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Abstract

Algebraic Riccati Equation (ARE) solutions play an important role in many optimal/suboptimal control problems. However, a key assumption in formulation and solution of the ARE is a certain ‘regularity condition’ on the feedthrough term D of the system. For example, formulation of the ARE requires nonsingularity of $D + D^T$ in positive real dissipative systems and, in the case of bounded real dissipative systems, one requires nonsingularity of $I - D^T D$. Note that for lossless systems $D + D^T = 0$, while for all-pass systems $I - D^T D = 0$; this rules out the formulation of the ARE. Noting that the ARE solutions are also extremal “storage functions” for dissipative systems, one can speak of storage function for the lossless case too. This contributed chapter formulates new properties of the ARE solution; we then show that this property is satisfied by the storage function for the lossless case too. The formulation of this property is via the set of trajectories of minimal dissipation. We show that the states in a first-order representation of this set satisfy *static* relations that are closely linked to ARE solutions; this property holds for the lossless case too. Using this property, we propose an algorithm to compute the storage function for the lossless case.

Chapter 5
New Properties of ARE Solutions for Strictly Dissipative and Lossless Systems

Chayan Bhawal, Sandeep Kumar, Debasattam Pal and Madhu N. Belur

Abstract Algebraic Riccati Equation (ARE) solutions play an important role in many optimal/suboptimal control problems. However, a key assumption in formulation and solution of the ARE is a certain 'regularity condition' on the feedthrough term D of the system. For example, formulation of the ARE requires nonsingularity of $D + D^T$ in positive real dissipative systems and, in the case of bounded real dissipative systems, one requires nonsingularity of $I - D^T D$. Note that for lossless systems $D + D^T = 0$, while for all-pass systems $I - D^T D = 0$; this rules out the formulation of the ARE. Noting that the ARE solutions are also extremal "storage functions" for dissipative systems, one can speak of storage function for the lossless case too. This contributed chapter formulates new properties of the ARE solution; we then show that this property is satisfied by the storage function for the lossless case too. The formulation of this property is via the set of trajectories of minimal dissipation. We show that the states in a first-order representation of this set satisfy *static* relations that are closely linked to ARE solutions; this property holds for the lossless case too. Using this property, we propose an algorithm to compute the storage function for the lossless case.

With best wishes to Harry L. Trentelman on the occasion of his 60th birthday.
The last author adds: "to my Ph.D. supervisor, who always has been very friendly, and imparted a sense of discipline and rigour in teaching and research during the course of my Ph.D. Through your actions, I learnt the meaning of 'zero tolerance' to careless and hasty work. Thanks very much for all the skills I imbibed from you: both technical and non-technical."

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5.1 Introduction

The algebraic Riccati equation (ARE) has found widespread application in many optimal and suboptimal control/estimation problems. For example, Kalman filter, LQ control, H_∞ and H_2 control; see [1, 11], for example. Since its introduction in control theory by Kalman, many conceptual and numerical methods to solve ARE have been developed [3, 11] for instance. In the context of dissipative systems, the ARE solutions are extremal storage functions of the system. More about the link between storage functions, dissipative systems and solvability of AREs can be found in [16, 18]. However, for a special class of dissipative systems, namely, conservative systems, the ARE does not exist. This happens due to the formulation of the ARE depending on a suitable regularity condition on the feedthrough term D of any input-state-output representation of a system. The precise form of the regularity condition depends on the supply rate function, with respect to which dissipativity holds. For example, in case of the "positive real supply rate," $u^T y$, where u is the input and y is the output of the system, existence of the corresponding ARE requires nonsingularity of $D + D^T$. Similarly, for the "bounded real supply rate," $u^T u - y^T y$, nonsingularity of $I - D^T D$ is required for existence of the corresponding ARE. Contrary to this regularity condition, systems that are conservative with respect to the positive real supply rate and the bounded real supply rate have $D + D^T = 0$ and $I - D^T D = 0$, respectively.[1] Hence, for such systems the regularity conditions are violated, and consequently, the corresponding ARE does not exist. In this chapter, we formulate new properties of the ARE solution in terms of the set of trajectories of "minimal dissipation" as formulated recently in [17]: for reasons we will elaborate later, we will call this set "a Hamiltonian system." We show that the ARE solution is closely linked to the static relations that hold between the states in a first-order representation of this set. We then show that this property is satisfied for the storage function for the conservative case too, though the ARE does not exist in this case. We use this result to develop an algorithm to compute the unique storage function for the conservative systems case.

We now elaborate further on the key property that the ARE solution satisfies: which we extend to the lossless case. The property is based on an observation concerning the relation between ARE solutions and Hamiltonian systems. It is well known that when the feedthrough term satisfies the regularity conditions, that is, when the ARE exists, the solutions to the ARE can be found using suitable invariant subspaces of a corresponding Hamiltonian matrix. Note that, in the singular cases (lossless/all-pass), the Hamiltonian matrix does not exist. Consequently, this method involving the invariant subspace fails to work for the singular cases. However, this same method, when viewed from a different perspective opens up a new way of computing the

[1] Lossless systems, with u input and y output, are conservative with respect to the "positive real supply rate" $u^T y$ and have $D + D^T = 0$. Similarly, all-pass systems are conservative with respect to the "bounded real supply rate" $u^T u - y^T y$. For all-pass systems $I - D^T D = 0$. Hence, all arguments about ARE solutions and storage functions made for lossless systems are applicable to all-pass systems as well.

ARE solutions, which extends naturally to the singular case, too. This new point of view stems from the fact that the first-order system defined by the Hamiltonian matrix associated to an ARE is nothing but a state representation of a system comprised of the "trajectories of minimal dissipation." Consequently, choosing an invariant subspace im $\begin{bmatrix} I \\ K \end{bmatrix}$ of the Hamiltonian matrix to get K as a solution to the ARE, can be viewed as obtaining a subsystem of the Hamiltonian system by *restricting* the trajectories to satisfy an extra set of equations as $z = Kx$, where x, z are state variables of the original system and its 'dual', respectively. The crucial fact about this new view-point is that, although, the Hamiltonian *matrix* and the ARE do not exist in the singular case, the Hamiltonian *system*, comprising of the trajectories of minimal system does exist. We show in this chapter that, in such cases too, the strategy of putting static relation $z = Kx$ leads to a storage function $x^T Kx$ to the original system.

