Stabilization, Pole Placement, and Regular Implementability

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***Abstract*—In this paper, we study control by interconnection of linear differential systems. We give necessary and sufficient conditions for regular implementability of a given linear differential system. We formulate the problems of stabilization and pole placement as problems of finding a suitable, regularly implementable sub-behavior of the manifest plant behavior. The problem formulations and their resolutions are completely representation free, and specified only in terms of the system dynamics. Control is viewed as regular interconnection. A controller is a system that constrains the plant behavior through a distinguished set of variables, namely, the control variables. The issue of implementation of a controller in the feedback configuration and its relation to regularity of interconnection is addressed. Freedom of disturbances in a plant and regular interconnection with a controller also turn out to be inter-related.**

***Index Terms*—Behaviors, controller implementation, interconnection, pole placement, regular implementability, stabilization.**

I. Introduction and Notation

In this paper, we discuss the issue of stabilization of linear dynamical systems. The problem is studied in the behavioral context and control is viewed as interconnection. This view of treating control problems has been used before in, for example, [2], [3], [7], [16], and [19], in an H_∞ control context in [1], [4], [5], [12]–[14], [17], and [18], for adaptive control in [9], and for distributed systems in [6]. In contrast to [19] where the problems of stabilization and pole placement were considered for the case that *all* system variables are available for interconnection (the so-called full information case), we work in the generality that we are allowed to use only some of the system variables for the purpose of interconnection. These variables are called the control variables. Restricting oneself to using only the control variables for interconnection introduces the issue of *implementability* into the control problem, see [18] and [9]. In the context of stabilization, an important role is played by the notion of *regular implementability*. We establish necessary and sufficient conditions for a given behavior to be regularly implementable. This result is then applied to solve the problems of stabilization and pole placement by interconnection.

The paper is structured as follows. We start with the notation that we use in this paper. A brief review of basic definitions and concepts of the behavioral approach forms the later part of this section. In Section II, we discuss the problem of restricting control to just the control variables. The relevant notions are introduced and we give necessary and sufficient conditions for regular implementability. Section III contains the main problems of this paper. These problems deal with shaping the trajectories of the to-be-controlled variables w, using the control variables c. We consider the problems of stabilization and pole placement. The main results of this section are two theorems that solve these problems. The proofs of the theorems are given in Section IV. As an illustration, in Section V we apply our main results to the case that the plant to-be-controlled is given in input–state–output representation. Implementation of a controller in a feedback configuration plays a very prominent role in control theory. This issue is addressed in Section VI. Finally, in Section VII, we give a motivation for the fact that in our problem formulations we restrict ourselves to regular interconnections.

We first discuss some of the notation to be used in this paper, and review some basic facts from the behavioral approach. We use the standard notation $\mathbb{R}^{\mathtt{n}}$ for the $\mathtt{n}$-dimensional real Euclidean space. Often, the notation $\mathbb{R}^{\mathtt{w}}$ is used if w denotes a typical element of that vector space, or a typical function taking its value in that vector space. Vectors are understood to be column vectors in equations. In text, however, we write them as row vectors.

The ring of (one-variable) polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$. $\mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_2}[\xi]$ denotes the set of matrices with $\mathtt{n}_1$ rows and $\mathtt{n}_2$ columns in which each entry is an element of $\mathbb{R}[\xi]$. We use the notation $\mathbb{R}^{\bullet\times\mathtt{n}_2}$ when the number of rows is unspecified.

In this paper, we deal with linear time-invariant differential systems, in short, linear differential systems. A linear differential system is defined as a dynamical system whose behavior $\mathfrak{B}$ is equal to the set of solutions of a set of higher order, linear, constant coefficient differential equations. More precisely, there exists a polynomial matrix $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that $\mathfrak{B} = \{w \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) | R(d/dt)w = 0\}$. Here, $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ denotes the space of locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$, and $R(d/dt)w = 0$ is understood to hold in the distributional sense. The set of linear differential systems with manifest variable w taking its value in $\mathbb{R}^{\mathtt{w}}$ is denoted by $\mathfrak{L}^{\mathtt{w}}$.

We make a clear distinction between the behavior as defined as the space of all solutions of a set of (differential) equations, and the set of equations itself. A set of equations in terms of which the behavior is defined, is called a *representation* of the behavior. Let $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ be a polynomial matrix. If a behavior $\mathfrak{B}$ is represented by $R(d/dt)w = 0$ then we call this a kernel representation of $\mathfrak{B}$. Further, a kernel representation is said to be *minimal* if every other kernel representation of $\mathfrak{B}$ has at least g rows. A given kernel representation, $R(d/dt)w = 0$, is minimal if and only if the polynomial matrix R has full-row rank. We speak of a system as the behavior $\mathfrak{B}$, one of whose

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representations is given by $R(d/dt)w = 0$ or just $Rw = 0$. The "d/dt" is often suppressed to enhance readability. We will also encounter behaviors $\mathfrak{B}$ with manifest variable w, that are represented by equations of the form $R(d/dt)w = M(d/dt)\ell$, in which an auxiliary, latent variable ℓ appears. Here, R and M are polynomial matrices with the same number of rows. Through such an equation, we can consider the subspace of all $w \in \mathcal{L}_1^{\rm loc}(\mathbb{R}, \mathbb{R}^{\tt w})$ for which there exists an $\ell \in \mathcal{L}_1^{\rm loc}(\mathbb{R}, \mathbb{R}^{\tt l})$ such that the equation holds. A technical detail is that, by itself, this subspace is not an element of $\mathfrak{L}^{\tt w}$, because it is not a closed subspace (closed in the topology of $\mathcal{L}_1^{\rm loc}(\mathbb{R}, \mathbb{R}^{\tt w})$). Therefore, we call $R(d/dt)w = M(d/dt)\ell$ a latent variable representation of $\mathfrak{B}$ if

$$\mathfrak{B} = \left\{ w \in \mathcal{L}_1^{\rm loc}(\mathbb{R}, \mathbb{R}^{\tt w}) | \exists\, \ell \in \mathcal{L}_1^{\rm loc}(\mathbb{R}, \mathbb{R}^{\tt l}) \text{ such that } R\left(\frac{d}{dt}\right) w = M\left(\frac{d}{dt}\right) \ell \right\}^{\rm closure}$$

where the closure is taken in the $\mathcal{L}_1^{\rm loc}$ topology. Then, by the elimination theorem (see [7, Ch. 6, Th. 6.2.6]), $\mathfrak{B} \in \mathfrak{L}^{\tt w}$.

