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Leader Selection for Minimum-Time Consensus in Multi-Agent Networks

Ameer K. Mulla, Deepak Anand, Debraj Chakraborty and Madhu N. Belur

***Abstract*— Leader selection for a single-leader, continuous-time multi-agent system, with single-integrator agents, is considered. A network of single integrator agents interact with each other according to the well studied asymptotic consensus law proposed by Olfati-Saber and Murray. In addition to the input prescribed by this consensus law, it is assumed that a bounded external input is allowed to act on *only one* (called the leader) of the agents. For each choice of leader, this bounded external input can be optimized to drive all the agents to a consensus state in the minimum possible time. This paper presents an algorithm for selecting a leader such that the time taken to reach consensus is the least among the minimum consensus times achievable by each leader. Recently developed Groebner basis based algorithms are used to calculate explicit set of polynomials which partitions each hyper-sphere in the state-space, centered at the origin; where each partition is identified with a particular leader node. The Groebner basis needs to be computed only once. To select the minimum time leader, these demarcating polynomials need to be evaluated at given initial condition exactly one time.**

I. INTRODUCTION

Consensus in multi-agent systems, interacting over a communication graph, has been studied elaborately in the recent past (e.g see [1] for a detailed review). One of the earliest models considers single integrator agents updating their states based on local errors, but in the process converging asymptotically on to the global average of the initial conditions [1]. Such systems can be modeled using the Laplacian of the communication graph and the speed of the convergence is usually characterized through the graph's algebraic connectivity. However, convergence on to the average consensus is asymptotic. Interestingly, in a related experiment, formations with local interaction laws converged after significantly long durations [2]. In this paper, we connect an external input to one node (called leader) of an n-agent system, which are otherwise interacting among themselves according to the basic Laplacian model of [3]. Clearly, even without this input, the agents would converge (albeit asymptotically) on to the average consensus point. However, if the collection is controllable from the node where we connect the input, then it is possible to force the entire collection to reach a consensus in finite (minimum) time. Fastest possible consensus is required in various fields such as synchronization of generators in power-grids [27], finite time load-balancing across servers [28], uniform cooling in thermal industries [29] etc.

Let the external input $u(t)$ be connected to the i^{th} agent. For a particular initial configuration of the multi-agent system, let the minimum time to consensus, among all possible inputs connected at the i^{th} agent, be denoted by T^i_{min}. We aim to connect the external input to the agent, which will yield the least minimum time to consensus, among all the agents. In other words, we need to indentify agent i such that

$$i^* = \arg\min_i T^i_{min}$$

Clearly, for a fixed communication topology, this choice is dependent on the initial configuration of the agents. It is well known that the controllability of the multi-agent system depends on the position of the leader in the graph [4]-[8]. We assume that the multi-agent system is controllable from at least one node, and connect the external input only to the nodes from which it is controllable.

Using recently developed Groebner basis based methods [22], we first present a method to explicitly compute a set of polynomial equations representing isochronous sets for each input position. These surfaces are characterized by points in the state space from where it takes identical time to optimally reach the origin. We show that the loci of pairwise intersections of the isochronous sets, corresponding to each input position, partition set of states with a constant norm, i.e. each hyper-sphere in the state-space centered at the origin, in to disjoint sets. The set of points in each partition corresponds to a unique choice of the leader that achieves consensus in minimum time. In other words, given any initial vector of values for the states of the multi-agent system, one needs to only check, which partition this vector belongs to. The corresponding partition, uniquely identifies which leader will enable the consensus in minimum possible time. The isochronous sets and thus the partitions can be computed offline. The polynomial equations of the partitioning manifolds have to be evaluated online to select the leader.

The optimal leader selection problem with various objectives such as controllability, robustness, coherence, convergence errors etc., have been studied in the literature. Leader selection for structural controllability is addressed in [4]-[8]. The problem of selecting multiple leaders for controllability of complex networks is studied in [4], while [5] uses graph-theoretic characteristics of a network such that the network is controllable by a given single-leader. In [6], sufficient graph condition for uncontrollability of a multi-agent system through leaders is given. A network equitable partition condition is also introduced in [6] to extend the controllability conditions to multi-leader case. Relaxed equitable partitions are used to translate controllability in terms of graph theory in [7]. A trade-off between number of leader nodes and control energy is given in [8]. A sub-modular approach to controllability problem with various performance criterion is discussed in papers like [9]-[12].

A. K. Mulla is with the Department of Electrical Engineering, Indian Institute of Technology Dharwad, India. Email: ameer@iitdh.ac.in.

D. Aannd, D. Chakraborty and M.N. Belur are with the Department of Electrical Engineering, Indian Institute of Technology Bombay, India. Email: {deepakanand,dc,belur}@ee.iitb.ac.in . This work was supported in part by Department of Science and Technology, Goverment of India and BRNS, India.

