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New results and methods in balancing/spectral-zero-interpolation based model order reduction

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Abstract— **This paper studies spectral-zero-interpolation based methods for model order reduction (MOR). We focus on symmetric passive systems in which we prove new results about spectral zeros and balancing of state systems using extremal Algebraic Riccati Equation (ARE) solutions. We first show that for symmetric state space systems, not just are the poles and system zeros interlaced, but in fact, the poles, *spectral zeros* and system zeros are interlaced too. In the context of positive real *balanced* state-space realization, we introduce a notion of 'quasi-balanced', which turns out to inter-relate various extremal Riccati equation solutions of the literature. Finally, using the interpretation that the extremal ARE solutions indicate the minimum/maximum energy considerations during charging/discharging processes, we propose methods to choose a suitable subset of spectral zeros at which the reduced order system should interpolate the original transfer function in order to have lower error with respect to the H-infinity, H-2 and Hankel norms.**

Keywords: spectral zeros, maximum and minimum Riccati equation solutions, error in approximation, balancing methods in MOR

1. INTRODUCTION

Modeling and control of physical systems are generally done by construction of suitable dynamical systems. Modeling complex dynamical systems results in a model that often has a very large order: it is essential to obtain a lower order approximation so that simulation and controller design is computationally feasible. There are many methods of obtaining a reduced order approximation from a larger order model [10], [4], [5], [3]: the methods depend crucially on the criteria of measuring the approximation error. It is generally required that the reduced model preserve the stability and passivity of the original system. Many methods for model reduction with stability and passivity preservation have been proposed, for example, in [16], [7], [17]. Many of these methods focus on a specific class of systems. Of particular importance for this paper is the approach presented in Antoulas [2] and Sorensen [18] where a technique with efficient numerical algorithms to perform model order reduction is proposed which guarantees preservation of passivity and stability. The approach proposed in Antoulas [2] is based on positive-real interpolation through Krylov projection methods. Interpolation based methods stand the advantage of good computational efficiency in obtaining the reduced order model. The reduced order model is obtained by interpolating a subset of the spectral zeros of the original system but the technique requires computation of the spectral zeros before hand. Sorensen [18] develops a method of model reduction that does not require explicit interpolation of the spectral zeros in the implementation. This method relies on computing a suitable basis for the invariant subspace of a Hamiltonian matrix associated with the system for computing the reduced order model. Different reduced order models (ROMs) result depending on the set of spectral zeros chosen for interpolation. Assuming n is the order of the full-order system, and k is the desired order of ROM, then a central question remains as how to choose the k spectral zeros from the n spectral zeros.

This paper addresses the question of how to choose k spectral zeros for the ROM from n original spectral zeros. A notion of 'residues' of spectral zeros and criteria for selection of dominant spectral zeros based on their residues were proposed in [12]. Loosely speaking, spectral zeros close to the imaginary axis, being dominant, suggest their *retention*. On the other hand, arguing by controllability/observability Gramian considerations, a small residue in the partial fraction expansion suggests *non-retention*. This paper makes this intuitive method more concrete. We first obtain and prove new results in the context of spectral zeros and balancing methods and then propose methods to choose the spectral zeros and analyze this choice with respect to various notions of error between the original system and the reduced order system.

The rest of the paper is organized as follows. Section 2 contains some preliminaries required for the paper. In Section 3 we present and prove some new results in spectral zeros based balancing. Section 4 lists the different notions of errors employed for comparing ROM with original system and two new criteria for selection of spectral zeros are proposed here. Some concluding remarks are given in Section 5.

2. PRELIMINARIES

In this paper we consider linear time-invariant dynamical system Σ with i/s/o representation (A, B, C, D) and transfer function $G(s)$.

$$\Sigma : \begin{matrix} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{matrix} \ , \ G(s) = C(sI - A)^{-1}B + D \quad (1)$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times p}$.

