# **Target Controllability of Structured Systems**

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Abstract-This paper deals with controllability of structured linear time-invariant systems. Controllability of large complex systems are often difficult to attain as a substantial part of nodes in the network require to be actuated and the amount of energy required is large. Moreover, often it is unnecessary to control the entire dynamics of the network: instead, only certain portions of the network need to be controlled. This concept is known as target control. In this paper, we address target control for large complex systems using their network topology. We prove that if there exists one numerical system in the family of the network that is target controllable, then almost all systems in the family are target controllable. This result thus concludes that target controllability is a generic property. Then we propose a bipartitematching based condition to determine target controllability of a subset of nodes in a single input network based on the generic rank of the controllable subspace. Finally, we provide experimental analysis to complement our results using real-world data sets.

# 1. INTRODUCTION

Complex dynamical systems are indispensable constituents in social, biological, and technological fields. Control of complex networks is extensively studied over the last few decades on account of its wide applicability [1]. Typically, these networks consist of a large number of nodes and the evolution of the internal state of a node is determined by its interaction pattern with other nodes in the network. The network is said to be *controllable* if the states of all the nodes can be driven to any specified state using an appropriate input. The large size of the network often makes it difficult or rather unnecessary to control the full network by virtue of the amount of actuation required and controlling some of the nodes may not be necessary. In many networks, it is not essential to control all the nodes (states) to achieve the desired result: controlling only a subset of nodes which is decisive to the specified task is sufficient [2]. For example, suppose a credit card company wants to promote its product on social network, it may only want to target individuals in a certain economic bracket. Controlling a subset of nodes in a network instead of the full network is referred to as target control [2].

Given a general network and a subset of nodes, there do not exist conditions to verify if the given node set is target controllable, except in two special cases: (1) tree networks [2] and (2) symmetric networks [3]. Our objective here is to find conditions for verifying the target controllability of a set of nodes in a general network. Direct application of existing control theory concepts in the analysis of complex networks has many challenges [1]. The combinatorial complexity of employing the known concepts and conditions to address various design and optimal selection problems makes it applicable only to small networks. Additionally, often the full system information is unknown in large complex networks and only the graph of the network is known: either it is not possible to measure the link weights of the graph or the link weights are time-varying [4]. Consequently, exact numerical calculations cannot be done for such systems and thus many control problems are approached in a *structural* framework. The strength of structural analysis is that it requires only the topology of the network to

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<sup>\*</sup>Department of Electrical Engineering, Indian Institute of Technology Bombay, India. Email: {kryashashwi,chaporkar, belur}@ee.iitb.ac.in. guarantee various system theoretic properties for a family of systems associated with the network. While there exist graph-theoretic conditions to verify system theoretic properties of networks, including controllability, decentralized control, and disturbance decoupling, using topological characteristics of the network [5], there is no known graph-theoretic condition to verify target controllability.

In this paper, we study target controllability of structured linear time-invariant (LTI) systems. We perform our analysis of target control using the structural framework. We first prove that target controllability is a *generic* property. Structural controllability of a network is a generic property does not guarantee target controllability to be generic. Then we propose a graph-theoretic condition to verify target controllability of a subset of states. In our analysis, we use the concepts of the *generic rank* of controllable subspace [6] and the *weighted cycle partition* condition to calculate the generic rank of the controllable subspace of a structured system [7].

The contributions of this paper are as follows:

- We prove that target controllability of structured systems is a generic property. Thus, if a subset of nodes of a network is target controllable, then the states corresponding to those nodes for *almost all* numerical systems in the family of the network can be driven to any desired state (numerical value) from any initial value using an appropriate input.
- We provide a graph-theoretic condition to identify a subset of controllable nodes in a large complex network when the number of inputs is one. Our condition serves two purposes: (i) it verifies whether a given set of nodes is target controllable, and (ii) it identifies a set of nodes in the network that are target controllable. Hence, one can compute the generic rank of a submatrix of the controllability matrix using the condition proposed in this paper.
- We perform experimental analysis complementing the proposed results on real-world data sets for four networks.