In [9], a leader-selection is proposed, using controllability-index as a performance criteria, for a multi-agent system for which the number of selected leaders is not sufficient for controllability. Minimizing convergence error through leader selection is studied in [11], [10]. A related work in [13] is to select leader to ensure coherence which is defined as ability of system to achieve consensus in the presence of noise. Also the leader selection problem is analyzed for robustness of the links between the agents in [14]. In [15], stochastically forced network was analyzed for mean-square deviation from the consensus using convex-relaxation of the boolean constraints of leader selection. In [16]-[19], energy constraints and robustness were used as deciding parameters for leader selection. In [18] and [19] information centrality is used as tool for deciding optimal leaders.

The speed of convergence on to consensus is recognized as an important criteria for leader selection in the literature, e.g. see [20], [21], where asymptotic approximations in terms of the Laplacian eigenvalues and eigenvectors are optimized. A related work to achieve consensus using series of stochastic matrices, in finite-time (not minimum) is shown in [26]. However, a method to select a leader so as to directly minimize the time required to achieve consensus seems to be unavailable.

The rest of the paper is organized as follows: In section II, graph preliminaries, time optimal feedback control synthesis and properties of isochronous sets in bang-bang control are reviewed. In section III setting of the minimum-time optimal control problem to multi-agent network is discussed and algorithms for minimum-time computation is given. In section IV application of leader selection algorithm is shown for practical problems. Finally we conclude in section V.

II. Preliminaries and Problem Formulation

Following are some standard notations used in the paper. Other notations will be defined as and when they appear:

- $\mathbb{Z}$ ($\mathbb{Z}^+$): Set of (positive) integers
- $\mathbb{Q}$: Set of rational numbers
- $\mathbb{R}$ ($\mathbb{R}^+$): Set of (non-negative) real numbers
- $\mathbb{R}^n$: n-dimensional euclidean space

A. Preliminaries

Consider a network of $n+1$ agents with single-integrator dynamics $\dot{x}_i = u_i$, for $i = 1,\cdots,n+1$ where $u_i \in \mathbb{R}$ is the input and $x_i \in \mathbb{R}$ is the state corresponding to i^{th} agent. The topology of the inter-agent communication network is given by a time-invariant undirected, weighted connected graph $\mathscr{G} = (\mathscr{V},\mathscr{E},\mathscr{A})$. Each node in the vertex set $\mathscr{V} = \{1,2,\ldots,n+1\}$ represents an agent. An edge $(i,j) \in \mathscr{E} \subset \mathscr{V}\times\mathscr{V}$ indicates the communication between i^{th} and j^{th} agent. The weight of the edge (i,j) is denoted by $a_{ij} \in \mathbb{R}^+$. If $(i,j) \in \mathscr{E}$, then $a_{ij} > 0$ and i^{th} and j^{th} agents are referred as neighbors of each other. Otherwise, if $(i,j) \notin \mathscr{E}$, then $a_{ij} = 0$. The set of neighbors of the i^{th} agent is represented by $\mathscr{N}_i$. The agents interact with each other through the control law $u_i = \sum_{j\in\mathscr{N}_i} a_{ij}(x_j - x_i)$. The adjacency matrix of $\mathscr{G}$ is denoted by $\mathscr{A} \in \mathbb{R}^{(n+1)\times(n+1)}$. The weighted degree of a node i is given as $d_i = \sum_{j=1}^{n+1} a_{ij}$, whereas the degree matrix $\mathscr{D}$ is defined as $\mathscr{D} := diag\{d_1, d_2, \cdots, d_{n+1}\}$. The Laplacian matrix is defined as $L := \mathscr{D} - \mathscr{A}$. For undirected and connected graphs, the Laplacian is symmetric with a simple zero eigenvalue and all other eigenvalues are real & positive [24]. The eigenvector corresponding to the zero eigenvalue is $\mathbf{1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T \in \mathbb{R}^{(n+1)}$ [24]. For control law u_i given above, it is well known that overall system is given by [3]

$$\dot{\mathbf{x}} = -L\mathbf{x} \tag{1}$$

where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \ldots & x_{n+1} \end{bmatrix}^T \in \mathbb{R}^{(n+1)}$ is the state vector for the system (1).

Definition 1: [24] Consensus is defined as the agreement of all agents to a particular value of state i.e. $\mathbf{x}(t) \to \alpha\mathbf{1}$ as $t \to \infty$ for some scalar $\alpha \in \mathbb{R}$ and $\mathbf{1} \in \mathbb{R}^{n+1}$.

Theorem 1: [24] System (1) achieves consensus asymptotically i.e., $\mathbf{x}(t) \to \alpha\mathbf{1}$ as $t \to \infty$, where $\alpha \in \mathbb{R}$ if and only if the communication graph $\mathscr{G}$ is connected.