In this paper, we avoid the issue of properness of rational representations (transfer functions). Hence, we often restrict ourselves to signals that are infinitely often differentiable functions of time. The space of functions that are infinitely often differentiable with domain $\mathbb{R}$ and co-domain $\mathbb{R}^{\tt w}$, is denoted by $C^\infty(\mathbb{R}, \mathbb{R}^{\tt w})$. Let $\mathfrak{B} \in \mathfrak{L}^{\tt w}$ be represented by the kernel representation $R(d/dt)w = 0$ with $\operatorname{rank}(R) < {\tt w}$ (which also means that it is under-determined). Then some components of $w = (w_1, w_2, \ldots, w_{\tt w})$ are unconstrained by the requirement $w \in \mathfrak{B}$. These components are termed as *inputs* or are said to be *free* (in the C^∞ sense, for the purpose of this paper). The maximum number of such components is called the *input cardinality* of $\mathfrak{B}$ (denoted as ${\tt m}(\mathfrak{B})$). Once ${\tt m}(\mathfrak{B})$ free components are chosen, the remaining ${\tt w} - {\tt m}(\mathfrak{B})$ components are determined up to a finite-dimensional affine subspace of $C^\infty(\mathbb{R}, \mathbb{R}^{{\tt w} - {\tt m}(\mathfrak{B})})$. These are called *outputs*, and the number of outputs is denoted by ${\tt p}(\mathfrak{B})$. Thus, possibly after a permutation of components, $w \in \mathfrak{B}$ can be partitioned as $w = (u, y)$, with the ${\tt m}(\mathfrak{B})$ components of u as inputs, and the ${\tt p}(\mathfrak{B})$ components of y as outputs. We say that (u, y) is an input–output partition of $w \in \mathfrak{B}$, with input u and output y. The input–output structure of $\mathfrak{B} \in \mathfrak{L}^{\tt w}$ is reflected in its kernel representations as follows. Suppose $R(d/dt)w = 0$ is a minimal kernel representation of $\mathfrak{B}$. Partition $R = [Q \ P]$, and accordingly $w = (w_1, w_2)$. Then $w = (w_1, w_2)$ is an i/o partition (with input w_1 and output w_2) if and only if P is square and nonsingular. In general, there exist many input–output partitions, but the integers ${\tt m}(\mathfrak{B})$ and ${\tt p}(\mathfrak{B})$ are invariants associated with a behavior. It can be verified that ${\tt p}(\mathfrak{B})$ is equal to the rank of the polynomial matrix in any (not necessarily minimal) kernel representation of $\mathfrak{B}$ (for details see [7]).

A behavior whose input cardinality is equal to 0 is called *autonomous*. An autonomous behavior $\mathfrak{B}$ is said to be stable, if for all $w \in \mathfrak{B}$, we have $w(t) \to 0$ as $t \to \infty$. In the context of stability, we often need to describe regions of the complex plane $\mathbb{C}$. We denote the closed right-half of the complex plane by $\mathbb{C}^+$ and the open left-half complex plane by $\mathbb{C}^-$. A polynomial matrix $R \in \mathbb{R}^{\bullet \times {\tt w}}[\xi]$ is called *Hurwitz* if $\operatorname{rank}(R(\lambda)) = {\tt w}$ for all $\lambda \in \mathbb{C}^+$. If $\mathfrak{B} \in \mathfrak{L}^{\tt w}$ is represented by $R(d/dt)w = 0$ then $\mathfrak{B}$ is stable if and only if R is Hurwitz.

For autonomous behaviors, we also speak about poles of the behavior. Let $\mathfrak{B} \in \mathfrak{L}^{\tt w}$ be autonomous. Then there exists an $R \in \mathbb{R}^{{\tt w} \times {\tt w}}[\xi]$ such that $\mathfrak{B}$ is represented minimally by $R(d/dt)w = 0$. Obviously, for any nonzero $\alpha \in \mathbb{R}$, αR also yields a kernel representation of $\mathfrak{B}$. Hence, we can choose R such that $\det(R)$ is a monic polynomial. This monic polynomial is denoted by $\chi_{\mathfrak{B}}$ and is called *the characteristic polynomial of* $\mathfrak{B}$. $\chi_{\mathfrak{B}}$ depends only on $\mathfrak{B}$, and not on the polynomial matrix R we used to define it: if R_1, R_2 both represent $\mathfrak{B}$ minimally then there exists a unimodular U such that $R_2 = UR_1$. Hence, if $\det(R_1)$ and $\det(R_2)$ are monic then $\det(R_1) = \det(R_2)$. The *poles* of $\mathfrak{B}$ are defined as the roots of $\chi_{\mathfrak{B}}$. Note that $\chi_{\mathfrak{B}} = 1$ if and only if $\mathfrak{B} = 0$. A behavior is stable if and only if all its poles are in $\mathbb{C}^-$.

We now discuss the issue of control as interconnection. A plant behavior (denote it by $\mathcal{P}$) consists of all trajectories satisfying a set of differential equations. One would like to restrict this space of trajectories to a desired subsystem, $\mathcal{K} \subset \mathcal{P}$. This restriction can be effected by increasing the number of equations that the variables of the plant have to satisfy. These additional laws themselves define a new system, called the controller (denoted by $\mathcal{C}$). The interconnection of the two systems (the plant and the controller) results in the controlled behavior $\mathcal{K}$. After interconnection, the variables have to satisfy the laws of *both* $\mathcal{P}$ and $\mathcal{C}$. More precisely, let $\mathcal{P} \in \mathfrak{L}^{\tt w}$ (the plant) and $\mathcal{C} \in \mathfrak{L}^{\tt w}$ (the controller). Then the *full interconnection* of $\mathcal{P}$ and $\mathcal{C}$ is defined as the system with behavior $\mathcal{P} \cap \mathcal{C}$. Note that $\mathcal{P} \cap \mathcal{C}$ is again an element of $\mathfrak{L}^{\tt w}$. A given behavior $\mathcal{K} \in \mathfrak{L}^{\tt w}$ is called *implementable with respect to* $\mathcal{P}$ *by full interconnection* if there exists a $\mathcal{C} \in \mathfrak{L}^{\tt w}$ such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$. The full interconnection of $\mathcal{P}$ and $\mathcal{C}$ is called *regular*, if

$${\tt p}(\mathcal{P} \cap \mathcal{C}) = {\tt p}(\mathcal{P}) + {\tt p}(\mathcal{C}).$$

Let $Rw = 0$ and $Cw = 0$ be minimal kernel representations of $\mathcal{P}$ and $\mathcal{C}$ respectively. Then the full interconnection of $\mathcal{P}$ and $\mathcal{C}$ is regular if and only if $\begin{bmatrix} R \\ C \end{bmatrix} w = 0$ is a minimal kernel representation of $\mathcal{P} \cap \mathcal{C}$. Detailed discussions on control as interconnection and regular interconnections can be found in [19]. Regular interconnections have also been of interest in [2] and [3].

Finally, we review the concept of controllability in the context of the behavioral approach. A behavior $\mathfrak{B} \in \mathfrak{L}^{\tt w}$ is *controllable* if for all $w_1, w_2 \in \mathfrak{B}$, there exists a $T \geq 0$ and a $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t + T) = w_2(t)$ for $t \geq 0$. A weaker notion is *stabilizability*, which is defined as follows. A behavior $\mathfrak{B}$ is stabilizable if for all $w_1 \in \mathfrak{B}$, there exists a $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < 0$, and $w(t) \to 0$ as $t \to \infty$. Thus every trajectory in a stabilizable behavior $\mathfrak{B}$ can be steered to 0, asymptotically.

Often, we encounter behaviors $\mathfrak{B} \in \mathfrak{L}^{\tt w}$ that are neither autonomous nor controllable. The *controllable part* of a behavior $\mathfrak{B}$ is defined as the largest controllable sub-behavior of $\mathfrak{B}$. This is denoted by $\mathfrak{B}_{\rm cont}$. A given $\mathfrak{B} \in \mathfrak{L}^{\tt w}$ can always be decomposed as $\mathfrak{B} = \mathfrak{B}_{\rm cont} \oplus \mathfrak{B}_{\rm aut}$, where $\mathfrak{B}_{\rm cont}$ is the (unique) controllable part of $\mathfrak{B}$, and $\mathfrak{B}_{\rm aut}$ is a (nonunique) autonomous sub-behavior of $\mathfrak{B}$. For details we refer to [7].

We shall relate the notions of stabilizability to that of interconnection. Interconnections and stabilizability in the behavioral context have been issues in many publications, see, for example, [16] and [3]. Also, [7] contains a detailed exposition. We need the following proposition from [19] that relates stabilizability and regular, full interconnection.