This paper is organized is as follows: Section 2 summarizes the related work. Section 3 presents few notations and preliminaries, and the problem formulation. Section 4 gives the main results of this paper. Section 5 presents the simulations and experimental results obtained. Section 6 gives the final concluding remarks of the paper and future directions.

# 2. RELATED WORK

Structural analysis is a widely used framework for the analysis of large-scale LTI dynamical systems for solving various controltheoretic problems. Specifically, structural analysis is used in optimal input selection [8], [9], [10], optimal feedback selection [11], [12], [13], and optimal selection of interconnection pattern [14], [15] before. Target controllability and actuator selection for target control are addressed using structural analysis in many papers. A *greedy algorithm* is proposed in [2] to find an approximate solution to the minimum input selection for target control. Note that, though [2] applied the greedy algorithm on various real-time systems, it does not provide any theoretical guarantee for the performance of the algorithm. A *preferential matching* based algorithm is given in [16] for finding the input nodes for target control: this algorithm is experimentally shown to perform better than the greedy algorithm given in [2]. Paper [17] proposed an algorithm for drug target control which is based on an *integer linear programming combined with an MCMC based sampling*.

Target control is also studied in the context of *strong structural controllability*. The papers [18] and [19] addressed target control for *strong structural controllability* of a network satisfying a special network topology. The notion of strong structural controllability of a network is same as conventional controllability of numerical systems. The amount of energy required to control a target node from a remote input is characterized in [20].

Note that, some of the papers discussed above consider networks with some special graph topology, and some provide heuristics for finding a suboptimal solution. The key limitation in the existing literature of target control is that there does not exist a necessary and sufficient condition to determine target controllability of a set of nodes. As a consequence of this limitation, only suboptimal solutions can be obtained for all optimization and design problems in target control. In this direction, a necessary and sufficient condition to verify target controllability of symmetric (undirected) networks, i.e.,  $\bar{A} = \bar{A}^{T}$  is given recently in [3]. The objective of this paper is to formulate a necessary and sufficient condition for determining the target controllability of a set of nodes in a general network ( $\bar{A}, \bar{B}$ ), when there are no topological assumptions.

#### 3. PROBLEM FORMULATION AND BACKGROUND

# A. Problem Formulation

Consider an LTI dynamical system  $\dot{x}(t) = Ax(t) + Bu(t)$ , y(t) = Cx(t), where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ , and  $C \in \mathbb{R}^{p \times n}$  denote the state, input, and output matrices, respectively. Also,  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ , and  $y \in \mathbb{R}^p$  denote the state, input, and output vectors of the system, respectively. Here  $\mathbb{R}$  denotes the set of real numbers. In a network consisting of nodes  $\{x_1, \ldots, x_n\}$ , target controllability aims to control a subset of nodes, say  $T = \{x_{i_1}, \ldots, x_{i_k}\} \subset \{x_1, \ldots, x_n\}$ . The target controllability problem for linear networks is a particular case of *output controllability* [21] and a generalization of the full controllability problem, for  $T = \{x_1, \ldots, x_n\}$ , which requires the control over the entire system [2]. To define target controllability, we first define the notion of output controllability below: for a given initial condition  $x_0$  and an input function u(t), the output at time t is denoted by  $y_u(t, x_0)$ .

**Definition 1.** [19] The system (A,B,C) is said to be output controllable if for all  $x_0 \in \mathbb{R}^n$  and  $\hat{y} \in \mathbb{R}^p$ , there exists an input function u and finite time  $t_f$  such that  $y_u(t_f, x_0) = \hat{y}$ .

Using Definition 1, the system (A,B,C) is output controllable if and only if the dimension of the *output controllable space*  $\mathcal{OC}(A,B,C)$  is p [22], where  $\mathcal{OC}(A,B,C) :=$  [*CB CAB CA*<sup>2</sup>*B*,...,*CA*<sup>*n*-1</sup>*B*]. Using output controllability, now we define target controllability of (A,B,T).

**Definition 2.** [2] The system (A,B,T) with  $T = \{x_{i_1},...,x_{i_k}\} \subset \{x_1,...,x_n\}$  is said to be target controllable if the system  $(A,B,C_T)$  is output controllable, where  $C_T$  is  $k \times n$  matrix whose  $j^{\text{th}}$  row equals the  $i_i^{\text{th}}$  row of the  $n \times n$  identity matrix for all j = 1,...,k.