A. Model order reduction problem

The problem of model order reduction is to approximate the dynamics of the system Σ represented by equation (1), to a lower dimension $k \ll n$ system $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}, \hat{D})$:

$$\hat{\Sigma} : \left\{ \begin{matrix} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \\ y(t) = \hat{C}\hat{x}(t) + \hat{D}u(t) \end{matrix} \right. , \quad \hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$$

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where $\hat{A} \in \mathbb{R}^{k\times k}, \hat{B} \in \mathbb{R}^{k\times p}, \hat{C} \in \mathbb{R}^{m\times k}, \hat{D} \in \mathbb{R}^{m\times p}$.

The input-output map of $\hat{\Sigma}$ should be close to Σ in an appropriate sense that is elaborated in the following section. Often, system features and structure such as stability, passivity, Hamiltonian structure and subsystem interconnectivity should be preserved in the reduced order system.

The following assumptions hold for the rest of this paper.

1) the system is SISO,
2) all eigenvalues of A are real and negative,
3) the system Σ is controllable, observable and passive.

B. Passivity and positive realness

Passive systems are a class of systems which absorb externally supplied energy, cannot generate energy, i.e. have no source within, but can store energy. Passivity of a system is defined as follows with respect to the external variables, input u and output y.

Definition 2.1. *[18] A system Σ is said to be passive if*

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, \mathrm{d}\tau \geqslant 0 \quad \textit{for all } t \in \mathbb{R} \textit{ and all } u \in \mathcal{L}_2(\mathbb{R}).$$

The system Σ is strictly passive if there exists $\delta > 0$ such that

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\mathrm{d}\tau \geqslant \delta \int_{-\infty}^{t} u(\tau)^T u(\tau)\mathrm{d}\tau \forall t \in \mathbb{R},\ u \in \mathcal{L}_2(\mathbb{R}).$$

For LTI systems, positive realness is linked to passivity.

Definition 2.2. *[1] A real rational transfer function $G(s)$ is said to be positive real if $G(s)$ satisfies:*

1) $G(s)$ is analytic for $Re(s) > 0$,
2) $G(s) + G(s)^* \geqslant 0$ for all $Re(s) > 0$

It is well-known that an LTI system Σ is passive if and only if its transfer function is positive real. In this paper, we assume strict passivity and this rules out spectral zeros on the imaginary axis: spectral zeros are defined next.

C. Spectral Zeros

The spectral zeros of a positive real systems with transfer function $G(s)$ are defined as $\lambda \in \mathbb{C}$ such that:

$$G(\lambda) + G(-\lambda)^T = 0.$$

Considering systems for which $(D + D^T)$ is invertible, the spectral zeros are the eigenvalues of the Hamiltonian matrix $H \in \mathbb{R}^{2n\times 2n}$ defined as:

$$H := \begin{bmatrix} A - B(D+D^T)^{-1}C & B(D+D^T)^{-1}B^T \\ -C^T(D+D^T)^{-1}C & -(A - B(D+D^T)^{-1}C)^T \end{bmatrix}. \quad (2)$$

Due to symmetry about the imaginary axis $j\mathbb{R}$, if H does not have $j\mathbb{R}$ eigenvalues, then of the $2n$ spectral zeros of the system, n-spectral zeros are in the $\mathbb{C}^-$ plane and their n mirror images in $\mathbb{C}^+$ plane.

We are interested in characterizing systems which are guaranteed to have real spectral zeros.

D. State Space Symmetric Realization

As in [1],[15], a minimal state space system (A, B, C, D) is said to be *state-space symmetric* if

$$A = A^T, \qquad C = B^T, \qquad D = D^T. \quad (3)$$

It may be noted that these systems are also called *internally symmetric* [22] and are distinct from externally symmetric systems where $G(s) = G(s)^T$. Of course, internally symmetric systems are also externally symmetric, but not conversely. These systems are part of a broader class of systems called relaxation systems [22]. These systems correspond to physical systems which have only one "type" of energy storage possibility, e.g. only potential energy or only kinetic energy, but not both. Another family of examples which have only one type of storage is that of RC or RL electrical networks. It has also been shown that systems with zeros interlacing poles (ZIP) also admit a state space symmetric realization [19].