Proposition 1: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ and let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$. Assume that $R(d/dt)w = 0$ is a kernel representation of $\mathcal{P}$. Then the following statements are equivalent:

1) $\mathcal{P}$ is stabilizable;
2) $\mathrm{rank}(R(\lambda)) = \mathrm{rank}(R)$ for all $\lambda \in \mathbb{C}^+$;
3) there exists a $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{K}$ is stable and implementable w.r.t. $\mathcal{P}$ by regular, full interconnection.

Furthermore, if the representation $R(d/dt)w = 0$ is minimal then any of the above statements is equivalent to

4) there exists a $C \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that $\begin{bmatrix} R \\ C \end{bmatrix}$ is nonsingular and Hurwitz.

In the aforementioned, $\mathrm{rank}(R(\lambda))$ is understood to be the rank of the complex matrix $R(\lambda)$, while $\mathrm{rank}(R)$ is the rank of the polynomial matrix R. We say that the controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ *stabilizes* $\mathcal{P}$, if the system obtained by the full interconnection of $\mathcal{P}$ and $\mathcal{C}$ is stable, and the interconnection is regular. Note that from the above proposition, if $\mathcal{P}$ is not stabilizable then there does not exist $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ which stabilizes $\mathcal{P}$.

Thus controlling a system means restricting the system behavior to a desired sub-behavior. Stability of the sub-behavior is usually the desired feature. An alternate feature is specifying the poles of the sub-behavior. For a given behavior $\mathcal{P}$, by placing the poles in a given region, we mean, finding a controller such that the fully interconnected system is autonomous, the poles of the corresponding controlled system are in the given region and the interconnection is regular. It was shown in [19] that if one does not require the interconnection to be regular, then the pole placement problem is essentially a triviality. The following proposition from [19] that relates controllability, regular full interconnection, and pole placement, will help us in solving the pole placement problem for the general case.

Proposition 2: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ and let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$. Assume that $R(d/dt)w = 0$ is a kernel representation of $\mathcal{P}$. Then, the following statements are equivalent:

1) $\mathcal{P}$ is controllable;
2) $\mathrm{rank}(R(\lambda)) = \mathrm{rank}(R)$ for all $\lambda \in \mathbb{C}$;
3) for all monic $r \in \mathbb{R}[\xi]$, there exists a $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that $\chi_{\mathcal{K}} = r$ and $\mathcal{K}$ is implementable w.r.t. $\mathcal{P}$ by regular, full interconnection.

Furthermore, if the representation $R(d/dt)w = 0$ is minimal then any of the above statements is equivalent to:

4) for all $r \in \mathbb{R}[\xi]$, there exists a $C \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that $\det \begin{bmatrix} R \\ C \end{bmatrix} = r$.

We shall also deal with systems in which the signal space comes as a product space, with the first component viewed as an observed, and the second as a to-be-deduced variable. We talk about observability (in such systems). Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1 + \mathtt{w}_2}$ with manifest variable $w = (w_1, w_2)$, w_2 is said to be *observable* from w_1 if $(w_1, w_2'), (w_1, w_2'') \in \mathfrak{B}$ implies $w_2' = w_2''$. Let

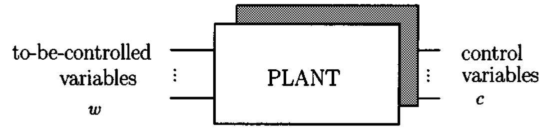

Fig. 1. Plant.

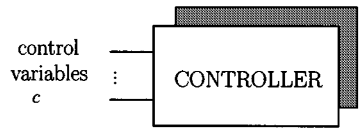

Fig. 2. Controller.

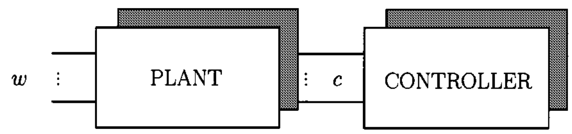

Fig. 3. The plant and controller after interconnection.

$R_1(d/dt)w_1 + R_2(d/dt)w_2 = 0$ be a kernel representation of $\mathfrak{B}$. Then observability of w_2 from w_1 is equivalent to $R_2(\lambda)$ having full column rank for all $\lambda \in \mathbb{C}$. The weaker notion of *detectability* is defined along similar lines. Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1 + \mathtt{w}_2}$, w_2 is said to be detectable from w_1 if $(w_1, w_2'), (w_1, w_2'') \in \mathfrak{B}$ implies $w_2'(t) - w_2''(t) \to 0$ as $t \to \infty$. In the aforementioned kernel representation, detectability of w_2 from w_1 is equivalent to $R_2(\lambda)$ having full-column rank for all $\lambda \in \mathbb{C}^+$. For details, see [7].

II. Regular Implementability

Suppose we have a plant to be controlled, with two types of variables, see Figs.1, 2, and 3. In the given plant, the variables whose trajectories we intend to shape (called the *to-be-controlled variables*), are denoted by w. These to-be-controlled variables can be controlled through a set of *control variables* c, over which we can "attach" a controller. These are the variables, that can be measured and/or actuated upon. Often we have some common components in w and c. We formulate the problem, however, for the general case, in which we have access to just the control variables c.

Before the controller acts, there are two behaviors of the plant that are relevant: $\mathcal{P}_{\text{full}}$ (called the *full plant behavior*) that formalizes the dynamics of the variables w and c, and the behavior $\mathcal{P}$ (called the *manifest plant behavior*) that formalizes the dynamics of the to-be-controlled variables w only. Thus

$$\begin{aligned} \mathcal{P}_{\text{full}} = \{ &(w,c) \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}+\mathtt{c}}) \mid (w,c) \\ &\text{satisfies the plant equations}\} \\ \mathcal{P} = \{ &w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) | \exists\, c \\ &\text{such that } (w,c) \in \mathcal{P}_{\text{full}}\}^{\text{closure}}. \end{aligned} \tag{1}$$

In this paper, we assume that the plant is a linear differential system, i.e., $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. The particular representation by which it is given, is immaterial to us. The manifest plant behavior $\mathcal{P}$ is obtained by *eliminating* c from $\mathcal{P}_{\text{full}}$, so, by the elimination theorem, $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$.

A controller restricts the trajectories that c can assume and is described by a *controller behavior* $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$:

$$\mathcal{C} = \left\{ c \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{c}}) \mid c \text{ satisfies the controller equations} \right\}.$$

The *full controlled behavior* $\mathcal{K}_{\text{full}}$ is obtained by the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ through the variable c and is defined as

$$\mathcal{K}_{\text{full}} = \{(w, c) \mid (w, c) \in \mathcal{P}_{\text{full}} \text{ and } c \in \mathcal{C}\}. \tag{2}$$

The *manifest controlled behavior* $\mathcal{K}$ is obtained by eliminating c from $\mathcal{K}_{\text{full}}$ and is defined as

$$\mathcal{K} = \{w \mid \exists\, c \in \mathcal{C} \text{ such that } (w, c) \in \mathcal{P}_{\text{full}}\}^{\text{closure}}. \tag{3}$$

In that case, we say that $\mathcal{K}$ is implemented by $\mathcal{C}$, or $\mathcal{C}$ implements $\mathcal{K}$ through c. A given $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is called *implementable with respect to* $\mathcal{P}_{\text{full}}$ *by interconnection through* c, if there exists a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$, such that $\mathcal{K}$ is implemented by $\mathcal{C}$. If it is clear from the context, we often suppress the specifications "w.r.t. $\mathcal{P}_{\text{full}}$" and "through c." An important issue is the question which $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ are implementable, i.e., for which $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ there exists a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that (3) holds. A crucial concept to answer this question is the notion of hidden behavior: the *hidden behavior* $\mathcal{N}$ is the behavior consisting of the plant trajectories that occur when the control variables are zero

$$\mathcal{N} = \left\{ w \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid (w, 0) \in \mathcal{P}_{\text{full}} \right\}. \tag{4}$$

We have access to only the control variables c—hence the notion of $\mathcal{N}$ being hidden from the control variables.