The notation used in the chapter is standard. The set $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. The set $\mathbb{R}[\xi]$ denotes the ring of polynomials in ξ with real coefficients. The set $\mathbb{R}^{\mathtt{w}\times\mathtt{p}}[\xi]$ denotes all $\mathtt{w} \times \mathtt{p}$ matrices with entries from $\mathbb{R}[\xi]$. We use $\bullet$ when a dimension need not be specified: for example, $\mathbb{R}^{\mathtt{w}\times\bullet}$ denotes the set of real constant matrices having $\mathtt{w}$ rows. $\mathbb{R}[\zeta, \eta]$ denotes the set of real polynomials in two indeterminates: ζ and η. The set of $\mathtt{w} \times \mathtt{w}$ matrices with entries in $\mathbb{R}[\zeta, \eta]$ is denoted by $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. The space $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ stands for the space of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$, and $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ stands for the subspace of all compactly supported trajectories in $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

This chapter is structured as follows: Sect. 5.2 summarizes preliminaries required in the chapter. New properties of ARE solutions are presented in Sect. 5.3. In Sect. 5.4, we formulate and prove new results that help computation of storage function K for conservative behaviors based on the notion of "trajectories of minimal dissipation." Section 5.5 uses the main result in Sect. 5.4 and proposes a numerical algorithm to compute storage function of conservative systems. Section 5.6 contains numerical examples to illustrate the main results. Some concluding remark is presented in Sect. 5.7.

5.2 Preliminaries

In this section, we give a brief introduction to various concepts that are required to formulate and solve the problem addressed in the chapter.

5.2.1 Behavior

We start with some essential preliminaries of the behavioral approach: a detailed exposition can be found in [12].

Definition 5.1 A linear differential behavior $\mathfrak{B}$ is defined as the subspace of infinitely often differentiable functions $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ consisting of all solutions to a set of linear ordinary differential equations with constant coefficients, i.e., for $R(\xi) \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$

$$\mathfrak{B} := \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{d}{dt}\right) w = 0 \right\}. \tag{5.1}$$

The variable w in Eq. (5.1) is called the *manifest variable* of the behavior $\mathfrak{B}$. We denote linear differential behaviors with $\mathtt{w}$ number of manifest variables as $\mathfrak{L}^{\mathtt{w}}$. Equation (5.1) is what we call a *kernel representation* of the behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and we sometimes also write $\mathfrak{B} = \ker R(\frac{d}{dt})$. We assume the polynomial matrix $R(\xi)$ has full row rank without loss of generality (see [12, Chap. 6]). This assumption guarantees existence of a nonsingular block $P(\xi)$ (after a permutation of columns, if necessary, with a corresponding permutation of the components of w) such that $R(\xi) = \begin{bmatrix} P(\xi) & Q(\xi) \end{bmatrix}$. Conforming to this partition of $R(\xi)$, partition w into $w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$, where it has been shown that u, y are the input and output of the behavior $\mathfrak{B}$ respectively: note that this partition is not unique. Such a partition is called an *input-output* partition of the behavior. The input-output partition is called *proper* if $P^{-1}Q$ is a matrix of *proper* rational functions. Although there are a number of ways in which the manifest variables can be partitioned as input and output, the number of components of the input depends only on $\mathfrak{B}$: we denote this number as $\mathtt{m}(\mathfrak{B})$, and call it the *input cardinality* of the behavior. The number of components in the output is called the *output cardinality* represented as $\mathtt{p}(\mathfrak{B})$. It is well known that $\mathtt{p}(\mathfrak{B}) = \operatorname{rank} R(\xi)$ and $\mathtt{m}(\mathfrak{B}) = \mathtt{w} - \mathtt{p}(\mathfrak{B})$.

In the behavioral approach, a system is nothing but its behavior: we use the terms behavior/system interchangeably. There are various ways of representing a behavior depending on how the system is modeled: a useful one is the *latent variable representation*: for $R(\xi) \in \mathbb{R}^{\bullet \times \mathtt{w}}$ and $M(\xi) \in \mathbb{R}^{\bullet \times \mathtt{m}}[\xi]$,

$$\mathfrak{B} := \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{there exists } \ell \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{m}}) \text{ such that } R\left(\frac{d}{dt}\right) w = M\left(\frac{d}{dt}\right) \ell \right\}.$$

Here ℓ is called a latent variable.

Controllability is another important concept required for this chapter.

Definition 5.2 A behavior $\mathfrak{B}$ is said to be *controllable* if for every pair of trajectories $w_1, w_2 \in \mathfrak{B}$ there exists $w_3 \in \mathfrak{B}$ and $\tau > 0$ such that

$$w_3(t) = \begin{cases} w_1(t) & \text{for } \quad t \leqslant 0, \\ w_2(t) & \text{for } \quad t \geqslant \tau. \end{cases}$$

We represent the set of all controllable behaviors with $\mathtt{w}$ variables as $\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. The familiar PBH rank test for controllability has been generalized: a behavior $\mathfrak{B}$ with minimal kernel representation $\mathfrak{B} = \ker R(\frac{d}{dt})$ is controllable if and only if $R(\lambda)$ has constant rank for all $\lambda \in \mathbb{C}$. One of the ways by which a behavior $\mathfrak{B}$ can be represented if (and only if) $\mathfrak{B}$ is controllable is the *image representation*: for $M(\xi) \in \mathbb{R}^{\mathtt{w} \times \mathtt{m}}[\xi]$

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$$\mathfrak{B} := \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{there exists } \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}}) \text{ such that } w = M\left(\frac{d}{dt}\right)\ell \right\}. \tag{5.2}$$

If $M(\xi)$ is such that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, then the image representation is said to be an *observable* image representation: this can be assumed without loss of generality (see [12, Sect. 6.6]).

5.2.2 Quadratic Differential Forms and Dissipativity

This section contains a brief review of Quadratic Differential Forms (QDFs): more on QDFs can be found in [18]. We often encounter quadratic expressions of derivatives of the manifest and/or latent variables of the behavior $\mathfrak{B}$. Two-variable polynomial matrices can be associated with such quadratic forms. Consider a two-variable polynomial matrix $\phi(\zeta, \eta) := \sum_{j,k} \phi_{jk}\zeta^j\eta^k \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ where $\phi_{jk} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. The QDF Q_ϕ induced by $\phi(\zeta, \eta)$ is a map $Q_\phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined as

$$Q_\phi(w) := \sum_{j,k} \left(\frac{d^j w}{dt^j}\right)^T \phi_{jk} \left(\frac{d^k w}{dt^k}\right).$$

Of course, when $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, then $Q_\Sigma(w) = w^T \Sigma w$. Using the definition of QDFs, we next define a dissipative system.

Definition 5.3 Consider $\Sigma = \Sigma^T \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and controllable $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. The system $\mathfrak{B}$ is said to be Σ-dissipative if

$$\int_{\mathbb{R}} Q_\Sigma(w)\, dt \geqslant 0 \quad \text{for every} \quad w \in \mathfrak{B} \cap \mathfrak{D}. \tag{5.3}$$

The function $Q_\Sigma(w)$ is also called the supply rate: it is the rate of supply of energy to the system. For simplicity, we also call Σ the supply rate. Equation (5.3) formalizes the notion that dissipative systems are such that the net energy exchange is always an *absorption* when the trajectories considered are those that start-from-rest and end-at-rest, i.e. compactly supported. The link with existence of a storage function is well known for the controllable system case: a controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is dissipative with respect to Σ if and only if there exists a quadratic differential form $Q_\psi(w)$ such that

$$\frac{d}{dt} Q_\psi(w) \leqslant Q_\Sigma(w) \text{ for all } w \in \mathfrak{B}.$$

The QDF Q_ψ is called a storage function for $\mathfrak{B}$ with respect to the supply rate Σ.