B. Problem Formulation

We study the problem of selecting a particular node as the leader, so as to minimize the time required to reach consensus under the influence of an external bounded input applied to the selected node. If the i^{th} node is selected as the leader, the dynamics of i^{th} agent is given as $\dot{x}_i = \sum\limits_{j\in\mathscr{N}_i} a_{ij}(x_j - x_i) + u$, where, $u \in U := \{u(t) \in \mathbb{R} : |\, u(t)\,| \leq 1\}$ is an external bounded input. For the remaining agents the dynamics remains $\dot{x}_k = \sum\limits_{j\in\mathscr{N}_k} a_{kj}(x_k - x_j)$ for $k = 1,...,n+1$, $k \neq i$. Thus the dynamics of the multi-agent system is

$$\dot{\mathbf{x}} = -L\mathbf{x} + e_i u;\ \mathbf{x}(0) = \mathbf{x_0} \tag{2}$$

where $\mathbf{x_0} \in \mathbb{R}^{(n+1)}$ is the initial condition and e_i is the i^{th} column of identity matrix of size $n+1$. For the choice of i^{th} agent as the leader, the external control of the leader $u_i^*(t)$ that achieves the consensus in minimum time $T_{min}(e_i, \mathbf{x}_0)$ is obtained as the solution of the following time optimal control problem

$$\begin{aligned} \min_{u(t)\in U} \quad & t \\ \text{subject to } & \dot{\mathbf{x}} = -L\mathbf{x} + e_i u; \\ & \mathbf{x}(0) = \mathbf{x_0}; \\ & \mathbf{x}(t) = \alpha\mathbf{1} \text{ for some } \alpha \in \mathbb{R} \end{aligned} \tag{3}$$

Let $T_{min}(e_i, \mathbf{x}_0) := \min_{u(t)\in U} t$ in the above problem. For every agent chosen as the leader, we get different value of $T_{min}(e_i, \mathbf{x}_0)$. Our objective is to choose an agent as the leader so that we get the smallest amongst all possible $T_{min}(e_i, \mathbf{x}_0)$ i.e. $i^* = \arg\min\limits_i T_{min}(e_i, \mathbf{x}_0)$ Note that, the choice of the leader that achieves minimum time consensus depends only on the initial conditions of the agents, $\mathbf{x}_0$. Formally, the problem of choosing the leader for minimum time consensus can be stated as

Problem 1: Identify i^{th} agent such that, $i^* = \arg\min\limits_{i\in\{1,\ldots,(n+1)\}} T_{min}(e_i, \mathbf{x_0})$ where $T_{min}(L, e_i, \mathbf{x_0})$ is the solution of the minimum time control problem (3).

We define $T^*_{min}(\mathbf{x_0}) := T_{min}(e_{i^*}, \mathbf{x_0})$. This problem appears hard at first sight, since it seems that one needs to solve $n+1$ time optimal state transfer problem for each initial condition $\mathbf{x_0}$. However, we show that, it becomes tractable through the use of the Groebner basis based algorithm proposed in [22]. Next, we briefly discuss about the isochronous sets arising in time optimal control [23], followed by a review of the algorithm from [22] in subsection II-D.

C. Null-controllable sets and Isochronous sets

Consider an n-dimensional LTI system given below:

$$\dot{\mathbf{x}} = A\mathbf{x} + bu; \ \mathbf{x}(0) = \mathbf{x_0} \tag{4}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $\mathbf{x}_0$ is the initial condition and $u \in U := \{u(t) \in \mathbb{R} : | u(t) | \leq 1 \, \forall t \in [0,\infty)\}$ is the set of admissible inputs. For (4) we define the *reachable set* as follows:

Definition 2: [25] The *reachable set* to origin at time $T > 0$ (denoted by $R_0(T)$) is the set of all initial states $\mathbf{x_0} \in \mathbb{R}^n$, which can be driven to origin using input $u(t) \in U$ in time T. That is, $R_0(T) = \{-\int_0^T e^{-A\tau} bd\tau \, : \, u(t) \in U\}$.

Definition 3: The *null-controllable region* (denoted by X_0) is the set of all the initial conditions that can be driven to origin using admissible input i.e. $u(t) \in U$.

Lemma 1: $X_0 = \bigcup_{t\in[0,\infty)} R_0(t)$.

Lemma 2: [22] For stable system, if $T_1 \leq T_2 \leq \cdots < \infty$, then $R_0(T_1) \subseteq R_0(T_2) \subseteq \cdots \subset X = \mathbb{R}^n$.

Lemma 3: [25] $R_0(T)$ is closed, convex and symmetric about the origin.

Lemma 4: [25] For each $\mathbf{b}_i, i = 1,...,n$, if $\mathbf{z}_0 \in \text{int}(R_0(T_1))$, the minimum time required to reach origin, $T_{min}(\Lambda, \mathbf{b}_i, \mathbf{z}_0) < T_1$.