The following proposition from [18] settles the question of implementability for a given $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. We refer to this proposition as the controller implementability theorem.

Proposition 3: Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be a given full plant behavior, and let $\mathcal{P}, \mathcal{N} \in \mathfrak{L}w$ be the manifest plant behavior and hidden behavior, respectively. Then $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is implementable w.r.t. $\mathcal{P}_{\text{full}}$ by interconnection through c if and only if $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$.

In addition to implementability issues, the hidden behavior $\mathcal{N}$ plays a role in observability and detectability of $\mathcal{P}_{\text{full}}$. It can be easily seen that, in $\mathcal{P}_{\text{full}}$, w is observable from c if and only if $\mathcal{N} = 0$, and w is detectable from c if and only if $\mathcal{N}$ is stable.

Roughly speaking, for a given $\mathcal{P}_{\text{full}}$ we want to find a controller $\mathcal{C}$ such that the manifest controlled behavior $\mathcal{K}$ has desired properties. However, we shall restrict ourselves to $\mathcal{C}$'s such that the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular. A motivation for this is provided in Section VII. The interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ through c is *regular* if

$$\mathtt{p}(\mathcal{K}_{\text{full}}) = \mathtt{p}(\mathcal{P}_{\text{full}}) + \mathtt{p}(\mathcal{C})$$

i.e., if the output cardinalities of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ add up to that of $\mathcal{K}_{\text{full}}$.

A given $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is called *regularly implementable* if there exists a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that $\mathcal{K}$ is implemented by $\mathcal{C}$, and if the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular. Similar to plain implementability, an important question is under what conditions a given sub-behavior $\mathcal{K}$ of $\mathcal{P}$ is regularly implementable. The following theorem is the main result of this section, and provides necessary and sufficient conditions for this.

Theorem 4: Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. Let $\mathcal{P}, \mathcal{N} \in \mathfrak{L}^{\mathtt{w}}$ be the corresponding manifest plant behavior and hidden behavior respectively. Let $\mathcal{P}_{\text{cont}}$ be the controllable part of $\mathcal{P}$. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Then, $\mathcal{K}$ is implementable w.r.t. $\mathcal{P}_{\text{full}}$ by regular interconnection through c if and only if the following conditions are satisfied:

$$\boxed{\begin{array}{l} \bullet\ \ \mathcal{N} \subset \mathcal{K} \subset \mathcal{P} \\ \bullet\ \ \mathcal{K} + \mathcal{P}_{\text{cont}} = \mathcal{P} \end{array}}$$

The previous theorem has two conditions. The first one is exactly the condition for implementability through c (as in the controller implementability theorem). The second condition formalizes the notion that the autonomous part of $\mathcal{P}$ is taken care of by $\mathcal{K}$. While the autonomous part of $\mathcal{P}$ is not unique, $\mathcal{P}_{\text{cont}}$ is. This makes verifying the regular implementability of a given $\mathcal{K}$ computable. As a consequence of this theorem, note that if $\mathcal{P}$ is controllable, then $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is regularly implementable if and only if it is implementable, see also the main results of [8].

III. Pole Placement and Stabilization

In this section, we discuss the problems of pole placement and stabilization. The problem statements and the theorems involve the behaviors of the plant, etc., which have been defined Section II.

Pole placement problem: Given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, find conditions under which there exists, and compute, for every monic $r \in \mathbb{R}[\xi]$, a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that

- the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular;
- the manifest controlled behavior $\mathcal{K}$ is autonomous and has characteristic polynomial r.

Suppressing the controller $\mathcal{C}$ from the problem formulation, the problem can alternatively be stated as:

Given $\mathcal{P}_{\text{full}}$, find conditions under which there exists, and compute, for every monic $r \in \mathbb{R}[\xi]$ a regularly implementable, autonomous $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that $\chi_{\mathcal{K}} = r$.

When the manifest controlled behavior $\mathcal{K}$ is only required to be stable, we refer to the problem as that of stabilization.

Stabilization problem: Given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, find conditions for the existence of, and compute $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that

- the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular;
- the manifest controlled behavior $\mathcal{K}$ is autonomous and stable.

Again, suppressing the controller $\mathcal{C}$ from the formulation, the stabilization problem can be restated as:

Given $\mathcal{P}_{\text{full}}$, find conditions for the existence of, and compute a behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ that is autonomous, stable and regularly implementable.

The main results of this section are the following theorems, which establish necessary and sufficient conditions for pole placement and stabilization.

Theorem 5: Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. For every monic $r \in \mathbb{R}[\xi]$, there exists a regularly implementable, autonomous $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that $\chi_{\mathcal{K}} = r$ if and only if $\mathcal{N} = 0$ and $\mathcal{P}$ is controllable, equivalently, if and only if

- in $\mathcal{P}_{\text{full}}$, w is observable from c;
- $\mathcal{P}$ is controllable.

Theorem 6: Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. There exists a regularly implementable, autonomous, stable $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ if and only if $\mathcal{N}$ is stable and $\mathcal{P}$ is stabilizable, equivalently, if and only if

- in $\mathcal{P}_{\text{full}}$, w is detectable from c;
- $\mathcal{P}$ is stabilizable.

Note that, neither in the problem formulations nor in the conditions appearing in Theorems 5 and 6, do representations of the given plant appear. Indeed, our problem formulations and their resolutions are completely *representation free*, and are formulated purely in terms of properties of the *behavior* $\mathcal{P}_{\text{full}}$. Thus, our treatment of the pole placement and stabilization problems is genuinely behavioral. Of course, theorems 5 and 6 are applicable to any particular representation of $\mathcal{P}_{\text{full}}$ as well. As an example, in Section V we treat the case that $\mathcal{P}_{\text{full}}$ is represented in input–state–output representation.

In both the stabilization problem and the pole placement problem, we have restricted ourselves to regular interconnections. We give an explanation for this in Section VII. At this point we note that if in the above problem formulations we replace "regularly implementable" by merely "implementable," then in the stabilization problem a necessary and sufficient condition for the existence of $\mathcal{K}$ is that $\mathcal{N}$ is stable (equivalently: in $\mathcal{P}_{\text{full}}$, w is detectable from c). In the pole placement problem, necessary and sufficient conditions are that $\mathcal{N} = 0$ (i.e., in $\mathcal{P}_{\text{full}}$, w is observable from c) and that $\mathcal{P}$ is not autonomous.

We close this section with some words on the case that, instead of only the behavior as $t \to \infty$ of the w-trajectories, we also want to modify the behavior of the c-trajectories in the controlled behavior. Given a full plant behavior $\mathcal{P}_{\text{full}}$, this leads to the problem of finding $\mathcal{K}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ that is stable and regularly implementable w.r.t. $\mathcal{P}_{\text{full}}$ (the full stabilization problem), and the problem of finding, for every monic polynomial $r \in \mathbb{R}[\xi]$, a $\mathcal{K}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ such that $\chi_{\mathcal{K}_{\text{full}}} = r$ and $\mathcal{K}_{\text{full}}$ is regularly implementable w.r.t. $\mathcal{P}_{\text{full}}$ (the full pole placement problem). These problems can be easily tackled by including the c-variable in the to-be-controlled variables, i.e., by introducing a new system with to-be-controlled variable $w' = (w, c)$, and control variable c. The details are left to the reader.