The notion of a storage function captures the intuition that the rate of increase of stored energy in a dissipative system is at most the power supplied. In this chapter, we

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shall be dealing with supply rates Q_Σ induced by real symmetric constant nonsingular matrices Σ only. We need a count of the number of positive eigenvalues (with multiplicities) of the symmetric matrix Σ: call this number the *positive signature* of the matrix Σ and denote it by $\sigma_+(\Sigma)$.

For a Σ-dissipative system, $\mathtt{m}(\mathfrak{B})$, the input cardinality of the behavior, cannot exceed the positive signature $\sigma_+(\Sigma)$ of the supply rate Σ i.e. $\mathtt{m}(\mathfrak{B}) \leqslant \sigma_+(\Sigma)$ (details in [18, Remark 5.11] and [19]). For this chapter, we restrict ourselves to the so-called *maximum input cardinality condition*, i.e.

$$\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma). \tag{5.4}$$

Given $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and a system described by the observable image representation $w = M(\frac{d}{dt})\ell$, the QDF $Q_\Sigma(w)$ can also be expressed as $Q_\Phi(\ell)$ in the latent variables induced by $\Phi(\zeta,\eta) \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\zeta,\eta]$ is given by

$$\Phi(\zeta,\eta) := M(\zeta)^T \Sigma M(\eta).$$

Conservative systems are a special class of dissipative systems and this work focusses on the conservative systems' case: this is when the algebraic Riccati equation does *not* exist.

Definition 5.4 Consider a symmetric and nonsingular matrix $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. The system $\mathfrak{B}$ is called Σ-conservative if

$$\int_{\mathbb{R}} Q_\Sigma(w)dt = 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

In order to simplify the exposition in this chapter, we shall be using the positive real supply rate $2u^T y$ i.e.

$$Q_\Sigma = \begin{bmatrix} u \\ y \end{bmatrix}^T \Sigma \begin{bmatrix} u \\ y \end{bmatrix} \text{induced by } \Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{5.5}$$

where u and y are the input and output of the system respectively. Systems conservative with respect to the positive real supply rate are known in the literature as lossless systems: see Footnote 1. We will be dealing with only lossless systems in this chapter. However, the results in the chapter can be extended to system conservative with respect to other supply rates also.

5.2.3 State Representation and Trimness

A *state variable representation* of a behavior $\mathfrak{B}$ is a latent variable representation where the latent variable x satisfies the *axiom of state:* whenever $(w_1, x_1),\ (w_2, x_2) \in$

$\mathfrak{B}_{\text{full}}$ and $x_1(0) = x_2(0)$, the *concatenation*[2] $(w_1, x_1) \wedge_0 (w_2, x_2)$ at $t = 0$ also satisfies the equations of $\mathfrak{B}_{\text{full}}$ in a weak/distributional sense. For such a system, we have a first-order description, called the state-space description:

$$E\frac{dx}{dt} + Fx + Gw = 0 \text{ where } E, F, G \text{ are constant real matrices.} \tag{5.6}$$

A state-space description is said to be *minimal* if the number of components in the state x is the minimum amongst all possible state representations. The number of states corresponding to a minimal state representation of $\mathfrak{B}$ is called the *McMillan degree* of the behavior $\mathfrak{B}$. When the state x is not minimal (but is observable from the system variable w), it is known that one or more components in x satisfy a static relation and the states are said to be *nontrim*. A more formal definition of *state trim* is presented next.

Definition 5.5 The state x in Eq. (5.6) is said to be trim if for every $a \in \mathbb{R}^{\mathtt{n}}$ there exist a $w \in \mathfrak{B}$ such that $x(0) = a$ and (w, x) satisfies Eq. (5.6).

The algorithm proposed in this chapter is based on this notion of state trimness. The static relation between the state x of the given lossless system and the "dual state" z of the adjoint system are used to find the unique storage function for the lossless case: see Theorem 5.13 below.

5.2.4 Minimal Polynomial Basis

This section contains a review of the notion of a minimal polynomial basis.

The *degree* of a polynomial vector $r(s) \in \mathbb{R}^{\mathtt{n}}[s]$ is the maximum degree among the $\mathtt{n}$ components of the vector. The degree of the zero polynomial and the zero vector in $\mathbb{R}^{\mathtt{n}}[s]$ is defined as $-\infty$.

For $R(s) \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}[s]$, the set of all polynomial vectors $v(s) \in \mathbb{R}^{\mathtt{m}}[s]$ that satisfy $R(s)v(s) = 0$ forms a vector space over the field of scalar rational functions. It is known from the literature that such a vector space admits a polynomial basis called the right nullspace basis of the polynomial matrix $R(s)$: see [8, Sect. 6.5.4]. There is a special nullspace basis called the *minimal polynomial basis* of the polynomial matrix $R(s)$ which is of importance to us in this chapter. Consider the polynomial matrix $R(s) \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}[s]$ of rank $\mathtt{n}$. Let the set $\{p_1(s), p_2(s), \ldots, p_{\mathtt{m}-\mathtt{n}}(s)\}$ be a nullspace basis of $R(s)$ ordered be their degrees $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_{\mathtt{m}-\mathtt{n}}$. The set $\{p_1(s), p_2(s), \ldots, p_{\mathtt{m}-\mathtt{n}}(s)\}$ is said to be a *minimal polynomial basis* of $R(s)$ if every other nullspace basis $\{q_1(s), q_2(s), \ldots, q_{\mathtt{m}-\mathtt{n}}(s)\}$ with degree $c_1 \leqslant c_2 \leqslant$

[2] For trajectories (w_1, x_1) and (w_2, x_2), their *concatenation at* t_0, denoted by $(w_1, x_1)\wedge_{t_0}(w_2, x_2)$, is defined as

$$(w_1, x_1) \wedge_{t_0} (w_2, x_2)(t) := \begin{cases} (w_1, x_1)(t) & \text{for } t < t_0 \\ (w_2, x_2)(t) & \text{for } t \geqslant t_0. \end{cases}$$

.

$\cdots \leqslant c_{\mathtt{m}-\mathtt{n}}$ is such that $d_i \leqslant c_i$, for$i = 1, 2, \ldots \mathtt{m} - \mathtt{n}$. The degrees of the vectors of minimal polynomial basis of $R(s)$ are called the *(Forney invariant) minimal indices* or *Kronecker indices* (more details in [8, Sect. 6.5.4]).

