = \frac{\Phi(\zeta,\eta)}{(\zeta+\eta)} = \frac{d(\zeta)n(\eta) + d(\eta)n(\zeta)}{(\zeta+\eta)}.$$

The storage function can be calculated by "polynomial long division technique", which is based on Euclidean division of polynomials. We state this as a result below.

Theorem 3.1. *Consider a Σ-lossless system with transfer function $G(s) = \frac{n(s)}{d(s)}$ with the controller canonical state space realization as:*

$$\dot{x} = Ax + Bu \qquad \text{and} \qquad y = Cx. \quad (9)$$

Construct the two variable polynomial $z_b(\zeta,\eta)$, induced by the Bezoutian of the polynomials $n(s)$ and $d(s)$ by

$$z_b(\zeta,\eta) := \frac{n(\zeta)d(\eta) + n(\eta)d(\zeta)}{\zeta+\eta} \quad (10)$$

Write $z_b(\zeta,\eta)$ as, $\begin{bmatrix} 1 \\ \zeta \\ \vdots \\ \zeta^{\mathtt{n}-1} \end{bmatrix}^T Z_b \begin{bmatrix} 1 \\ \eta \\ \vdots \\ \eta^{\mathtt{n}-1} \end{bmatrix}$, where $Z_b \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Then, $x^T Z_b x$ is the unique storage function for the Σ-lossless system with state space description (9), i.e. $\frac{d}{dt} x^T Z_b x = 2uy$.

The proof is skipped due to shortage of space and can be found in an extended journal version of this paper. We give a brief intuitive reason behind the result. The behavior

$\mathfrak{B}$ of the system in equation (9) admits an observable image representation $w = \begin{bmatrix} d(\frac{d}{dt}) \\ n(\frac{d}{dt}) \end{bmatrix} \ell$. Using equation (5) and (10), the storage function expressed in the latent variable ℓ is $Q_{z_b}(\ell)$. Since the vector $[\ell \ \dot{\ell} \ \cdots \ \ell^{(n-1)}]^T$ represents the state x of the system in the canonical basis, we have $x^T Z_b x = Q_{z_b}(\ell)$, which is the unique storage function.

Remark 3.2. The conventional Bezoutian of two polynomials $p(x)$ and $q(x)$ is defined as $z_b(x,y) := \frac{p(x)q(y)-p(y)q(x)}{x-y}$. There is a change in sign between this conventional Bezoutian definition and the one defined in equation (10): this is due to the special structure of lossless transfer functions elaborated below. In any lossless transfer function $G(s) = \frac{n(s)}{d(s)}$, when the order of the system is even then $n(s)$ is an odd polynomial i.e. $n(-s) = -n(s)$ and $d(s)$ is even polynomial i.e. $d(-s) = d(s)$. The converse is true when the order of the system is odd. Thus our definition is same as the conventional if we substitute $(x = -\zeta;\ y = \eta)$ when the order of the system is even and $(x = \zeta;\ y = -\eta)$ when the order of the system is odd.

Though Theorem 3.1 involves *bivariate* polynomial manipulation, Algorithm 4.1 in Section 4 implements the Bezoutian based method of Theorem 3.1 using only *univariate* polynomial operations. The algorithm is similar to Euclidean long division. Write $\Phi(\zeta,\eta) = \phi_0(\eta) + \zeta\phi_1(\eta) + \ldots + \zeta^{\mathtt{n}}\phi_{\mathtt{n}}(\eta)$. Then the storage function $\Psi(\zeta,\eta) = \psi_0(\eta) + \zeta\psi_1(\eta) + \ldots + \zeta^{\mathtt{n}-1}\psi_{\mathtt{n}-1}(\eta)$ can be computed by the following recursion with $k = 1,\ldots,\mathtt{n}-1$:

$$\psi_0(\xi) := \frac{\phi_0(\xi)}{\xi}, \qquad \psi_k(\xi) := \frac{\phi_k(\xi) - \psi_{k-1}(\xi)}{\xi} \tag{11}$$

B. Partial fraction expansion based method

The transfer function of a lossless SISO system $G(s) = \frac{n(s)}{d(s)}$ can be considered as the driving point impedance function of a physically realizable LC network. Since the system is lossless, the poles of the system are all on the imaginary axis. Hence the transfer function $G(s)$ can be expanded into its partial fractions[3] as

$$G(s) = r_\infty s + \frac{r_0}{s} + \sum_{i=1}^{\mathtt{m}} \frac{r_i s}{s^2 + \omega_i^2} \tag{12}$$

where $r_\infty, r_0 \geqslant 0, r_1, r_2, \ldots, r_{\mathtt{m}} > 0$ and each $\omega_i > 0$.

