Port selection for optimal charging/discharging and external communication in single-integrator graphs networks

Sudarshan Pulapadi * **Sandeep Kumar** ** **Madhu N. Belur** *
Narayan Rangaraj ***

* *Department of Electrical Engineering, Indian Institute of Technology Bombay.* sudarshanp@ee.iitb.ac.in
** *Department of Electrical Engineering, Indian Institute of Technology Bombay;ACEM, DRDO, Nasik, India.* sandeepkumardrdo@gmail.com
*** *Industrial Engineering and Operations Research, Indian Institute of Technology Bombay.* narayan.rangaraj@iitb.ac.in

Abstract: This paper studies optimal input port selection for simple undirected graphs with an objective to control the graph externally with least amount of supplied energy. Given a graph with n-nodes, we address the problem of selecting k out of these for external input with this minimization objective. We formulate a resistive-capacitive (RC) network analogy of such single integrator multiagent networks, allowing us to consider the given problem as that of optimal selection of the input ports w.r.t. the energy required for charging/discharging of the RC circuit. We set up a link between these optimal port locations and values in the Fiedler vector of the corresponding graph Laplacian matrix L and other eigenvectors of L. This paper contains new results involving passivity, Hamiltonian matrix and the Algebraic Riccati equation in the context of RC networks associated with such graphs. We link this formulation to optimal node(s) identification for optimal external communication with a multi-agent network of single integrator systems.

Keywords: Multi-agent systems, algebraic Riccati equation, graph-Laplacian

1. INTRODUCTION

Often in a connected network of multiagent systems, it is essential to communicate with one or more agents (from an external source) and these communicated agents are then able to communicate within the network. In such a problem, it is important to be able to identify which node to communicate to from the external source. The objective being that least amount of communication energy is utilized in this communication. The connectivity of a graph and the 'central node' (with respect to a suitable notion of centrality) would appear to play a key role and this paper investigates if there is a relation between the algebraic connectivity, component values of Fiedler vector, and other eigenvectors of the graph Laplacian, and the port that is best with respect to energy needed to achieve an objective: for example, consensus, or other formations, or an average over all formations.

This paper makes the above problem concrete by associating a resistor-capacitor (RC) network with a given undirected graph and this corresponds to a network of multi-agents with each agent modelled as a single-integrator. Our work also applies to more general diffusion problems where the interaction with neighbours can be viewed as diffusion of heat or information. This is made precise in the problem formulation section below.

The paper is organized as follows. The next section elaborates on the problem formulation and a summary of related work in the literature. Section 3 contains preliminaries that are essential for stating the main results, which are stated in Section 4. We elaborate on some examples of graphs in Section 5; this section also contains some key inferences between graph Laplacian eigenvector properties and optimal port locations for charging/discharging. The proofs of the main results are pursued in Section 6. Some concluding remarks are summarized in Section 7. The rest of this section contains notation.

The set $\mathbb{R}$ stands for the field of real numbers, and $\mathbb{R}^n$ stands for n-tuples, stacked as a column vector. Vector norm considered in this paper is the standard Euclidean 2-norm. The unit-sphere in $\mathbb{R}^n$ consists of all vectors of unit 2-norm, and this set is denoted by $\mathbb{S}^{n-1}$.

2. PROBLEM FORMULATION AND RELATED WORK

In this paper we study the effect of charging and discharging patterns for the single integrator network on selection of input ports and the performance is compared with the algebraic graph theoretic properties of the corresponding structure. This problem is linked to optimal actuator placement in multi-agent systems, where each agent corresponds to a node in the network and the input corresponds to energy given to control node potentials.

Optimality of port placement is loosely linked with placement of *one* port at the 'central node': the notion of centrality defined appropriately. We use energy as the parameter such as maximum energy that can be extracted from the system or, analogously, minimum energy that has to be provided to the system to reach a desired state. This is obtained using extremal solutions of the ARE: $K_{\max}$ and $K_{\min}$; more about this is reviewed in Section 6. We relate the ARE solution based optimal port locations with graph properties of the system like eigenvectors of the graph-Laplacian and graph-centrality.

Three types of optimization play a central role in this paper: we summarize these here, while Sections 3 and 6 contain more about this. The first two types have been addressed in various papers: these two optimization play a role in the third one: that being the focus of this paper.

(1) *[Optimize over trajectories]* Given a state vector $a \in \mathbb{R}^n$, minimize the energy supplied to the system, over all input trajectories that take the state to a. Analogously, maximizing the energy extracted across all trajectories that start from a state-vector a. This gives rise to Algebraic Riccati Equation solutions: $K_{\min}$ and $K_{\max}$; see equation (9).
(2) *[Optimize over directions in $\mathbb{R}^n$]* Given a state space system, across all vectors $a \in \mathbb{S}^{n-1}$ (the unit sphere in $\mathbb{R}^n$), consider the vector that needs most energy for charging to reach this vector: this corresponds to the eigenvector corresponding to maximum eigenvalue $\lambda_{\max}(K_{\max})$ defined in equation (9): call this vector as the 'hardest' direction. Similarly, $\lambda_{\min}(K_{\max})$ and the corresponding eigenvector is the direction 'easiest' to charge. The eigenvectors corresponding to maximum and minimum eigenvalues of $K_{\min}$ are interpreted as 'easiest' and 'hardest' to *discharge*. It is well-known that the trace (scaled by n) of a symmetric and positive semidefinite matrix K is the average of the quantity $a^T Ka$ for $a \in \mathbb{S}^{n-1}$: see [11], for example.
(3) *[Optimize over input-port(s)]* Given a graph G_n with n vertices, and a fixed number of input-ports p with $p < n$, identifying the optimal location of the p ports such that an appropriate measure is minimized: this measure can be any of
$$\begin{aligned}&\lambda_{\max}(K_{\max}), \lambda_{\min}(K_{\max}), \text{trace}(K_{\max}),\\ &\lambda_{\max}(K_{\min}), \lambda_{\min}(K_{\min}), \text{ or trace}(K_{\min}).\end{aligned} \tag{1}$$
As mentioned above, the third one above is the focus of this paper, but this third one builds on existing results for the earlier two. For easier reference, these parameters with a brief meaning are summarized in Table 2.