5.3 The Algebraic Riccati Equation (ARE) and Hamiltonian Systems

With a proper input-output partition (u, y), a controllable dissipative behavior $\mathfrak{B}$ admits the following minimal i/s/o representation.

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du, \qquad A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, \quad B,\ C^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}} \text{ and } D \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}} \tag{5.7}$$

with (C, A) observable. We assume here that the number of input $\mathtt{m}(\mathfrak{B})$ = number of output $\mathtt{p}(\mathfrak{B})$: this assumption is in view of the maximum input cardinality condition and $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. The storage functions of a dissipative behavior are closely related to the algebraic Riccati inequality (ARI) and the Hamiltonian matrix. One of the results relating LMI, controllable behavior and storage function is the *Kalman–Yakubovich–Popov (KYP)* lemma: details in [6, Sect. 5.6]. For easy reference, we present the KYP lemma, in a behavioral context, as a proposition next.

Proposition 5.6 *A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, with a controllable and observable minimal i/s/o representation as in Eq. (5.7), is Σ-dissipative if and only if there exists a solution $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ to the LMI*

$$\begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -(D + D^T) \end{bmatrix} \leqslant 0. \tag{5.8}$$

For systems with $D + D^T > 0$, the Schur complement with respect to $D + D^T$ in LMI (5.8) results in the algebraic Riccati inequality

$$A^T K + KA + (KB - C^T)(D + D^T)^{-1}(B^T K - C) \leqslant 0. \tag{5.9}$$

The corresponding equation to the inequality (5.9) is called the algebraic Riccati equation (ARE). Symmetric solutions to the ARE have a one-to-one correspondence to $\mathtt{n}$-dimensional invariant subspaces of the matrix below (details in [10, Theorem 3.1.1]).

$$\mathscr{H} = \begin{bmatrix} A - B(D + D^T)^{-1}C & B(D + D^T)^{-1}B^T \\ -C^T(D + D^T)^{-1}C & -A^T + C^T(D + D^T)^{-1}B^T \end{bmatrix} \tag{5.10}$$

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The matrix $\mathscr{H}$ is known as the Hamiltonian matrix. Every $\mathtt{n}$-dimensional $\mathscr{H}$ invariant subspace spanned by columns of $\begin{bmatrix} I \\ K \end{bmatrix}$ corresponding to a suitably chosen set of eigenvalues of $\mathscr{H}$, provides a solution K to the ARE.

The detailed procedure to find the solution to the ARE from $\mathtt{n}$-dimensional eigenspaces of the Hamiltonian matrix can be found in [4]. We provide a brief review of the procedure next. In the lines of [10] and [13, Definition 5.1.1], we define a Lambda set (Λ) to define the partition of eigenvalues of the Hamiltonian matrix $\mathscr{H}$. $\bar{\Lambda}$ denotes the set of complex conjugates of the elements in Λ.

Definition 5.7 Consider an even polynomial $p(\xi) \in \mathbb{R}[\xi]$ with no roots on the imaginary axis. A set of complex numbers $\Lambda \subset$ roots (p) is called a Lambda set of the roots of p if the following conditions are satisfied:

1. $\Lambda = \bar{\Lambda}$
2. $\Lambda \cap (-\Lambda) = \emptyset$
3. $\Lambda \cup (-\Lambda) =$ roots of $p(\xi)$ (counted with multiplicity)

Condition 1 in Definition 5.7 implies that the Lambda set should contain conjugate pairs of complex roots of $p(\xi)$. By condition 2, polynomial $p(\xi)$ should not have any roots on the imaginary axis.

In this chapter, we use the word Lambda set with respect to the eigenvalues of a matrix to mean the Lambda set corresponding to the roots of the characteristic polynomial of the matrix. Constructing Lambda set from the set of eigenvalues of $\mathscr{H}$ (spec($\mathscr{H}$)), we find the solutions to the ARE. This is a well-known result in the literature [10] and we present it as a proposition here.

Proposition 5.8 *Consider a minimal i/s/o system given by Eq.* (5.7) *and the algebraic Riccati equation* $A^T K + KA + (KB - C^T)(D + D^T)^{-1}(B^T K - C) = 0$. *The corresponding Hamiltonian matrix* $\mathscr{H} \in \mathbb{R}^{2\mathtt{n}\times 2\mathtt{n}}$ *is given by Eq.* (5.10)*. Assume that the Hamiltonian matrix* $\mathscr{H}$ *has no eigenvalues on the imaginary axis and define* Λ *to be a Lambda set of spec* $(\mathscr{H})$*. Let the* $\mathtt{n}$*-dimensional* $\mathscr{H}$*-invariant subspace corresponding to the Lambda set* Λ *be*

$$S_\Lambda := im \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \text{ where } X_1, X_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$$

Then, X_1 *is invertible and* $K := X_2 X_1^{-1}$ *is a real symmetric solution to the ARE.*

The solutions to the ARE are storage functions $x^T K x$ of the behavior $\mathfrak{B}$ with x the state in i/s/o representation (Eq. (5.7)).

In order to describe the algorithm and the main results of the chapter, we need the definition of the orthogonal complement of a behavior $\mathfrak{B}$.

Definition 5.9 Consider a controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and a symmetric $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. The Σ-orthogonal complement behavior $\mathfrak{B}^{\perp_\Sigma}$ of $\mathfrak{B}$ is defined as

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$$\mathfrak{B}^{\perp_\Sigma} := \left\{ v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \int_{-\infty}^{\infty} v^T \Sigma w \, dt = 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D} \right\}.$$

It is well known that an i/s/o representation of $\mathfrak{B}$ (with $w = (u, y) \in \mathfrak{B}$) gives one for $\mathfrak{B}^{\perp_\Sigma}$: see [18, Sect. 10]. If $\dot{x} = Ax + Bu$, $y = Cx + Du$ is a minimal i/s/o representation of $\mathfrak{B}$, then (with respect to the positive real supply rate), a minimal i/s/o representation $\mathfrak{B}^{\perp_\Sigma}$ (with $v \in \mathfrak{B}^{\perp_\Sigma}$, $v = (u, y)$) is

$$\dot{z} = -A^T z + C^T u \quad \text{and} \quad y = B^T z - D^T u. \tag{5.11}$$

For a given behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and supply rate Σ, we call $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ a *Hamiltonian system* and denote it by $\mathfrak{B}_{\text{Ham}}$: see Remark 5.10 below for a brief background. It has been shown in [17] that these trajectories are *trajectories of minimal dissipation* for the given supply rate. The first-order representation for this set has a good structure: this has been used in [15] for example. Define $E := \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $H := \begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & D + D^T \end{bmatrix}$. A (possibly nonminimal) first-order representation of $\mathfrak{B}_{\text{Ham}}$ is given by

$$\left(\frac{d}{dt} E - H \right) \begin{bmatrix} x \\ z \\ y \end{bmatrix} = 0. \tag{5.12}$$

Define $R(\xi) := (\xi E - H)$; we call $R(\xi)$ a 'Hamiltonian pencil'.