In this paper, we study target controllability of large-scale networks using their graph topology. Specifically, we assume that matrices *A* and *B* are given only in their structured forms denoted by  $\bar{A}$  and  $\bar{B}$ , respectively. Here,  $\bar{A} \in \{0, \star\}^{n \times n}$  and  $\bar{B} \in \{0, \star\}^{n \times m}$ are such that the 0 entries are fixed zeros and the  $\star$  entries are indeterminate. The structured LTI system  $(\bar{A}, \bar{B})$  represents a class of LTI systems (A, B) such that

$$A_{ij} = 0 \text{ whenever } \bar{A}_{ij} = 0, \text{ and}$$
  

$$B_{ij} = 0 \text{ whenever } \bar{B}_{ij} = 0.$$
 (1)

Any (A,B) that satisfies (1) is called a *numerical realization* of the structured system  $(\bar{A},\bar{B})$ . Now we define target controllability of a structured system.

**Definition 3.** In a structured system  $(\overline{A}, \overline{B})$ , a set of states T satisfies target controllability if there exists at least one numerical realization (A, B) such that T is target controllable.

In this paper, we use structural analysis of LTI systems to analyze target control problem in networks. The target control problem in the structured systems is formulated below.

**Problem 1.** Given a structured system  $(\overline{A}, \overline{B})$  and a subset of nodes T, obtain a graph-theoretic conditions to verify target controllability of T in  $(\overline{A}, \overline{B})$ .

A graph-theoretic condition for determining target controllability of a network helps in identifying a set of nodes in the network that are target controllable, and also in verifying whether a given set of nodes is target controllable or not.

#### B. Preliminaries and Terminology

Structural controllability characterizes complete controllability of the network and there exist necessary and sufficient conditions for verifying structural controllability of a network using graphtheoretic conditions [23], algebraic theory [24], and flow networks [10]. Given a structured system  $(\bar{A}, \bar{B}, \bar{C})$  we first define the *state digraph*  $\mathcal{D}(\bar{A}) := (V_X, E_X)$ , where  $V_X = \{x_1, \ldots, x_n\}$  and an edge  $(x_j, x_i) \in E_X$  if  $\bar{A}_{ij} = \star$ . Thus a directed edge  $(x_j, x_i)$  exists if state  $x_j$  can directly *influence* state  $x_i$ . The system *digraph*  $\mathcal{D}(\bar{A}, \bar{B}) :=$  $(V_X \cup V_U, E_X \cup E_U)$ , where  $V_U = \{u_1, \ldots, u_m\}$  and an edge  $(u_j, x_i) \in$  $E_U$  if  $\bar{B}_{ij} = \star$ . Thus a directed edge  $(u_j, x_i)$  exists if input  $u_j$  can *actuate* state  $x_i$ . Using  $\mathcal{D}(\bar{A}, \bar{B})$ , Lin in [23] introduced a topology referred as *cacti* and gave a necessary and sufficient condition to determine structural controllability of a system  $(\bar{A}, \bar{B})$ . The graphtheoretic equivalent of this condition is given using the concepts of *accessibility* and *dilation*.

**Definition 4.** [23] A state node  $x_i \in V_X$  in the digraph  $\mathcal{D}(\bar{A}, \bar{B})$  is said to be accessible if there exists a directed path in  $\mathcal{D}(\bar{A}, \bar{B})$  from some input node  $u_j \in V_U$  to the state node  $x_i$ . Further, a structured system  $(\bar{A}, \bar{B})$  is said to be accessible if all state nodes in  $\mathcal{D}(\bar{A}, \bar{B})$  are accessible.