Definition 4: The *isochronous set* at time T, (denoted by $\mathscr{C}(T)$) is the set of all initial conditions from which, minimum time required to drive the states to origin is T.

Lemma 5: The isochronous set at time T is the boundary of the reachable set at time T. ($\mathscr{C}(T) = \partial R_0(T)$)

Proof: The boundary of the reachable set at time T i.e. $\partial R_0(T)$ is the set of *all* the initial conditions for which there exists unique bang-bang control with $n-1$ switches that drives the states to origin at time T [25]. Further, such a control is time-optimal control [22]. Thus, $\partial R_0(T) \in \mathscr{C}(T)$. From principle of optimality, it is well known that, the time optimal control is necessarily unique bang-bang with at-most $n-1$ switches, which implies $\mathscr{C}(T) = \partial R_0(T)$. ■

Using Lemma 2 and 5, the following result follows:

Lemma 6: For a system of the form (4), if $T_1 < T_2$, then $\mathscr{C}(T_1)$ is encompassed by $\mathscr{C}(T_2)$.

D. Characterization of Isochronous Sets

Consider a controllable LTI system of the form (4):

Assumptions

- A has distinct negative rational eigenvalues i.e. $\lambda_i(A) \in \mathbb{Q}\setminus\{0\}$ for $i = 1,\cdots,n$ and $\lambda_i \neq \lambda_j$ for $i \neq j$.
- (A,b) pair is controllable.

Since A is assumed to have distinct real eigenvalues, A can be diagonalized using similarity transform. Thus, without loss of generality, A is assumed to be diagonal. Let $T_{min}(A,b,\mathbf{x}_0)$ denote the minimum time required for driving the states of (2) from $\mathbf{x}_0$ to the origin. From Lemma 5, it is clear that, $\mathscr{C}(T) = \partial R_0(T)$. Thus, characterization of $\partial R_0(T)$, automatically characterizes $\mathscr{C}(T)$. Characterization of isochronous set at time T, requires computation of the initial states $\mathbf{x}_0$ for which $T_{min}(A,b,\mathbf{x}_0) = T$. By Pontryagin's Maximum Principle (PMP), the time optimal control for state transfer in system (2), switches between the extremes of the admissible values (± 1) and at most $n-1$ times [22]. The following function characterizes the states that can be driven to the origin using bang-bang input (± 1) with $n-1$ switches.

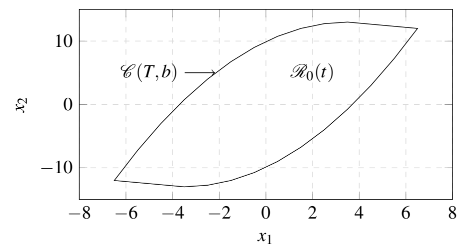


Fig. 1. Formation of isochronous set for (6) with $T = 1.61$).

$$F^{\pm}(t_1,\cdots,t_n) = \pm(-\int_0^{t_1} + \int_{t_1}^{t_2} \ldots + (-1)^n \int_{t_{n-1}}^{t_n} \quad)e^{-A\tau}bd\tau$$

with $0 \leq t_1 \leq \ldots \leq t_n < \infty$ and the sign depends on the sign of the initial input $u(0)$. Let b_j be the j^{th} element of vector b and $f_j^{\pm}$ is $\frac{1}{b_j}$ times the j^{th} component of $F^{\pm}(t_1,\cdots,t_n)$. Then,

$$x_j = b_j \cdot f_j^{\pm}(e^{-\lambda_j t_1}, e^{-\lambda_j t_2} \cdots e^{-\lambda_j t_n}) \text{ for } j = 1,...,n+1$$
$$\text{with } 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < \infty \tag{5}$$

Using these notations, an algorithm to characterize $\partial R_0(T)$, and hence $\mathscr{C}(T)$ is proposed in [22].

Example 2: Consider an LTI system of form (4) satisfying assumptions in II-D, with

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6}$$

The isochronous set obtained, using the Algorithm proposed in [22], for $T = 1.61$ is shown in Figure 1.

We will show later that, for a multi-agent system of the form (2), we can obtain transformed system of the form (4) satisfying the assumptions on A. The choice of i^{th} agent, $i = 1,...,n+1$, as the leader gives different b's in (4) (say b_i). The states of the multi-agent system can be driven to consensus (equivalently, the states of the transformed system are driven to origin) in finite time using i^{th} agent as the leader if and only if the system is controllable from i^{th} agent. If there are p such agents, there are p isochronous sets at time T. Let the set of all states with norm r be denoted by $\mathscr{S}(r)$, i.e. $\mathscr{S}(r) := \mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|| = r \in \mathbb{R}, r \geq 0$. Observe that, for each $r \geq 0$, $\mathscr{S}(r)$ is an $n-1$ dimensional hypersphere of radius r, centered at the origin of the state-space $\mathbb{R}^n$ We will show that, the locus of pairwise intersections of the isochronous sets for different b_i's partitions $\mathscr{S}(r), \forall r > 0$ in the state-space of the transformed system in to multiple disjoint sets. For the initial conditions in each such partitions, there is a unique b_i (and hence the unique leader) that drives the states to the origin (hence, achieves the consensus) in