As defined in [22, Proposition 4], a system Σ with transfer function $G(s)$ and state space realization (A, B, C, D) is said to be *state space symmetric realizable* if there exists a nonsingular matrix $T > 0$ such that

$$A = T^{-1}A^T T \leqslant 0, \qquad B = T^{-1}C^T, \qquad D = D^T.$$

The transfer function $G(s)$ of a state space symmetric realizable system can be written as

$$G(s) = g_\infty + \sum_{k=1}^{k=n} \frac{g_k}{s + \lambda_k} \quad (4)$$

where $g_\infty > 0$, $g_k > 0$ and $0 \leqslant \lambda_1 < \cdots < \lambda_n$. The above system has following symmetric state space realization

$$A = \mathrm{diag}(-\lambda_1, -\lambda_2, ..., -\lambda_n),$$
$$B^T = C = [g_1^{\frac{1}{2}}\ g_2^{\frac{1}{2}} \cdots g_n^{\frac{1}{2}}], \qquad D = g_\infty.$$

Symmetric state-space systems have been well-studied. A class of well-studied systems with collocated actuator and sensor [20], [9], [11] also utilize such symmetry $B = C^T$. Collocated sensors and actuators in decentralized control systems reduce the complexity and hence are economically appealing. In other words, symmetric state-space systems are an important class of systems and their study is motivated by many practical situations.

E. Algebraic Riccati Equation

The algebraic Riccati equation (ARE) for a system Σ in i/s/o realization A, B, C, D with respect to the passivity supply rate is

$$A^T K + KA + (KB - C^T)(D + D^T)^{-1}(B^T K - C) = 0. \quad (5)$$

The system Σ is positive real if and only if there exists a positive definite solution $K = K^T$ to the above equation. The set of ARE solutions is known to be a bounded and finite set with a maximum $K_{\max}$ and a minimum $K_{\min}$:

$$0 < K_{\min} \leqslant K \leqslant K_{\max}\ .$$

The solutions of the ARE in equation (5) can be computed from the $n-$dimensional invariant subspace of the associated Hamiltonian matrix, H as follows:.

$$H \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} R \quad \text{and define} \quad K := YX^{-1} \quad (6)$$

where $X, Y \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$ is a diagonal matrix with n eigenvalues of the Hamiltonian matrix, i.e. the n-spectral zeros. Each solution K can be associated with n-spectral zeros chosen from the set of $2n$ spectral zeros. Each solution K of the ARE gives rise to what we call a 'closed loop state transition matrix' A_K corresponding to the H-invariant subspace dictated by the $n-$chosen spectral zeros: we define A_K as

$$A_K := A - B(D + D^T)^{-1}C + B(D + D^T)^{-1}B^T K. \quad (7)$$

The eigenvalues of the state transition matrix A_K are the n spectral zeros chosen for construction of K. When either n stable or n anti-stable spectral zeros are chosen, we get an extremal solution of the ARE, i.e.

$$H\begin{bmatrix} X_+ \\ Y_+ \end{bmatrix} = \begin{bmatrix} X_+ \\ Y_+ \end{bmatrix} R_+ \text{ and } H\begin{bmatrix} X_- \\ Y_- \end{bmatrix} = \begin{bmatrix} X_- \\ Y_- \end{bmatrix} R_- \quad (8)$$

where $X_\pm, Y_\pm \in \mathbb{R}^{n\times n}$; $Re(\sigma(R_+)) > 0; Re(\sigma(R_-)) < 0$ Then $K_{\max} = Y_+ X_+^{-1}$ and $K_{\min} = Y_- X_-^{-1}$.

F. Model order reduction by spectral zeros interpolation

Antoulas [2] and Sorensen [18] propose algorithms for passivity preserving model reduction by interpolation of spectral zeros and subspace interpolation: we review this next.

