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Algorithmic Issues in the Synthesis of Dissipative Systems

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ABSTRACT

In this paper we discuss algorithmic issues that arise in the problem of synthesis of dissipative systems. We deal with linear differential systems that can be controlled only through a restricted set of variables called the control variables. The main feature of this paper is that we assume the system dynamics to be specified in the most general form: a latent variable representation. Starting from such a representation, we provide concrete algorithms that finally fetch a controller to implement the desired behavior. Many other peripheral algorithmic issues that crop up are also studied.

Keywords: Behaviors, dissipativity, strict dissipativity, quadratic differential forms, algorithms, storage function, linear matrix inequalities.

1. INTRODUCTION AND NOTATION

The synthesis of dissipative system behaviors wedged in between two given behaviors and satisfying certain maximality requirements has been studied in [1, 2]. In these references, necessary and sufficient conditions were given for the existence of such behaviors. In this paper we deal with algorithms for the verification of these existence conditions. Further, for the case that these conditions are satisfied, we describe constructive algorithms to compute such controlled behaviors. We also consider issues concerning the synthesis of *strictly* dissipative behaviors and the related algorithms.

The paper is structured as follows. In the remainder of this section we introduce the necessary notation. In Section 2 we deal with linear differential systems, and review the problem of synthesis of dissipative systems. We discuss only the basic definitions there in order to be able to formulate the synthesis problem that has been treated in

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full detail in [1]. After a brief motivation for this problem we move over to a review of quadratic differential forms in Section 3. Section 4 contains additional material necessary for stating the main propositions from [1] and [3] on the dissipative synthesis problem and the strictly dissipative synthesis problem. These propositions follow in Section 5. In that section we also give a step-by-step procedure both for the verification of the existence conditions, and for the construction of a required controlled behavior (in case the conditions are satisfied). Some of these steps require auxiliary algorithms themselves and these related algorithms have been collected in Section 6. Section 7 deals with algorithms related to the concept of orthogonality. Storage functions play a central role in the theory of dissipative systems and their computation can be cast into solving a linear matrix inequality, and into a spectral factorization problem. This has been studied in Sections 8 and 9, respectively.

The notation we use is standard. We use $\mathbb{R}$ to denote the field of real numbers and $\mathbb{C}$ to denote the complex plane. $\mathbb{R}^{\mathrm{n}}$ and $\mathbb{R}^{\mathrm{n}_1 \times \mathrm{n}_2}$ are the obvious extensions to vectors and matrices respectively. When specification of the row dimension is unnecessary, or if the context clarifies it, we use $\mathbb{R}^{\bullet \times \mathrm{n}_2}$. We typically use the superscript $^{\mathrm{w}}$ (for example, $\mathbb{R}^{\mathrm{w}}$) when a generic element w has w components. $\mathbb{R}[\xi]$ denotes the ring of polynomials in the indeterminate ξ with coefficients in $\mathbb{R}$, while $\mathbb{R}[\zeta, \eta]$ is the corresponding ring in two (commutative) indeterminates. We use $\mathbb{R}^{\mathrm{w} \times \mathrm{w}}[\xi]$ and $\mathbb{R}^{\mathrm{w} \times \mathrm{w}}[\zeta, \eta]$ to denote the sets of matrices with entries from the above rings. The space of infinitely often differentiable functions with domain $\mathbb{R}$ and co-domain $\mathbb{R}^{\mathrm{n}}$ is denoted by $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathrm{n}})$, and its subspace of compactly supported elements by $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathrm{n}})$. When the domain and co-domain are clear from the context we shall use $\mathfrak{D}$ instead of $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathrm{n}})$. We use $\mathrm{rowdim}(M)$ to indicate the row dimension of a matrix M and just $\dim(M)$ if M is a vector or a square matrix. We frequently need to stack matrices with the same column dimension into a column. When we do this within text, for improved readability we use the operator 'col', i.e. $\mathrm{col}(M_1, M_2) := [M_1^T \; M_2^T]^T$.

2. BEHAVIORS

The *behavior* $\mathfrak{B}$ of a linear differential system is a subspace $\mathfrak{B} \subseteq \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathrm{w}})$ such that, for some polynomial matrix $R \in \mathbb{R}^{\mathrm{g} \times \mathrm{w}}[\xi]$, we have $\mathfrak{B} = \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})^{\mathrm{w}}) \middle| R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0 \right\}$. If this holds, then R is said to induce a *kernel representation* of $\mathfrak{B}$.

We use $\mathfrak{L}^{\mathrm{w}}$ to denote the set of such behaviors. We shall often restrict ourselves to a subset of $\mathfrak{L}^{\mathrm{w}}$, namely the controllable behaviors. Roughly speaking, a controllable behavior is a behavior in which for any two of its elements there exists a third element which coincides with the first one on the past and the second one on the future (for details, see [4]). $\mathfrak{L}^{\mathrm{w}}_{\mathrm{cont}}$ denotes this subset of controllable behaviors. Given a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$, if $w \in \mathfrak{B}$ then it is possible to choose some of the components of w to be any function in $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})$. The maximal number of such components (called free components) is called the *input cardinality* of $\mathfrak{B}$, and is denoted by $\mathtt{m}(\mathfrak{B})$. Given a nonsingular matrix $\Sigma \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ such that $\Sigma = \Sigma^T$, we denote its signature by $\mathrm{sign}(\Sigma) = (\sigma_+(\Sigma), \sigma_-(\Sigma))$, where, $\sigma_+(\Sigma)$ and $\sigma_-(\Sigma)$ are the number of positive and negative eigenvalues of Σ, respectively. Σ defines a quadratic form $w^T\Sigma w$ on $\mathbb{R}^{\mathtt{w}}$. We call a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ *dissipative* with respect to the quadratic form $w^T\Sigma w$ (or, Σ-dissipative) if $\int_{\mathbb{R}} w^T\Sigma w dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. Half-line dissipativity on $\mathbb{R}_-$ plays an important role in stability issues: $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is called Σ-dissipative *on* $\mathbb{R}_-$ if $\int_{-\infty}^{0} w^T\Sigma w dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. Strict dissipativity is defined as follows. $\mathfrak{B}$ is *strictly* Σ-dissipative if there exists an $\epsilon > 0$ such that $\int_{\mathbb{R}} w^T\Sigma w dt \geq \epsilon \int_{\mathbb{R}} w^T w dt$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. Strict dissipativity on $\mathbb{R}_-$ is defined analogously. Additional material on dissipativity with respect to general quadratic differential forms follows in the next section, Section 3.

The main purpose of this paper is to describe algorithms concerning the following two problems. Let $\Sigma = \Sigma^T \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ be nonsingular and let $\mathcal{N}, \mathcal{P} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ be such that $\mathcal{N} \subseteq \mathcal{P}$. The first problem whose algorithmic issues we want to study is the problem of *dissipative system synthesis*: find $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ such that:

- $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$,
- $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$,
- $\mathtt{m}(\mathcal{K}) = \sigma_+(\Sigma)$.

The second problem for which we want to develop algorithms is the problem of *strictly dissipative system synthesis*: find such a $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ that is *strictly* Σ-dissipative on $\mathbb{R}_-$, instead of just Σ-dissipative on $\mathbb{R}_-$.

We call a $\mathcal{K}$ having these desired properties a controlled behavior. $\mathcal{P}$ is called the plant behavior and is specified by a plant to be controlled. We wish to restrict this plant behavior to a desired sub-behavior $\mathcal{K}$ by means of control. Imposing $\mathcal{N} \subseteq \mathcal{K}$ is equivalent to ensuring that $\mathcal{K}$ is *implementable* through a set of variables called the *control variables*, which are different from the w-variables that we are actually interested in controlling. The controlled behavior is a result of interconnection of the plant with the controller as depicted in Figure 1. This has been studied in [1] and we will return to this issue later in this section.

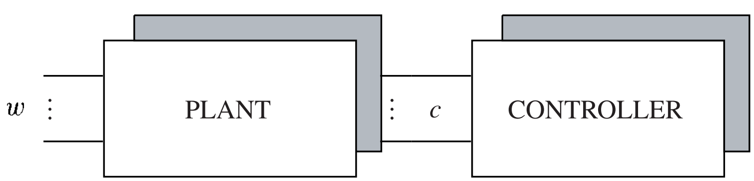

Fig. 1. Plant and controller interconnection.

$\mathcal{K}$ being Σ-dissipative is the basic control design specification, and depending on the particular choice of Σ, this has important consequences. For example, if $w = (d,f)$ and if Σ is chosen as $\Sigma = \begin{bmatrix} I_{\mathtt{d}} & 0 \\ 0 & -I_{\mathtt{f}} \end{bmatrix}$ then we exactly obtain the familiar $\mathcal{H}_\infty$-disturbance attenuation design problem. Half-line dissipativity takes care of the required stability of f when the exogenous disturbances d are equal to zero. This has been explained in detail in [2]. It has also been shown there that a system being passive is equivalent to the behavior being dissipative with respect to Σ of the form $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. Hence designing a controller that renders a system passive is equivalent to the above synthesis problem with this particular Σ. The input-cardinality condition on $\mathcal{K}$ is a *liveness* requirement. In the $\mathcal{H}_\infty$ problem it assures that the exogenous disturbances remain free. Also in the synthesis of passive systems it has an important interpretation, see [2].

We now introduce some additional notions that are needed to state the main propositions from [1] and [3] on the existence of $\mathcal{K}$ satisfying the conditions of the dissipative system synthesis problem, and the strictly dissipative system synthesis problem formulated above. After stating these propositions, we will deal with algorithmic issues to compute a required $\mathcal{K}$.