In the context of multi-agent systems, a special role is played by the vector $\tilde{\mathbb{1}}$ defined as the vector with all components equal to one: we call this consensus. In order to be able to compare with other quantities (like eigenvalues of symmetric matrices), we consider $\tilde{\mathbb{1}}$: but after normalizing it to unit 2-norm. Define $\mathbb{1}$ as the vector in $\mathbb{R}^n$ with all components positive and equal and having unit 2-norm. In addition to the quantities listed in equation (1), $\mathbb{1}^T K_{\max} \mathbb{1}$ and $\mathbb{1}^T K_{\min} \mathbb{1}$ also play a role. Using the fact that $K_{\min}$ and $K_{\max}$ are symmetric and positive semidefinite matrices, and using Courant-Fischer-Weyl min-max theorem, we have
$$\lambda_{\min}(K_{\max}) \leqslant \mathbb{1}^T K_{\max} \mathbb{1} \leqslant \lambda_{\max}(K_{\max}).$$
Thus, while energy to reach consensus is an important measure to optimize, energy for worst case scenario $\lambda_{\max}(K_{\max})$, and average energy (trace$(K_{\max})$ (averaged over $\mathbb{S}^{n-1}$) are important too. For ease of reference, the various quantities are summarized in Table 2.

Optimization in port/actuator placement has been of interest in other papers too: amongst them, we list here the ones relatively more relevant to our work. Minimal actuator placement for minimum control effort to achieve a given state transfer has been presented in [19]: here the authors consider a *bound* on the control-effort and, further, the notion of energy is different from $u^T y$, as used in our paper: the controllability Gramian turns out to play a key role there. In [3], a network approximation scheme for minimizing the global energy cost for multi-agent systems to achieve consensus over undirected network topologies is proposed: the focus being on *distributed* control and hence on the proposed approximation and its analysis. With robustness and response to stochastic disturbance inputs as the focus, the problem of optimal leader selection in a multiagent network has been studied in [15] and [9]. From a relatively more graph-theoretic/'generic' view, minimum input selection with respect to controllability of the network has been pursued in [13] and [7]: these papers focus on generic systems results (with a prespecified graph structure) and not on the amount of control effort. In our paper we propose the selection of the optimal input ports from the Fielder vector (and other eigenvectors) of the Laplacian matrix. For a requirement where k-input ports to be selected, we propose a way to select the best combination of k input-ports, with 'best' defined using the quantities listed in equation (1) above. The notion of power used in this paper is $u^T y$ and this allows use of passivity results: in the context of RC circuits constructed from graph networks, we formulate new results regarding solutions of Algebraic Riccati Equations (ARE) using the Laplacian matrix structure, and use these results to link optimal port selection to Fiedler vector and other eigenvectors of the Laplacian matrix, thus linking ARE solutions and algebraic-graph theory.

3. PRELIMINARIES

In this section we cover preliminaries that are essential to state the main results (in Section 4), further preliminaries are pursued in Section 6.

3.1 The graph-Laplacian and algebraic connectivity

We deal with graphs that are undirected: this is viewed as current flow (and more generally diffusion) allowed in both directions along each edge. Further, we consider only simple graphs, i.e. graphs with no self-loops. We denote the undirected graph $G(V,E)$, where E is the set of all edges and V is set of all vertices with $|V| = n$. Though many of the results in this paper are valid for *weighted* graphs, we focus on unweighted graphs; each edge weight is assigned value 1. For the graph G (also denoted by G_n, to indicate the number of vertices), we construct its adjacency matrix A, which corresponds to a symmetric $n \times n$ matrix, with the (i, j)-th entry A_{ij} defined as 1 if and only if there exists an edge between i^{th} and j^{th} node. Since G_n does not contain any self loops, all diagonal entries of A are zero.

For a graph G_n the Laplacian matrix L is defined as $L := D - A$, where D is a diagonal matrix, with each diagonal entry corresponding to the degree of the node. The Laplacian of an undirected graph is a symmetric positive semidefinite matrix, and denote its real and non-negative eigenvalues (sorted) as follows:
$$0 = \lambda_n \leqslant \lambda_{n-1} \leqslant \cdots \leqslant \lambda_2 \leqslant \lambda_1. \tag{2}$$
The second smallest eigenvalue λ_{n-1} captures the 'connectivity' of this graph and called the Fiedler value, λ_{n-1} is called the algebraic connectivity of G_n. The eigenvector corresponding to λ_{n-1} is called the Fiedler vector and plays a central role in our paper; this vector has been well studied in various graph connectivity studies, in spectral partitioning on graphs: see [5,1], for example.

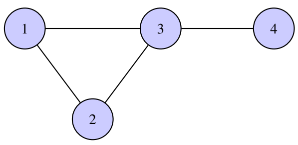
Optimal actuator placement can be considered as identifying the most influential node of the graph, see [2] for example. Some of the centrality measures defined for a graph are betweenness [10], closeness [17], eigenvector centrality, Local Fiedler Vector Centrality (LFVC, [6]) and Laplacian centrality [16]. LFVC and Laplacian centrality use entries in the Fiedler vector to determine the centre of the graph: 'node centrality' is defined as
$$\text{node-LFVC}(i) = \sum_{j \in N_i} (y_i - y_j)^2 \tag{3}$$
where y_i is i^{th} entry of Fiedler vector (normalized to length 1) and N_i is set of neighbouring nodes of i. It is known that the node corresponding to the smallest magnitude in the Fiedler vector

has the highest node-LFVC. Our inferences in Section 5 pursue further along this direction using eigenvector corresponding to λ_{n-1} for the best *single* port, and more generally, eigenvector corresponding to λ_{n-p} for the best p-ports: see Observations 5.1 and 5.2.