$$\min_i T_{min}(A, b_i, \mathbf{x}_0)$$

III. LEADER-SELECTION ALGORITHM

Our objective is to choose an agent as the leader such that

$$i^* = \arg \min_{i \in \{1,\dots,(n+1)\}} \min_{u(t)\in U} t$$

subject to $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(t) = \alpha \mathbf{1}$ for some $\alpha \in \mathbb{R}$. We use the isochronous sets characterized in Section II to achieve the objective. In system (2) the system matrix is $-L$ which is symmetric. One can diagonalize $-L$, using the matrix $E := [v_1, v_2, \cdots, v_{n+1}] \in \mathbb{R}^{(n+1)\times(n+1)}$ where $(-L)v_i = \lambda_i v_i$ with $v_i{}^T v_j = 0$ for $i \neq j$ and $||v_i|| = 1$. We assume eigenvalues are arranged as $\lambda_1 \le \lambda_2 \le \lambda_2 \le \cdots \lambda_n < \lambda_{n+1} = 0$. From $Lv_{n+1} = 0 = L\mathbf{1}$, we have $v_{n+1} = \frac{1}{\sqrt{n+1}}\mathbf{1}$.

In order to apply the discussion in previous section onto system (2), it must have all non-zero eigenvalues. Thus, we transform $\mathbf{x}$ as $\mathbf{x} = P\mathbf{z}$ where $P = [v_1, \cdots, v_n] \in \mathbb{R}^{(n+1)\times n}$. Define $\langle w_1 \rangle := span\{w_1\}$ where w_1 is any vector and let $\mathbf{1}^{\perp} := \langle v_1, \cdots, v_n \rangle$ be the n-dimensional subspace of $\mathbb{R}^{(n+1)}$ orthogonal to $\langle \mathbf{1} \rangle$. This transformation removes the zero eigenvalue. The transformed dynamics is given as

$$\dot{\mathbf{z}} = \Lambda \mathbf{z} + \mathbf{b}_i u; \;\; \mathbf{z}(0) = \mathbf{z}_0 \tag{7}$$

where $\mathbf{z} = [z_1 \cdots z_n]^T$, $u \in U := \{u(t) \in \mathbb{R} \mid u(t) \mid\le 1\}$, $\Lambda = P^T(-L)P = \text{diag}\{\lambda_1, \cdots, \lambda_n\} \in \mathbb{R}^{n\times n}$, $\mathbf{b}_i = P^T e_i \in \mathbb{R}^n$ and $\mathbf{z}_0 = P^T \mathbf{x}_0$. Let $\mathbf{b}_i(j)$ $j = 1,...,n$ denote the j^{th} element of $\mathbf{b}_i$.

Lemma 7: The system (L, e_i) is controllable if and only if $\mathbf{b}_i(j) \neq 0$ for $j \in \{1, \cdots, n\}$.

Proof: Given (L, e_i) is controllable $\Leftrightarrow (E^T LE, E^T e_i) = (diag\{-\Lambda, 0\}, [\mathbf{b}_i, *]^T)$ is controllable. Since $diag\{-\Lambda, 0\}$ is diagonal, none of the components of $[\mathbf{b}_i, *]^T$ can be zero i.e. $\mathbf{b}_i(j) \neq 0$ for all $j \in \{1, \cdots n\}$). ■

The eigenvalues of Laplacian are real. But for the purpose of computations, taking finite precision one can approximate the irrational eigenvalues by an arbitrarily close rational number.

Remark 1: Consensus of system (2) i.e. $\mathbf{x}(t) \to \alpha \mathbf{1}$ as $t \to \infty$, for some $\alpha \in \mathbb{R}$; is equivalent to driving the system (7) to the origin i.e. $\mathbf{z}(t) \to 0$ as $t \to \infty$ [8].

After the transformation, the leader-selection problem can be rewritten as:

Problem 2: Find

$$i^* = \arg \min_{i\in\{1,\dots,n+1\}} T_{min}(\Lambda, \mathbf{b}_i, \mathbf{z}_0)$$

where $T_{min}(\Lambda, \mathbf{b}_i, \mathbf{z}_0)$ is the solution of the following minimization problem

$$\begin{aligned} T_{min}(\Lambda, \mathbf{b}_i, \mathbf{z}_0) = \min_{u(\tau)\in U\ \forall \tau \in [0,t]} \quad & t \\ \text{subject to} \quad & \dot{\mathbf{z}} = \Lambda\mathbf{z} + \mathbf{b}_i u; \\ & \mathbf{z}(0) = \mathbf{z}_0; \\ & \mathbf{z}(t) = 0 \end{aligned}$$