We have defined a behavior as the kernel of a polynomial differential operator. Often, we encounter behaviors that are not represented as a kernel. Let $R, M \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, and

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \,\middle|\, \exists\, \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{1}) \text{ such that } R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell \right\}.$$

By the *elimination theorem* (see [4], Chapter 6) the set defined above is indeed a behavior in the sense we have defined. A representation of $\mathfrak{B}$ like the one above is called a *latent variable representation* (with ℓ as the latent variable). The *full behavior* $\mathfrak{B}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+1}$ is the set of all (w, ℓ) that satisfy the equation $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$. Further, controllable behaviors admit latent variable representations of a special kind: namely latent variable representations with $R(\xi) = I$, i.e. $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) | \exists \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{1})$ such that $w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell\}$. Such representations are called *image representations*. In a latent variable representation the latent variable ℓ is said to be *observable* from the manifest variable w if $(w, \ell_1), (w, \ell_2) \in \mathfrak{B}_{\text{full}}$ implies $\ell_1 = \ell_2$. In this case we refer to the representation as an observable latent variable representation of $\mathfrak{B}$. When a behavior $\mathfrak{B}$ is not controllable, we often deal with its *controllable part*: $\mathfrak{B}_{\text{cont}}$. $\mathfrak{B}_{\text{cont}}$ is the largest controllable behavior contained in $\mathfrak{B}$.

A latent variable representation of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called a state representation if the latent variable (denoted here by x) has the *property of state*, that is, if (w_1, x_1), $(w_2, x_2) \in \mathfrak{B}_{\text{full}}$ are such that $x_1(0) = x_2(0)$ then $(w_1, x_1) \wedge (w_2, x_2)$, their concatenation at $t = 0$, satisfies the equations of $\mathfrak{B}_{\text{full}}$ in a *weak* sense (i.e., in a distributional sense). We call such an x a state for $\mathfrak{B}$. A state map for $\mathfrak{B}$ is a differential operator $X\left(\frac{d}{dt}\right)$ (induced by $X \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$) such that $X\left(\frac{d}{dt}\right)w$ is a state for $\mathfrak{B}$. A state map $X \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ is minimal if every other state map has at least as many rows as those of X, and this minimal number of rows is called the *McMillan degree* of $\mathfrak{B}$, denoted by $\mathtt{n}(\mathfrak{B})$. A minimal state map $X\left(\frac{d}{dt}\right)$ is also *trim*, i.e. for all $a \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})}$, there exists a $w \in \mathfrak{B}$ such that $\left(X\left(\frac{d}{dt}\right)w\right)(0) = a$. This is equivalent to the behavior $X\left(\frac{d}{dt}\right)\mathfrak{B}$ being trim: a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called *trim* if for all $a \in \mathbb{R}^{\mathtt{w}}$ there exists a $w \in \mathfrak{B}$ such that $w(0) = a$.

We now return to the problem we are studying. The plant in Figure 1 consists of variables w (the to-be-controlled variables) and c (the control variables). Usually, we have *the full plant behavior* $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ given by a latent variable representation:

$$R_w\left(\frac{d}{dt}\right)w + R_c\left(\frac{d}{dt}\right)c + R_\ell\left(\frac{d}{dt}\right)\ell = 0. \tag{1}$$

Here, ℓ is a latent variable that comes in as a result of the modeling process of the plant. $\mathcal{P}$ is the manifest plant behavior and is the set of trajectories the w variable can assume. $\mathcal{P}$ is obtained from $\mathcal{P}_{\text{full}}$ by eliminating c. For the part concerning algorithms, we will try as much as possible to assume that $\mathcal{P}_{\text{full}}$ is represented in this most general form [Equation (1)], and to describe algorithms starting from this representation.

A controller brings about a restriction in the plant behavior by introducing additional laws. This restriction is brought about through *only* the control variables c, that is, we are trying to shape the w trajectories through the c variables. A given behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is called *implementable* if it can be obtained from $\mathcal{P}_{\text{full}}$ by putting restrictions on c, i.e. if there exists $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that:

$$\mathcal{K} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) | \exists\ c \text{ such that } (w, c) \in \mathcal{P}_{\text{full}} \text{ and } c \in \mathcal{C}\}. \tag{2}$$

To what extent this is possible is expressed by the condition $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$. Here $\mathcal{N}$ is called the hidden behavior and is defined as:

$$\mathcal{N} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) | \exists\ \ell \text{ such that } (w, 0, \ell) \text{ satisfies Equation (1)}\}.$$

In other words, $w \in \mathcal{N} \Leftrightarrow (w, 0) \in \mathcal{P}_{\text{full}}$. It has been proven in [1] that $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is implementable if and only if $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$. This is the reason why we have the condition $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$ in the problem formulation.

3. QUADRATIC DIFFERENTIAL FORMS

This section contains a brief review of bilinear differential forms and quadratic differential forms. Let $\Phi \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$. Such a polynomial matrix can be expressed as a finite sum $\Phi(\zeta, \eta) = \Sigma_{k,\ell \geq 0} \Phi_{k\ell} \zeta^k \eta^\ell$ with $\Phi_{k\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$ its coefficient matrices.

Let $\mathfrak{B}_1 \in \mathfrak{L}^{\mathtt{w}_1}$ and $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}_2}$. Then, Φ induces the map $L_\Phi : \mathfrak{B}_1 \times \mathfrak{B}_2 \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by $L_\Phi(w_1, w_2) := \sum_{k,\ell \geq 0} \left(\frac{\mathrm{d}^k}{\mathrm{d}t^k} w_1\right)^T \Phi_{k\ell}\left(\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} w_2\right)$. We call this the *bilinear differential form (BDF)* on $\mathfrak{B}_1 \times \mathfrak{B}_2$ induced by Φ and denote it by $L_\Phi|_{\mathfrak{B}_1 \times \mathfrak{B}_2}$. When $\mathtt{w}_1 = \mathtt{w}_2 = \mathtt{w}$ and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, Φ also induces the map $Q_\Phi : \mathfrak{B} \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by $Q_\Phi(w) := L_\Phi(w, w)$. We call this map the *quadratic differential form (QDF)* on $\mathfrak{B}$ induced by Φ and denote it by $Q_\Phi|_{\mathfrak{B}}$. Define the * operator by $\Phi^*(\zeta, \eta) := \Phi(\eta, \zeta)^T$. When considering QDF's it is sufficient to consider Φ's that are *symmetric*, that is, those that satisfy $\Phi = \Phi^*$. For a symmetric Φ, we also speak of the degree of Φ, which is the highest power of ζ (and η) that appears in Φ with a nonzero coefficient. Henceforth, unless otherwise specified, the two variable polynomial matrices that will appear in this paper shall be assumed to be symmetric. For QDF's we have the important notion of non-negativity. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$. We call the QDF Q_Φ *non-negative on* $\mathfrak{B}$ (and denote it by $Q_\Phi|_{\mathfrak{B}} \geq 0$) if $Q_\Phi(w)(t) \geq 0$ for all $t \in \mathbb{R}$ and $w \in \mathfrak{B}$. The quadratic form $\mathbb{R}^{\mathtt{w}}$ induced by the matrix $S = S^T \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ is a special case of a QDF. We shall also use $|w|_S^2$ to denote $w^T S w$, and when $S = I$ the subscript is often dropped.

If $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ then Φ (assumed symmetric) can be expressed as $\Phi(\zeta, \eta) = F_+^T(\zeta) F_+(\eta) - F_-^T(\zeta) F_-(\eta)$, with $F = \mathrm{col}(F_+, F_-) \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$, such that $F\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathfrak{B}$ is trim. Such a factorization of Φ is called a *canonical factorization on* $\mathfrak{B}$. It yields the signature and the rank of $Q_\Phi|_{\mathfrak{B}}$ by defining $\mathrm{sign}(Q_\Phi|_{\mathfrak{B}}) := (\mathrm{rowdim}(F_-), \mathrm{rowdim}(F_+))$ and $\mathrm{rank}(Q_\Phi|_{\mathfrak{B}}) := \mathrm{rowdim}(F)$. $Q_\Phi|_{\mathfrak{B}}$ can then be expressed as $Q_\Phi(w) = \left|F_+\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w\right|^2 - \left|F_-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w\right|^2$, for $w \in \mathfrak{B}$.

4. DISSIPATIVITY

In Section 2 we have reviewed dissipativity with respect to QDF's of the form $|w|_\Sigma^2$, with $\Sigma = \Sigma^T$ constant. Dissipativity with respect to arbitrary QDF's is defined as follows. Let $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. $\mathfrak{B}$ is said to be *dissipative* with respect to Q_Φ (or briefly, Φ-dissipative) if $\int_{\mathbb{R}} Q_\Phi(w)\, dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. $\mathfrak{B}$ is said to be Φ-dissipative on $\mathbb{R}_-$ if $\int_{-\infty}^0 Q_\Phi(w)\, dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. Dissipativity on $\mathbb{R}_+$ is defined analogously.

For a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ and $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$, we say that $\Psi = \Psi^* \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ induces a *storage function* Q_Ψ for $\mathfrak{B}$ with respect to Q_Φ if the *dissipation inequality* $\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) \leq Q_\Phi(w)$ is satisfied for all $w \in \mathfrak{B}$. It has been shown in [5] that existence of a storage function is equivalent to Φ-dissipativeness of $\mathfrak{B}$. Moreover, $\mathfrak{B}$ is Φ-dissipative on $\mathbb{R}_-$ if and only if there exists a storage function Q_Ψ that, in addition, satisfies $Q_\Psi|_{\mathfrak{B}} \geq 0$. Analogously, $\mathfrak{B}$ is Φ-dissipative on $\mathbb{R}_+$ if and only if there exists a nonpositive storage function Q_Ψ. In [6] it was established that every storage function is a state function, i.e. if $X \in \mathbb{R}^{\mathtt{n} \times \mathtt{w}}[\xi]$ induces a state map for $\mathfrak{B}$, then associated with any storage function Q_Ψ there exists a $K \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$ such that $Q_\Psi(w) = \left|X\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w\right|_K^2$ for all $w \in \mathfrak{B}$. Storage functions are not unique. However, there exist a maximal one Q_{Ψ^+} and a minimal one Q_{Ψ^-} between which every storage function Q_Ψ lies, that is, for all $w \in \mathfrak{B}$: $Q_{\Psi^-}(w) \leq Q_\Psi(w) \leq Q_{\Psi^+}(w)$.