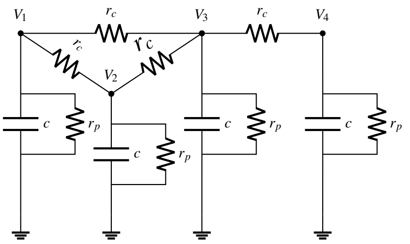
3.2 Undirected graphs, RC circuits and multi-agent networks

Given an undirected graph, we define an RC electrical network. Each edge is replaced with an 'interconnecting' resistance r_c. Each node has a potential with respect to the ground (say, node 0): this is the voltage across a capacitor, with a parasitic resistance r_p; this models capacitor leakage. For ease of exposition, we assume all capacitances as 1 F. We consider the circuit to be charged by one or more current sources, each with a series source resistance r_s: this is connected to one or more nodes in the graph. As an example, see graph G1 as shown in the Figure 1a and its RC equivalent representation in Figure 1b. The graph with input nodes, say node-3 and node-4, and analogous RC circuit with current sources is shown in Figure 2b.

The analysis of RC networks with one or more charging input-ports applies to a network of multi-agents, with each agent modeled as a single integrator, and diffusion captured as a flow across neighbouring agents. As in our work, the Laplacian matrix (despite the leakage due to the parasitic resistance) plays a key role in any diffusion problems arising in multi-agent networks. The charging of the RC network using a current source is analogous to external communication to one or more ports: the objective is to identify one or more nodes that help minimize external communication energy.



(a) Graph G1



(b) RC network of G1

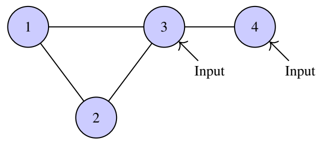
Fig. 1. Graph, multi-agent network and analogous RC circuit

4. MAIN RESULTS

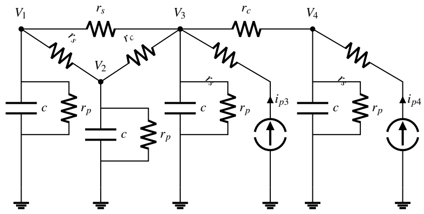
This section contains main results of this paper. For ease of exposition, the theorems are stated here, while the proofs are in Section 6.4.

4.1 State space model for RC network

The next result, though a straightforward application of circuit theory, states that the RC circuit with input-ports constructed from the graph (see Figure 2, for example) has a state space with a so-called *internal symmetric structure* (see [18]). This structure helps other results in our paper, and also helps in, for example, model order reduction of large order ZIP systems: see again [18].



(a) Graph G1 with two inputs (for external communication)



(b) Analogous RC network of G1 with ports at nodes 3 & 4

Fig. 2. Multi-agent receiving external communication and analogous RC circuit with input ports

Table 1: Symbols and their meaning

Symbol	Definition
G_n	A general graph with n nodes
P_n	The path-graph on n nodes
S_n	The star-graph on n nodes
c	Capacitance (assumed to be 1 F for ease of exposition)
r_c	Connecting resistance
r_p	Leakage resistance across capacitor
r_s	Source resistance
$\lambda_i(P)$	i-th largest eigenvalue of the matrix P (assuming P has real eigenvalues
L	Laplacian of a graph
v_{g_i}	Fiedler and 'generalized'-Fiedler vectors

Table 2: Notation, objectives, meaning

Parameter	Definition, interpretation
$\mathbb{R}^n$	The real vector-space with n-components stacked as a column vector
$\mathbb{S}^{n-1}$	The unit-sphere in $\mathbb{R}^n$, containing all vectors with Euclidean norm = 1; each vector is a formation (normalized to length 1)
$\mathbb{1}$	The unit-vector in $\mathbb{S}^{n-1}$ with all components equal and positive; suggests consensus
$K_{\max}$	Maximum solution of the passivity Algebraic Riccati Equation (ARE) (equation (6), Section 6.2)
$K_{\max}$	Minimum solution of the passivity Algebraic Riccati Equation (ARE) (equation (6), Section 6.2)
$v^T K_{\max} v$	Minimum energy required to reach formation-vector v
$v^T K_{\min} v$	Maximum energy extractable from formation-vector v
$\mathbb{1}^T K_{\max} \mathbb{1}$	Minimum energy required to reach consensus
$\mathbb{1}^T K_{\min} \mathbb{1}$	Maximum energy extractable <u>from</u> consensus
$\lambda_{\max}(K_{\max})$	Minimum energy to reach the <u>hardest</u> $v \in \mathbb{S}^{n-1}$
$\lambda_{\min}(K_{\max})$	Minimum energy to reach the <u>easiest</u> $v \in \mathbb{S}^{n-1}$
$\lambda_{\max}(K_{\min})$	Maximum energy extractable from the <u>easiest</u> $v \in \mathbb{S}^{n-1}$
$\lambda_{\min}(K_{\min})$	Maximum energy extractable from the <u>hardest</u> $v \in \mathbb{S}^{n-1}$
$\frac{\text{trace}(K_{\max})}{n}$	Average over all vectors $v \in \mathbb{S}^{n-1}$ of the minimum energy required to reach a vector v
$\frac{\text{trace}(K_{\min})}{n}$	Average over all vectors $v \in \mathbb{S}^{n-1}$ of the maximum energy extractable from a vector v

Theorem 4.1. Consider an undirected graph G_n with n nodes and p input-ports: at nodes denoted by $(j_1, j_2, \cdots, j_p)$ with $j_k \in \{1,2,\ldots,n\}$, and let L denote the corresponding graph Laplacian matrix. Introduce the ground-node, and construct the corresponding RC network of the graph with each edge as a resistance r_c, each node connected to the ground-node with capacitance $c = 1$ F and with parasitic resistance r_p. Suppose input at each of the p nodes is a current source through input source resistance r_s.
Then, the system $\dot{x} = Ax + Bu$ and $y = Cx + Du$, with state as the voltages at nodes 1,2,...,n, the input u and y as the port current and voltage across the current source respectively, admits an internally symmetric state space realization and has system matrices as follows:

$$\begin{aligned} A &= \frac{-L}{cr_c} - \frac{I_n}{cr_p}, \\ B(j_k,k) &= \begin{cases} 1 & \textit{for } j_k \in \{j_1, j_2, \ldots, j_p\}, \\ 0 & \textit{otherwise,} \end{cases} \\ C &= B^T \ \textit{ and } \ D = r_s I_p. \end{aligned} \tag{4}$$

See Section 6.4 for the proof. Note that the fact that the input to the system is the port currents and the system output is the corresponding port-voltages: this gives a special structure to the system, which we exploit in this paper, namely a collocated sensor-actuator system having a symmetric state-space realization.