Given a set of nodes, presence of a node set  $Z \subset V_X$  such that its neighborhood node set T(Z) (where node  $x_i \in T(Z)$ , if there exists a directed edge from  $x_i$  to a node in Z in  $\mathcal{D}(\bar{A},\bar{B})$ ), satisfying |T(Z)| < |Z| is called as dilation. Accessibility and no-dilation condition of all the state nodes is necessary and sufficient for structural controllability [23]. Now we define bipartite graph and perfect matching. A bipartite graph  $G_B := (V_B \cup \tilde{V}_B, \mathcal{E}_B)$  satisfies  $V_B \cap \tilde{V}_B = \emptyset$  and  $\mathcal{E}_B \subseteq V_B \times \tilde{V}_B$ . A matching  $M_B$  is a collection of edges  $M_B \subseteq \mathcal{E}_B$  such that no two edges in  $M_B$  share a common end point. For  $|V_B| \leq |\tilde{V}_B|$  ( $|V_B| \ge |\tilde{V}_B|$ , resp.), if  $|M_B| = |\tilde{V}_B|$  ( $|M_B| = |V_B|$ , resp.), then  $M_B$  is said to be a perfect matching, where |D| denotes the cardinality of a set D.

# 4. MAIN RESULTS: GENERICITY AND SOLUTION TO THE TARGET CONTROL PROBLEM

Consider a structured system  $(\overline{A}, \overline{B})$ . We first prove that target controllability in structured systems is a generic property. We use algebraic analysis employed before in [24] for proving this. Let the

number of nonzero entries in  $\overline{A}$  be N and the number of nonzero entries in  $\overline{B}$  be M. Associated with  $(\overline{A}, \overline{B})$  there is a parameter space  $\mathbb{R}^{N+M}$  such that every set of (N+M) values represents a data point  $d \in \mathbb{R}^{N+M}$ . As a result, every numerical realization (A, B) of  $(\overline{A}, \overline{B})$  is uniquely determined by a data point d.

Consider  $(\bar{A}, \bar{B})$  as matrices whose entries are from a ring of polynomials in (N+M) variables  $\mathbb{R}[\lambda]$ , where  $\lambda = (\lambda_1, \dots, \lambda_{N+M})$ is the list of nonzero entries. Consider polynomials  $\psi_i \in \mathbb{R}[\lambda]$ . A variety  $V \subset \mathbb{R}^{N+M}$  is the set of common zeros of a finite number of polynomials  $\psi_i, \dots, \psi_g$ . Further, V is said to be a proper variety if  $V \neq \mathbb{R}^{N+M}$ , and nontrivial if  $V \neq \emptyset$ . Let  $\Pi(\bar{A}, \bar{B})$  be a property on  $(\bar{A}, \bar{B})$ . Then,  $\Pi$  is a function from  $\mathbb{R}^{N+M}$  to the set  $\{0, 1\}$ , such that  $\Pi(d) = 0$  means  $\Pi$  fails at d and  $\Pi(d) = 1$  means  $\Pi$  holds at d. Let Ker  $\Pi := \{v : \Pi(v) = 0\}$ . The following definition holds.

**Definition 5.** A property  $\Pi$  is generic relative to the proper variety V if Ker  $\Pi \subset V$  and  $\Pi$  is generic if such a V exists.

Thus if  $\Pi$  is a generic property, then all points in  $\mathbb{R}^{N+M}$  at which the property  $\Pi$  fails to lie in a thin<sup>1</sup> set. As a result, if  $\Pi(d) = 0$ , then an arbitrarily small perturbation of *d* will result in some *d'* such that  $\Pi(d') = 1$ . Using these, we now prove that target controllability is a generic property.

**Theorem 1.** In the structured system  $(\bar{A}, \bar{B})$ , a set of states  $T = \{x_{i_1}, \ldots, x_{i_k}\}$  is target controllable if and only if all numerical realizations (A, B) of  $(\bar{A}, \bar{B})$  that are target uncontrollable lie on a proper variety in  $\mathbb{R}^{N+M}$ .

Proof of Theorem 1 is omitted in the interest of space. Theorem 1 thus concludes that target controllability is a generic property. In a network specified by  $(\bar{A}, \bar{B})$ , if there exists one system in the family of the network that is target controllable, then *almost all* systems are target controllable.