Finally, we review the notion of Σ-orthogonality of two behaviors. $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ are called Σ-*orthogonal* if $\int_{\mathbb{R}} w_1^T \Sigma w_2 dt = 0$ for all $(w_1, w_2) \in (\mathfrak{B}_1 \times \mathfrak{B}_2) \cap \mathfrak{D}$. For such $\mathfrak{B}_1$ and $\mathfrak{B}_2$ there exists a $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ such that $\frac{\mathrm{d}}{\mathrm{d}t} L_\Psi(w_1, w_2) = w_1^T \Sigma w_2$ for all $(w_1, w_2) \in \mathfrak{B}_1 \times \mathfrak{B}_2$. We call this BDF L_Ψ the $[(\mathfrak{B}_1, \mathfrak{B}_2); \Sigma]$ *adapted bilinear differential form*. The Σ-*orthogonal complement* $\mathfrak{B}^{\perp_\Sigma}$ of a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is defined as follows:

$$\mathfrak{B}^{\perp_\Sigma} := \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \,\middle|\, \int_{\mathbb{R}} w^T \Sigma v dt = 0 \text{ for all } v \in \mathfrak{B} \cap \mathfrak{D} \right\}. \tag{3}$$

For $\Sigma = I$ we obtain the ordinary orthogonal complement of $\mathfrak{B}$, and this behavior is denoted by $\mathfrak{B}^\perp$.

5. PROBLEM SOLUTION AND CONSTRUCTION OF $\mathcal{K}$

Equipped with what has been described up to now, we state the following proposition from [1] which gives necessary and sufficient conditions for the existence of a behavior $\mathcal{K}$ satisfying the properties required in the dissipative system synthesis problem stated in section 2. Given $\mathcal{N}$ and $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with $\mathcal{N} \subseteq \mathcal{P}$, let $L_{\Psi_{(\mathcal{N}, \mathcal{P}^{\perp_\Sigma})}}$ be the $[(\mathcal{N}, \mathcal{P}^{\perp_\Sigma}); \Sigma]$ adapted bilinear differential form.

Proposition 1 *A behavior $\mathcal{K} \in \mathcal{L}^{\mathtt{w}}_{\text{cont}}$ with the properties required in the dissipative system synthesis problem exists if and only if the following conditions are satisfied:*

1. *$\mathcal{N}$ is Σ-dissipative,*
2. *$\mathcal{P}^{\perp_\Sigma}$ is $(-\Sigma)$-dissipative,*
3. *the coupling QDF*

$$Q_{\text{cpl}}(w_1, w_2) := Q_{\Psi^+_{\mathcal{N}}}(w_1) - Q_{\Psi^-_{\mathcal{P}^{\perp_\Sigma}}}(w_2) + 2L_{\Psi_{(\mathcal{N}, \mathcal{P}^{\perp_\Sigma})}}(w_1, w_2) \tag{4}$$

satisfies $Q_{\text{cpl}}|_{\mathcal{N} \times \mathcal{P}^{\perp_\Sigma}} \geq 0$.

Here, $\Psi^+_{\mathcal{N}}, \Psi^-_{\mathcal{P}^{\perp_\Sigma}} \in \mathcal{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ induce the largest and the smallest storage functions for $\mathcal{N}$ and $\mathcal{P}^{\perp_\Sigma}$ as Σ-dissipative and $(-\Sigma)$-dissipative systems, respectively.

Necessary and sufficient conditions for the *strictly* dissipative system synthesis problem (stated in Section 2) are given in the following proposition from [3].

Proposition 2 *A behavior $\mathcal{K} \in \mathcal{L}^{\mathtt{w}}_{\text{cont}}$ with the properties required in the strictly dissipative system synthesis problem exists if and only if the following conditions* *are satisfied:*

1. $\mathcal{N}$ *is strictly* Σ*-dissipative,*
2. $\mathcal{P}^{\perp_\Sigma}$ *is strictly* $(-\Sigma)$*-dissipative,*
3. *the coupling QDF* Q_{cpl} *defined in equation* (4) *satisfies the following two properties:*

   (i) $Q_{\text{cpl}}|_{\mathcal{N}\times\mathcal{P}^{\perp_\Sigma}} \geq 0$ *and*
   (ii) $\text{rank}(Q_{\text{cpl}}|_{\mathcal{N}\times\mathcal{P}^{\perp_\Sigma}}) = \mathtt{n}(\mathcal{N}) + \mathtt{n}(\mathcal{P})$.

It has been explained in [1] and [3] that the conditions on the coupling QDF in the above propositions is akin to the coupling of the solutions of the algebraic Riccati equations that first appeared in [7].

We now give a step-by-step procedure to compute $\mathcal{K}$. We split the algorithm into two parts. First we deal with verification of the conditions for the existence of $\mathcal{K}$. Next we look at computation of a suitable $\mathcal{K}$ and of a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that implements $\mathcal{K}$ with respect to $\mathcal{P}_{\text{full}}$.

5.1. Verification of the Conditions

Step 1. Given $\mathcal{N}$ and Σ, verify if $\mathcal{N}$ is (strictly) Σ-dissipative. An algorithm for this will be described in Section 8. When $\mathcal{N}$ is given in image representation $w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$, then Σ-dissipativity of $\mathcal{N}$ is equivalent to non-negativity the $M^T(-i\omega)\Sigma M(i\omega)$ for all $\omega \in \mathbb{R}$. For $\mathcal{N}$ expressed in more general representations, we refer to algorithm 8 in Section 8. If dissipativity holds, we compute the maximal storage function $Q_{\Psi^+_{\mathcal{N}}}$. Algorithms for this also will be described in Section 8. For the *strict* dissipativity synthesis problem, we use a modification of algorithm 8 (in Section 8) as explained in the remark following it, to compute the maximal ϵ_1 such that $\mathcal{N}$ is dissipative with respect to $\Sigma - \epsilon_1 I$. This ϵ_1 will be used in Step 8 of the present procedure.

Step 2. Given $\mathcal{P}$ and Σ, compute a representation of $\mathcal{P}^{\perp_\Sigma}$. If $\mathcal{P}$ is represented by an observable image representation $w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$ then $\mathcal{P}^{\perp_\Sigma}$ is given in kernel representation $M^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)\Sigma w = 0$. For more general representations of $\mathcal{P}$, we refer to the remark after lemma 6 in Section 7, specifically Equation (10), to compute a representation of $\mathcal{P}^{\perp_\Sigma}$.

Step 3. Given $\mathcal{P}^{\perp_\Sigma}$ and Σ, verify if $\mathcal{P}^{\perp_\Sigma}$ is (strictly) $(-\Sigma)$-dissipative. If this dissipativity fails, the algorithm ends, since $(-\Sigma)$-dissipativity of $\mathcal{P}^{\perp_\Sigma}$ is a necessary condition. If dissipativity holds, compute the minimal storage function $Q_{\Psi^-_{\mathcal{P}^{\perp_\Sigma}}}$. We also compute the maximal ϵ_2 such that $\mathcal{P}^{\perp_\Sigma}$ is dissipative with respect to $-\Sigma - \epsilon_2 I$. This ϵ_2 will also be used in Step 8 of the present procedure.

Step 4. Given $\Sigma, \mathcal{N}$, and $\mathcal{P}^{\perp_\Sigma}$, compute the $[(\mathcal{N}, \mathcal{P}^{\perp_\Sigma}); \Sigma]$-adapted bilinear differential form $L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}}$. If $\mathcal{N}$ and $\mathcal{P}^{\perp_\Sigma}$ are given by observable image representations

$w_1 = M_1(\frac{d}{dt})\ell_1$ and $w_2 = M_2(\frac{d}{dt})\ell_2$, respectively, then the associated two-variable polynomial matrix should be taken as $\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}(\zeta,\eta) = \frac{M_1^T(\zeta)\Sigma M_2(\eta)}{\zeta+\eta}$. Note that $\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\Sigma})}$ need not be symmetric. For algorithms about carrying out this computation we refer to Sections 6.5 and 7.

Step 5. Verify the non-negativity of the coupling QDF $Q_{\rm cpl}$ defined in Equation (4) for $w_1 \in \mathcal{N}$ and $w_2 \in \mathcal{P}^{\perp_\Sigma}$. To verify the non-negativity of this QDF we factor it canonically and then check if $\sigma_-(Q_{\rm cpl}|_{\mathcal{N}\times\mathcal{P}^{\perp_\Sigma}}) = 0$. This can be done using state maps as explained in Step 6 below. Non-negativity of the QDF is necessary for the existence of a $\mathcal{K}$.

5.2. Computation of a Controlled Behavior $\mathcal{K}$

Step 6. Compute matched pairs of minimal state maps $(X_\mathcal{N}, Z_\mathcal{N})$ and $(X_\mathcal{P}, Z_\mathcal{P})$ for $(\mathcal{N},\mathcal{N}^\perp)$ and $(\mathcal{P},\mathcal{P}^\perp)$, respectively. A definition of matched pairs and an algorithm for computing them will be given in Subsection 7.1. If $Z_\mathcal{N}(\frac{d}{dt})$ is a state map for $\mathcal{N}^\perp$ then $Z_\mathcal{N}(\frac{d}{dt})\Sigma$ is a state map for $\mathcal{N}^{\perp_\Sigma}$. Similarly, $Z_\mathcal{P}(\frac{d}{dt})\Sigma$ is a state map for $\mathcal{P}^{\perp_\Sigma}$.

Step 7. As described in Section 4 the fact that every storage function is a state function allows us to use a state map of a behavior to associate a constant matrix to a storage function of the behavior. An adapted bilinear differential form also turns out to be a bilinear function of the states of the two Σ-orthogonal behaviors (see [1], corollary 11). We use the state maps obtain in Step 6 to compute the matrices $K_\mathcal{N}^+$, $K^-_{\mathcal{P}^{\perp_\Sigma}}$ and L corresponding to $Q_{\Psi^+_\mathcal{N}}$, $Q_{\Psi^-_{\mathcal{P}^{\perp_\Sigma}}}$ and $L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp_\pm})}}$, respectively. Procedures to compute the matrices $K_\mathcal{N}^+$ and $K^-_{\mathcal{P}^{\perp_\Sigma}}$ will be discussed in Subsection 6.4 and computation of L will be discussed in Section 7. We form the matrix $K = K^T$ defined by:

$$K = \begin{bmatrix} K_\mathcal{N}^+ & L \\ L^T & -K^-_{\mathcal{P}^{\perp_\Sigma}} \end{bmatrix}.$$

Using K we compute $Q_{\rm cpl}$, which is induced by $\Psi_{\rm cpl} \in \mathbb{R}^{2\mathtt{w}\times 2\mathtt{w}}[\zeta,\eta]$ defined by:

$$\Psi_{\rm cpl}(\zeta,\eta) = \begin{bmatrix} X_\mathcal{N}(\zeta) \\ Z_\mathcal{P}(\zeta) \end{bmatrix}^T K \begin{bmatrix} X_\mathcal{N}(\eta) \\ Z_\mathcal{P}(\eta) \end{bmatrix}.$$

Non-negativity of the QDF $Q_{\rm cpl}$ on $\mathcal{N} \times \mathcal{P}^{\perp_\Sigma}$ is then equivalent to $K \geq 0$. The conditions (i) and (ii) on this QDF appearing in proposition 2 on the *strict* dissipativity synthesis problem, are equivalent to $K > 0$. We continue this algorithm only for the case $K > 0$, although for the existence of a *nonstrictly* dissipative $\mathcal{K}$, the condition $K \geq 0$ is sufficient.