IV. Proofs of the Main Results

In this section, we will give proofs of Theorems 4–6. We state and prove a few lemmas first. The following lemma is used to prove Theorem 4. It gives a necessary and sufficient condition for a given $\mathcal{K}$ to be regularly implementable by full interconnection (see also [10]).

Lemma 7: Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathcal{P}_{\text{cont}}$ be its controllable part. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Then, the following two statements are equivalent:

1) $\mathcal{K}$ is regularly implementable w.r.t. $\mathcal{P}$ by full interconnection;
2) $\mathcal{K} + \mathcal{P}_{\text{cont}} = \mathcal{P}$.

Proof: It is easily proven that, for any unimodular $U \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$, $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is regularly implementable w.r.t. $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ if and only if $U(d/dt)\mathcal{K}$ is regularly implementable w.r.t. $U(d/dt)\mathcal{P}$. Also, $\mathcal{K} + \mathcal{P}_{\text{cont}} = \mathcal{P}$ if and only if $U(d/dt)\mathcal{K} + (U(d/dt)\mathcal{P})_{\text{cont}} = U(d/dt)\mathcal{P}$. Hence, without loss of generality we can assume that $\mathcal{P}$ is represented by $R(d/dt)w = 0$, with R in Smith form: $R = [D \quad 0]$ with D a nonsingular, diagonal matrix. Accordingly, partition $w = (w_1, w_2)$. This immediately yields $\mathcal{P}_{\text{cont}}$, the controllable part of $\mathcal{P}$

$$\mathcal{P}_{\text{cont}} = \left\{(0, w_2) \mid w_2 \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2})\right\}.$$

[(1)⇒(2)]: Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ be a controller that regularly implements $\mathcal{K}$. Let $C_1 w_1 + C_2 w_2 = 0$ be a minimal kernel representation of $\mathcal{C}$. Then, since the interconnection is regular, $\mathcal{K}$ is represented minimally by

$$\begin{bmatrix} D & 0 \\ C_1 & C_2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0.$$

As a result, note that C_2 has full-row rank. We need to show $\mathcal{P} \subset \mathcal{K} + \mathcal{P}_{\text{cont}}$. Let $(w_1, w_2) \in \mathcal{P}$. Let $\bar{w}_2 \in C^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2})$ be such that $C_1 w_1 + C_2 \bar{w}_2 = 0$. Obviously, $(w_1, w_2) = (w_1, \bar{w}_2) + (0, w_2 - \bar{w}_2) \in \mathcal{K} + \mathcal{P}_{\text{cont}}$. The converse inclusion, $\mathcal{K} + \mathcal{P}_{\text{cont}} \subset \mathcal{P}$, is immediate.

[(2)⇒(1)]: Let $\mathcal{K}$ satisfy $\mathcal{K} + \mathcal{P}_{\text{cont}} = \mathcal{P}$. Assume $\mathcal{K}$ is represented minimally by $K_1 w_1 + K_2 w_2 = 0$. Now, note that $(w_1, w_2) \in \mathcal{P}$ if and only if there exists an $\ell \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2})$ such that $(w_1, w_2) = (w_1, \ell) + (0, w_2 - \ell)$, with $K_1 w_1 + K_2 \ell = 0$. Using this, it is immediate that

$$\begin{bmatrix} D & 0 \\ K_1 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -K_2 \end{bmatrix}\ell$$

is a latent variable representation of $\mathcal{P}$, with latent variable ℓ. To eliminate ℓ, premultiply K_2 by a unimodular matrix $U \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, such that $UK_2 = \begin{bmatrix} 0 \\ K_{22} \end{bmatrix}$ with K_{22} full-row rank. Correspondingly, let $UK_1 = \begin{bmatrix} K_{11} \\ K_{21} \end{bmatrix}$. Then

$$\begin{bmatrix} D & 0 \\ K_{11} & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$$

is a (nonminimal) kernel representation of $\mathcal{P}$. Since $[D \;\; 0]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$ is also a kernel representation of $\mathcal{P}$, there exists a polynomial matrix K'_{11} such that $K_{11} = K'_{11}D$. Since U is unimodular, $\mathcal{K}$ is also represented minimally by

$$\begin{bmatrix} K'_{11}D & 0 \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0. \tag{5}$$

We will now prove that K'_{11} is, in fact, unimodular. Since $\mathcal{K} \subset \mathcal{P}$, there exists $[F_1 \; F_2] \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that

$$[F_1 \quad F_2]\begin{bmatrix} K'_{11}D & 0 \\ K_{21} & K_{22} \end{bmatrix} = [D \quad 0].$$

Equating, we get that $F_2 K_{22} = 0$, whence $F_2 = 0$. Consequently, $F_1 K'_{11} D = D$. Because D is nonsingular, we get $F_1 K'_{11} = I$, so K'_{11} is left invertible. Further, since (5) is a minimal representation, K'_{11} has full-row rank, and hence it is unimodular. This implies that $\mathcal{K}$ is also represented minimally by $\begin{bmatrix} D & 0 \\ K_{21} & K_{22} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$. Now define the controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ by $K_{21} w_1 + K_{22} w_2 = 0$. Since K_{22} has full-row rank, the interconnection is indeed regular. □

It follows from this lemma that if $\mathcal{P}$ is controllable, then every sub-behavior $\mathcal{K}$ of $\mathcal{P}$ is regularly implementable (by full interconnection), see also [19].

For a given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ it is important to know how to compute its input and output cardinalities from the parameters of the representation in which it is given. The following lemma solves this problem for the case of latent variable representations.

Lemma 8: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be represented by the latent variable representation

$$R\left(\frac{d}{dt}\right)w + M\left(\frac{d}{dt}\right)\ell = 0 \tag{6}$$

with latent variable ℓ. Then, we have

$$\mathrm{p}(\mathfrak{B}) = \mathrm{rank}([R \quad M]) - \mathrm{rank}(M). \tag{7}$$

Proof: Let U be a unimodular polynomial matrix such that $UM = \begin{bmatrix} M_1 \\ 0 \end{bmatrix}$, with M_1 having full-row rank. Partitioning $UR = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ compatibly, we get that $R_2 w = 0$ is a kernel representation of $\mathfrak{B}$. Thus we get that $\mathrm{rank}\left(\begin{bmatrix} R_1 & M_1 \\ R_2 & 0 \end{bmatrix}\right) = \mathrm{rank}(R_2) + \mathrm{rank}(M_1) = \mathrm{p}(\mathfrak{B}) + \mathrm{rank}(M)$. This proves our claim. □

The following lemma establishes an important link in implementation issues. We use this lemma in the proofs of the stabilization and the pole placement results.

Lemma 9: Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be given. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ be such that $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$. Then, the following statements are equivalent:

2) $\mathcal{K}$ is regularly implementable w.r.t. $\mathcal{P}_{\text{full}}$ through c;
3) $\mathcal{K}$ is regularly implementable w.r.t. $\mathcal{P}$ by full interconnection.

Proof: Let $R_1 w + R_2 c = 0$ be a minimal kernel representation of $\mathcal{P}_{\text{full}}$. After premultiplication by a suitable unimodular matrix, $\mathcal{P}_{\text{full}}$ is represented by

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & 0 \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0$$

with R_{12} having full-row rank.