Without loss of generality, we assume the transfer function $G(s)$ is proper and therefore $r_\infty = 0$. The physical realization of transfer function in equation (12) in an LC network can be done as shown in Figure 1. We address the case of no pole at the origin in Theorem 3.3 below.

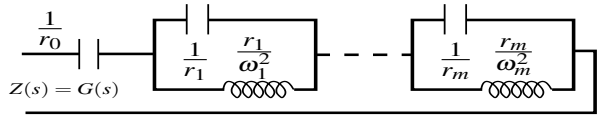

Fig. 1. LC realization of the partial fractions in Foster-I form

Theorem 3.3. *Consider a lossless system with transfer function*

$$G(s) = \sum_{i=1}^{\mathtt{m}} \frac{r_i s}{s^2 + \omega_i^2}, \ \textit{where } r_1, r_2, \ldots, r_{\mathtt{m}} > 0, \ \textit{and each } \omega_i > 0,$$

with a minimal state space representation given by

$$A = \text{diag } (A_1, A_2, ..., A_{\mathtt{m}}) \ \textit{where } A_i = \begin{bmatrix} 0 & -r_i \\ \frac{\omega_i^2}{r_i} & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} r_1 & 0 & r_2 & 0 & \cdots & r_{\mathtt{m}} & 0 \end{bmatrix}^T$$
$$C = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Then, the unique storage function is $x^T K x$ with the symmetric matrix $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ defined by

$$K = \text{diag } (K_1, K_2, \ldots, K_{\mathtt{m}}) \ \textit{where } K_i = \text{diag } (r_i^{-1}, r_i\omega_i^{-2}).$$

The proof is skipped due to lack of space and can be found in an extended journal version of this paper. The case of a lossless system with a pole at the origin i.e. with $G(s) = \frac{r_0}{s} + \sum_{i=1}^{\mathtt{m}} \frac{r_i s}{s^2+\omega_i^2}$ where $r_0, r_1, \ldots, r_{\mathtt{m}} > 0$, each $\omega_i > 0$ is easy to deal with. For such a system the storage function is given by diag $\left(r_0^{-1}, K\right)$ where K is given by Theorem 3.3. The result in Theorem 3.3 is used to develop Algorithm 4.2, which is referred to as the partial fraction expansion based algorithm.

C. Static relations extraction based method

This method requires the notion of the Σ-orthogonal complement of a behavior $\mathfrak{B}$: see [15] for details.

Definition 3.4. *Consider $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and a nonsingular, symmetric $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. The Σ-orthogonal complement $\mathfrak{B}^{\perp\Sigma}$ of $\mathfrak{B}$ is defined as*

$$\mathfrak{B}^{\perp\Sigma} := \{v \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid \int_{-\infty}^{\infty} v^T \Sigma w \, dt = 0 \ \textit{for all } w \in \mathfrak{B} \cap \mathfrak{D}\}.$$

Let (A,B,C,D) be a minimal state space representation of the behavior $\mathfrak{B}$ and let x be the state of the system. Suppose the McMillan degree of $\mathfrak{B}$ is $\mathtt{n}$. One of the minimal state space representation of $\mathfrak{B}^{\perp\Sigma}$, for Σ corresponding to the passivity supply rate (see equation (6)) i.e. $w^T\Sigma w = 2u^T y$, is $(-A^T, C^T, B^T, -D^T)$. Let z be the state of the Σ-orthogonal complement behavior $\mathfrak{B}^{\perp\Sigma}$ [15, Section 10].

Without elaborating much on the interpretation of $\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma}$ as a Hamiltonian system/trajectories of minimal dissipation, we review some properties of $\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma}$. Call $\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma}$ as $\mathfrak{B}_{\text{Ham}}$, a Hamiltonian behavior. In general, it can be shown following similar arguments as in [12] that $\mathfrak{B}_{\text{Ham}}$ admits a first order kernel representation of the form

$$R\left(\frac{d}{dt}\right)\begin{bmatrix} x \\ z \\ y \end{bmatrix} = 0 \quad \text{with} \quad R(\xi) := \xi E - H \tag{13}$$
$$\text{where } E := \begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } H := \begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & D+D^T \end{bmatrix}.$$