Step 8. For the strict dissipativity synthesis problem we need some additional work. We use the ϵ_1 and ϵ_2 computed in Step 1 and 3, respectively. Take $0 < \epsilon < \min(\epsilon_1, \epsilon_2)$ and recompute $\Psi^+_{\mathcal{N},\epsilon}$ and $\Psi^-_{\mathcal{P}^{\perp_\Sigma},\epsilon}$ that induce storage functions for $\mathcal{N}$ and $\mathcal{P}^{\perp_\Sigma}$ as

$\Sigma - \epsilon I$ and $-\Sigma - \epsilon I$-dissipative systems, respectively. The matrix K_ϵ is recomputed this way and ϵ is chosen sufficiently small so that $K_\epsilon > 0$ too. The fact that there exists such an $\epsilon > 0$ has been studied in [3]. The precise choice of ϵ depends on the eigenvalues of K. We proceed further with this ϵ. For the nonstrict dissipativity synthesis problem one can continue the rest of this section assuming $\epsilon = 0$.

Step 9. Factor

$$|w_3|^2_{\Sigma-\epsilon I} - \frac{\mathrm{d}}{\mathrm{d}t}\left|\left[\begin{array}{c} Z_{\mathcal{N}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\Sigma w_3 \\ X_{\mathcal{P}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w_3 \end{array}\right]\right|^2_{K_\epsilon^{-1}} = \left|F^+\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w_3\right|^2 - \left|F^-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w_3\right|^2$$

for $w_3 \in \mathcal{P} \cap \mathcal{N}^{\perp_\Sigma}$, with $F\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)(\mathcal{P} \cap \mathcal{N}^{\perp_\Sigma})$ trim, where $F = \mathrm{col}(F^+, F^-)$. Trimness and canonical factorization will be discussed in Subsections 6.2 and 6.3.

Step 10. Define $\mathcal{F}'$ as the behavior $\left\{w_3 \in \mathcal{P} \cap \mathcal{N}^{\perp_\Sigma} \middle| F^-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w_3 = 0\right\}$. We are actually interested in $\mathcal{F} := \mathcal{F}'_{\mathrm{cont}}$, the controllable part of $\mathcal{F}'$. An algorithm to compute a representation for the controllable part of a behavior will be discussed in Subsection 6.1.

Step 11. The behavior $\mathcal{K}$ defined by $\mathcal{K} := \mathcal{N} + \mathcal{F}$ satisfies all the conditions in the problem formulation. The proof of this is the subject of [1] for the nonstrictly dissipative case and of [3] for the strictly dissipative case.

5.3. Computation of a Controller

Step 12. It remains to find a controller $\mathcal{C} \in \mathfrak{L}^{\mathrm{c}}$ that implements $\mathcal{K} \in \mathfrak{L}^{\mathrm{w}}$ with respect to $\mathcal{P}_{\mathrm{full}}$. Given $\mathcal{K}$ and $\mathcal{P}_{\mathrm{full}}$, we define a controller $\mathcal{C}^0 \in \mathfrak{L}^{\mathrm{c}}$ by:

$$\mathcal{C}^0 := \{c \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{c}}) | \exists\, w \text{ such that } (w,c) \in \mathcal{P}_{\mathrm{full}} \text{ and } w \in \mathcal{K}\}. \tag{5}$$

It has been proved in [8] that this $\mathcal{C}^0$ implements $\mathcal{K}$ if and only if $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$. The controller $\mathcal{C}^0$ has been called the *canonical controller.*

Step 13. Let $\mathcal{P}_{\mathrm{full}}$ be given in latent variable representation $R_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w + R_c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)c + R_\ell\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell = 0$ with latent variable ℓ. Then the hidden behavior $\mathcal{N}$ is given by the latent variable representation $R_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w + R_\ell\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell = 0$. Equation (9) in subsection 7.2 is useful to compute a latent variable representation of $\mathcal{N}''$ such that $\mathcal{N}^{\perp_\Sigma} = \mathcal{N}''_{\mathrm{cont}}$. We define $\mathcal{F}'' := \mathcal{P} \cap \ker\left(F^-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right) \cap \mathcal{N}'$, and $\mathcal{F}''$ has the following latent variable representation (latent variables (c_1, ℓ_1, ℓ_2) and manifest variable w_1):

$$\begin{bmatrix} R_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ F^-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ I \\ 0 \end{bmatrix} w_1 = \begin{bmatrix} -R_c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & -R_\ell\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Sigma^{-1}R_w^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 0 & 0 & R_\ell^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix} \begin{bmatrix} c_1 \\ \ell_1 \\ \ell_2 \end{bmatrix}. \tag{6}$$

By statement 1 of lemma 4 in Subsection 6.1 below, we infer that $\mathcal{F}''_{\mathrm{cont}} = \mathcal{F}$. We define $\mathcal{K}' := \mathcal{N} + \mathcal{F}''$ and using statement 2 of the same lemma, we obtain $\mathcal{K}'_{\mathrm{cont}} = \mathcal{K}$.

We get the following latent variable representation for $\mathcal{K}'$ with manifest variable w_2: $R_w\left(\frac{d}{dt}\right)w_2 = R_w\left(\frac{d}{dt}\right)w_1 - R_\ell\left(\frac{d}{dt}\right)\ell_3$ where $w_1 \in \mathcal{F}''$.

Step 14. We use $\mathcal{K}'$ to compute a latent variable representation for the canonical controller $\mathcal{C}' \in \mathfrak{L}^{\mathrm{c}}$ similar to Equation (5), but with $\mathcal{K}$ replaced by $\mathcal{K}'$. Note that because the equations of $\mathcal{P}_{\text{full}}$ were included in Equation (6), we also have $\mathcal{N} \subseteq \mathcal{K}' \subseteq \mathcal{P}$. Hence $\mathcal{C}'$ obtained here is a canonical controller that implements $\mathcal{K}'$. Equations (5) and (6) result in the following latent variable representation of $\mathcal{C}'$:

$$\begin{bmatrix} -R_c\left(\frac{d}{dt}\right) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} c = \begin{bmatrix} R_w\left(\frac{d}{dt}\right) & 0 & 0 & R_\ell\left(\frac{d}{dt}\right) & 0 & 0 & 0 \\ R_w\left(\frac{d}{dt}\right) & -R_w\left(\frac{d}{dt}\right) & R_\ell\left(\frac{d}{dt}\right) & 0 & 0 & 0 & 0 \\ 0 & R_w\left(\frac{d}{dt}\right) & 0 & 0 & R_c\left(\frac{d}{dt}\right) & R_\ell\left(\frac{d}{dt}\right) & 0 \\ 0 & F^-\left(\frac{d}{dt}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 & \Sigma^{-1}R_w^T\left(-\frac{d}{dt}\right) \\ 0 & 0 & 0 & 0 & 0 & 0 & R_\ell^T\left(-\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} w_2 \\ w_1 \\ \ell_3 \\ \ell_4 \\ c_1 \\ \ell_1 \\ \ell_2 \end{bmatrix}.$$

In this latent variable representation of $\mathcal{C}'$ the manifest variable is c, and the latent variables are $\ell_1, \ell_2, \ell_3, \ell_4, c_1, w_1$ and w_2. We now take $\mathcal{C} := \mathcal{C}'_{\text{cont}}$, the controllable part of $\mathcal{C}'$, and it turns out that $\mathcal{C}$ implements $\mathcal{K}$. This has been shown in the remark following lemma 5 in Subsection 6.1. Computation of the controllable part of a behavior is done using the algorithm discussed in the same subsection.

This completes the algorithm to compute a controller that renders a plant dissipative or strictly dissipative.

6. RELATED ALGORITHMS

In the previous section we have given a broad outline of the procedure to compute a controlled behavior. The necessary auxiliary algorithmic issues are discussed in the present and the following sections.

6.1. Controllable Part of a Behavior

We often encounter the situation when a behavior is not controllable but we are interested in its controllable part. For $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$, we already defined its controllable part $\mathfrak{B}_{\text{cont}}$ to be the largest controllable behavior contained in $\mathfrak{B}$. It has the same input cardinality as $\mathfrak{B}$. Let $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$ have a minimal kernel representation $R\left(\frac{d}{dt}\right)w = 0$. (A kernel representation of $\mathfrak{B}$ induced by $R \in \mathbb{R}^{\mathrm{p}\times\mathrm{w}}[\xi]$ is said to be *minimal* if every other kernel representation of $\mathfrak{B}$ has at least p rows. The representation is minimal if and only if $\operatorname{rank}(R) = \mathrm{p}$.) For such a minimal kernel representation of $\mathfrak{B}$,

$\mathtt{m}(\mathfrak{B}) = \mathtt{w} - \mathrm{rank}(\mathfrak{R}) = \mathtt{w} - \mathtt{p}$. We compute an image representation for $\mathfrak{B}_{\mathrm{cont}}$ as follows. We write R in its Smith form using unimodular matrices $U, V \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ such that $URV = [S \quad 0]$ with S nonsingular and diagonal. We define $M \in \mathbb{R}^{\mathtt{w} \times \mathtt{m}(\mathfrak{B})}[\xi]$ as the last $\mathtt{m}(\mathfrak{B})$ columns of V, that is, $M := V\begin{bmatrix} 0 \\ I \end{bmatrix}$ where the identity matrix I has dimension $\mathtt{m}(\mathfrak{B})$. This image representation is observable. More generally, let $\mathfrak{B}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{w}+1}$ and let $(\mathfrak{B}_{\mathrm{full}})_{\mathrm{cont}}$ have an image representation (with latent variable k):

$$\begin{bmatrix} w \\ \ell \end{bmatrix} = \begin{bmatrix} M_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ M_\ell\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix} k.$$

If $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is obtained from $\mathfrak{B}_{\mathrm{full}}$ by eliminating ℓ, then $\mathfrak{B}_{\mathrm{cont}}$ has an image representation $w = M_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)k$. But this representation need not be observable.