1) Spectral zero interpolation method: Antoulas [2] proposed and proved that if a subset $(\mathscr{S}_{\hat{G}})$ of the spectral zeros $(\mathscr{S}_G)$are interpolated (preserved) in the reduced model, then the reduced model would also be passive.

Proposition 2.1. *[2, Lemma 4.1] If $\mathscr{S}_{\hat{G}} \subset \mathscr{S}_G$ and also that $\hat{G}(\lambda) = G(\lambda)$ for all $\lambda \in \mathscr{S}_{\hat{G}}$ and that $\hat{G}$ is a minimum degree rational interpolation of the values of G in the set $\mathscr{S}_{\hat{G}}$. Then the reduced system $\hat{\Sigma}$ with transfer function $\hat{G}$ is both stable and passive.*

This method requires computation of $2k$ spectral zeros to be interpolated. Given $2k$ spectral zeros $s_1, \ldots, s_{2k}$ the projection matrices are constructed as follows:

$$\tilde{V} = [(s_1 I_n - A)^{-1}B \cdots (s_k I_n - A)^{-1}B]$$
$$\tilde{W} = [(s_{k+1} I_n - A)^{-1}B \cdots (s_{2k} I_n - A)^{-1}B]. \quad (9)$$

Proposition 2.2. *The reduced system $\hat{\Sigma}$ obtained by projection as*

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV \quad (10)$$

where $V = \tilde{V}$ and $W = \tilde{W}(\tilde{V}^T\tilde{W})^{-1}$, is both stable and passive and interpolates the transfer function of Σ at points s_i [2, Proposition 4.1]:

$$\hat{G}(s_i) = G(s_i), \quad \text{for } i = 1, 2, ..., 2k.$$

2) Retention of minimum dissipation trajectories: Trentelman et al [21] presented a behavioral approach to passivity preserving model order reduction. Their method is based on construction of a basis for a selected invariant subspace corresponding to the chosen spectral zeros as presented in [18]. Their paper introduced the notion of '*sub-behavior of minimal dissipation*' of a system, where it was shown that if a particular part of this sub-behavior of minimal dissipation is inherited by the reduced system, then the reduced system is passive and stable. It also showed that by interpolation of spectral zeros in the ROM, the corresponding parts of the '*sub-behavior of minimal dissipation*' of the system are inherited by the ROM.

In order to explain further things in our paper, for conceptual simplicity, we propose a modified subspace based interpolation algorithm (Algorithm 2.1) for MOR using the invariant subspace of the Hamiltonian matrix H.

Algorithm 2.1 Modified subspace-interpolation algorithm

Input: Original system $\Sigma : (A, B, C, D)$, $k \ll n$.
Output: Reduced system $\hat{\Sigma}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$.

1: Compute the k-dimensional H-invariant stable subspace of H as:

$$H\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} R_- \text{ where } X, Y \in \mathbb{R}^{n\times k};$$
$$Re(\lambda) < 0 \text{ and } \lambda \in \sigma(R_-)$$

2: Compute $X^T Y =: QS^2Q^T$ with Q-orthogonal, S-diagonal.

3: Construct projection matrices

$$V := XQS^{-1} \text{ and } W := YQS^{-1}$$

4: The reduced order system is obtained as

$$\hat{A} = W^T AV; \ \hat{B} = W^T B; \ \hat{C} = CV; \hat{D} = D$$

3. New results in spectral zero based balancing

In this section we first present some new results of positive real balancing in systems with state space symmetric realization, then we introduce positive real quasi-balancing.

Definition 3.1. *[3, Section 7.5.4] A positive real system Σ with i/s/o representation (A, B, C, D) is said to be in positive real balanced realization if the extremal solutions of the ARE, $K_{\max}$ and $K_{\min}$, are related as*

$$K_{\max} = K_{\min}^{-1}.$$

Using the above preliminaries, the following result follows by straight-forward verification of balancing.