((1)⇒(2)) By the elimination theorem, $R_{21} w = 0$ is a minimal kernel representation of $\mathcal{P}$. Since $\mathcal{K}$ is implementable w.r.t. $\mathcal{P}_{\text{full}}$ by regular interconnection through c, there exists a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ which regularly implements $\mathcal{K}$. Let $Cc = 0$ be a minimal kernel representation of $\mathcal{C}$. Hence, $\mathcal{K}_{\text{full}}$ is represented minimally by

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0.$$

By definition, $\mathcal{K}$ is the behavior obtained from $\mathcal{K}_{\text{full}}$ by eliminating c. Now, define $\mathcal{C}' \in \mathfrak{L}^{\mathtt{w}}$ by the following latent variable representation (here, the latent variable is c):

$$\begin{bmatrix} R_{11} & R_{12} \\ 0 & C \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0.$$

Note that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}'$. Hence, $\mathcal{K}$ is implementable w.r.t. $\mathcal{P}$ by full interconnection. We claim that this interconnection is regular, i.e., $\mathrm{p}(\mathcal{P}) + \mathrm{p}(\mathcal{C}') = \mathrm{p}(\mathcal{K})$. Indeed, by Lemma 8

$$\begin{aligned} \mathrm{p}(\mathcal{K}) &= \mathrm{rank} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & 0 \\ 0 & C \end{bmatrix} - \mathrm{rank} \begin{bmatrix} R_{12} \\ 0 \\ C \end{bmatrix} \\ &= \mathrm{rank}(R_{21}) + \mathrm{rank}\left(\begin{bmatrix} R_{11} & R_{12} \\ 0 & C \end{bmatrix}\right) \\ &\quad - \mathrm{rank}\left(\begin{bmatrix} R_{12} \\ C \end{bmatrix}\right) \\ &= \mathrm{p}(\mathcal{P}) + \mathrm{p}(\mathcal{C}'). \end{aligned}$$

Thus, we have shown that $\mathcal{K}$ is regularly implementable w.r.t. $\mathcal{P}$ by full interconnection.

[(2)⇒(1)]: Assume that $\mathcal{K}$ is regularly implementable w.r.t. $\mathcal{P}$ by full interconnection, and that $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$. We shall show that $\mathcal{K}$ is implementable w.r.t. $\mathcal{P}_{\text{full}}$ by regular interconnection through c. Again, $R_{21} w = 0$ is a minimal kernel representation of $\mathcal{P}$. Let $\mathcal{C}_0 \in \mathfrak{L}^{\mathtt{w}}$ be a controller that, w.r.t. $\mathcal{P}$, regularly implements $\mathcal{K}$ by full interconnection, and let $C_0 w = 0$ be a minimal kernel representation of $\mathcal{C}_0$. Hence

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & 0 \\ C_0 & 0 \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0 \tag{8}$$

is a minimal kernel representation of $\mathcal{K}_{\text{full}}$. Using this $\mathcal{C}_0$, we shall construct a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ which implements $\mathcal{K}$ w.r.t. $\mathcal{P}_{\text{full}}$ by regular interconnection through c.

The hidden behavior $\mathcal{N}$ is represented by the kernel representation: $\begin{bmatrix} R_{11} \\ R_{21} \end{bmatrix} w = 0$. This representation does not need to be minimal. Let $Gw = 0$ be a minimal kernel representation of $\mathcal{N}$. Then there exists a unimodular matrix $[U_1 \ U_2] \in \mathbb{R}^{\bullet \times \bullet}[\xi]$, partitioned conformably, such that

$$\begin{bmatrix} R_{11} \\ R_{21} \end{bmatrix} = [U_1 \quad U_2] \begin{bmatrix} G \\ 0 \end{bmatrix}. \tag{9}$$

Note that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}_0$, hence $\mathcal{N} \subset \mathcal{K}$ implies $\mathcal{N} \subset \mathcal{C}_0$. This means that there exists an $F \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ such that the kernel representations of $\mathcal{N}$ and $\mathcal{C}_0$ are related as in the following equation:

$$F \begin{bmatrix} R_{11} \\ R_{21} \end{bmatrix} = C_0.$$

Using (9), we get $C_0 = F U_1 G$. Also let the inverse of the unimodular matrix $[U_1 \ U_2]$ be $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ and let this be further partitioned conforming with the blocks $\begin{bmatrix} R_{11} \\ R_{21} \end{bmatrix}$ as follows:

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} R_{11} \\ R_{21} \end{bmatrix} = \begin{bmatrix} G \\ 0 \end{bmatrix}.$$

We now claim that $\mathcal{K}$ is implemented by regular interconnection with respect to $\mathcal{P}_{\text{full}}$ by the controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ defined by $FU_1V_{11}R_{12}c = 0$. Indeed, we have the following equality:

$$\begin{bmatrix} G & V_{11}R_{12} \\ 0 & V_{21}R_{12} \\ 0 & FU_1V_{11}R_{12} \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} & 0 \\ V_{21} & V_{22} & 0 \\ FU_1V_{11} & FU_1V_{12} & -I \end{bmatrix} \times \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & 0 \\ FU_1G & 0 \end{bmatrix}. \quad (10)$$

Note that in this equality, the matrix in the middle is unimodular. Further, the matrix on the right is the same as that in (8). Hence, the matrix on the left also yields a minimal kernel representation of $\mathcal{K}_{\text{full}}$. Consequently, the matrix consisting of the third block row of the matrix on the left has full-row rank. Since the first two block rows of the matrix on the left yield a minimal kernel representation of $\mathcal{P}_{\text{full}}$, while the third block row yields a minimal kernel representation of $\mathcal{C}$, the interconnection of $\mathcal{C}$ and $\mathcal{P}_{\text{full}}$ is regular. This yields the conclusion that $\mathcal{K}$ is implementable w.r.t. $\mathcal{P}_{\text{full}}$ by regular interconnection through c. □

Using the above lemmas we prove Theorem 4.

Proof of Theorem 4: (if) We assume that $\mathcal{K}$ is such that $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$ and $\mathcal{K}+\mathcal{P}_{\text{cont}} = \mathcal{P}$. Using Lemma 7, we infer that the second condition is equivalent to regular implementability of $\mathcal{K}$ by full interconnection with $\mathcal{P}$. Using Lemma 9 and the first condition, we infer that $\mathcal{K}$ is regularly implementable w.r.t. $\mathcal{P}_{\text{full}}$ through c.

(only if) Since $\mathcal{K}$ is regularly implementable w.r.t. $\mathcal{P}_{\text{full}}$ through c, it follows that $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$. Now using lemma 9, this $\mathcal{K}$ is regularly implementable w.r.t. $\mathcal{P}$ by full interconnection. Hence, using Lemma 7, we have that $\mathcal{K} + \mathcal{P}_{\text{cont}} = \mathcal{P}$. This completes the proof. □

Equipped with these lemmas, we prove the pole placement theorem, Theorem 5, as follows.

Proof of Theorem 5: ($\Rightarrow$) Assume that for every $r \in \mathbb{R}[\xi]$, there exists a regularly implementable $\mathcal{K}$ such that $\chi_{\mathcal{K}} = r$. By Lemma 9, $\mathcal{K}$ is implementable w.r.t. $\mathcal{P}$ by regular, full interconnection. Since r is arbitrary, from Proposition 2 it follows that $\mathcal{P}$ is controllable. From the controller implementability theorem, Proposition 3, we obtain $\mathcal{N} \subset \mathcal{K}$. By taking $r = 1$, we get $\mathcal{K} = 0$, and hence, $\mathcal{N} = 0$ as well.