We now state some results concerning the controllable part of a behavior. These results were used in Section 5 in the computation of controllers.

Lemma 3 *Let $\mathfrak{B}^1, \mathfrak{B}^2 \in \mathfrak{L}^{\mathtt{w}}$ be such that $\mathfrak{B}^1 \subseteq \mathfrak{B}^2$ and $\mathtt{m}(\mathfrak{B}^1) = \mathtt{m}(\mathfrak{B}^2)$. Then we have $\mathfrak{B}^1_{\mathrm{cont}} = \mathfrak{B}^2_{\mathrm{cont}}$.*

The above lemma is just another way of stating the following. Let $R_1, R_2 \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}[\xi]$ and $F \in \mathbb{R}^{\mathtt{g} \times \mathtt{g}}[\xi]$. Suppose $FR_1 = R_2$ and suppose F is nonsingular. Though they need not have full rank, R_1 and R_2 have the same rank because F is nonsingular. Hence the kernels of $R_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ and $R_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ have the same input cardinality. The above lemma asserts that their controllable parts are equal. The following lemma relates to how intersection and addition of behaviors affects the controllable parts.

Lemma 4 *Let $\mathfrak{B}^1, \mathfrak{B}^2 \in \mathfrak{L}^{\mathtt{w}}$. Then*

1. $(\mathfrak{B}^1 \cap \mathfrak{B}^2)_{\mathrm{cont}} = (\mathfrak{B}^1_{\mathrm{cont}} \cap \mathfrak{B}^2_{\mathrm{cont}})_{\mathrm{cont}}$,
2. $(\mathfrak{B}^1 + \mathfrak{B}^2)_{\mathrm{cont}} = \mathfrak{B}^1_{\mathrm{cont}} + \mathfrak{B}^2_{\mathrm{cont}}$.

We use this lemma to prove the following result concerning implementability. We need the following result to justify why we could proceed with $\mathcal{K}'$ instead of $\mathcal{K}$ while constructing a controller in Subsection 5.3.

Lemma 5 *Let $\mathcal{P}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. Let $\mathcal{C}^1$, $\mathcal{C}^2 \in \mathfrak{L}^{\mathtt{c}}$ implement $\mathcal{K}^1$, $\mathcal{K}^2 \in \mathfrak{L}^{\mathtt{w}}$ respectively. Then $\mathcal{C}^1_{\mathrm{cont}} = \mathcal{C}^2_{\mathrm{cont}}$ implies that $\mathcal{K}^1_{\mathrm{cont}} = \mathcal{K}^2_{\mathrm{cont}}$. In particular, if $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ implements $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ and $\mathcal{K}$ is controllable, then $\mathcal{C}_{\mathrm{cont}}$ also implements $\mathcal{K}$.*

Using this lemma we shall explain why in Subsection 5.3 it is possible to find a canonical controller $\mathcal{C}'$ using $\mathcal{K}'$ and then to take $\mathcal{C} := \mathcal{C}'_{\mathrm{cont}}$, instead of using $\mathcal{K}$ to obtain the canonical controller $\mathcal{C}^0$. Comparing Equations (5) (in Subsection 5.3) and (2) (in Section 2), we note that $\mathcal{K}$ and $\mathcal{C}$ have switched roles. From Equation 5, we can say $\mathcal{K}$ implements its canonical controller $\mathcal{C}^0$ through $\mathcal{P}_{\mathrm{full}}$. Similarly, $\mathcal{K}'$ implements its canonical controller $\mathcal{C}'$. Using lemma 5 with the roles of $\mathcal{C}$ and $\mathcal{K}$ reversed, we obtain that $\mathcal{K}'_{\text{cont}} = \mathcal{K} \Rightarrow \mathcal{C}'_{\text{cont}} = \mathcal{C}^0_{\text{cont}}$. Further, because $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$ and $\mathcal{N} \subseteq \mathcal{K}' \subseteq \mathcal{P}$, $\mathcal{C}^0$ implements $\mathcal{K}$, and $\mathcal{C}'$ implements $\mathcal{K}'$. Hence using lemma 5 again, we infer that both $\mathcal{C}^0$ and $\mathcal{C}^0_{\text{cont}}$ implement $\mathcal{K}$. Thus it is sufficient to start with $\mathcal{K}'$, obtain $\mathcal{C}'$ and then take $\mathcal{C} := \mathcal{C}'_{\text{cont}}$ as a controller that implements $\mathcal{K}$ through $\mathcal{P}_{\text{full}}$. The important reason that this has worked is that the desired $\mathcal{K}$ is controllable.

6.2. Trimness

We discuss here some algorithmic issues related to the notion of trimness that has been defined in Section 2. It can be shown that a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is trim if and only if:

$$(\eta \in \mathbb{R}^{\mathtt{w}} \text{ and } \eta^T w = 0 \text{ for all } w \in \mathfrak{B}) \Rightarrow \eta = 0. \tag{7}$$

Let $\mathcal{N}_0$ denote the zeroth order annihilators (i.e., static relations) of $\mathfrak{B}$: $\mathcal{N}_0 := \{n_0 \in \mathbb{R}^{1\times\mathtt{w}} | n_0\mathfrak{B} = 0\}$. $\mathcal{N}_0$ can be computed as follows. Let $\mathfrak{B}$ be given by a minimal kernel representation $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0$. We assume R is row reduced. (If R is not row reduced then we premultiply R by a suitable unimodular matrix to obtain row reducedness.) We refer to [9, Section 6.3] for a definition of row reducedness. One can show that $\mathfrak{B}$ is trim if and only if there are no zeroth order rows in R. If there are any zeroth order rows then these rows generate $\mathcal{N}_0$. Equation (7) is equivalent to $\mathcal{N}_0 = 0$. If $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is not trim, there exist a matrix $S \in \mathbb{R}^{\mathtt{w}\times\mathtt{v}}$ (with $\mathtt{v} < \mathtt{w}$) and a *trim* behavior $\mathfrak{B}' \in \mathfrak{L}^{\mathtt{v}}$ such that $\mathfrak{B} = S\mathfrak{B}'$. Such an S and $\mathfrak{B}'$ can be computed as follows. Construct $N_0 \in \mathbb{R}^{\bullet\times\mathtt{w}}$ from the zeroth order rows of R. Then we take for S a matrix whose columns form a basis for the kernel of N_0. S has full column rank (say $\mathtt{v}$) and we compute a left inverse $S^\dagger$ of S and define $\mathfrak{B}' \in \mathfrak{L}^{\mathtt{v}}$ by $\mathfrak{B}' = S^\dagger\mathfrak{B}$. If $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is given by an observable image representation, then checking the trimness of $\mathfrak{B}$ can be reformulated into checking the rank of a certain constant matrix as explained below.

6.3. Canonical Factorization of QDF's

In Section 3 canonical factorization of QDF's was defined. Here we discuss how we obtain such a factorization. We first deal with one variable polynomial matrices. For an $F \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$, we define the constant matrix $\text{mat}(F)$, called the *coefficient matrix* of F, as follows. We write out the finite sum $F(\xi) = \sum_{k\geq 0} F_k\xi^k$ and define $\text{mat}(F) = [F_0\ F_1 \cdots F_k \cdots]$. For this constant matrix $\text{mat}(F)$ we have $F(\xi) = \text{mat}(F)\ \text{col}(I, I\xi, \ldots, I\xi^k, \ldots)$ where the I's are identity matrices of dimension $\mathtt{q}$. Trimness of the behavior $\mathfrak{B}$ with image representation $w = F\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$ is equivalent to linear independence over $\mathbb{R}$ of the rows of F. This is further equivalent to full row rank of $\text{mat}(F)$.

We now come to two variable polynomial matrices. Given a $\Phi \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta, \eta]$, we use a similar procedure to define its *coefficient matrix* $\text{mat}(\Phi)$. We first write out the finite sum $\Phi(\zeta,\eta) = \sum_{k,\ell\geq 0} \Phi_{k\ell}\zeta^k\eta^l$ and then form the infinite block matrix $\mathrm{mat}(\Phi)$ with $\Phi_{k\ell}$ at the $(k+1,\ell+1)^{th}$ position, that is,

$$\mathrm{mat}(\Phi) = \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \Phi_{0\ell} & \cdots \\ \Phi_{10} & \Phi_{11} & \cdots & \Phi_{1\ell} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Phi_{k0} & \Phi_{k1} & \cdots & \Phi_{k\ell} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \ddots \end{bmatrix}. \tag{8}$$

Note that only a finite number of entries of $\mathrm{mat}(\Phi)$ are nonzero. $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is symmetric if and only if $\mathrm{mat}(\Phi)$ is. In this case we factor $\mathrm{mat}(\Phi)$ into $\Gamma_+^T\Gamma_+ - \Gamma_-^T\Gamma_-$ with $\mathrm{col}(\Gamma_+,\Gamma_-)$ surjective. We have $\mathrm{sign}(\mathrm{mat}(\Phi)) = (\mathrm{rowdim}(\Gamma_-), \mathrm{rowdim}(\Gamma_+)) = \mathrm{sign}(Q_\Phi)$. More generally, when we have a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ with an observable image representation $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$, we define $\Phi' \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\zeta,\eta]$ by $\Phi'(\zeta,\eta) = M^T(\zeta)\Phi(\zeta,\eta)M(\eta)$. Then $Q_\Phi(w) = Q_{\Phi'}(\ell)$ whenever $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$. We can factorize Φ' canonically on $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^l)$ as, say, $Q_{\Phi'}(\ell) = \left|F'_+(\frac{\mathrm{d}}{\mathrm{d}t})\ell\right|^2 - \left|F'_-(\frac{\mathrm{d}}{\mathrm{d}t})\ell\right|^2$. By defining F_+ and F_- by $F_+ = F'_+M^\dagger$ and $F_- = F'_-M^\dagger$ where $M^\dagger$ is a polynomial left inverse of M, $Q_\Phi(w) = \left|F_+(\frac{\mathrm{d}}{\mathrm{d}t})w\right|^2 - \left|F_-(\frac{\mathrm{d}}{\mathrm{d}t})w\right|^2$ is then a canonical factorization of Q_Φ on $\mathfrak{B}$. This has been discussed in [5].