Call $R(\xi)$ a "Hamiltonian pencil". For a lossless behavior $\mathfrak{B}$, a first order kernel representation of the Hamiltonian system $\mathfrak{B}_{\text{Ham}}$ is

$$\begin{bmatrix} \xi I_{\mathtt{n}} - A & 0 & -B \\ 0 & \xi I_{\mathtt{n}} + A^T & -C^T \\ -C & B^T & 0 \end{bmatrix}\begin{bmatrix} x \\ z \\ y \end{bmatrix} = 0. \tag{14}$$

It turns out that when a behavior $\mathfrak{B}$ is lossless, then $\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma} = \mathfrak{B}$ ([2, Lemma 11] for lossless case), and hence the McMillan degree of $\mathfrak{B}_{\text{Ham}} = \mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma}$ is $\mathtt{n}$. However, the Hamiltonian behavior in equation (14) has $2\mathtt{n}$ states and hence

[3]The residues in this expansion are assumed non-negative primarily to make contact with LC realization studies, where the residues affect the inductance and capacitance parameters. In our case, this is closely linked to positive definiteness of the obtained storage function and plays no further role. We do not dwell further on this.

x and z satisfy static relations amongst each other. The next theorem helps extract the static relations of the first order representation (14) of $\mathfrak{B}_{\text{Ham}}$ and in the process yields the storage function for the lossless behavior $\mathfrak{B}$.

Theorem 3.5. *Consider a lossless, controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with its Hamiltonian pencil represented by equation (13). Then, there exists a unique $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}_{\text{sym}}$ such that*

$$\frac{d}{dt}x^TKx = 2u^Ty \qquad \textit{for all } \begin{bmatrix} u \\ y \end{bmatrix} \in \mathfrak{B}. \tag{15}$$

if and only if

$$\textit{rank} \begin{bmatrix} sI-A & 0 & -B \\ 0 & sI+A^T & -C^T \\ -C & B^T & 0 \\ -K & I & 0 \end{bmatrix} = \textit{rank} \begin{bmatrix} sI-A & 0 & -B \\ 0 & sI+A^T & -C^T \\ -C & B^T & 0 \end{bmatrix}. \tag{16}$$

The proof to Theorem 3.5 can be found in an extended journal version of this paper. Note that introduction of the new rows $\begin{bmatrix} -K & I & 0 \end{bmatrix}$ in $R(\xi)$ is viewed as a controller which *restricts* the plant behavior $\ker(R(\frac{d}{dt}))$ to a desired behavior by introducing new laws on the plant behavior. Interestingly, for lossless systems, using equation (16), we conclude that $\begin{bmatrix} -K & I & 0 \end{bmatrix}$ is in the row-span of the polynomial matrix $R(\xi)$. Hence the new rows $\begin{bmatrix} -K & I & 0 \end{bmatrix}$ does not actually introduce any further laws on the behavior $\mathfrak{B}_{\text{Ham}}$.

4. ALGORITHM

In this section we present three algorithms based on the results developed in Section 3.

Algorithm 4.1 is based on the Bezoutian of polynomials described in Section 3-A. We use a form of the long division technique to compute the unique symmetric matrix K that induces the storage function of the lossless system. The algorithm takes the transfer function of the lossless system as the input.

Algorithm 4.1 Bezoutian based algorithm.

Input: Transfer function of a lossless system $G(s) = \frac{n(s)}{d(s)}$ of order $\mathtt{n}$ and $G(s)$ proper.
Output: $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ with x^TKx the storage function.

1: Extract coefficients of the polynomials $n(s)$ and $d(s)$ into arrays $N \in \mathbb{R}^{1\times\mathtt{n}}$ and $D \in \mathbb{R}^{1\times(\mathtt{n}+1)}$ with constant term coefficient first.
2: Equate the length of arrays N and D by suffixing a zero element to array N i.e. $N(\mathtt{n}+1) := 0$.
3: Using equation (10) compute the Bezoutian coefficient matrix $Z_b := N^TD + D^TN \in \mathbb{R}^{(\mathtt{n}+1)\times(\mathtt{n}+1)}$
4: Implement the division in first equation of (11) by constructing a row vector from the first row of Z_b
$F_{\text{old}} := \begin{bmatrix} Z_b(1, 2:\mathtt{n}+1) & 0 \end{bmatrix} \in \mathbb{R}^{1\times(\mathtt{n}+1)}$ and $F_{\text{new}} := F_{\text{old}}$.
5: The following iteration is essentially the division in second equation of (11).
6: **for** $i = 2, \ldots, \mathtt{n}$ **do**
7: $r := Z_b(i,:) - F_{\text{new}}(i-1,:)$
8: $F_{\text{new}} := \begin{bmatrix} F_{\text{old}} & \\ r(2:\mathtt{n}+1) & 0 \end{bmatrix}$ and $F_{\text{old}} \leftarrow F_{\text{new}}$
9: **end for**
10: The storage function x^TKx induced by the matrix K is then given by $K := F_{\text{new}}(1:\mathtt{n}, 1:\mathtt{n})$

Algorithm 4.2 is based on partial fraction expansion described in Section 3-B. The algorithm takes transfer function of a lossless system as input and gives a unique $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}_{\text{sym}}$ which induces the storage function of that system as output.