Target controllability has implications on various other properties of the system, such as maximum controllable subspace [6], [7], and structural output controllability [21]. For structured systems that are structurally uncontrollable, the concept of controllable subspace is well studied (see [6], [7], [25], and [26]). Hosoe in [6] provided a graph-theoretic condition to find the generic rank of the controllable subspace of a structured system  $(\bar{A}, \bar{B})$ . We present below Hosoe's result.

**Proposition 1.** [6, Theorem 1] Consider a structured system  $(\overline{A}, \overline{B})$ . Assume that the structured system  $(\overline{A}, \overline{B})$  is accessible. Then the generic rank of the controllability matrix of  $(\overline{A}, \overline{B})$  is given by

$$d_c(\bar{A},\bar{B}) = \max_{G_S \in \mathcal{G}} \{ |E(G_S)| \},\$$

where  $\mathcal{G}$  denotes the set of subgraphs of  $\mathcal{D}(\bar{A},\bar{B})$  which is defined by  $\mathcal{G} = \{G' \subset \mathcal{D}(\bar{A},\bar{B}) : G' \text{ consists of node disjoint cycles and atmost m simple<sup>2</sup> directed paths of <math>\mathcal{D}(\bar{A},\bar{B})$ . Also, the paths start form some input node in  $V_U$  and there are no common nodes in two paths}.

Note that Hosoe's theorem applies only to a structured system  $(\bar{A}, \bar{B})$  that is accessible. If some state node is inaccessible, then it is not controllable as well. Without loss of generality, we assume that all the state nodes are accessible in the structured system  $(\bar{A}, \bar{B})$  we analyze for target control.

Using Proposition 1, a *cycle-partition* based polynomial-time algorithm is given in [7] to find the generic rank of the controllability matrix of  $(\bar{A}, \bar{B})$ . Many papers on target control apply

Proposition 1 and the algorithm in [7] to obtain suboptimal solutions to various optimization problems in target control by finding upper and lower bounds [2]. While the condition in [6] provides a condition to compute the generic rank of the controllability matrix of a structured system, it does not yield insight on finding the generic rank of a submatrix of the controllability matrix. Finding generic rank of a submatrix of the controllability matrix. Finding generic rank of a submatrix of the controllability matrix of a structured system is a long-standing open problem [7]. Note that, verifying target controllability of states in set T (Problem 1) is equivalent to verifying whether the submatrix of the controllability matrix corresponding to the nodes in T has generic rank full. In this paper, we propose a graph-theoretic condition to verify if a subset of nodes is target controllable. Due to the equivalence of the two problems, this condition also suffices to compute the generic rank of a submatrix of the controllability matrix.

We use Proposition 1 to find the maximum cardinality of a subset of target controllable nodes, i.e., generic rank. However, the set of target controllable nodes not necessarily be spanned by any graph pattern, specified in Proposition 1. Note that, the set of target controllable nodes T of  $(\bar{A}, \bar{B})$  satisfies  $|T| \leq d_c(\bar{A}, \bar{B})$ , since all the minors of the controllability matrix of dimension greater that  $d_c(\bar{A}, \bar{B})$  is identically zero. In Figure 1 we illustrate an example showing that the set of nodes in T need not be spanned by a cacti structure or disjoint paths and cycles for target controllability of a network. The structured matrices  $(\bar{A}, \bar{B})$  for the network given in



Fig. 1: A single-input structured system  $(\bar{A}, \bar{B})$ . The generic rank  $d_c(\bar{A}, \bar{B})$  of this system is 5. For  $T = \{x_1, x_3, x_4, x_6, x_8\}$  (nodes shown in blue colour) the network is shown to be target controllable.

Using indeterminate entries this can be rewritten as

The controllability matrix C for  $(\bar{A}, \bar{B})$  given above is

$$\mathcal{C}(\bar{A},\bar{B}) = \begin{bmatrix} b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1b_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1a_2b_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1a_2a_3b_1 & a_1a_2a_3a_4b_1 & 0 & 0 \\ b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_5b_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_5a_6b_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_5a_6a_7b_2 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consider the 5×5 submatrix of  $C(\bar{A},\bar{B})$ , denoted as  $C_T(\bar{A},\bar{B})$ , corresponding to states  $T = \{x_1, x_3, x_4, x_6, x_8\}$ . Then

$$\mathcal{C}_{T}(\bar{A},\bar{B}) = \begin{bmatrix} b_{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1}a_{2}b_{1} & 0 & 0 \\ 0 & 0 & 0 & a_{1}a_{2}a_{3}b_{1} & a_{1}a_{2}a_{3}a_{4}b_{1} \\ 0 & a_{5}b_{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{5}a_{6}a_{7}b_{2} & 0 \end{bmatrix}.$$