6.4. Storage and State

Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ be Φ-dissipative. The existence of a storage function Q_Ψ has been discussed in Section 4. We now relate Q_Ψ to the state of the behavior. It has been established in [6] that every storage function Q_Ψ is a function of the state of $\mathfrak{B}$. In particular, if $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{w}}[\xi]$ induces a minimal state map for $\mathfrak{B}$, Ψ can be expressed as $\Psi(\zeta,\eta) = X^T(\zeta)KX(\eta)$ for a suitable symmetric matrix $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$. Such a K can be computed as follows. We first obtain a factorization of Ψ, such that it is canonical on $\mathfrak{B}$, into $\Psi(\zeta,\eta) = F^T(\zeta)K'F(\eta)$ with $\dim(K') = \mathrm{rank}(\Psi|_{\mathfrak{B}}) = \mathtt{r}$ (say). Let $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ be an observable image representation of $\mathfrak{B}$. We compute $\mathrm{mat}(FM)$ and $\mathrm{mat}(XM)$, the coefficient matrices associated with FM and XM. Since $X(\frac{\mathrm{d}}{\mathrm{d}t})\mathfrak{B}$ is trim, $\mathrm{mat}(XM)$ has full row rank. Let $S \in \mathbb{R}^{\bullet\times\mathtt{n}}$ be a right inverse of $\mathrm{mat}(XM)$. The K we are looking for can be defined as $K := S^T\mathrm{mat}(FM)^TK'\,\mathrm{mat}(FM)S$.

6.5. Storage Function for Lossless Behaviors

Losslessness is a special case of dissipativity. Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ is called *Φ-lossless* if $\int_{\mathbb{R}} Q_\Phi(w) = 0$ for all $w \in \mathfrak{B}\cap\mathfrak{D}$. We explore the construction of the storage function when the behavior is lossless. The storage function in this case is essentially unique, i.e. any two storage functions coincide on the given behavior. The algorithm to compute the storage function involves straightforward manipulations of one variable polynomial matrices.

Assume initially $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Given $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, we define $\partial\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ by $\partial\Phi(\xi) := \Phi(-\xi, \xi)$. It has been shown in [5] that Φ-losslessness of $\mathfrak{B}$ is equivalent to $\partial\Phi = 0$, and to the existence of a $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that $\frac{d}{dt}Q_\Psi(w) = Q_\Phi(w)$ for all $w \in \mathfrak{B}$. We now discuss the computation of such a Ψ. Let the operator $\bullet : \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta] \to \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta]$ be defined as $\overset{\bullet}{P}(\zeta,\eta) := (\zeta+\eta)P(\zeta,\eta)$. It is easily seen that $\frac{d}{dt}Q_\Psi(w) = Q_{\overset{\bullet}{\Psi}}(w)$. Also, it is clear that for a $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, there exists a $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that $\overset{\bullet}{\Psi} = \Phi$ if and only if $\partial\Phi = 0$. The following algorithm gives such a Ψ. Write $\Phi(\zeta,\eta) = \phi_0(\eta) + \zeta\phi_1(\eta) + \cdots + \zeta^n\phi_n(\eta)$. Then $\Psi(\zeta,\eta) = \psi_0(\eta) + \zeta\psi_1(\eta) + \cdots + \zeta^{n-1}\psi_{n-1}(\eta)$ is computed by the following recursion:

$$\psi_0(\xi) = \frac{\phi_0(\xi)}{\xi}; \qquad \psi_k(\xi) = \frac{\phi_k(\xi) - \psi_{k-1}(\xi)}{\xi} \quad \text{for } k = 1, \ldots, n-1.$$

Another method to compute Ψ using the associated coefficient matrix $\mathrm{mat}(\Phi)$ of Φ and solving a linear matrix equation will be described in Section 8.

We now consider the case of Σ-losslessness of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$, with $\mathfrak{B}$ represented by an observable image representation $w = M\left(\frac{d}{dt}\right)\ell$, and $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. We define $\Phi' \in \mathbb{R}^{1\times 1}[\zeta,\eta]$ by $\Phi'(\zeta,\eta) := M^T(\zeta)\Sigma M(\eta)$ and find a $\Psi' \in \mathbb{R}^{1\times 1}[\zeta,\eta]$ such that $\overset{\bullet}{\Psi'} = \Phi'$. Next we obtain Ψ from Ψ' by using a polynomial left inverse $M^\dagger \in \mathbb{R}^{1\times\mathtt{w}}[\xi]$ of M as follows: $\Psi(\zeta,\eta) := M^\dagger(\zeta)^T\Psi'(\zeta,\eta)M^\dagger(\eta)$.

7. ORTHOGONALITY

In this section we discuss computational issues related to orthogonality of two behaviors. We call $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ *orthogonal* (and denote it by $\mathfrak{B}_1 \perp \mathfrak{B}_2$) if $\int_{\mathbb{R}} w_1^T w_2 dt = 0$ for all $w_1 \in \mathfrak{B}_1 \cap \mathfrak{D}$ and $w_2 \in \mathfrak{B}_2 \cap \mathfrak{D}$. If $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ are given by observable image representations $w_1 = M_1\left(\frac{d}{dt}\right)\ell_1$ and $w_2 = M_2\left(\frac{d}{dt}\right)\ell_2$ respectively, then $\mathfrak{B}_1 \perp \mathfrak{B}_2$ if and only if $M_1^T(-\xi)M_2(\xi) = 0$. For such $\mathfrak{B}_1$ and $\mathfrak{B}_2$ there exists an *adapted bilinear form* L_Ψ, that is, there exists $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that $\frac{d}{dt}L_\Psi(w_1, w_2) = w_1^T w_2$ for $(w_1, w_2) \in \mathfrak{B}_1 \times \mathfrak{B}_2$. L_Ψ is again a function of the states of $\mathfrak{B}_1$ and $\mathfrak{B}_2$. If $X_1 \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{w}}[\xi]$ and $X_2 \in \mathbb{R}^{\mathtt{n}_2\times\mathtt{w}}[\xi]$ induce minimal state maps for $\mathfrak{B}_1$ and $\mathfrak{B}_2$ with McMillan degrees $\mathtt{n}_1$ and $\mathtt{n}_2$, respectively, then there exists $L \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_2}$ such that $\Psi(\zeta,\eta) = X_1^T(\zeta)LX_2(\eta)$. Computation of the L here can be reduced to the case of losslessness by noting that orthogonality of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is the same as losslessness of $\mathfrak{B}_1 \times {}_2$ with respect to $\Phi = \frac{1}{2}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. Here I and 0 are the identity matrix and the zero matrix of dimension $\mathtt{w}$.

7.1. Orthogonal Complement and Matched State Maps

We now discuss more about the orthogonal complement of a behavior. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$. Let $\mathfrak{B}^{\perp}$ be its orthogonal complement as defined in Equation (3) with $\Sigma = I$. If $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ is represented by minimal kernel and observable image representations $R\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)w = 0$ and $w = M\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)\ell_1$ respectively, then $\mathfrak{B}^{\perp}$ is represented by the minimal kernel representation $M^T\big(-\frac{\mathrm{d}}{\mathrm{d}t}\big)w = 0$ and the observable image representation $w = R^T\big(-\frac{\mathrm{d}}{\mathrm{d}t}\big)\ell_2$. Of course, $\mathfrak{B}$ and $\mathfrak{B}^{\perp}$ are orthogonal behaviors, and hence we can construct the adapted bilinear differential form L_{Ψ} for $\mathfrak{B}$ and $\mathfrak{B}^{\perp}$. We can also express this L_{Ψ} as a function of the states of $\mathfrak{B}$ and $\mathfrak{B}^{\perp}$. It is easy to verify that $\mathfrak{B}$ and $\mathfrak{B}^{\perp}$ have the same McMillan degree. Further, in this case the constant matrix L happens to be invertible and we can modify one of the two minimal state maps to obtain a matched pair of state maps. (X, Z) is said to be a *matched pair* of minimal state maps for $(\mathfrak{B}, \mathfrak{B}^{\perp})$ if $\frac{\mathrm{d}}{\mathrm{d}t}\big(X\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)w_1\big)^T Z\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)w_2 = w_1^T w_2$ for all $(w_1, w_2) \in \mathfrak{B} \times \mathfrak{B}^{\perp}$. The fact that L_{Ψ} is a state function can also be used to compute a matched pair of minimal state maps for $\mathfrak{B}$ and $\mathfrak{B}^{\perp}$ as follows. Define $\Gamma(\zeta, \eta)$ by $\overset{\bullet}{\Gamma}(\zeta, \eta) = R^T(-\zeta)M(\eta)$ and compute $\Gamma(\zeta, \eta) = \frac{R^T(-\zeta)M(\eta)}{\zeta+\eta}$. Then $\frac{\mathrm{d}}{\mathrm{d}t}L_{\Gamma}(\ell_1, \ell_2) = \big(R^T\big(-\frac{\mathrm{d}}{\mathrm{d}t}\big)\ell_1\big)^T\big(M\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)\ell_2\big)$ for all $\ell_1 \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{p}(\mathfrak{B})})$ and $\ell_2 \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{m}(\mathfrak{B})})$. Factor Γ as $\Gamma(\zeta, \eta) = Z^T(\zeta)X(\eta)$ with the rows of X and Z linearly independent over $\mathbb{R}$. Such a factorization is done using the coefficient matrix of $\Gamma(\zeta, \eta)$. We first factor $\mathrm{mat}(\Gamma)$ into $\tilde{Z}^T\tilde{X}$ with $\tilde{Z}$ and $\tilde{X}$ surjective, and then define $X \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}) \times \mathtt{m}(\mathfrak{B})}[\xi]$ and $Z \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}) \times \mathtt{p}(\mathfrak{B})}[\xi]$ by $\mathrm{mat}(X) = \tilde{X}$ and $\mathrm{mat}(Z) = \tilde{Z}$. We have then obtained a matched pair of minimal state maps (X, Z) that act on the latent variables ℓ_1 and ℓ_2 of $\mathfrak{B}$ and $\mathfrak{B}^{\perp}$, respectively. In terms of the original variables, this yields matched minimal state maps $X\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)M^{\dagger}\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)w_1$ and $Z\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)(R^{\dagger})^T\big(-\frac{\mathrm{d}}{\mathrm{d}t}\big)w_2$ for $(\mathfrak{B}, \mathfrak{B}^{\perp})$, where $M^{\dagger}$ is a polynomial left inverse of M, and $R^{\dagger}$ is a polynomial right inverse of R. More on this can be found in [5] (Section 10). Given $\Sigma \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ nonsingular and symmetric, the Σ-orthogonal complement $\mathfrak{B}^{\perp_{\Sigma}}$ of $\mathfrak{B}$ can be computed from $\mathfrak{B}^{\perp}$ by noting that $\mathfrak{B}^{\perp_{\Sigma}} = (\Sigma\mathfrak{B})^{\perp} = \Sigma^{-1}\mathfrak{B}^{\perp}$.