An important point to note is that the eigenvectors of the graph Laplacian L are the *same* as the eigenvectors of A, and the eigenvalues of A and L are related by a simple relation. For a connected graph G_n, the vector $\mathbb{1}$ is an eigenvector of both L and A, and this consensus vector corresponds the steady state capacitor voltage configuration: when there is no leakage across the capacitor. Further, when studying convergence to the steady state vector $\mathbb{1}$, the Fiedler vector of L, say v_{n-1}, is the slowest mode of the system dynamics. We mentioned in Section 2 about the magnitude of the components in the Fiedler vector helping in identifying the 'central node'. We refer to the eigenvector v_{n-2} corresponding to λ_{n-2} as the 'second-slowest' mode; the eigenvector v_{n-3} corresponding to λ_{n-3} as the 'third-slowest' mode. Note that, when the eigenvalues are distinct, these modes are perpendicular to each other. We pursue this argument further in Section 5.

The rest of this section deals with solutions of the Algebraic Riccati Equation (ARE) corresponding to power supply: $i^T v$, with i the currents supplied at the ports (input), and v the voltages across the current source (output):

$$AK + KA + (KB - B)(D + D^T)^{-1}(B^T K - B^T) = 0.$$

The rest of this paper contains new results and some observations between the Laplacian matrix eigenvectors and the ARE solution properties (see Table 2). The next result considers the special case when the capacitor has *no leakage*, i.e. $r_p = \infty$. In this case, all Algebraic Riccati Equations solutions have consensus as their common eigenvector. While more details of

Theorem 4.2. Consider an undirected and connected graph G_n on n nodes and the corresponding RC circuit as described in Theorem 4.1. Suppose the capacitors have no leakage, i.e. parasitic resistance $r_p = \infty$. Then, $\mathbb{1}$ is an eigenvector of every ARE solution and 1 *is the corresponding eigenvalue.*

See Section 6.4 for the proof. As summarized in Tables 1 and 2, when considering minimum energy required to reach consensus or any other formation vector say v, the ARE solution $K_{\max}$, the maximum of all the ARE solutions, plays a key role: $v^T K_{\max} v$ is this energy. Similarly, $v^T K_{\min} v$, is the maximum energy available for extraction from the vector v. The next result states that for case of the no-leakage in capacitor the consensus vector is the easiest to reach (amongst all vectors on the unit-sphere $\mathbb{S}^{n-1}$) and also the hardest to 'discharge', again amongst all vectors on the unit-sphere $\mathbb{S}^{n-1}$: note that the capacitance is assumed to be 'normalized' to 1 F.

Theorem 4.3. Consider an undirected and connected graph G_n on n-nodes and construct the corresponding RC-circuit as in Theorem 4.1. Assume the capacitors have no leakage, i.e. parasitic resistance $r_p = \infty$. Let $K_{\min}$ and $K_{\max}$ be the minimum and maximum ARE solutions. Then,

$$\lambda_{\min}(K_{\max}) = 1 = \lambda_{\max}(K_{\min}).$$

See Section 6.4 for the proof. We note that for the case when $r_p < \infty$, i.e. for the case that there is nonzero leakage across each capacitor, then the consensus vector $\mathbb{1}$ is no longer an eigenvector. However, for the special case when *all* nodes are input-ports, then the following result states that $\mathbb{1}$ is an eigenvector of each ARE solution.

Theorem 4.4. Consider system as described in Theorem 4.1, with parasitic resistance $0 < r_p < \infty$. Suppose each of the n-nodes is an input port. Then $\mathbb{1}$ is an eigenvector of every solution of the ARE

for the system.

See Section 6.4 for the proof.

5. EXAMPLES AND KEY INFERENCES

This section pursues the relation between optimal port location and magnitude of the Fiedler vector components. We noted in Section 2 about the link between the centrality of a node in a graph and the magnitude of the component of the Fiedler vector v_{n-1}, the eigenvector corresponding to $\lambda_{n-1}(L)$. As mentioned after Theorem 4.1, the eigenvectors of the graph Laplacian corresponding to λ_{n-1}, λ_{n-2}, λ_{n-3}, called as say, v_{n-1}, v_{n-2}, v_{n-3}, are the slowest mode, second-slowest mode, third-slowest mode, respectively. Assume each of these vectors are normalized to unit 2-norm.

While perpendicularity of these modes is assured due to the symmetry of matrices A and L (assuming distinctness of the eigenvalues), the magnitude of the components in v_{n-p}, the location of the smallest p of them, signify the best p locations. More precisely, if p optimal ports have to be found (w.r.t. $\lambda_{\min}(K_{\max})$ or trace$(K_{\min})$), then one identifies the p-smallest components in v_{n-p}, and these p-nodes are the optimal port locations w.r.t. the criteria. Perhaps this is linked to the Courant-Fischer-Weyl min-max theorem for symmetric matrices, with the Laplacian matrix revealing the slowest, second-slowest, third-slowest modes of the RC-network. This is the key observation that we pursue further in this section.

We present observations based on various graphs and consider the corresponding RC-networks as described in Theorem 4.1. The experiments have been performed using tools developed in Python and Scilab.

Consider the graph shown in Figure 3a. Our first observation based on the examples is that the best single port choice w.r.t. average energy tallies with the definition of centrality of node w.r.t. node-LFVC (defined in equation (3).)

Observation 5.1. For a single port network, the trace of $K_{\min}$ is minimum if the selected port corresponds to minimum absolute value in Fiedler vector.

The next observation is for the case of p-ports.

Observation 5.2. For a given network in which p ports are to be identified, considering the p^{th} slowest eigenvector of the graph Laplacian provides the optimal p-locations of ports: these nodes correspond to the p smallest absolute value components of the eigenvector.

Tables 3 and 4 compare results of different parameters described in Table 2 for the Graph G2 (in Figure 3a), computed for $r_p = 500\ \Omega$ and $r_s = 5\ \Omega$, $r_c = 1\ \Omega$ (and $c = 1$ F. We see that the minimum absolute value of the Fiedler vector components corresponds to energy extracted from consensus for the node with 1 port. This result compares with the definition of centrality defined using LFVC in equation (3). Further, in both tables, the correlation between actual optimal port location (using the Riccati equation solution $K_{\min}$) and generalized-Fiedler vector component values is visible: this has been shown in bold font for easy of readability.