Lemma 3.2. *A positive real system in state space symmetric realization is positive real balanced.*

A. Positive real quasi-balancing

Due to spectral zeros interpolation, there is a certain balancing with respect to the extremal solutions of ARE in the ROMs, we define them as positive real quasi-balancing.

Definition 3.3. *A positive real system Σ is said to be in positive real quasi-balanced form if one of the extremal positive definite solutions of the ARE is identity. Positive real quasi-balanced Form-I if $K_{\max} = I$ and positive real quasi-balanced Form-II if $K_{\min} = I$.*

Lemma 3.4. *A strictly passive system Σ, with i/s/o realization (A, B, C, D) can be transformed into a positive real quasi-balanced realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ such that*

1) ***Form-I:*** *$K_{\max} = I$, if $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are constructed by Algorithm 2.1 from the H-invariant subspace of the n–anti-stable spectral zeros $(real(\lambda_i) > 0)$, and*
2) ***Form-II:*** *$K_{\min} = I$, if $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are constructed by Algorithm 2.1 from the H-invariant subspace of the n-stable spectral zeros $(real(\lambda_i) < 0)$*

The proof follows by a careful use of the matrices defined in Algorithm 2.1 while defining the model order reduction technique, and hence is skipped.

A system Σ represented by (A, B, C, D) in positive real quasi-balanced Form-I realization $K_{\max} = I$ can be converted to positive real quasi-balanced Form-II realization $K_{\min} = I$ with i/s/o $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ by basis transformation matrix T such that $\tilde{A} = T^{-1}AT$, $\tilde{B} = T^{-1}B$, $\tilde{C} = CT$, $\tilde{D} = T$ where $T = P^{-1}$ with P being the Cholesky factor of $K_{\max}$, i.e. $P^T P := K_{\max}$, with P upper-triangular and having positive diagonal entries.

The following lemma is helpful for proving our main result about spectral zeros being real (Theorem 3.6).

Lemma 3.5. *Consider the function*

$$f(x) := \sum_{i=1}^{n} \frac{q_i}{p_i - x^2}$$

with $q_i, p_i > 0$ for $i = 1, \ldots, n$, and $p_i \neq p_j$ for $i \neq j$. Then, $f(x)$ has only real roots.

The proof of the above lemma is skipped due to space constraints. The lemma is crucially used to prove the theorem below: one of the main results of this paper.

Theorem 3.6. *A SISO system Σ which admits a state space symmetric realization (A, B, C, D) has, not just all poles and zeros as real, but also all spectral zeros as real. Assume the system poles $-p_i$, system zeros $-z_i$ and stable spectral zeros $-s_i$ are indexed such that:*

$$p_1 < p_2 < \cdots < p_n, \quad z_1 < z_2 < \cdots < z_n, \quad s_1 < s_2 < \cdots < s_n$$

and assume, without loss of generality, $p_1 < z_1$. Then,

$$p_1 < s_1 < z_1 < p_2 < s_2 < \cdots s_n < z_n \ .$$

In other words, not just are the poles and zeros interlaced, but between every pair of pole-zero, there is also a stable spectral zero.

The proof uses the previous lemmas and Bolzano's theorem about change of signs of a polynomial function over an interval when the interval contains a root of the polynomial. We skip the proof for paucity of space.

4. Notions of error in model order reduction

The degree of closeness of the reduced system $\hat{\Sigma}$ with the original system Σ is measured by the respective system outputs over a range of inputs. Consider Figure 1.

The transfer function from u to δ_{error} is $G(s) - \hat{G}(s)$. There are many metrics one can use to evaluate the error system,

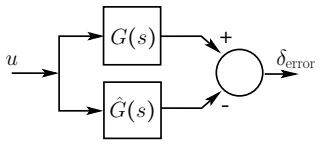

Fig. 1: Block diagram for error

three of the popular metrics are: $\mathcal{H}_\infty$-norm, $\mathcal{H}_2$-norm and Hankel-norm.