For  $a_i$ 's = 1, for all  $i \in \{1, ..., 7\}$  and  $b_j = 1$ , for j = 1, 2,

 $<sup>^{1}\</sup>mathrm{A}$  nontrivial algebraic variety is 'thin' and a set of Lebesgue measure zero.

 $<sup>^{2}</sup>$ A directed path in a directed graph is said to be simple if there are no cycles in it.

rank of  $C_T(\bar{A},\bar{B})$  is 5. Thus there exists a numerical realization such that *T* is target controllable. This implies that the network  $(\bar{A},\bar{B})$  is target controllable for  $T = \{x_1, x_3, x_4, x_6, x_8\}$ . Notice that  $d_c(\bar{A},\bar{B}) = 5$  and the states  $\{x_1, x_3, x_4, x_6, x_8\}$  are not spanned by disjoint cacti or disjoint sets of paths and cycles.

Using the generic rank  $d_c(\bar{A},\bar{B})$ , we construct a system specific bipartite graph  $\mathcal{B}(\bar{A},\bar{B},T) := (\hat{V}_T \cup V_T, \mathcal{E}_T)$ , where  $\hat{V}_T := \{x_{i_1}, \ldots, x_{i_k}\} = T$ ,  $V_T = \{1, \ldots, d_c(\bar{A},\bar{B})\}$  and  $(x_{i_a}, j) \in \mathcal{E}_T$  if there exists a directed path in  $\mathcal{D}(\bar{A},\bar{B})$  form some input node  $\hat{u}$  to state node  $x_{i_a}$  of length j.



Fig. 2: Bipartite graph  $\mathcal{B}(\bar{A},\bar{B},T)$ with  $T = \{x_1, x_3, x_4, x_6, x_8\}$  for the structured system given in Figure 1. Here,  $d_c(\bar{A},\bar{B}) = 5$ . The bipartite graph  $\mathcal{B}(\bar{A},\bar{B},T)$  has a perfect matching corresponding to the subset of nodes  $T = \{x_1, x_3, x_4, x_6, x_8\}$ .

Using  $\mathcal{B}(\bar{A}, \bar{B}, T)$ , we have the following result.

**Theorem 2.** Consider a structured system  $(\bar{A}, \bar{B})$  and let  $d_c(\bar{A}, \bar{B})$ be the generic rank of the controllability matrix of  $(\bar{A}, \bar{B})$ . Let  $\bar{B}$ be a single-input matrix. Then, the bipartite graph  $\mathcal{B}(\bar{A}, \bar{B}, T)$  has a perfect matching if  $(\bar{A}, \bar{B})$  is target controllable with respect to a set of states  $T = \{x_{i_1}, \dots, x_{i_k}\}$ , where  $k \leq d_c(\bar{A}, \bar{B})$ .

The proof of Theorem 2 is omitted in the interest of space.

Using Theorem 2 now we present Algorithm 1 to verify target controllability of a structured system, which also suffices to compute the generic rank of a submatrix of the controllability matrix.

Algorithm 1 Pseudo-code to verify target controllability of a structured system

**Input:** Structured system  $(\bar{A}, \bar{B})$  and subset of nodes *T*, where  $T \subset \{x_{i_1}, \ldots, x_{i_k}\}$ 

Output: Returns set T is target controllable or not

- 1: Compute the generic rank of the controllability matrix of  $(\bar{A}, \bar{B})$ , say  $d_c(\bar{A}, \bar{B})$
- Construct the bipartite graph \$\mathcal{B}(\bar{A}, \bar{B}, T)\$
   if \$k > d\_c(\bar{A}, \bar{B})\$ then
- 4: T is not target controllable