The various graphs and corresponding plots are shown in Figures 3-6.

6. FURTHER PRELIMINARIES AND PROOFS OF THEOREMS 4.1-4.4

In this section we prove the main results: Theorems 4.1-4.4. For the proof, we need additional preliminaries which are reviewed next.

6.1 Spectral zeros

The spectral zeros of a positive real systems with transfer function $G(s)$ are defined as $\lambda \in \mathbb{C}$ such that $\det\left(G(\lambda) + G(-\lambda)^T\right) = 0$. Consider a minimal state space realization of $G(s)$ as $\dot{x} = Ax + Bu$ and $y = Cx + Du$. Considering systems for which $(D + D^T)$ is invertible, the spectral zeros are the eigenvalues of the Hamiltonian matrix $H \in \mathbb{R}^{2n\times 2n}$ defined as:

$$H := \begin{bmatrix} A - B(D+D^T)^{-1}C & B(D+D^T)^{-1}B^T \\ -C^T(D+D^T)^{-1}C & -(A-B(D+D^T)^{-1}C)^T \end{bmatrix}. \quad (5)$$

Due to symmetry about the imaginary axis $j\mathbb{R}$, if H does not have $j\mathbb{R}$ eigenvalues, then of the $2n$ spectral zeros of the system, n-spectral zeros are in the $\mathbb{C}^-$ plane and their n mirror images in $\mathbb{C}^+$ plane.

6.2 Algebraic Riccati Equation

The algebraic Riccati equation (ARE) for the above system with respect to the so-called 'passivity supply rate', i.e. power defined as $u^T y$, is

$$A^T K + KA + (KB - C^T)(D + D^T)^{-1}(B^T K - C) = 0. \quad (6)$$

The system Σ is positive real if and only if there exists a positive definite solution $K = K^T$ to the above equation. The set of ARE solutions is known to be a bounded and finite set with a maximal $K_{\max}$ and a minimal $K_{\min}$ and every ARE solution K satisfying $0 \leqslant K_{\min} \leqslant K \leqslant K_{\max}$. The solutions of the ARE in equation (6) can be computed from the $n-$dimensional invariant subspace of the associated Hamiltonian matrix H as follows:.

$$H\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} R \quad \text{and define} \quad K := YX^{-1} \quad (7)$$

where $X, Y \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$ is an upper-triangular matrix with n eigenvalues of the Hamiltonian matrix, i.e. the n-spectral zeros. When either n stable or n anti-stable spectral zeros are chosen, we get an extremal solution of the ARE, i.e.

$$H\begin{bmatrix} X_+ \\ Y_+ \end{bmatrix} = \begin{bmatrix} X_+ \\ Y_+ \end{bmatrix} R_+ \text{ and } H\begin{bmatrix} X_- \\ Y_- \end{bmatrix} = \begin{bmatrix} X_- \\ Y_- \end{bmatrix} R_- \quad (8)$$

where $X_\pm, Y_\pm \in \mathbb{R}^{n\times n}$; $Re(\sigma(R_+)) > 0$, $Re(\sigma(R_-)) < 0$. Then $K_{\max} = Y_+ X_+^{-1}$ and $K_{\min} = Y_- X_-^{-1}$.

6.3 Optimum charging/discharging

The extremal positive definite solutions of the ARE $K_{\min}$ and $K_{\max}$ have special significance in terms of the energy dissipation by the system. For a given $a \in \mathbb{R}^n$, consider $\mathfrak{B}_a$, the set of all continuous system trajectories (u, x, y) satisfying equation (4) with $x(0) = a$. Then,

$$\begin{aligned} a^T K_{\max} a &= \inf_{\substack{(u,x,y)\in\mathfrak{B}_a \\ x(-\infty)=0}} \int_{-\infty}^{0} uy\, dt, \\ a^T K_{\min} a &= \sup_{\substack{(u,x,y)\in\mathfrak{B}_a \\ x(\infty)=0}} -\int_{0}^{\infty} uy\, dt. \end{aligned} \quad (9)$$

Thus $a^T K_{\max} a$ is the minimum (or more precisely, infimum) energy required to reach a state $x(0) = a$ from state of rest $x(-\infty) = 0$. As in the RC-network we are dealing with, the states are the voltages of each node, $a^T K_{\max} a$ is the minimum energy required to charge the circuit up to node voltages given by $a = [v_1, v_2, \cdots, v_n]$. This is called 'optimal charging' of the system. Similarly, $a^T K_{\min} a$ is the maximum energy that can be extracted as the system is brought to rest $x(\infty) = 0$ starting from state $x(0) = a$.

6.4 Proofs of main results

In this section we prove the results stated in Section 4.

Proof of Theorm 4.1

Consider an undirected graph G with n-nodes and Laplacian matrix given by $L \in \mathbb{R}^{n\times n}$. Construct the corresponding RC network with each edge resistance as r_c and node capacitances to ground as c with parasitic resistance r_p as illustrated in Figure 1b.

Considering $x = [v_1, \cdots, v_n]^T$-as the node voltages, it follows using nodal analysis that $c\dot{x} = (-\frac{1}{r_c}L - \frac{1}{r_p}I_n)x$. If one applies inputs at p-nodes in the undirected graph G, and if we represent the inputs as currents $(i_{x_1}, \cdots, i_{x_p})$ at the nodes $(x_1, x_2, \cdots, x_p)$ through a source resistance r_s as illustrated in Figure 2b, then the node analysis obtained above is modified for only for the input nodes. If $j = x_k$ is an input node with applied input i_k, then the node analysis of j-th node can be given as (if node j is connected by t-edges to nodes by:$(x_{e_1}, \cdots, x_{e_t})$ as given by the j-th row of L):

$$c\dot{v}_j = -\frac{1}{r_c}(tv_j - v_{e_1} - \cdots - v_{e_t}) - \frac{v_j}{r_p} + i_j \quad (10)$$