Remark 5.10 In classical optimal control theory, given a quadratic cost functional, the system of trajectories satisfying the corresponding Euler–Lagrange (EL) equation can be considered a Hamiltonian system. Further, the trajectories are called stationary with respect to this cost: see [14, Sect. 4] for example. The EL equation with respect to the integral of QDF Q_Σ turns out to be $\partial\Phi(\frac{d}{dt})\ell := M(-\frac{d}{dt})^T \Sigma M(\frac{d}{dt})\ell = 0$ with $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$. For $\ell^\star$ satisfying this system of equations, $w^\star := M(\frac{d}{dt})\ell^\star$ turns out to be stationary with respect to $w^T \Sigma w$: see [14, Proposition 4.1]. For a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and its orthogonal complement $\mathfrak{B}^{\perp_\Sigma}$, it is shown in [7, Theorem 3.3] that $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma} = M(\frac{d}{dt})\ker\partial\Phi(\frac{d}{dt})$; with this background, we call $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ a Hamiltonian behavior and the matrix pencil $R(\xi)$ related to the first-order representation of $\mathfrak{B}_{\text{Ham}}$, a *Hamiltonian pencil*.

Corresponding to a Λ-set of the eigenvalues of $\mathscr{H}$, we associate a behavior $(\mathfrak{B}_{\text{Ham}})_\Lambda \in \mathfrak{L}^{\mathtt{w}}$ such that $(\mathfrak{B}_{\text{Ham}})_\Lambda$ contains (possibly polynomial times) exponential trajectories with the time-exponent λ_i an element in Λ. Further $(\mathfrak{B}_{\text{Ham}})_\Lambda$ is a sub-behavior of $\mathfrak{B}_{\text{Ham}}$, i.e., all the trajectories in $(\mathfrak{B}_{\text{Ham}})_\Lambda$ are trajectories in $\mathfrak{B}_{\text{Ham}}$. This notion has been used elsewhere too. For example, for Λ-set corresponding to the $\mathtt{n}$ eigenvalues of $\mathscr{H}$ in $\mathbb{C}^+$, the corresponding $(\mathfrak{B}_{\text{Ham}})_\Lambda = (\mathfrak{B}_{\text{Ham}})_{antistab}$ as defined

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in [17, Theorem 3.6]. The same notion has also been used in [15, Sect. 3]. We present a theorem next which shows the relations between Hamiltonian systems and storage functions of a behavior. Some of the equivalences are known. This theorem is the one we extend to the lossless case in Theorem 5.13 below.

Theorem 5.11 *Consider a controllable, strictly dissipative behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with minimal state-space representation as in Eq. (5.7) and McMillan degree* $\mathtt{n}$. *The corresponding Hamiltonian behavior $\mathfrak{B}_{Ham} = ker R(\frac{d}{dt})$ where $R(\xi) := \xi E - H \in \mathbb{R}^{(2\mathtt{n}+\mathtt{p})\times(2\mathtt{n}+\mathtt{p})}$ is the Hamiltonian pencil defined in Eq. (5.12). Suppose $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is a solution to the ARE corresponding to the behavior $\mathfrak{B}$. Then, the following statements hold.*

1. *The Hamiltonian behavior $\mathfrak{B}_{Ham}$ is autonomous, i.e.* $\det R(\xi) \neq 0$.
   In fact $\deg \det R(\xi) = 2\mathtt{n}$.
2. $\frac{d}{dt} x^T K x = 2u^T y$ *for all* $\begin{bmatrix} u \\ y \end{bmatrix} \in (\mathfrak{B}_{Ham})_{\Lambda}$.
3. *rank* $\begin{bmatrix} & R(\xi) & \\ -K & I & 0 \end{bmatrix} = \text{rank } R(\xi) = 2\mathtt{n}+\mathtt{p}$.
4. *rank* $\begin{bmatrix} & R(\lambda) & \\ -K & I & 0 \end{bmatrix} = \text{rank } R(\lambda) < 2\mathtt{n}+\mathtt{p}$ *for each* $\lambda \in \Lambda(\text{roots } det\ R(\xi))$.

Proof Statement **1** is trivial and so we do not dwell on it further: see [18, Sect. 4]. The polynomial matrix $R(\xi)$ is full row rank and hence **3** is true. Statement **2** has been proved in [18, Theorem 4.8]. Hence, we proceed to prove Statement 4.

4: In order to prove 4 of Theorem 5.11, we first prove that

$$\ker \begin{bmatrix} & R(\lambda) & \\ -K & I & 0 \end{bmatrix} = \ker R(\lambda) \text{ for any } \lambda \in \Lambda \text{ (roots det } R(\xi)) = \Lambda(\text{spec}(\mathscr{H})).$$

Of course $\ker \begin{bmatrix} & R(\lambda) & \\ -K & I & 0 \end{bmatrix} \subseteq ker\, R(\lambda)$ holds and the reverse inclusion requires to be proved.

Conversely, let $v \in \ker R(\lambda)$. Hence v is an eigenvector[3] of $R(\xi)$ corresponding to eigenvalue λ. By Proposition 5.8 we have $\begin{bmatrix} I \\ K \\ 0 \end{bmatrix}$ spans the eigenspace of $R(\xi)$. Hence $v \in \text{span} \begin{bmatrix} I \\ K \\ 0 \end{bmatrix}$. It is obvious that $\begin{bmatrix} -K \\ I \\ 0 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} I \\ K \\ 0 \end{bmatrix}$. Hence $[\,-K\ I\ 0\,]\,v = 0$. Thus we conclude that $\ker R(\lambda) \subseteq \ker \begin{bmatrix} & R(\lambda) & \\ -K & I & 0 \end{bmatrix}$, and this proves *equality* of the kernels. This proves that the ranks are equal. Hence 4 follows. This completes the proof of Theorem 5.11. □

[3] As in [5], for a square and nonsingular polynomial matrix $R(s)$, we call the values of $\lambda \in \mathbb{C}$ at which rank of $R(\lambda)$ drops the *eigenvalues* of the polynomial matrix $R(s)$ and we call the vectors in the nullspace of $R(\lambda)$ the *eigenvectors* of $R(s)$ corresponding to λ.

5.4 Storage Functions for Lossless Systems

Due to the condition $D + D^T = 0$ for lossless systems, Proposition 5.8 cannot be used to find storage functions of lossless systems. However, for lossless systems, the LMI (5.8) still exists with equality and solution to this LME can be interpreted as storage functions even in the absence of the ARI and Hamiltonian matrix. The LME is equivalent to solving the following matrix equations.

$$A^T K + KA = 0 \quad \text{and} \quad B^T K - C = 0 \tag{5.13}$$

For a lossless behavior $\mathfrak{B}$, the first-order representation of the Hamiltonian system $\mathfrak{B}_{\text{Ham}}$ is

$$\begin{bmatrix} \xi I_n - A & 0 & -B \\ 0 & \xi I_n + A^T & -C^T \\ -C & B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ y \end{bmatrix} = 0. \tag{5.14}$$

Our main result (Theorem 5.13) below uses the nontrimness aspect in the states above. A special case of [2, Lemma 11] relates to trimness: we state this as a proposition below for easy reference.