7.2. Latent Variable Representations for $\mathcal{N}^{\perp_{\Sigma}}$ and $\mathcal{P}^{\perp_{\Sigma}}$

In this subsection we address the following problem: given the full plant behavior $\mathcal{P}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, represented by a latent variable representation $R_w\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)w + R_c\big(\frac{\mathrm{d}}{\mathrm{d}t}\big) + R_{\ell}\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)\ell = 0$ (with latent variable ℓ), and a symmetric nonsingular weighting matrix Σ, compute a representation for $\mathcal{N}^{\perp_{\Sigma}}$ (where $\mathcal{N}$ is the hidden behavior associated with $\mathcal{P}_{\mathrm{full}}$) and $\mathcal{P}^{\perp_{\Sigma}}$ (where $\mathcal{P}$ is the manifest plant behavior associated with $\mathcal{P}_{\mathrm{full}}$). In order to solve this problem, we first formulate and prove a general result on the orthogonal complements of behaviors in relation with elimination of variables.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ be a system behavior with manifest variable (w_1, w_2). Let $\mathcal{P}_{w_1}(\mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}_1}$ be defined as the behavior obtained from $\mathfrak{B}$ by eliminating w_2:

$$\mathcal{P}_{w_1}(\mathfrak{B}) := \{w_1 \mid \exists\, w_2 \text{ such that } (w_1, w_2) \in \mathfrak{B}\}.$$

Let $\mathcal{N}_{w_1}(\mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}_1}$ be the behavior 'hidden from w_2': $\mathcal{N}_{w_1}(\mathfrak{B}) := \{w_1 \mid (w_1, 0) \in \mathfrak{B}\}$.

Lemma 6 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$. Then the following statements hold:*

1. $\mathcal{P}_{w_1}(\mathfrak{B}_{\text{cont}}) = \mathcal{P}_{w_1}(\mathfrak{B})_{\text{cont}}$,
2. $\mathcal{P}_{w_1}(\mathfrak{B}_{\text{cont}})^{\perp} = \mathcal{N}_{w_1}(\mathfrak{B}^{\perp}_{\text{cont}})_{\text{cont}}$.

We remark here that when we have an image representation $w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$ of a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, we need to ensure that $M(\lambda)$ has constant rank for all $\lambda \in \mathbb{C}$, before we deduce that $M^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0$ is a kernel representation of $\mathfrak{B}^{\perp}$. But this is not necessary when starting from a kernel representation. More precisely, if $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ has a kernel representation $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0$, then we directly obtain $w = R^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$ as an image representation of $\mathfrak{B}^{\perp}_{\text{cont}}$. We use this remark in addition to the above lemma to compute representations for the behaviors $\mathcal{N}^{\perp_\Sigma}$ and $\mathcal{P}^{\perp_\Sigma}$ from $\mathcal{P}_{\text{full}}$ and Σ.

Let $\mathcal{P}_{\text{full}}$ be given by the latent variable representation (latent variable ℓ): $R_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w + R_c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)c + R_\ell\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell = 0$. Then the hidden behavior $\mathcal{N}$ has latent variable representation $R_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w + R_\ell\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell = 0$. Assume $\mathcal{N}$ is controllable. Let $\mathfrak{B}$ be the full (w, ℓ) behavior of this latent variable representation. Then, $\mathfrak{B}^{\perp}_{\text{cont}}$ has image representation

$$\begin{bmatrix} w \\ \ell \end{bmatrix} = \begin{bmatrix} R_w^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ R_\ell^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix} k$$

(with latent variable k). Application of the lemma yields that $\mathcal{N}^{\perp}$ is the controllable part of $\mathcal{N}' \in \mathfrak{L}^{\mathtt{w}}$ represented by the latent variable representation: $w = R_w^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)k$ and $0 = R_\ell^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)k$, with latent variable k. We use $\mathcal{N}^{\perp_\Sigma} = \Sigma^{-1}\mathcal{N}^{\perp}$ to obtain a latent variable representation for $\mathcal{N}'' \in \mathfrak{L}^{\mathtt{w}}$ (with latent variable k)

$$\begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} \Sigma^{-1}R_w^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ R_\ell^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix} k \tag{9}$$

which yields $\mathcal{N}^{\perp_\Sigma} = \mathcal{N}''_{\text{cont}}$.

Next, we compute a representation of $\mathcal{P}^{\perp_\Sigma}$. This time, for $\mathfrak{B}$ take the full (w, c, ℓ) behavior of the kernel representation: $R_w\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w + R_c\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)c + R_\ell\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell = 0$. Then clearly $\mathcal{P} = \mathcal{P}_w(\mathfrak{B})$. Also, $\mathfrak{B}^{\perp}_{\text{cont}}$ is represented in image representation by

$$\begin{bmatrix} w \\ c \\ \ell \end{bmatrix} = \begin{bmatrix} R_w^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ R_c^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ R_\ell^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix} k$$

(with latent variable k). Again assuming $\mathcal{P}$ is controllable, we get $\mathcal{P}^{\perp} = \mathcal{N}_w(\mathfrak{B}^{\perp}_{\text{cont}})_{\text{cont}}$, where $\mathcal{N}_w(\mathfrak{B}^{\perp}_{\text{cont}})$ has latent variable representation: $w = R_w^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)k$, $0 = R_c^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)k$ and $0 = R_\ell^T\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)k$. Consequently, $\mathcal{P}^{\perp_\Sigma} = \mathcal{P}'_{\text{cont}}$

with $\mathcal{P}'$ the behavior defined in latent variable representation (latent variable k) by

$$\begin{bmatrix} w \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \Sigma^{-1} R_w^T(-\frac{\mathrm{d}}{\mathrm{d}t}) \\ R_c^T(-\frac{\mathrm{d}}{\mathrm{d}t}) \\ R_\ell^T(-\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} k. \tag{10}$$

8. LMI'S AND STORAGE FUNCTIONS

We now turn to the important problem of computing a storage function when a behavior is dissipative. We first consider the case $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. We wish to find, for a given $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, a solution $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ to $Q_{\overset{\bullet}{\Psi}}(w) \leq Q_\Phi(w)$ for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. In this section we shall write this inequality of QDF's as $\overset{\bullet}{\Psi} \leq \Phi$. This inequality of QDF's can be made into a matrix inequality problem involving real constant matrices as follows. Let $\mathtt{n}$ denote the degree of Φ (i.e., the highest power of ζ or η in Φ, as explained in Section 3). $\overset{\bullet}{\Psi} \leq \Phi$ implies that the degree of Ψ must be less than $\mathtt{n}$. Let $\mathfrak{M}^{k\times k}$ denote the $(k(k+1)/2)$-dimensional real vector space consisting of the real symmetric $k \times k$ matrices. Let $\text{mat}(\Phi)_{\mathtt{n}\times\mathtt{n}} \in \mathfrak{M}^{(\mathtt{n}+1)\mathtt{w}\times(\mathtt{n}+1)\mathtt{w}}$ denote the truncation of the coefficient matrix of Φ to its first $(\mathtt{n}+1)\mathtt{w} \times (\mathtt{n}+1)\mathtt{w}$ rows and columns, and similarly let $\text{mat}(\Psi)_{(\mathtt{n}-1)\times(\mathtt{n}-1)} \in \mathfrak{M}^{\mathtt{nw}\times\mathtt{nw}}$ denote the corresponding truncation to the first $\mathtt{nw} \times \mathtt{nw}$ rows and columns of Ψ. Now, the operator $^\bullet$ acting on symmetric elements of $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ of degree less than $\mathtt{n}$ corresponds to a linear mapping from $\mathfrak{M}^{\mathtt{nw}\times\mathtt{nw}}$ to $\mathfrak{M}^{(\mathtt{n}+1)\mathtt{w}\times(\mathtt{n}+1)\mathtt{w}}$ and we denote this operator by $L_\bullet$. We describe the precise way in which $L_\bullet$ acts. Let $I_{\mathtt{nw}}$ be the identity matrix of dimension $\mathtt{nw}$ and let $X \in \mathfrak{M}^{\mathtt{nw}\times\mathtt{nw}}$. We have

$$L_\bullet(X) = \begin{bmatrix} I_{\mathtt{nw}} \\ 0 \end{bmatrix} X [0 \quad I_{\mathtt{nw}}] + \begin{bmatrix} 0 \\ I_{\mathtt{nw}} \end{bmatrix} X [I_{\mathtt{nw}} \quad 0]. \tag{11}$$

The first term corresponds to a right shifted version of X and the second corresponds to a down shifted version, and the rest of the matrix gets padded with zeros of suitable size. The inequality $\overset{\bullet}{\Psi} \leq \Phi$ is equivalent to the matrix inequality

$$L_\bullet(\text{mat}(\Psi)_{(\mathtt{n}-1)\times(\mathtt{n}-1)}) \leq \text{mat}(\Phi)_{\mathtt{n}\times\mathtt{n}}. \tag{12}$$

The problem of computing Ψ hence reduces to solving an LMI. It is possible to use standard routines in the LMI toolbox to look for the maximum or the minimum of all the solutions. This maximum or minimum is useful for checking half-line dissipativity on $\mathbb{R}_-$ or $\mathbb{R}_+$, respectively. Alternatively, one can add $X \leq 0$ or $X \geq 0$ to the inequality, Equation (12).