5: else if  $k \leq d_c(\bar{A}, \bar{B})$  then

6:if  $\mathcal{B}(\bar{A}, \bar{B}, T)$  has a perfect matching then7:T is target controllable8:else9:T is not target controllable

10: end if 11: end if

If  $k > d_c(\bar{A}, \bar{B})$ , then all  $k \times k$  minors of the controllability matrix is zero and hence the set T is not target controllable. On the other hand, if  $k \leq d_c(\bar{A}, \bar{B})$ , then we construct the bipartite graph  $\mathcal{B}(\bar{A}, \bar{B}, T)$ . Note that all the state nodes that are matched in the bipartite graph  $\mathcal{B}(\bar{A}, \bar{B}, T)$  are accessible. Further, all these state nodes can be controlled using an input in such a way that no two nodes need to be actuated at the same instant of time. In other words, not only that all the states of the state nodes in a matching in  $\mathcal{B}(\bar{A}, \bar{B}, T)$  can be controlled, but their differences can also be controlled. This is due to the fact that no two nodes in a matching in  $\mathcal{B}(\bar{A}, \bar{B}, T)$  receive the same input at the same instant of time. Thus all states nodes that correspond to a perfect matching in  $\mathcal{B}(\bar{A}, \bar{B}, T)$ can be driven to any desired state. Notice that for  $T = \{x_1, ..., x_n\}$  existence of perfect matching in  $\mathcal{B}(\bar{A}, \bar{B}, T)$  is equivalent to the accessibility and the no-dilation condition (since the generic rank is *n* in that case). Further, the matching condition given in this paper boils down to the directedpath based necessary and sufficient condition given in [2] for structured systems whose state digraph is a tree. This implies that the matching condition given in this paper holds for the conditions that are given in the literature for special case or graph topology. In Remark 1, we relate our results to the known conditions for special graph topologies, specifically tree networks given in [2].

**Remark 1.** For directed tree networks, the bipartite matching-based condition given in this paper reduces to the k-walk condition given in [2] that prove that one input node can control a set of target nodes if the path length to each of the target node is unique.

### 5. EXPERIMENTAL RESULTS AND DISCUSSION

In this section, we present an experimental analysis of our results using real-world networks. All the experiments are done for the single-input case. Given a structured system  $(\bar{A}, \bar{B})$ , we perform the following steps.

- Step 1: select a subset of states T for target controllability
- Step 2: compute the generic rank  $d_c(\bar{A}, \bar{B})$  of  $(\bar{A}, \bar{B})$
- *Step 3*: verify if *T* is target controllable using the perfect matching condition of  $\mathcal{B}(\bar{A}, \bar{B}, T)$  given in Algorithm 1
- Step 4: assign all nonzero entries in  $(\bar{A}, \bar{B})$  as 1 to obtain a numerical realization (A, B)
- Step 5: construct the restricted controllability matrix  $C_T(A, B)$
- Step 6: check if  $C_T(A,B)$  is full rank, else perturb some nonzero entry in (A,B) to a randomly generated value
- Step 7: repeat Step 6 until  $C_T(A, B)$  becomes full rank

Once a numerical realization (A,B) is obtained for which  $C_T(A,B)$  is full rank, it is guaranteed that states in set T can be driven from any initial state to any desired final state in finite time for some input. The final state of rest of the states can be any finite value which is not of interest. Without loss of generality, we set the initial state of all the states in the system (A,B) to zero. To design the input for driving the states in T to a desired final state  $x_{t_f}$ , we implement the following equation.

$$u(t) = B^{\mathsf{T}} e^{A^{\mathsf{T}} t} C^{\mathsf{T}} (CW(t_f) C^{\mathsf{T}})^{-1} x_{t_f}$$
(2)

Here  $W(t) = \int_0^t e^{A\tau} BB^{\tau} e^{A^{\tau} \tau} d\tau$ ,  $x_{t_f}$  is the desired final state for states in *T* and  $t_f > 0$ . Note that, Eq. (2) gives the minimum energy input for states in *T* to reach  $x_{t_f}$  from 0 in time  $t_f$ . Thus  $x(t_f) = x_{t_f}$  for states in *T*. We conduct experiments on 4 real-world networks. Next, we describe the real-world networks and show the time response of states and the corresponding input design in Figure 3.