Proposition 5.12 *Consider a Σ-dissipative behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and its orthogonal complement behavior $\mathfrak{B}^{\perp_\Sigma}$ with supply rate induced by the nonsingular matrix Σ of Eq. (5.5) (i.e. the positive real supply rate). Assume the behavior satisfies the maximum input cardinality (Eq. (5.4)). Then the following are equivalent.*

1. *$\mathfrak{B}$ is lossless.*
2. $\mathfrak{B} = \mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma} = \mathfrak{B}^{\perp_\Sigma}$

Since the McMillan degree of $\mathfrak{B}$ is $\mathtt{n}$, from Proposition 5.12, we infer that McMillan degree of the Hamiltonian behavior $\mathfrak{B}_{\text{Ham}}$ is also $\mathtt{n}$. However, the Hamiltonian behavior in Eq. (5.14) has $2\mathtt{n}$ states and hence $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma} = \mathfrak{B}_{\text{Ham}}$ is not state trim, i.e., there is a static relationship between state x and the dual state z. The next theorem helps extract the static relations of the first-order representation (5.14) of behavior $\mathfrak{B}_{\text{Ham}}$ and in the process yields the unique storage function for the lossless behavior $\mathfrak{B}$.

Theorem 5.13 *Consider a controllable, lossless behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with minimal state-space representation as in Eq. (5.7). The McMillan degree of $\mathfrak{B}$ is $\mathtt{n}$. The corresponding Hamiltonian behavior $\mathfrak{B}_{Ham} = \ker R(\frac{d}{dt})$ where $R(\xi) := \xi E - H$ is the Hamiltonian pencil described in Eq. (5.12) with $D + D^T = 0$. Then the following statements hold.*

1. *The Hamiltonian behavior $\mathfrak{B}_{Ham}$ is not autonomous, i.e. $\det R(\xi) = 0$.*
2. *There exists a unique symmetric matrix $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ that satisfies*

$$\frac{d}{dt} x^T K x = 2u^T y \quad \textit{for all} \quad \begin{bmatrix} u \\ y \end{bmatrix} \in \mathfrak{B}_{Ham} = \mathfrak{B}. \tag{5.15}$$

3. *There exists a unique symmetric matrix* $K \in \mathbb{R}^{n \times n}$ *that satisfies*

$$rank \begin{bmatrix} & R(\xi) & \\ -K & I & 0 \end{bmatrix} = rank\, R(\xi). \tag{5.16}$$

Proof Statement 1 is well known and details on it can be found in [7, 14] for example. Statement 2 shows the existence of a storage function and this has been dealt with in [18, Remark 5.9]. Hence we prove 3 next.

3: We prove Eq. (5.16) of Theorem 5.13 here.
Using 2 of Theorem 5.13, we have

$$\frac{d}{dt} x^T K x = 2u^T y \quad \text{i.e.} \quad \dot{x}^T K x + x^T K \dot{x} = 2u^T y$$

Using system Eq. (5.7) of behavior $\mathfrak{B}$, we have

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \text{for each} (x, u) \text{ satisfying system equations.}$$

Since (A, B) is controllable and u is input to the system, there is a system trajectory (x, u) that passes through each (x_0, u_0) for $x_0 \in \mathbb{R}^n$ and $u_0 \in \mathbb{R}^m$. Hence

$$\begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & 0 \end{bmatrix} = 0$$

Therefore, we infer that

$$A^T K + KA = 0 \quad \text{and} \quad B^T K - C = 0 \tag{5.17}$$

It is known from [18, Sect. 10] that

$$\frac{d}{dt} x^T z = 2u^T y = \frac{d}{dt} x^T K x \text{ which evaluates to } \dot{x}^T z + x^T \dot{z} - \dot{x}^T K x - x^T K \dot{x} = 0.$$

Using the Eqs. (5.7) and (5.11), we have

$$\begin{aligned} (Ax + Bu)^T z + x^T(-A^T z + C^T u) - (Ax + Bu)^T K x - x^T K (Ax + Bu) = 0 \\ \text{i.e.} \quad u^T B^T z + x^T C^T u - x^T (A^T K + KA) x - u^T B^T K x - u^T B^T K x = 0 \end{aligned}$$

Using Eq. (5.17), we have

$$2u^T B^T (z - Kx) = 0 \tag{5.18}$$

Equation (5.18) is satisfied for all system trajectories and at every time instant. This proves that $B^T(z - Kx) = 0$. We crucially use (A, B) controllability and (C, A) observability, together with Eq. (5.17) to conclude that $z - Kx = 0$ is satisfied over *all* system trajectories. Thus we proved that adding the laws $\begin{bmatrix} -K & I & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ y \end{bmatrix}$ to the system equations $R(\frac{d}{dt}) \begin{bmatrix} x \\ z \\ y \end{bmatrix}$ imposes *no further restriction* on $\mathfrak{B}$. This proves that rank $\begin{bmatrix} & R(\xi) & \\ -K & I & 0 \end{bmatrix} =$ rank $R(\xi)$, and thus completes the proof of Theorem 5.13. $\square$

The next corollary states that Conditions 2 and 3 of Theorem 5.13 are equivalent. This equivalence condition is used to develop an algorithm to compute the storage function of a lossless behavior $\mathfrak{B}$.

Corollary 5.14 *Consider a controllable, lossless behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with minimal state-space representation as in Eq. (5.7). Let the McMillan degree of $\mathfrak{B}$ be* $\mathtt{n}$. *Consider the corresponding Hamiltonian behavior $\mathfrak{B}_{Ham} = \ker R(\frac{d}{dt})$ where $R(\xi) := \xi E - H$ is the Hamiltonian pencil described in Eq. (5.12) with $D + D^T = 0$. Then a necessary and sufficient condition for $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ to be a storage function for $\mathfrak{B}$ is*

$$\text{rank} \begin{bmatrix} & R(\xi) & \\ -K & I & 0 \end{bmatrix} = \text{rank } R(\xi). \tag{5.19}$$

Proof (Necessity) This follows from Statements 2 and 3 of Theorem 5.13.
(Sufficiency) We assume a symmetric matrix $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ satisfies Eq. (5.19) and show that K satisfies Eq. (5.15) i.e. K induces the storage function for $\mathfrak{B}$. Using Eq. (5.19), behavior $\mathfrak{B}_{\text{Ham}}$ has trajectories that satisfy $z = Kx$. By definition of "dual states," the relation between "states" and its "dual states" is

$$\frac{d}{dt} x^T z = 2u^T y \quad \text{i.e.} \quad \frac{d}{dt} x^T K x = 2u^T y.$$

Hence K satisfies Eq. (5.15) if and only if K satisfies Eq. (5.19). This completes the proof of Corollary 5.14. $\square$

Using Corollary 5.14, we conclude that $[-K \quad I \quad 0]$ is in the row span of the polynomial matrix $R(\xi)$. The corollary guarantees that $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ serves as the unique storage function of the lossless behavior $\mathfrak{B}$. In the next section, we present an algorithm to find the unique storage function of the lossless behavior $\mathfrak{B}$ using the fact that $[-K \quad I \quad 0]$ is in the row span of $R(\xi)$.