Given $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, the problem of computing storage functions for an arbitrary $\mathfrak{B} in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ can be reduced to the case $\mathfrak{B} = \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{l})$ by using an observable image representation $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ and proceeding with $\Phi' \in \mathbb{R}^{1\times 1}[\zeta,\eta]$ defined by $\Phi'(\zeta,\eta) := M(\zeta)^T\Phi(\zeta,\eta)M(\eta)$.

When a controllable behavior is represented by a kernel representation, it is possible to use LMI's to determine dissipativity and we explore this issue now. The following lemma is stated here for easy reference and is an easy consequence of proposition 3.2 of [5] and the dissipation inequality on a behavior.

Lemma 7 *Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ and let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ be given by the kernel representation $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$. $\mathfrak{B}$ is Φ-dissipative if and only if there exist $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, $F \in \mathbb{R}^{\mathtt{w}\times\bullet}[\zeta,\eta]$ and $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that*

$$(\zeta+\eta)\Psi(\zeta,\eta) = \Phi(\zeta,\eta) + F(\zeta,\eta)R(\eta) + R^T(\zeta)F^T(\eta,\zeta) - D^T(\zeta)D(\eta)$$

We note that it is possible to estimate the degrees of Ψ and F in the equation above. (As mentioned in Section 3, the degree of a symmetric two-variable polynomial matrix is the highest power of ζ and/or η having a nonzero coefficient.) Let $\mathtt{n}_\Phi$ be the degree of $\Phi(\zeta,\eta)$ and let $\mathtt{n}_\mathrm{R}$ be the degree of $R(\xi)$. Then, there exists a $\Psi(\zeta,\eta)$ with degree at most $\mathtt{n}_\Phi + n_R - 1$. For such a Ψ, there exists an $F(\zeta,\eta)$ that has at most degree $\mathtt{n}_\Phi + n_R$ in ζ and at most degree $\mathtt{n}_\Phi$ in η. We use the above lemma and this estimate of degrees in the following algorithm. We first describe how the operation of multiplication by a polynomial matrix can be written in terms of the associated coefficient matrix. Given a polynomial matrix $R \in \mathbb{R}^{\mathtt{v}\times\mathtt{w}}[\xi]$, and a two variable polynomial matrix $F \in \mathbb{R}^{\mathtt{w}\times\mathtt{v}}[\zeta,\eta]$ we shall relate the coefficient matrices associated with $F(\zeta,\eta)R(\eta)$ and $F(\zeta,\eta)$. Let $R(\xi) = R_0 + \xi R_1 + \cdots + \xi^{\mathtt{n}_\mathrm{R}}R_{\mathtt{n}_\mathrm{R}}$ for $R_i \in \mathbb{R}^{\mathtt{v}\times\mathtt{w}}$. Define $\widehat{R} \in \mathbb{R}^{(\mathtt{n}_\Phi+1)\mathtt{v}\times(n_\Phi+n_R+1)\mathtt{w}}$ as follows:

$$\widehat{R} := \underbrace{\begin{bmatrix} R_0 & R_1 & \cdots & R_{\mathtt{n}_\mathrm{R}} & 0 & \cdots & 0 \\ 0 & R_0 & R_1 & \cdots & R_{\mathtt{n}_\mathrm{R}} & \ddots & 0 \\ \vdots & \ddots & \cdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & R_0 & R_1 & \cdots & R_{\mathtt{n}_\mathrm{R}} \end{bmatrix}}_{(\mathtt{n}_\Phi+n_R+1)\mathtt{w} \text{ columns}} (\mathtt{n}_\Phi + 1)\mathtt{v} \text{ rows} .$$

Let $\widetilde{F}$ denote the truncation of $\mathrm{mat}(F)$ to its first $(\mathtt{n}_\Phi + n_R + 1)\mathtt{w}$ rows and first $(\mathtt{n}_\Phi + 1)\mathtt{v}$ columns. By mere multiplication of $F(\zeta,\eta)R(\eta)$ we obtain that $\widetilde{F}\widehat{R}$ is equal to the truncation of $\mathrm{mat}(F(\zeta,\eta)R(\eta))$ to its first $(\mathtt{n}_\Phi + n_R + 1)\mathtt{w}$ rows and columns. Define the linear mapping $L_R : \mathbb{R}^{(\mathtt{n}_\Phi+n_R+1)\mathtt{w}\times(n_\Phi+1)\mathtt{v}} \longrightarrow \mathbb{R}^{(\mathtt{n}_\Phi+n_R+1)\mathtt{w}\times(n_\Phi+n_R+1)\mathtt{w}}$ by $L_R(Y) := Y\widehat{R}$. It follows that $L_R(\widetilde{F}) = \mathrm{mat}(F(\zeta,\eta)R(\eta))_{(\mathtt{n}_\Phi+n_R)\times(n_\Phi+n_R)}$. Also, $\mathrm{mat}(R^T(\zeta)F^T(\eta,\zeta)) = (\mathrm{mat}(F(\zeta,\eta)R(\eta)))^T$ and their truncations to their first

$(\mathtt{n}_\Phi + n_R + 1)\mathtt{w}$ rows and columns are equal to $L_R(\mathrm{mat}(F))^T$. We use this in the following algorithm.

Algorithm 8 *Data:* $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ *and* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ *described by a minimal kernel representation:* $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0$.
Output: Whether $\mathfrak{B}$ *is* Φ*-dissipative, and if it is, a storage function.*

- Given $\mathfrak{B} = \ker\left(R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$, the existence of a storage function Q_Ψ for $\mathfrak{B}$ as a Φ-dissipative system is equivalent to the existence of $F \in \mathbb{R}^{\mathtt{w}\times\bullet}[\zeta.\eta]$ and $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that:

$$(\zeta+\eta)\Psi(\zeta,\eta) = \Phi(\zeta,\eta) + F(\zeta,\eta)R(\eta) + R^T(\zeta)F^T(\eta,\zeta) - D^T(\zeta)D(\eta).$$

- In terms of the associated constant matrices,

$$\begin{aligned}\mathrm{mat}(\overset{\bullet}{\Psi}) &= \mathrm{mat}(\Phi) + \mathrm{mat}(\mathrm{F}(\zeta,\eta)R(\eta)) + \mathrm{mat}(R^T(\zeta)F^T(\eta,\zeta)) \\ &\quad - \mathrm{mat}(D^T(\zeta)D(\eta)).\end{aligned}$$

- Solving the above *nonlinear* equation can be done by computing an X and a Y that solve the following *linear inequality*:

$$L_\bullet(X) - L_R(Y) - L_R(Y)^T \leq \mathrm{mat}(\Phi)_{(n_\Phi+n_R)\mathtt{w}\times(n_\Phi+n_R)\mathtt{w}}.$$

  We can use the LMI toolbox to check the existence of solutions, and a storage function can be found from X as discussed before. Half-line dissipativity is studied by further imposing sign-definiteness on X as was done for the case $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$.

The above algorithm has two important extensions. Firstly, the case of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ being given by a latent variable representation $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$ is easily reformulated so that the above algorithm can be used. Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be given. We define $\Phi' \in \mathbb{R}^{(\mathtt{w}+1)\times(\mathtt{w}+1)}[\zeta,\eta]$ by $\Phi' = \begin{bmatrix}\Phi & 0\\ 0 & 0\end{bmatrix}$. We consider $\mathfrak{B}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{w}+1}$ given by the kernel representation: $R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w - M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell = 0$ and note that Φ-dissipativity of $\mathfrak{B}$ and Φ'-dissipativity of $\mathfrak{B}_{\mathrm{full}}$ are equivalent. We then use the previous algorithm to check Φ'-dissipativity of $\mathfrak{B}_{\mathrm{full}}$.

A second extension is the case of strict dissipativity. This extension of the above algorithm has been used in Subsection 5.2. We have defined strict dissipativity for the case that $\Phi(\zeta,\eta)$ is constant, that is, $\Phi(\zeta,\eta) = \Sigma$ for some $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Given such a Σ, we replace $\Phi(\zeta,\eta)$ in the above algorithm by $\Sigma - \epsilon I$ with ϵ as an additional variable. We also add the constraint $\epsilon > 0$ and then compute solutions to the modified LMI. This modified problem has solutions if and only if $\mathfrak{B}$ is strictly dissipative with respect to $Q_\Sigma(w) = w^T\Sigma w$. Moreover, if this LMI has solutions, one can maximize ϵ and thus find the maximum ϵ such that $\mathfrak{B}$ is dissipative with respect to $\Sigma - \epsilon I$.

The use of LMI's in order to find storage functions for systems in state space representation has been studied in the context of $\mathcal{H}_\infty$ control by many authors, for example, [10–12].

9. SPECTRAL FACTORIZATION

In this final section we remark very briefly that there is a close relation between spectral factorization and storage functions. This relation was discussed in detail in Section 5 of [5]. Let $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ and let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ have an observable image representation $w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$. Define $\Phi' \in \mathbb{R}^{1 \times 1}[\zeta, \eta]$ by $\Phi'(\zeta, \eta) := M^T(\zeta)$ $\Phi(\zeta, \eta)M(\eta)$. Then, $\mathfrak{B}$ being Φ-dissipative is equivalent to $\partial\Phi'(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. This is equivalent to spectral factorizability of $\partial\Phi'(\xi)$ into $F^T(-\xi)F(\xi)$. $F \in \mathbb{R}^{1 \times 1}[\xi]$ is said to be a spectral factor of $\partial\Phi'$. Among all the spectral factors, particular choices of F results in the extremal storage functions. Further, under additional assumptions, it is possible to determine half-line dissipativity by checking the sign definiteness of a certain Pick matrix. This has been studied in [5] and in [13]. The use of spectral factorization methods in $\mathcal{H}_\infty$ control has also been pursued in [14–17].

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