Combining the equations 10 and 11, the node analysis of the complete RC network can be given by:

$$c\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \left(-\frac{L}{r_c} - \frac{I_n}{r_p}\right)\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} \quad (11)$$

where $i_x = 0$ if x is not a input node, and $i_x = i_k$ if x-is the k-th input node. The equation (11) can be represented as:

$$\dot{x} = Ax + Bu \quad (12)$$

Table 3: Result of parameter values for one input port in different node pairs of graph G2 of Figure 3a

Node	Fiedler vector	$\mathbb{1}^T K_{\min} \mathbb{1}$	$\mathbb{1}^T K_{\max} \mathbb{1}$	$\lambda_{\min}(K_{\min})$	$\lambda_{\max}(K_{\min})$	$\lambda_{\min}(K_{\max})$	$\lambda_{\max}(K_{\max})$	trace($K_{\min}$)	trace($K_{\max}$)
1	**70.094**	0.555	**1.869**	5.34×10^{-11}	0.557	**1.797**	$\mathbf{1.22 \times 10^{10}}$	**0.615**	$\mathbf{1.224 \times 10^{10}}$
2	40.932	0.574	1.81	5.10×10^{-10}	0.574	1.742	1.24×10^9	0.604	1.257×10^9
3	*5.259*	**0.586**	1.773	1.40×10^{-10}	**0.586**	1.707	4.034×10^9	0.597	4.4047×10^9
4	14.776	0.577	1.799	$\mathbf{1.92 \times 10^{-09}}$	0.577	1.732	3.23×10^8	0.602	3.289×10^8
5	18.147	0.582	1.785	3.06×10^{-10}	0.582	1.719	1.796×10^9	0.6	1.827×10^9
6	26.856	0.581	1.787	8×10^{-10}	0.581	1.72	6.742×10^8	0.6	6.799×10^8
7	45.988	0.562	1.847	2.93×10^{-10}	0.563	1.777	1.965×10^9	0.611	2.015×10^9

Table 4: Result of parameter values for two port input in different nodes of graph G2 of Figure 3a

Nodes	$\mathbb{1}^T K_{\min} \mathbb{1}$	$\mathbb{1}^T K_{\max} \mathbb{1}$	$\lambda_{\min}(K_{\min})$	$\lambda_{\max}(K_{\min})$	$\lambda_{\min}(K_{\max})$	$\lambda_{\max}(K_{\max})$	trace($K_{\min}$)	trace($K_{\max}$)
3, 4	0.678	1.513	2.30×10^{-06}	0.678	1.474	3.0×10^5	0.715	3.15×10^5
3, 5	0.68	1.499	1.38×10^{-05}	0.681	1.469	$5.64 \times x10^4$	0.71	7.2×10^4
3, 6	0.681	1.499	1.56×10^{-05}	0.682	1.467	$4.73 \times x10^4$	0.712	6.2×10^4
4, 5	0.675	1.523	1.37×10^{-05}	0.675	1.481	$5.92 \times x10^4$	0.718	7.95×10^4
4, 6	0.679	1.509	4.00×10^{-07}	0.679	1.473	$1.99 \times x10^6$	0.723	2.03×10^6
5, 6	0.678	1.509	4.00×10^{-06}	0.678	1.474	2.02×10^5	0.715	2.19×10^5
2, 3	0.68	1.501	6.37×10^{-08}	0.68	1.471	1.15×10^7	0.722	1.15×10^7
3, 7	0.677	1.513	4.00×10^{-06}	0.678	1.476	1.93×10^5	0.738	2.14×10^5
2, 4	0.68	1.495	$\mathbf{1.75 \times 10^{-05}}$	0.68	1.471	4.61×10^4	0.736	6.29×10^4
2, 5	0.682	1.489	9.60×10^{-06}	0.682	1.466	7.56×10^4	0.731	9.1610^4
4, 7	0.674	1.523	2.00×10^{-07}	0.675	1.482	3.13×10^6	0.749	3.19×10^6
5, 7	0.674	1.522	4.10×10^{-06}	0.674	1.483	2.08×10^5	0.741	2.71×10^5
2, 6	**0.683**	1.485	1.17×10^{-05}	**0.683**	1.464	4.82×10^4	0.733	7.56×10^4
6, 7	0.666	1.55	5.71×10^{-09}	0.667	1.499	1.05×10^8	0.734	1.06×10^8
1, 3	0.676	1.513	6.09×10^{-08}	0.676	1.479	1.19×10^7	0.747	1.20×10^7
1, 4	0.675	1.508	3.40×10^{-06}	0.676	1.48	2.52×10^5	0.76	2.90×10^5
2, 7	0.678	1.493	4.10×10^{-06}	0.678	1.474	1.87×10^5	0.758	2.01×10^5
1, 5	0.678	1.501	2.80×10^{-06}	0.678	1.474	3.01×10^5	0.755	3.64×10^5
1, 6	0.679	1.489	1.50×10^{-06}	0.679	1.472	5.15×10^5	0.757	5.28×10^5
1, 2	0.655	**1.577**	*1.78×10^{-09}*	0.656	**1.523**	$\mathbf{3.85 \times 10^8}$	0.743	$\mathbf{3.86 \times 10^8}$
1, 7	0.673	1.507	2.90×10^{-06}	0.673	1.485	3.0×10^5	**0.781**	4.18×10^5

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, u \in \mathbb{R}^p$-scaled input currents with $A = (-\frac{1}{r_c c}L - \frac{1}{r_p c}I_n)$, and entries of B defined by $B(i,j) = 1$ when $i = x_k$ and $j = k$, and 0 otherwise. Further, the input $u = \frac{1}{c}\left[i_{x_1}, \ldots, i_{x_p}\right]^T$. Considering the voltage across the input source current as the output, the output equation can be expressed as:

$$y = Cx + Du, \tag{13}$$

with $C = B^T$ and $D = r_s I_p$, thus completing the proof. □

Proof of Theorem 4.2 Consider a system with n-nodes and p-inputs with parasitic resistance $r_p = \infty$. From Theorem 4.1 we have

$$A = A^T = -\frac{1}{r_c c}L \text{ and } \mathbb{1}^T A = A\mathbb{1} = 0$$

$$B = C^T, \ D = r_s I_p \ \text{ which gives } (D + D^T)^{-1} = \frac{1}{2r_s}I_p.$$