$\underline{\mathcal{H}_\infty - \textbf{norm}}$: The $\mathcal{H}_\infty$-norm of a stable transfer function $G(s)$ is defined as:

$$\|G(s)\|_{\mathcal{H}_\infty} := \max_{\omega \in \mathbb{R}} \|G(j\omega)\|_2 \tag{11}$$

where $\|\cdot\|_2$ denotes the induced 2-norm of a matrix.

$\underline{\mathcal{H}_2 - \textbf{norm}}$: The $\mathcal{H}_2$-norm of a stable transfer function $G(s)$ is defined as:

$$\|G(s)\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(j\omega)\|_F^2 d\omega \right)^{\frac{1}{2}} \tag{12}$$

where $\|\cdot\|_F$ denotes the Frobenius[1] norm of a matrix.

Hankel norm: The Hankel norm of a stable transfer function $G(s)$ is defined as:

$$\|G(s)\|_{\mathcal{H}nkl} := \left(\sup_{u \in \mathcal{L}_2 \setminus \{0\}} \frac{\int_0^\infty y^2 dt}{\int_{-\infty}^0 u^2 dt} \right)^{\frac{1}{2}} \tag{13}$$

and is known to be equal to $(\lambda_{\max}(PQ))^{\frac{1}{2}}$ where P and Q are respectively the controllability and observability Gramians of a given state space system.

A. Criterion for selection of spectral zeros

The closeness of the reduced model with the original system depends on choice of spectral zeros ($2k$ spectral zeros from $2n$-spectral zeros) selected for interpolation. Antoulas [2], Sorensen [18] and Trentelman et al [21] did not consider the question of a criterion for selection of spectral zeros. We illustrate this selection problem with an example. Consider a 4th order system given by the transfer function

$$G(s) = \frac{(s+1)(s+3)(s+5)(s+7)}{(s+2)(s+4)(s+6)(s+8)}.$$

The spectral zeros for the system are: $s_1 = \pm 7.20$; $s_2 = \pm 5.31$; $s_3 = \pm 3.41$; $s_4 = \pm 1.54$. We construct order 3 approximation of the above system by using algorithm 2.1. In total, 4 number of ROMs of order 3 each are possible by choosing set of 3 spectral zeros from set of 4. Figure 2 shows the bode plot of the original system and its order 3 ROMs, where SYS123 means ROM constructed by interpolating spectral zeros s_1, s_2, s_3. From Figure 2, it is evident that the choice of spectral zeros for interpolation is very critical for the closeness of the ROM to the original system

B. Spectral zeros and system poles

The spectral zeros of a strictly passive system with a biproper transfer function $G(s)$ lie within the range of the

[1]The Frobenius norm of a matrix is defined as the square-root of the sum of the squares of all matrix-entries' magnitudes.

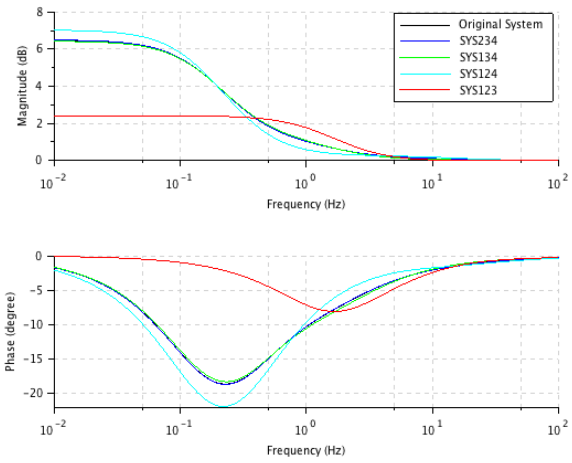

Fig. 2: Bode plot for original system and ROMs

poles and zeros. If we write the transfer function $G(s)$ in terms of its partial fractions then one can relate each spectral zero with a pole.