($\Leftarrow$) Assume that $\mathcal{P}$ is controllable, and that $\mathcal{N} = 0$. Since $\mathcal{P}$ is controllable, for any monic $r \in \mathbb{R}[\xi]$, there exists a $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ that is implementable w.r.t. $\mathcal{P}$ by regular, full interconnection, with $\chi_{\mathcal{K}} = r$ (see Proposition 2). Further, $\mathcal{N} \subset \mathcal{K}$, and hence, using Lemma 9, we conclude that $\mathcal{K}$ is implementable w.r.t. $\mathcal{P}_{\text{full}}$ by regular interconnection through c. This proves the pole placement theorem. □

Finally, we give a proof of the stabilization theorem. This also makes use of Theorem 4.

Proof of Theorem 6: ($\Rightarrow$) Assume that $\mathcal{K}$ is implementable w.r.t. $\mathcal{P}_{\text{full}}$ by regular interconnection through c, and that it is stable. We need to show that $\mathcal{N}$ is stable and $\mathcal{P}$ is stabilizable. By Proposition 3, $\mathcal{N} \subset \mathcal{K}$, and hence, $\mathcal{N}$ is stable.

We now show that $\mathcal{P}$ is stabilizable. Using Lemma 9, we first deduce that $\mathcal{K}$ is implementable w.r.t. $\mathcal{P}$ by regular, full interconnection. Now using Proposition 1, we obtain that $\mathcal{P}$ must be stabilizable.

($\Leftarrow$) We need to show that if $\mathcal{N}$ is stable and $\mathcal{P}$ is stabilizable, then there exists a stable $\mathcal{K}$ which is implementable w.r.t. $\mathcal{P}_{\text{full}}$ by regular interconnection through c. Since $\mathcal{P}$ is stabilizable, by Proposition 1 there exists a stable $\mathcal{K}' \in \mathfrak{L}^{\mathtt{w}}$ which is regularly implementable by full interconnection with $\mathcal{P}$. Using Lemma 7, it follows that $\mathcal{K}' + \mathcal{P}_{\text{cont}} = \mathcal{P}$. Now define $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ by $\mathcal{K} = \mathcal{N} + \mathcal{K}'$. Because $\mathcal{K} + \mathcal{P}_{\text{cont}} = \mathcal{P}$ and $\mathcal{N} \subset \mathcal{K}$, we use theorem 4 to infer that this $\mathcal{K}$ is regularly implementable w.r.t. $\mathcal{P}_{\text{full}}$ through c. Since $\mathcal{K}'$ and $\mathcal{N}$ are stable, $\mathcal{K}$ also is stable. This concludes the proof of the theorem. □

V. An Example: The State Space Case

As an illustration, in this section we briefly explain how the main results of this paper can be applied when the full plant behavior $\mathcal{P}_{\text{full}}$ is represented by an input–state–output representation, i.e.,

$$\mathcal{P}_{\text{full}} = \left\{ (x,u,y) \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{n}+\mathtt{m}+\mathtt{p}}) \mid \frac{d}{dt}x = Ax + Bu \text{ and } y = Cx + Du \right\}. \quad (11)$$

We take $w = x$ and $c = (u,y)$. Consequently, the hidden behavior $\mathcal{N}$ is given by $\mathcal{N} = \{x \mid (d/dt)x = Ax \text{ and } Cx = 0\}$. Clearly, $\mathcal{N}$ is stable if and only if $\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}$ has full-column rank for all λ in $\mathbb{C}^+$. This coincides with detectability of the pair (C,A). Also, $\mathcal{N} = 0$ if and only if this matrix has full-column rank for all $\lambda \in \mathbb{C}$. This coincides with observability of the pair (C,A).

The manifest behavior is equal to $\mathcal{P} = \{x \mid \exists\ u \text{ such that } (d/dt)x = Ax + Bu\}$. We show that $\mathcal{P}$ is controllable (stabilizable) if and only if the pair (A,B) is controllable (stabilizable). Let $F \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ be a nonsingular matrix such that $FB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$, with B_1 having full-row rank. Let F be partitioned into $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ depending on the size of B_1. Then $\mathcal{P}$ is given by the minimal kernel representation $(F_2A - F_2(d/dt))x = 0$.

Obviously, for any $\lambda \in \mathbb{C}$ we have $\operatorname{rank}([A-\lambda I \quad B]) = \operatorname{rank}(B_1) + \operatorname{rank}(F_2A - F_2\lambda)$. Hence the pair (A,B) is controllable (stabilizable) if and only if $\mathcal{P}$ is controllable (stabilizable).

This gives us the conclusion: pole placement is possible if and only if (A,B) is controllable and (C,A) is observable. Similarly, stabilization is possible if and only if (A,B) is stabilizable, and (C,A) is detectable. These conclusions coincide with classical results about state space systems, see, for example, [11, p. 252], [15, p. 66], and the references therein.

VI. Input–Output Partition

In the classical view of control, a controller is, in general, considered to be a feedback processor that generates control inputs for the plant on the basis of measured outputs of the plant. In our set-up, controller behaviors are obtained directly from the full plant. It is important to know a priori when such controlled behavior is implementable by a *feedback processor*. Results on this have been obtained in [14], [19], and [18]. We extend these results for the problems considered in this paper.

Our first result states that if $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is regularly implementable with respect to $\mathcal{P}_{\text{full}}$ through c, and is autonomous (so, in particular, if it is stable or has prescribed characteristic polynomial), then for any controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that regularly implements $\mathcal{K}$ there exists a partition of the control variable c such that the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is, in fact, a feedback interconnection:

Theorem 10: Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ be autonomous and regularly implementable through c, and let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ be a controller that regularly implements $\mathcal{K}$. Then, possibly after permuting its components, there exists a partition of c into $c = (y, u_1, u_2)$ such that

ii) for $(w, y, u_1, u_2) \in \mathcal{P}_{\text{full}}$, (u_1, u_2) is input and (w, y) is output;
iii) for $(y, u_1, u_2) \in \mathcal{C}$, (y, u_2) is input and u_1 is output;
iv) for $(w, y, u_1, u_2) \in \mathcal{K}_{\text{full}}$, u_2 is input and (w, y, u_1) is output.

Proof: Let $R_1 w + R_2 c = 0$ be a minimal kernel representation of $\mathcal{P}_{\text{full}}$. Let U be a unimodular matrix such that $UR_1 = \begin{bmatrix} G \\ 0 \end{bmatrix}$, with G full-row rank. Accordingly partition $UR_2 = \begin{bmatrix} R_{21} \\ R_{22} \end{bmatrix}$. Then $\begin{bmatrix} G & R_{21} \\ 0 & R_{22} \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0$ is a minimal kernel representation of $\mathcal{P}_{\text{full}}$. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ regularly implement $\mathcal{K}$. Assume $Cc = 0$ is a minimal kernel representation of $\mathcal{C}$. Then a minimal kernel representation of the corresponding $\mathcal{K}_{\text{full}}$ is given by

$$\begin{bmatrix} G & R_{21} \\ 0 & R_{22} \\ 0 & C \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0. \tag{12}$$

The submatrix $\begin{bmatrix} R_{22} \\ C \end{bmatrix}$ has full-row rank, hence (possibly after a permutation of its columns, and accordingly, of the components of c), there exists a partition of this submatrix into $\begin{bmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{bmatrix}$ such that $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ is square and nonsingular. Due to the nonsingularity, again after possibly permuting the columns, we can partition

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

with P_{11} and P_{22} square and nonsingular. Such a partition exists because of Lagrange's formula which expresses the determinant as a sum of the products of the determinants of its minors of suitable dimensions.