5.5 Lossless System's Storage Function: Algorithmic Aspects

Algorithm 5.5.1 is based on extraction of static relations in first order representation of the Hamiltonian behavior $\mathfrak{B}_{\mathrm{Ham}}$ described in Sect. 5.4. The Hamiltonian pencil $R(\xi)$ is an input to the algorithm and a unique symmetric matrix K that induces storage function of the lossless behavior is the output.

Algorithm 5.5.1 Static relations extraction-based algorithm.

Input: $R(\xi) := \xi E - H \in \mathbb{R}[\xi]^{(2\mathrm{n}+\mathrm{p})\times(2\mathrm{n}+\mathrm{p})}$, a rank $2\mathrm{n}$ polynomial matrix.
Output: $K \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ with $x^T K x$ the storage function.

1: Calculate a minimal polynomial nullspace basis of $R(\xi)$.
2: *Result*: A full column rank polynomial matrix $M(\xi) \in \mathbb{R}[\xi]^{(2\mathrm{n}+\mathrm{p})\times\mathrm{p}}$.
3: Partition $M(\xi)$ as $\begin{bmatrix} M_1(\xi) \\ M_2(s) \end{bmatrix}$ where $M_1(\xi) \in \mathbb{R}[\xi]^{2\mathrm{n}\times\mathrm{p}}$.
4: Calculate a minimal polynomial nullspace basis of $M_1(\xi)^T$.
5: *Result*: A full column rank polynomial matrix $N(\xi) \in \mathbb{R}[\xi]^{2\mathrm{n}\times(2\mathrm{n}-\mathrm{p})}$.
6: Partition $N(\xi) = \begin{bmatrix} N_{11} & N_{12}(\xi) \\ N_{21} & N_{22}(\xi) \end{bmatrix}$ with $N_{11},\ N_{21} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$. (See Theorem 5.15 below)
7: The storage function $x^T K x$ induced by the symmetric matrix K is given by

$$K := -N_{11}N_{21}^{-1} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$$

Using the partition of the various matrices in the Algorithm 5.5.1, we state the following result about the unique storage function for a lossless behavior.

Theorem 5.15 *Consider $R(\xi) := \xi E - H \in \mathbb{R}[\xi]^{(2\mathrm{n}+\mathrm{p})\times(2\mathrm{n}+\mathrm{p})}$ as defined in Eq. (5.12) constructed for the lossless behavior $\mathfrak{B} \in \mathfrak{L}^{2\mathrm{p}}_{\mathrm{cont}}$. Let $M(\xi) \in \mathbb{R}[\xi]^{(2\mathrm{n}+\mathrm{p})\times\mathrm{p}}$ be any minimal polynomial nullspace basis (MPB) for $R(\xi)$. Partition $M = \begin{bmatrix} M_1(\xi) \\ M_2(\xi) \end{bmatrix}$ with $M_1 \in \mathbb{R}[\xi]^{2\mathrm{n}\times\mathrm{p}}$. Let $N(\xi)$ be any MPB for $M_1(\xi)^T$. Then, the following statements are true.*

1. *The first n (Forney invariant) minimal indices of $N(\xi)$ are 0, i.e. first n columns of $N(\xi)$ are constant vectors.*
2. *Partition N into $\begin{bmatrix} N_1 & N_2(\xi) \end{bmatrix}$ with $N_1 \in \mathbb{R}^{2\mathrm{n}\times\mathrm{n}}$ and further partition $N_1 = \begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix}$ with $N_{21} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$. Then,*
   a. *N_{21} is invertible.*
   b. *$K := -N_{11}N_{21}^{-1}$ is symmetric.*
   c. *$x^T K x$ is the unique storage function for $\mathfrak{B}$, i.e. $\frac{d}{dt}x^T K x = 2u^T y$ for all system trajectories.*

Proof **1**: Using Statement 1 of Theorem 5.13, we have $\det R(\xi) = 0$. Hence there exists a nullspace $M(\xi)$ of $R(\xi)$. Since rank $R(\xi) = 2\mathtt{n}$ where $\mathtt{n}$ is the McMillan degree of the behavior $\mathfrak{B}$ and $R(\xi) \in \mathbb{R}^{(2\mathtt{n}+\mathtt{p})\times(2\mathtt{n}+\mathtt{p})}[\xi]$, we have that the minimal polynomial basis $M(\xi) \in \mathbb{R}^{(2\mathtt{n}+\mathtt{p})\times \mathtt{p}}[\xi]$.
Using Corollary 5.14, we have $\begin{bmatrix}-K & I & 0\end{bmatrix}$ is in the row span of $R(\xi)$. Therefore,

$$\begin{bmatrix}-K & I & 0\end{bmatrix} M(\xi) = 0 \qquad \text{i.e.} \qquad \begin{bmatrix}-K & I & 0\end{bmatrix}\begin{bmatrix}M_1(\xi)\\ M_2(\xi)\end{bmatrix} = 0, \text{ where } M_1 \in \mathbb{R}[\xi]^{2\mathtt{n}\times\mathtt{p}}$$

This implies that

$$\begin{bmatrix}-K & I\end{bmatrix}\begin{bmatrix}M_1(\xi)\end{bmatrix} = 0 \qquad \text{i.e.} \qquad M_1(\xi)^T\begin{bmatrix}-K\\ I\end{bmatrix} = 0$$

The nullspace of $M_1(\xi)^T$ must have $\mathtt{n}$ constant polynomial vectors. Hence the first $\mathtt{n}$ (Forney invariant) minimal indices are 0. This proves 1 of Theorem 5.15.
2: Here we prove 2 of Theorem 5.15.
The minimal nullspace basis of $M_1(\xi)^T$ is the columns of $N(\xi) \in \mathbb{R}[\xi]^{2\mathtt{n}\times(2\mathtt{n}-\mathtt{p})}$. Partition N into $\begin{bmatrix}N_1 & N_2(\xi)\end{bmatrix}$ with $N_1 \in \mathbb{R}^{2\mathtt{n}\times\mathtt{n}}$ and further partition $N_1 = \begin{bmatrix}N_{11}\\ N_{21}\end{bmatrix}$ with $N_{21} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Further

$$\text{span}\begin{bmatrix}N_{11}\\ N_{21}\end{bmatrix} = \text{span}\begin{bmatrix}-K\\ I\end{bmatrix}.$$

This proves that N_{21} is invertible and therefore $K = -N_{11}N_{21}^{-1}$. The entire proof is based on Theorem 5.13 and Corollary 5.14, hence the symmetric matrix K found by Algorithm 5.5.1 induces storage function of the lossless behavior $\mathfrak{B}$ i.e. $\frac{d}{dt}x^T K x = 2u^T y$ for all system trajectories. Hence 2 of Theorem 5.15 follows. This completes the proof of Theorem 5.15. □

Algorithm 5.5.1 is based on computation of nullspace basis of polynomial matrices. Computation of nullspace basis of a polynomial matrix can be done by block Toeplitz matrix algorithm: more details can be found in [9, 20].