Algorithm 4.2 Partial fraction expansion algorithm.

Input: Transfer function of the lossless system $G(s)$.
Output: $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ with x^TKx the storage function.

1: Calculate the partial fraction expansion of $G(s)$.
2: $G(s) = G_1(s) + G_2(s) + \cdots + G_{\mathtt{m}}(s)$ (say) where each $G_i(s) = \frac{r_i s}{s^2+\omega_i^2}$, $i = 1, \cdots, \mathtt{m}$ and $\omega_i > 0$.
3: For each $G_i(s)$, obtain (A_i, B_i, C_i) triple, where $A_i \in \mathbb{R}^{2\times2}$, $B_i \in R^{2\times1}$ and $C_i \in R^{1\times2}$ using Theorem 3.3.
4: Obtain K_i from each triple (A_i, B_i, C_i) using Theorem 3.3.
5: The storage function x^TKx for the lossless system is induced by the matrix $K := \text{diag}\,(K_1, K_2, \ldots, K_{\mathtt{m}}) \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.

Algorithm 4.3 is based on extraction of static relations in first order representation of the Hamiltonian behavior $\mathfrak{B}_{\text{Ham}}$ described in Section 3-C. The algorithm takes as input the Hamiltonian pencil $R(\xi)$ and gives a unique $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}_{\text{sym}}$ that induces storage function of the lossless behavior.

Algorithm 4.3 Static relations extraction based algorithm.

Input: Recall $R(\xi) := \xi E - H \in \mathbb{R}[\xi]^{(2\mathtt{n}+\mathtt{p})\times(2\mathtt{n}+\mathtt{p})}$, a rank $2\mathtt{n}$ polynomial matrix.
Output: $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ with x^TKx the storage function.

1: Compute a MPB of $R(\xi)$. Result: A full column rank polynomial matrix $M(\xi) \in \mathbb{R}[\xi]^{(2\mathtt{n}+\mathtt{p})\times\mathtt{p}}$.
2: Partition $M(\xi)$ as $\begin{bmatrix} M_1(\xi) \\ M_2(s) \end{bmatrix}$ where $M_1(\xi) \in \mathbb{R}[\xi]^{2\mathtt{n}\times\mathtt{p}}$.
3: Compute a MPB of $M_1(\xi)^T$. Result: A full column rank polynomial matrix $N(\xi) \in \mathbb{R}[\xi]^{2\mathtt{n}\times(2\mathtt{n}-\mathtt{p})}$.
4: Partition $N(\xi) = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$ with $N_{11}, N_{21} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. (See Theorem 4.1 below)
5: The storage function x^TKx induced by the symmetric matrix K is given by $K := -N_{11}N_{21}^{-1} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$

Using the partition of the various matrices in the Algorithm 4.3, we state the following result about the unique storage function for a lossless behavior.

Theorem 4.1. *Consider $R(\xi) := \xi E - H \in \mathbb{R}[\xi]^{(2\mathtt{n}+\mathtt{p})\times(2\mathtt{n}+\mathtt{p})}$ as defined in equation (13) constructed for the lossless behavior $\mathfrak{B} \in \mathfrak{L}^{2\mathtt{p}}_{\text{cont}}$. Let $M(\xi) \in \mathbb{R}[\xi]^{(2\mathtt{n}+\mathtt{p})\times\mathtt{p}}$ be any minimal polynomial nullspace basis (MPB) for $R(\xi)$. Partition $M = \begin{bmatrix} M_1(\xi) \\ M_2(\xi) \end{bmatrix}$ with $M_1 \in \mathbb{R}[\xi]^{2\mathtt{n}\times\mathtt{p}}$. Let $N(\xi)$ be any MPB for $M_1(\xi)^T$. Then, the following statements are true.*