$$G(s) = 1 + \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \cdots + \frac{k_n}{s+p_n}; \;\; k_i > 0 \qquad (14)$$

The zeros z_i of the system are decided by the poles p_i and the partial fraction coefficient k_i. The spectral zeros of the system lie between each pole-zero pair i.e. $z_i < s_i < p_i$. There are $2n$ spectral zeros- $(s_1, s_2, \ldots, s_n; s_i < 0)$ and their mirror images $(-s_1, -s_2, \ldots, -s_n)$. One can relate each of spectral with the associated pole and the partial fraction coefficient. This helps in associating the state with the spectral zero.

Example 4.1. *Consider an order 4 system Σ with biproper transfer function $G(s)$:*

$$G(s) = 1 + \frac{k_1}{s+1000} + \frac{k_2}{s+100} + \frac{k_3}{s+10} + \frac{k_4}{s+1}; k_i > 0$$

When we keep $k_1 = k_2 = k_3 = k_4 = 1$, then the stable spectral zeros are $s_1 = -1000.50, s_2 = -100.50, s_3 = -10.49, s_4 = -1.38$ Now if we represent the transfer in symmetric state space realization:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1000 & 0 & 0 & 0 \\ 0 & -100 & 0 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} x(t) + 1.u(t) \end{aligned} \qquad (15)$$

For the system given in Example 4.1, one can associate spectral zeros (s_1, s_2, s_3, s_4) with the states (x_1, x_2, x_3, x_4). This will help in constructing a mode-wise criteria for selection of spectral zeros. The system is strictly passive when none of the spectral zeros are on the imaginary axis. The system becomes lossless when all the spectral zeros are on the imaginary axis. As the spectral zeros are moved away from the imaginary axis the system becomes more dissipative.

C. Minimum dissipation criterion for spectral zeros selection

The approach of MOR presented in [21], by retaining sub-behaviour of minimal dissipation can also be interpreted as deleting those part of the original system along which a relatively large amount of dissipation takes place. The extremal positive definite solutions of the ARE $K_{\min}$ and $K_{\max}$ have special significance in terms of the energy dissipation by the system. For a given $a \in \mathbb{R}^n$, consider $\mathfrak{B}_a$, the set of all continuous system trajectories (u, x, y) satisfying equation (1) with $x(0) = a$. Then,

$$a^T K_{\max} a = \inf_{\substack{(u,x,y)\in\mathfrak{B}_a, \\ x(-\infty)=0}} \int_{-\infty}^{0} 2uy \, dt, \qquad (16)$$

$$a^T K_{\min} a = \sup_{\substack{(u,x,y)\in\mathfrak{B}_a, \\ x(\infty)=0}} \int_{0}^{\infty} -2uy \, dt. \qquad (17)$$

Thus $a^T K_{\max} a$ is the minimum energy required to reach a state $x(0) = a$ from state of rest $x(-\infty) = 0$ and $a^T K_{\min} a$ is the maximum energy that can be extracted as the system is brought to rest $x(\infty) = 0$ from state $x(0) = a$. The states are generally the energy storage elements, for example in electrical networks capacitors (C) and inductors (L) store energy and they define the states. Now, the energy required to reach a state is the sum of the energy stored in the state and the energy dissipated during reaching the state. Similarly, energy that can be extracted from a state is the difference of the energy stored in the state and the energy dissipated during reaching the state of rest. Therefore, if the given system is in state space symmetric realization, then one can utilize either of $K_{\max}$ or $K_{\min}$ to construct mode-wise measures of the dissipation of a system. Since, $K_{\max} = Y_+ X_+^{-1} > I$ where the columns of $\left[\begin{smallmatrix} X_+ \\ Y_+ \end{smallmatrix}\right]$ are unit 2–norm length, X_+ becomes ill-conditioned as system dissipation increases. Hence, computation of $K_{\max}$ becomes error prone. Therefore, we propose minimum dissipation criterion for selection of spectral zeros based on $K_{\min}$, wherein the least dissipative spectral zeros are to be chosen for interpolation for constructing ROM. We define two new "measures" for choosing the minimum dissipating spectral zero. We use the state space symmetric realization as relate the spectral zeros with the states.