• Social network: social networks are ever increasing and controlling a set of states or users in the social network is of importance for purposes like a targeted advertisement. To demonstrate this, we consider a *retweet network* where states correspond to users who interact using retweet or mention [27]. We select 5 users to form *T* and design the required input as described in Eq. (2) based on the obtained numerical realization (A, B) corresponding to a perfect matching of  $\mathcal{B}(\bar{A}, \bar{B}, T)$ . Figure 3a shows the time response of the selected 5 users with respect to the input given. The solid lines correspond to the selected 5 users, while the dashed lines correspond to other randomly selected users.

• *Biological network*: several studies have focused on controllability of biological networks involving interaction between different biomolecules to identify drug targets which affect human health. We use the example of *Figeys protein* interaction in humans [28]. Steering of specific states (proteins) can be done by the action of



Fig. 3: Time response of target controlled states (solid line) and not controlled states (dashed line) governed by the input design for 4 real-world networks. (a) Re-tweet network for which 5 states are selected for controllability with initial state as 0. The final states of the selected states are desired as 100. Time responses of controlled states, as well as uncontrolled states, are shown on top while input design is shown at the bottom. (b) Protein-protein interaction network for which 4 states are selected for target controllability with initial state as 0 and all controlled states are steered to final state 100 (top). The corresponding input design is given below the state time response. (c) Airport network for which 4 states are selected for target controllability (solid line) while other states are not of interest to be driven to any desired state. The final states of controllable states are desired as 100 which is achieved as shown in Figure 3c. The corresponding input design is below the state time response. (d) IEEE 39-bus power system network for which 5 states are selected for target controllability. The final state of all controlled states is desired as 100. Figure 3d shows that the targeted states are steered to the desired state using the input shown below.

drugs on these proteins. The interactions between proteins define the system dynamics whereas each protein corresponds to a state in the system. We select 4 proteins to form set *T* such that the bipartite graph  $\mathcal{B}(\bar{A}, \bar{B}, T)$  has a perfect matching and then obtain a numerical realization (*A*, *B*) for which  $C_T(A, B)$  is full rank. Figure 3b shows that the selected proteins are steered to the desired final state 100 in finite time for the input designed using Eq. (2).

• Infrastructure: target control is highly beneficial in the field of infrastructure networks such as a road network or an airport network since complete controllability of these networks incurs a large operational cost. We consider the network of airports [29] where each airport corresponds to a state while the network dynamics is governed by the flights interconnecting these airports. To depict target controllability, we select 4 airports as set T and obtain a numerical realization (A, B) corresponding to a perfect matching in  $\mathcal{B}(\bar{A}, \bar{B}, T)$ . Figure 3c shows the time response of transition of airports in T to the desired state 100 while other airports which were not of interest are steered to a finite value.

• Power system network: often a particular subset of a power system network is of more importance and is required to be less prone to failure, thus target controllability can save the unnecessary cost of controlling the full network which may be very large. We use the IEEE 39-bus electric power system [30] which is a highly studied model in power systems. Each state in the network corresponds to a bus and the system dynamics is based on the wiring between the different bus. Figure 3d shows that the states in T are steered to the value 100.

# 6. CONCLUSION

In this paper, we considered the target controllability of a network, i.e., a family of structured systems. We first showed that target controllability is a generic property. As a result, if a system (A, B) satisfies target controllability, then almost all systems that lie in the same family as that of the network of (A, B) are also target controllable. Subsequently, we presented a new graphtheoretic condition to determine target controllability of a network. The proposed condition is a bipartite matching-based condition and using this condition target controllability can be determined in  $O(n^{2.5})$  operations. We also show that the proposed condition helps us to compute the generic rank of a submatrix of the controllability matrix. While we prove that the bipartite-matching based condition is necessary for target controllability, we present reasons why this condition is believed to be also sufficient. Finally, we presented experimental analysis of our results on real-world networks. Proving the sufficiency of the matching-based condition and extending the results to a multi-input case is part of future work. Minimum input selection for target control in systems with general topology is also part of future work.

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