We define a set $B_c := \{\mathbf{b}_i : (\Lambda, \mathbf{b}_i) \text{ is controllable}\}$. Denote the cardinality of B_c by $|B_c|$. To choose the optimal leader, we need to identify i^* such that

$$T_{min}(\Lambda, \mathbf{b}_{i^*}, \mathbf{z}_0) = \min_{\mathbf{b_i}\in B_c} T_{min}(\Lambda, \mathbf{b}_i, \mathbf{z}_0)$$

One way to identify such i^* is to compute $T_{min}(\Lambda, \mathbf{b}_i)$ for each choice of $\mathbf{b}_i \in B_c$ and choose the one with least value of $T_{min}(\Lambda, \mathbf{b}_i)$. However this requires substantial computation online, i.e. after the initial condition is known, making it intractable for most realistic situations.

An important point to note here is that, as shown in (5), the elements of $\mathbf{b}_i$ appear as the constant multipliers in the parametric representation of the states. We define new states as $y_j := \frac{z_j}{\mathbf{b}_i(j)}$ where $\mathbf{b}_i(j) \neq 0$ from Lemma 7. For system (7), we can rewrite equation (5) as

$$y_j = f_j^{\pm}(e^{-\lambda_j t_1}, e^{-\lambda_j t_2}, \cdots, e^{-\lambda_j t_n}) \quad \text{where } 1 \le t_1 \le \cdots \le t_n < \infty \tag{8}$$

Observe that, the parametric representation y_j is independent of $\mathbf{b}_i$. The parametric representation in terms of $\mathbf{z}$ for different $\mathbf{b}_i$'s can be obtained by appropriate substitutions for y_j's.

Next we see how to exploit isochronous sets characterized in Section (II-D) for solving our problem. The expressions for isochronous sets in terms of y_j's can be computed using Algorithm given in [22]. The isochronous sets for each $\mathbf{b}_i$ are computed easily by appropriate substitution. Note that for $|B_c| = p$ we have $2p$ such elements (two for each choice of $\mathbf{b}_i$). Let the two isochronous surfaces at time t for $\mathbf{b}_i$ be denoted by $\mathscr{C}_i^+(t)$ and $\mathscr{C}_i^-(t)$. Similarly, let reachable set at time t for $\mathbf{b}_i$ be denoted by $R_0^i(t)$.

Theorem 3: If $|B_c| = p$, the loci of pairwise intersections of the isochronous sets for $t \in [0, \infty)$, (i.e. $\bigcup_{t\in[0,\infty)} \left(\mathscr{C}_i^{\pm}(t) \cap \mathscr{C}_j^{\pm}(t)\right)$ for $1 \le i,j \le p$, $i \neq j$), partition $\mathscr{S}(r) \subset \mathbb{R}^n$, in $\binom{2p}{2} - p$ parts, for each $r > 0$.

Proof: When $|B_c| = p$, there are $2p$ expressions representing isochronous surfaces at time t corresponding to i^{th} agent as the leader, $i = 1, \cdots, p$. These isochronous surfaces $\mathscr{C}_i^{\pm}(t)$ in terms of t and the states $\mathbf{z}$ can be computed using the method discussed in Section II-D. Note that, $\mathscr{C}_i^{\pm}(0) = \mathbf{0}$ i.e. the origin. Recall that, $R_0^i(t)$ is symmetric around origin and $\mathscr{C}_i^{\pm}(t)$ collectively form the boundary of $R_0^i(t)$. Thus, isochronous set is also symmetric about origin. Since each of $\mathscr{C}_i^{\pm}(t)$, $i = 1,...,n$ is continuous in t, the intersection of any two isochronous sets for $t \in [0, \infty)$ forms two surfaces passing through origin, namely, $S_{ij}^1 := \left(\mathscr{C}_i^+(t) \cap \mathscr{C}_j^+(t)\right) \cup \left(\mathscr{C}_i^-(t) \cap \mathscr{C}_j^-(t)\right)$ and $S_{ij}^2 := \left(\mathscr{C}_i^+(t) \cap \mathscr{C}_j^-(t)\right) \cup \left(\mathscr{C}_i^-(t) \cap \mathscr{C}_j^+(t)\right)$ for t varying from 0 to ∞. For each $r \in \mathbb{R}$, $r > 0$, the surfaces S_{ij}^1 and S_{ij}^2 partition $\mathscr{S}(r)$, in 4 partitions. The proof of the theorem follows by induction over p on this base case. ■

Theorem 4: For each partition, there is a unique choice of the leader that can drive the systems to consensus in minimum time.