5.6 Examples

In this section, we consider two examples: one in which we have strict dissipativity and another in which we have losslessness. We use Algorithm 5.5.1 for calculating K for the lossless case.

Example 5.16 In this example, we illustrate the conditions in Theorem 5.11. Consider a strictly dissipative behavior $\mathfrak{B}$ with transfer function $G(s) = \frac{s+2}{s+1}$. A minimal i/s/o representation of the system is $\dot{x} = -x + u$ and $y = x + u$. The Hamiltonian pencil for the behavior $\mathfrak{B}$ as obtained from Eq. (5.12) is

$$R(\xi) = \begin{bmatrix} \xi + 1 & 0 & -1 \\ 0 & \xi - 1 & -1 \\ -1 & 1 & -2 \end{bmatrix}$$

Hence $\det R(\xi) = 4 - 2\xi^2 \neq 0$, $\deg \det R(\xi) = 2$ and $R(\xi) \in \mathbb{R}^{3\times 3}[\xi]$ i.e. Hamiltonian system is autonomous. The roots of $\det R(\xi) = \{-\sqrt{2}, \sqrt{2}\}$. Following Definition 5.7, two Lambda sets can be formed $\Lambda_1 = \{-\sqrt{2}\}$ and $\Lambda_2 = \{\sqrt{2}\}$. For Λ_1, the storage function $K_{\Lambda_1} = 0.171$. Notice that

$$\operatorname{rank} \begin{bmatrix} -\sqrt{2}+1 & 0 & -1 \\ 0 & -\sqrt{2}-1 & -1 \\ -1 & 1 & -2 \end{bmatrix} = 2 \text{ and } \operatorname{rank} \begin{bmatrix} -\sqrt{2}+1 & 0 & -1 \\ 0 & -\sqrt{2}-1 & -1 \\ -1 & 1 & -2 \\ -0.171 & 1 & 0 \end{bmatrix} = 2.$$

It can be verified that the storage function for Lambda set Λ_2 is $K_{\Lambda_2} = 5.828$ and it also satisfies the conditions in Theorem 5.11. Consider any other arbitrary value of K which is not a solution to the ARE corresponding to the behavior $\mathfrak{B}$. Say $K = 1$ then

$$\operatorname{rank} \begin{bmatrix} -\sqrt{2}+1 & 0 & -1 \\ 0 & -\sqrt{2}-1 & -1 \\ -1 & 1 & -2 \\ -1 & 1 & 0 \end{bmatrix} = 3.$$

Hence for any other arbitrary value of K, $\operatorname{rank} \begin{bmatrix} & R(\xi) & \\ -K & I & 0 \end{bmatrix} \neq \operatorname{rank} R(\xi)$.

Next we consider transfer function of a *lossless* behavior $\mathfrak{B}$ that brings out the use of Theorem 5.13. In order to calculate the storage function K we use Algorithm 5.5.1.

Example 5.17 Consider a lossless behavior $\mathfrak{B}$ with transfer function $G(s) = \frac{s}{s^2+1}$. A minimal i/s/o representation of the behavior is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{and} \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x + 0\,u$$

The Hamiltonian pencil for the behavior $\mathfrak{B}$ as obtained from Eq. (5.12) is

$$R(\xi) = \begin{bmatrix} \xi & -1 & 0 & 0 & 0 \\ 1 & \xi & 0 & 0 & -1 \\ 0 & 0 & \xi & -1 & 0 \\ 0 & 0 & 1 & \xi & -1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \qquad \text{and one can check that } \det R(\xi) = 0.$$

Thus the behavior $\mathfrak{B}_{\text{Ham}}$ is not autonomous. We next calculate the storage function using Algorithm 5.5.1.

1. A minimal polynomial nullspace basis (MPB) of $R(\xi)$ is $M(\xi) = \begin{bmatrix} 1 \\ \xi \\ 1 \\ \xi \\ 1+\xi^2 \end{bmatrix}$.
2. Partitioning $M(\xi)$ by Step 3 of Algorithm 5.5.1, we have: $M_1(\xi) = \begin{bmatrix} 1 \\ \xi \\ 1 \\ \xi \end{bmatrix}$.
3. MPB of $M_1(\xi)^T$ is $N(\xi) = \begin{bmatrix} -4 & -\sqrt{2} & -3\xi \\ \sqrt{2} & -4 & 3 \\ 4 & \sqrt{2} & -3\xi \\ -\sqrt{2} & 4 & 3 \end{bmatrix}$.
4. Using Step 6 of the same algorithm, we partition $N(\xi)$. Hence $N_{11} = \begin{bmatrix} -4 & -\sqrt{2} \\ \sqrt{2} & -4 \end{bmatrix}$ and $N_{21} = \begin{bmatrix} 4 & \sqrt{2} \\ -\sqrt{2} & 4 \end{bmatrix}$.
5. Therefore, the matrix $K = -N_{11}N_{21}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ induces the storage function of the lossless behavior $\mathfrak{B}$.

It can be verified that rank $\begin{bmatrix} & R(\xi) & \\ -K & I & 0 \end{bmatrix} = \text{rank } R(\xi) = 4$.
With any arbitrary $K = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (say), we will have rank $\begin{bmatrix} & R(\xi) & \\ -K & I & 0 \end{bmatrix} = 5$. Hence for arbitrary K, rank $\begin{bmatrix} & R(\xi) & \\ -K & I & 0 \end{bmatrix} \neq \text{rank } R(\xi)$.

5.7 Concluding Remarks

This chapter dealt with the formulation of new properties of the ARE solution for the case when the equation exists: namely, when regularity conditions on the feedthrough term are satisfied. These results were extended to the case when the ARE does not exist: for example, the lossless case. For this case, the "ARE" solution is the storage function, which is unique for the lossless case. We formulated an algorithm that computes this storage function. The algorithm was developed exploiting the fact that the states in the Hamiltonian system (corresponding to a conservative behavior) are not trim. Static relations of the form $z = Kx$ helped to extract this nontrimness and hence led to a storage function $x^T Kx$ to the original system.

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