The Algebraic Riccati Equation for the passivity supply rate is given as:

$$AK + KA + (KB - B)(D + D^T)^{-1}(B^T K - B^T) = 0.$$

Multiplying $\mathbb{1}$ on both sides we get:

$$\mathbb{1}^T AK\mathbb{1} + \mathbb{1}^T KA\mathbb{1} + \mathbb{1}^T(KB - B)(D + D^T)^{-1}(B^T K - B^T)\mathbb{1} = 0,$$
$$\text{and hence } \mathbb{1}^T(KB - B)(D + D^T)^{-1}(B^T K - B^T)\mathbb{1} = 0,$$
$$\text{which gives } \mathbb{1}^T(KB - B)I_p(B^T K - B^T)\mathbb{1} = 0.$$

Simplifying further, we get $\mathbb{1}^T(KB - B)(B^T K - B^T)\mathbb{1} = 0$ which implies that $(B^T K - B^T)\mathbb{1} = 0$. Use this in what follows: multiply the ARE by $\mathbb{1}$ on the right to get:

$$AK\mathbb{1} + KA\mathbb{1} + (KB - B)(D + D^T)^{-1}(B^T K - B^T)\mathbb{1} = 0,$$

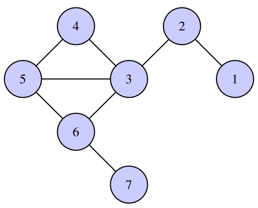
which results in $AK\mathbb{1} = 0$. Since $\mathbb{1}$ is the only (independent) vector in the kernel of A, the vector $\mathbb{1}$ is an eigenvector of the ARE solution K also.

Let λ be the eigenvalue of K corresponding to eigenvector $\mathbb{1}$, i.e. $K\mathbb{1} = \lambda\mathbb{1}$, then using $(B^T K - B^T)\mathbb{1} = 0$ (derived above): we get $B^T K\mathbb{1} = B^T\mathbb{1}$ and this proves $\lambda B^T\mathbb{1} = B^T\mathbb{1}$, and hence $\lambda = 1$ This proves that the eigenvalue corresponding to eigenvector $\mathbb{1}$ of any storage function K is 1. □

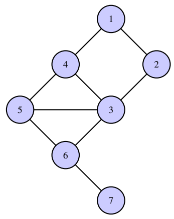
Proof of Theorem 4.3 Consider a system with n-nodes and p-inputs with parasitic resistance $r_p = \infty$. From Theorem 4.1, we have $A = A^T, B = C^T, D = D^T$ and hence the system is a so-called state space symmetrical realization. Using [12, Lemma 3.2], a passive system in minimal state space symmetrical realization is also 'positive real balanced', i.e. $K_{\max} = K_{\min}^{-1}$.

From Theorem 4.2, we know that maximal and minimal storage functions $K_{\max}$ and $K_{\min}$ both have eigenvalue 1, and moreover, the corresponding eigenvector is $\mathbb{1}$.
Since $K_{\max}$ and $K_{\min}$ are positive definite and $K_{\max} = K_{\min}^{-1}$ therefore eigenvalue $\lambda = 1$ corresponds to the largest eigenvalue of $K_{\min}$ and the smallest eigenvalue of $K_{\max}$. □

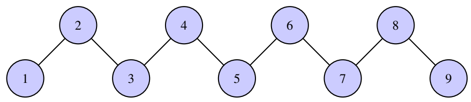
Proof of Theorem 4.4 Consider a system with n-nodes with parasitic resistance $0 < r_p < \infty$ and input applied to all the n-



(a) Graph G2

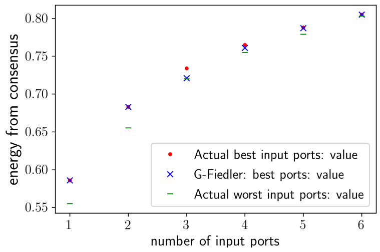


(b) Graph G3

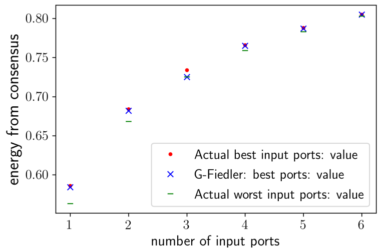


(c) Graph G4: path graph P_9

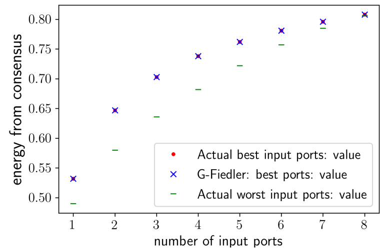
Fig. 3. Undirected graphs



(a) Graph G2 (Figure 3a)

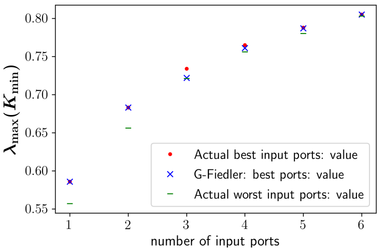


(b) Graph G3 (Figure 3b)

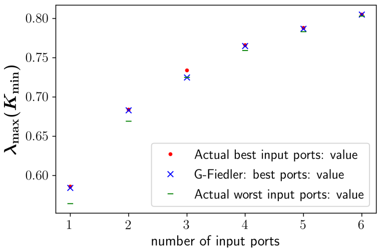


(c) Graph G4=P_9 (Figure 3c)

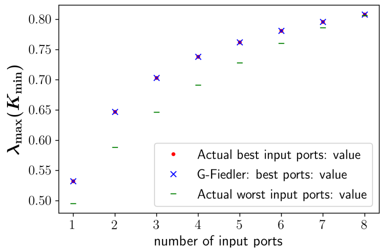
Fig. 4. Maximum energy extractable from consensus: $\mathbb{1}^T K_{\min} \mathbb{1}$