Proof: Let P_i, $i = 1, \cdots, \binom{2p}{2} - p$ denote the partitions of $\mathscr{C}(r) \in \mathbb{R}^n$, for some $r > 0$, formed due to the loci of pairwise intersections of isochronous sets. Since the boundary of P_i is formed by the intersection of two isochronous sets, in each partition, for any time t there exists an agent j such that, $R_0^j(t) \cap P_i \supset R_0^k(t) \cap P_i$ for $k \in \{1, \cdots, p\}, k \neq j$. Using Lemma 2, we can show that $R_0^j(\tau) \cap P_i \supset R_0^k(\tau) \cap P_i$ for all $\tau > t$. From Lemma 4, it follows that, for an initial condition, $\mathbf{z}_0 \in P_i$,

$$T_{min}(-L, \mathbf{b}_j, \mathbf{z}_0) = \min_{k=1,\dots,n} T_{min}(-L, \mathbf{b}_k, \mathbf{z}_0)$$

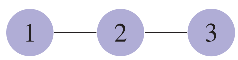
or equivalently, choice of j^{th} agent as the leader achieves consensus in minimum time. ■

Remark 2: Since $R_0^i(t)$ (and hence $\mathscr{C}_i(t)$) is symmetric about the origin, it can be shown that, if j^{th} agent is an optimal leader for a partition P_k with some $\mathbf{z}_0 \in P_k$, then j^{th} agent is also an optimal leader for the partition $P_{k'}$ such that $-\mathbf{z}_0 \in P_{k'}$.

Remark 3: In the examples demonstrated below, the loci of pair-wise intersections of the isochronous surfaces partition $\mathscr{S}(r) \subset \mathscr{R}^n$, for all $r > 0$ identically. That is, the loci partition the complete state-space, $\mathbb{R}^n$, in $\binom{2p}{2} - p$ parts. It is conjectured that, this property is due to the symmetry of the positions of the corresponding leaders in the graph.

IV. APPLICATION OF LEADER-SELECTION ALGORITHM & EXAMPLES

Example 5: Let us consider the following graph



For this graph the Laplacian matrix is given as

$$L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{9}$$

Let the initial condition be $\mathbf{x_0} = [-2.937\ 1.633\ 1.3048]^T$. After projecting out the zero eigenvalue (as shown in section III) the dynamics of form (7) is obtained with

$$\Lambda = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \tag{10}$$

After transformation $\mathbf{z_0} = [2\ 3]^T$. The input matrix for various choices of the leader node is shown below as columns of matrix B, where i^{th} column corresponds to i^{th} node as leader (i.e. the i^{th} column is $\mathbf{b}_i$).

$$B = \begin{bmatrix} -0.408 & 0.802 & -0.408 \\ -0.707 & 0 & 0.707 \end{bmatrix} \tag{11}$$

Parametric representation of states of the system is given as

$$F_2^{\pm}(t_1,t_2) = \begin{bmatrix} \mathbf{z}(1) \\ \mathbf{z}(2) \end{bmatrix} = \begin{bmatrix} \pm\mathbf{b}_i(1)(-\frac{2}{3}\theta_1^3 + \frac{1}{3}\theta_2^3 + \frac{1}{3}) \\ \pm\mathbf{b}_i(2)(-2\theta_1 + \theta_2 + 1) \end{bmatrix}$$

Using substitution $y_j = \frac{z_j(0)}{\mathbf{b_i}(j)}$ we obtain

$$\begin{aligned} y_1 \mp (-\frac{2}{3}\theta_1^3 + \frac{1}{3}\theta_2^3 + \frac{1}{3}) &= 0 \text{ and} \\ y_2 \mp (-2\theta_1 + \theta_2 + 1) &= 0 \end{aligned}$$

From the diagonal structure of Λ, it is clear that $(\Lambda, \mathbf{b_2})$ is uncontrollable. Form the set of controllable inputs choices i.e. $B_c = \{\mathbf{b}_1, \mathbf{b}_3\}$. In this case the isochronous sets intersect at $y_1 = 0$ and $y_2 = 0$. From here we get four partitions corresponding to the four quadrants of z_1z_2-plane as shown in Figure 2. We see that $\mathbf{z_0}$ lies in the region corresponding to $\mathbf{b_3}$. So node 3 is optimal leader.

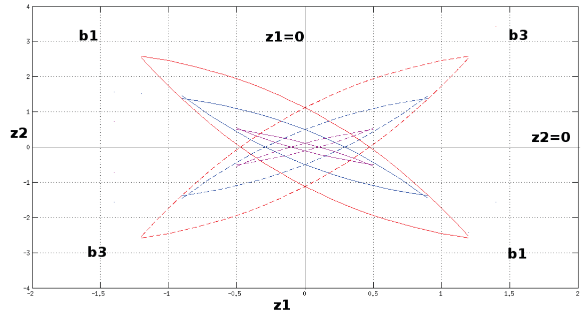
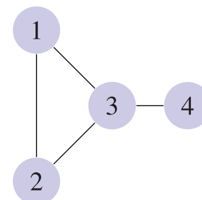