1) *The first $\mathtt{n}$ (Forney invariant) minimal indices of $N(\xi)$ are 0, i.e. first $\mathtt{n}$ columns of $N(\xi)$ are constant vectors.*
2) *Partition N into $\begin{bmatrix} N_1 & N_2(\xi) \end{bmatrix}$ with $N_1 \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n}}$ and further partition $N_1 = \begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix}$ with $N_{21} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Then N_{21} is invertible and $K := -N_{11}N_{21}^{-1}$ is the storage function for $\mathfrak{B}$, i.e. $\frac{d}{dt}x^TKx = 2u^Ty$ for all system trajectories.*

Algorithm 4.3 is based on computation of nullspace basis of polynomial matrices. Computation of nullspace basis of

a polynomial matrix can be done by block Toeplitz matrix algorithm: more details can be found in [8] and [16].

5. Comparison of the Numerical Methods

This section presents results of investigation on the performance of the three algorithms described in Section 4. The experiments were carried out on an Intel Core i3 computer at 3.30 GHz with 4 GB RAM using Ubuntu 14.04 LTS operating system. The relative machine precision is $\varepsilon \approx 10^{-16}$. Open source numerical computational package Scilab 5.5 has been used to implement the algorithms.

Randomly generated transfer functions of lossless systems are used to test the algorithms. Data for computation time and error for each order has been averaged over three randomly generated transfer functions with different seed values. To negate the effect of CPU delays the computation time to calculate K for each transfer function is further averaged over hundred iterations.

A. Computation time

The plot in Figure 2 shows the time taken by each algorithm to compute the matrix K for lossless systems of different orders. The Bezoutian long division method and the partial fraction expansion based method take relatively less computation time compared to static relations extraction based method.

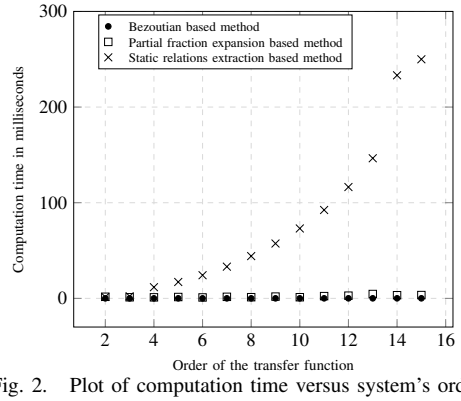

Fig. 2. Plot of computation time versus system's order.

B. Computation error

As discussed in Section 2-E, the matrix K that induces the storage function for a lossless system must satisfy equation (8). Hence define the error associated with the computation of K as

$$\mathrm{Err}(K) = \left\| \begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & 0 \end{bmatrix} \right\|_2 . \qquad (17)$$

We compute $\mathrm{Err}(K)$ for randomly generated lossless systems. Figure 3 shows a comparison of the error associated in computation of K with the three algorithms presented in the paper. From the plot we infer that the Bezoutian based method is marginally better than the partial fraction expansion based method and the static relations extraction based method.

6. Concluding remarks

This paper dealt with computation of storage functions for systems that do not admit an ARE/ARI: more specifically for systems that are conservative. While we dealt with only the lossless case, the results are applicable in more generality when the feedthrough term D loses traditional regularity conditions (needed for formulation of the ARE).

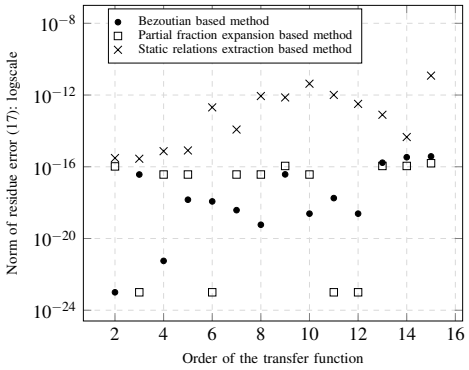

Fig. 3. Plot of error residue versus system's order.

We formulated new results (whose proofs can be found in an extended journal version of this paper) that the storage function for the lossless case satisfies. One result is based on the Bezoutian polynomial (Theorem 3.1), the second one based on partial fraction expansion and LC realization (Theorem 3.3) and the third result using static relations extraction from a Hamiltonian pencil (Theorem 3.5). Note that Theorem 3.1 and 3.3 are for lossless SISO systems. Theorem 3.5 is applicable to lossless MIMO systems as well. Algorithms arising out of these results were formulated and compared with respect to computation time and numerical accuracy: this comparison has been plotted in Figures 2 and 3. This paper focussed more on theoretical properties of the storage function and just 'proof-of-concept' algorithms.

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