1) Measure I (α_i): When the system is positive real balanced $K_{\max} = K_{\min}^{-1}$, the columns of $K_{\min}$ give a measure of the dissipation for the corresponding spectral zero linked with the pole as A is diagonal. Now, if we chose to construct the state transition matrix $A_{K_{\min}}$ with respect to $K_{\min}$ as defined in equation (7), then the eigenvalues of $A_{K_{\min}}$ are the n-stable spectral zeros. We use the eigenvectors of $A_{K_{\min}}$ to define a *mode-wise dissipativity measure* α_i.

Definition 4.2. *For a given strictly passive system Σ with biproper transfer function $G(s)$ and symmetric state space realization (A, B, C, D), with $(v_1, v_2, \ldots, v_n)$ being the 2–norm unit length eigenvectors of $A_{K_{\min}}$, the mode-wise dissipativity measure of each stable spectral zero s_i, is defined as α_i:*

$$\alpha_i := v_i^T K_{\min} v_i.$$

A higher value of α_i indicates the corresponding spectral zero s_i is less dissipative as more energy can be extracted

from the corresponding state. The criteria says that k spectral zeros with largest α_i to be chosen for $k-$order reduced model.

2) Measure II (β_i): We compute the weighted inner product of each column of $K_{\min}$ with respect to the system poles and define it as *mode-wise normalized energy measure* β_i for each spectral zero s_i.

Definition 4.3. *For given strictly passive system Σ with biproper transfer function $G(s)$ and symmetric state space realization (A, B, C, D), such that the state x_i corresponds to the pole p_i and spectral zero pair $(s_i, -s_i)$, the mode-wise normalized energy measure is defined as β_i:*

$$\beta_i := |p_i| \ \|K_{\min} e_i\|_2^2$$

where e_i is the i^{th} column of the identity matrix.

Similarly, here also higher β_i means the corresponding spectral zero is less dissipative. Hence, for $k-$order reduced model spectral zeros with the largest β_i are to be chosen.

5. Conclusion

We studied the class of symmetric passive systems and obtained new results on spectral zeros and on balancing of state-space realizations of such systems. We introduced positive real quasi-balancing and showed in Lemma 3.4 that a system can be transformed to positive real quasi-balanced form using MOR algorithm by interpolation of spectral zeros. We also inter-related the matrices that help transform between positive real quasi-balanced realizations Form-I and Form-II. Then in Theorem 3.6 we proved that a SISO system which admits state space symmetric realization has only real spectral zeros and further that each spectral zero lies between a pole-zero pair: this was one of the main results of this paper.

In the context of model order reduction from a large original order n to a smaller order $k \ll n$, we proposed methods to choose k spectral zeros from n based on the *relative* dissipativity of each spectral zero. We formulated two measures of dissipativity of each spectral zero: mode-wise dissipativity measure α_i and mode-wise normalized energy measure β_i. Preliminary analysis of the choices of spectral zeros with respect to these measures using error norms: $\mathcal{H}_\infty$, $\mathcal{H}_2$ and the Hankel norms have shown encouraging results. A thorough analysis about the suitability of the two measures with respect to the particular error norms is a matter of future work.

A few remarks about computational complexity. It was shown in [18] about how the subspace interpolation does *not* require all spectral zeros (and the eigenvectors) to be computed, but just the spectral zeros that are to be retained in the ROM. This was through the computation of the so-called *partial* Schur form instead of the full Schur form. In order to choose the spectral zeros to be retained, only the measures proposed in our paper need to be estimated instead of the eigenvectors corresponding to these spectral zeros. Methods to efficiently estimate these measures for each spectral zero needs further investigation.

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