Fig. 2. Intersection of isochronous sets for 3-node path graph

Example 6: Consider the following graph



For the above graph, the projected dynamics of form (7) is given by $\Lambda = diag\{-4, -3, -1\}$. And the choices for the input matrix ($\mathbf{b}_i$'s) are given as columns of matrix B below.

$$B = \begin{bmatrix} 0.288 & 0.288 & -0.866 & 0.288 \\ 0.707 & -0.707 & 0 & 0 \\ -0.408 & -0.408 & 0 & 0.8165 \end{bmatrix}$$

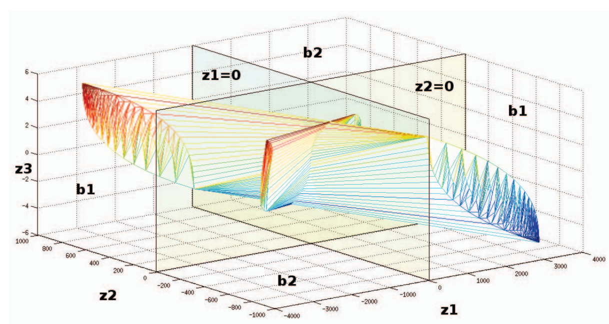


Fig. 3. Intersection of isochronous sets corresponding to Example 6.

The set of columns of B such that $(\Lambda, \mathbf{b}_i)$ is controllable is $B_c = \{\mathbf{b}_1, \mathbf{b}_2\}$. After solving for the intersection of isochronous sets for this case we get four regions partitioned by z_2z_3-plane and z_1z_3-plane as shown in Figure 6. There are two possible leaders: agent 1 and 2. The time optimal leader corresponding to each partition is marked in Figure 6. Let the given initial condition be $\mathbf{x}_0 = [2.0017\ -2.2409\ -0.8660\ 1.1052]^T$. The corresponding $\mathbf{z}_0 = [1\ 3\ 1]^T$. We see that this $\mathbf{z}_0$ lies in region corresponding to node 2. Thus node 2 is the optimal leader.
To verify our results, we can use the computation discussed in Section II-D, to actually calculate the minimum-time feedback control for each leader. Clearly this is not required to apply our algorithm for choosing the best leader, and is included here just for illustration purposes. For initial condition, $\mathbf{x_0} = [2.0017\ -2.2409\ -0.8660\ 1.1052]^T$ (and the corresponding $\mathbf{z_0} = [1\ 3\ 1]^T$), we get $T_{min}(\Lambda, \mathbf{b_1}, \mathbf{z_0}) =$

2.27 seconds, whereas $T_{min}(\Lambda, \mathbf{b_2}, \mathbf{z_0}) = 2.049$ seconds. This verifies that the time optimal leader for the given initial condition is node 2. When initial condition $\mathbf{x_0} = [1.872\ -0.9564\ -2.598\ 1.68]^T$ then $T_{min}(\Lambda, \mathbf{b_1}, \mathbf{z_0}) = 1.39$ seconds and $T_{min}(\Lambda, \mathbf{b_2}, \mathbf{z_0}) = 2.07$ seconds. So for this case node 1 is the optimal leader.

A method to synthesize time optimal feedback control using implicit expression for the switching surfaces was given in [22]. Using this method, we compute the time optimal feedback for the leader agent (node 2). The time domain plots of the resulting trajectories of the agents, with initial condition $\mathbf{x_0} = [2.0017\ -2.2409\ -0.8660\ 1.1052]^T$, achieving consensus in minimum time are presented in figure 4. It is seen from the figure that consensus is achieved in computed minimum time $T^*_{min}(L, \mathbf{x_0}) = 2.049$.

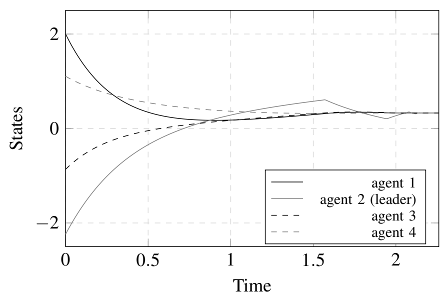


Fig. 4. Simulation plot for Example 6

V. Conclusion

In this paper, we propose a novel algorithm for leader selection in multi-agent networks based on the minimum time required to reach consensus. Groebner basis calculation is resource intensive and its complexity is the sole limitation of the proposed algorithm. However, in this case, we show that we need to compute the Groebner basis only once. Online computation (after knowing the initial condition) involves evaluation of a set of polynomials to determine the partition in which the initial condition belongs to. The time optimal leader is identified uniquely from this partition. The actual time optimal feedback control to be implemented by the selected leader can be computed using previously developed techniques. Due to the complexity of the associated Groebner basis computation, this method is currently limited to small groups of agents, and efforts to reduce complexity are subjects of ongoing and future research.

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