(a) Graph G2

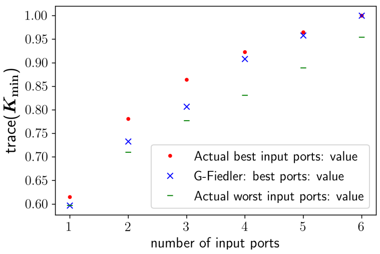


(b) Graph G3

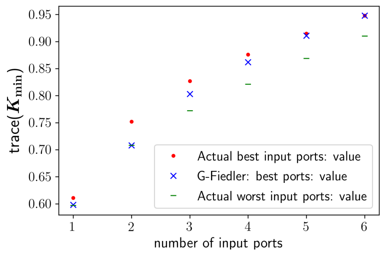


(c) Graph G4=P_9

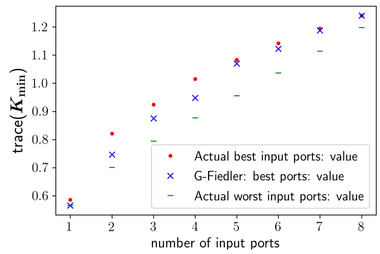
Fig. 5. Value of maximum energy extractable from the best formation vector $v \in \mathbb{S}^{n-1}$: $\lambda_{\max}(K_{\min})$



(a) trace($K_{\min}$) of G2



(b) trace($K_{\min}$) for G3



(c) trace($K_{\min}$) for G4=P_9

Fig. 6. Value of 'average' energy extractable across all formation vector $v \in \mathbb{S}^{n-1}$: trace$(K_{\min})$

nodes, then from Theorem 4.1 we have

$$A = A^T = -\frac{1}{r_c c} L - \frac{1}{r_p c} I_n \text{ which gives } A\mathbb{1} = -\frac{1}{r_p c}\mathbb{1}.$$

Further, $B = C^T = I_n$ and $D = r_s I_n$ and this implies $(D + D^T)^{-1} = \frac{1}{2r_s} I_n$. Multiplying $\mathbb{1}$ on both sides, we get:

$$\mathbb{1}^T AK\mathbb{1} + \mathbb{1}^T KA\mathbb{1} + \mathbb{1}^T (KB - B)(D + D^T)^{-1}(B^T K - B^T)\mathbb{1} = 0,$$
$$\implies -\tfrac{2}{r_p c}\mathbb{1}^T K\mathbb{1} + \mathbb{1}^T (KB - B)(D + D^T)^{-1}(B^T K - B^T)\mathbb{1} = 0,$$
$$\text{and hence } \frac{1}{2r_s}\mathbb{1}^T (K - I_n)(K - I_n)^T \mathbb{1} = \frac{2}{r_p c}\mathbb{1}^T K\mathbb{1}.$$

Since K is symmetric and positive definite, we use its so-called 'square root' $\sqrt{K}$, the unique symmetric and positive definite matrix whose square equals K, to get: $\mathbb{1}^T (K - I_n)(K - I_n)\mathbb{1} = \frac{4r_s}{r_p}\mathbb{1}^T \sqrt{K}\sqrt{K}\,\mathbb{1}$ which gives $(K - I_n)\mathbb{1} = \sqrt{\frac{4r_s}{r_p}}\sqrt{K}\,\mathbb{1}$. Using this next in the expression obtained by multiplying the ARE by $\mathbb{1}$:

$$AK\mathbb{1} + KA\mathbb{1} + (KB - B)(D + D^T)^{-1}(B^T K - B^T)\mathbb{1} = 0,$$

gives $(-\frac{1}{r_c c}L - \frac{1}{r_p c}I_n)K\mathbb{1} - \frac{1}{r_p c}K\mathbb{1} + \frac{2}{r_p c}K\mathbb{1} = 0$ and this implies $\frac{1}{c r_c} LK\mathbb{1} = 0$. This proves that $\mathbb{1}$ is an eigenvector of every ARE solution K. $\square$

7. CONCLUDING REMARKS

In this section we summarize the key contributions in this paper. We considered the problem of choosing the best (one or more) agent(s)/port(s) to communicate to externally, so that the energy required to communicate is minimized. This problem was made concrete by considering an RC analogue circuit for a given undirected graph G_n. The state-space symmetry was brought out in Theorem 4.1.

When dealing further with the problem of optimal port selection, multiple optimizing aspects were considered, we list them again:

•: Given a state-vector v and specified charging/discharging port locations, $v^T K_{\max} v$ is the minimum net energy required to be supplied to reach this state: minimum across all possible input profiles that take the state from 0 to v. (See equation (9).)

•: The minimum eigenvalue $\lambda_{\min}(K_{\max})$ is the minimum energy required to be provided across all vectors $v \in \mathbb{S}^{n-1}$ of unit-length, and in a sense the corresponding eigenvector is the best vector charging-wise. Due to $K_{\min} = K_{\max}^{-1}$, the eigenvectors corresponding to $\lambda_{\min}(K_{\min})$ and $\lambda_{\max}(K_{\max})$ are the same, and hence the corresponding 'best'/'worst' association to eigenvectors of $K_{\min}$: see Table 2.

•: Given a number of ports, say p, the problem considered in this paper is that of identifying which ports are ideal with respect to a criteria: $\lambda_{\max}(K_{\max})$, $\lambda_{\min}(K_{\max})$, $\lambda_{\max}(K_{\min})$, $\lambda_{\min}(K_{\min})$, or alternatively, the average of all vectors $v \in \mathbb{S}^{n-1}$, i.e. the trace of the matrix: $K_{\max}$ while charging and $K_{\min}$ while discharging. We proved new results using the symmetric state space realization properties between ARE solutions, common eigenvectors, eigenvalues and consensus directions.

We brought out how the Laplacian's eigenvectors provide the choices for the optimal port locations: while centrality of the node in a graph was linked to the Fiedler eigenvector in the literature, the choice of 2 or more ports gets analogously linked with eigenvectors corresponding to third-smallest, fourth-smallest, etc. Examples in Section 4 turn out to validate this observation. The advantage of this connection is that Laplacian matrix eigenvectors are easier to compute and have stable algorithms due to the matrix symmetry, unlike computation of Hamiltonian matrix eigenvectors and/or Riccati equation solutions.

A future direction is to pursue further links between centrality with respect to other notions and optimal charging/discharging port locations.

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