Summarizing, partitioning $c = (y, u_1, u_2)$, we have now found the following minimal representation of $\mathcal{K}_{\text{full}}$:

$$\begin{bmatrix} G & * & * & * \\ 0 & P_{11} & P_{12} & Q_1 \\ 0 & P_{21} & P_{22} & Q_2 \end{bmatrix} \begin{bmatrix} w \\ y \\ u_1 \\ u_2 \end{bmatrix} = 0$$

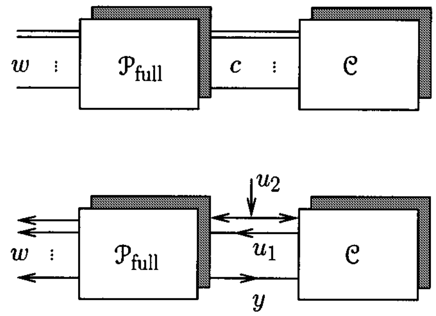

Fig. 4. Feedback interconnection of $\mathcal{P}$ and $\mathcal{C}$.

with the $*$'s denoting the corresponding blocks of R_{21}. Note that if $\mathcal{K}$ is autonomous, so is $\mathcal{N}$. Since $Gw = 0$ is a minimal kernel representation of $\mathcal{N}$, G must be square and nonsingular. From the construction of the blocks in the above representation, we can infer that in $\mathcal{K}_{\text{full}}$, u_2 is input and (w, y, u_1) is output; in $\mathcal{P}_{\text{full}}$ (u_1, u_2) is input and (w, y) is output; and in $\mathcal{C}$ (y, u_2) is input and u_1 is output. $\square$

As a special case, when $\mathcal{K}_{\text{full}}$ is autonomous, there are no inputs and the matrix in (12) is square and nonsingular. The partitioning still works, except that we interpret u_2 as having zero components. Fig. 4 depicts how the control variables are partitioned into inputs and outputs in order to implement the controller behavior in a feedback configuration.

The above theorem *assigns* an input–output partition without modifying the controller itself. Often, we are not allowed to choose an input–output partition, because we are given *a priori* that some variables are sensors, while others are actuators. Hence, necessarily, the sensors are plant outputs and should, correspondingly, be controller inputs. The actuators, then, are inputs to the plant. In the following theorem we show that if our plant $\mathcal{P}_{\text{full}}$ has an a priori given input–output structure with respect to sensors and actuators, and if $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is regularly implementable and autonomous, then $\mathcal{K}$ can be regularly implemented by a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that takes the sensors as input, and actuates part of the plant actuators. Since $\mathcal{K}_{\text{full}}$ is again not necessarily autonomous, some control variables remain free. These can be interpreted as plant actuators which are not being used for the control of the to-be-controlled variables.

Theorem 11: Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}+\mathtt{u}}$ with to-be-controlled variable w and control variable $c = (y, u)$. Assume, in $\mathcal{P}_{\text{full}}$, u is input and (w, y) is output. Then, for every regularly implementable, autonomous $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$, there exist a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that implements $\mathcal{K}$ through c, and a partition $u = (u_1, u_2)$ such that

- in $\mathcal{C}$, (y, u_2) is input and u_1 is output;
- in $\mathcal{K}_{\text{full}}$, u_2 is input and (w, y, u_1) is output.

Proof: The proof of this theorem closely mimics the proof of the previous theorem. Let $\mathcal{P}_{\text{full}}$ be represented by the minimal kernel representation

$$\begin{bmatrix} G & R_{21y} & R_{21u} \\ 0 & R_{22y} & R_{22u} \end{bmatrix} \begin{bmatrix} w \\ y \\ u \end{bmatrix} = 0$$

with G square and nonsingular (again because $\mathcal{N}$ is autonomous). Let $C_y y + C_u u = 0$ be a minimal kernel representation of a controller $\mathcal{C}' \in \mathfrak{L}^{\mathtt{c}}$ that regularly implements $\mathcal{K}$ through c. Hence, we have $\mathcal{K}_{\text{full}}$ given by the following minimal kernel representation

$$\begin{bmatrix} G & R_{21y} & R_{21u} \\ 0 & R_{22y} & R_{22u} \\ 0 & C_y & C_u \end{bmatrix} \begin{bmatrix} w \\ y \\ u \end{bmatrix} = 0.$$

The submatrix $\begin{bmatrix} R_{22y} & R_{22u} \\ C_y & C_u \end{bmatrix}$ has full-row rank. Further, R_{22y} is square and nonsingular because (w, y) is output in $\mathcal{P}_{\text{full}}$. This implies that $\begin{bmatrix} R_{22y} \\ C_y \end{bmatrix}$ has full-column rank. Hence, it is possible to partition (possibly after a permutation) u into $u = (u_1, u_2)$ such that $\mathcal{K}_{\text{full}}$ is represented as follows:

$$\begin{bmatrix} G & R_{21y} & R_{21u_1} & R_{21u_2} \\ 0 & R_{22y} & R_{22u_1} & R_{22u_2} \\ 0 & C_y & C_{u_1} & C_{u_2} \end{bmatrix} \begin{bmatrix} w \\ y \\ u_1 \\ u_2 \end{bmatrix} = 0 \tag{13}$$

with $\begin{bmatrix} R_{22y} & R_{22u_1} \\ C_y & C_{u_1} \end{bmatrix}$ square and nonsingular. This lets us choose u_2 as input to $\mathcal{K}_{\text{full}}$ and the rest of the variables as output. In order to have u_1 as output of the controller, we require that C_{u_1} be nonsingular. From $\mathcal{C}'$, we shall construct a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ to obtain the necessary nonsingularity. We have that $\begin{bmatrix} R_{22u_1} \\ C_{u_1} \end{bmatrix}$ has full column rank. Hence there exists a $T \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that $\det(C_{u_1} + TR_{22u_1}) \neq 0$. Once such a T is found, we define $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ by the kernel representation $(C_y + TR_{22y})y + (C_{u_1} + TR_{22u_1})u_1 + (C_{u_2} + TR_{22u_2})u_2 = 0$, with output u_1 and input (y, u_2). This $\mathcal{C}$ implements $\mathcal{K}_{\text{full}}$ regularly. This completes the proof. $\square$

In general, the controller transfer functions obtained in the above two theorems, are nonproper. In [19], it has been argued that many applications of control do not require this properness condition on the controller transfer functions, but that the properness condition is, nevertheless, a very important special case.

VII. Disturbances and Regular Interconnection

In Section III, we have formulated the problems of stabilization and pole placement for a given plant $\mathcal{P}_{\text{full}}$ with to-be-controlled variable w and control variable c. In most system models, an unknown external disturbance variable, d, also occurs. The stabilization problem is then to find a controller acting on c such that whenever $d(t) = 0$ $(t \geq 0)$, we have $w(t) \to 0$ $(t \to \infty)$. Typically, the disturbance d is assumed to be free, in the sense that *every* C^∞ function d is compatible with the equations of the model. As an example, think of a model of a car suspension system given by $R_1(d/dt)w + R_2(d/dt)c + R_3(d/dt)d = 0$, where d is the road profile as a function of time. In the stabilization problem, one puts $d = 0$ and solves the stabilization problem for the full plant $\mathcal{P}_{\text{full}}$ represented by $R_1(d/dt)w + R_2(d/dt)c = 0$. In doing this, one should make sure that the stabilizing controller $\mathcal{C}$: $C(d/dt)c = 0$, when connected to the actual model, *does not put restrictions on* d. The notion of regular interconnection captures this, explained as follows.

Consider the full plant behavior $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. An *extension* of $\mathcal{P}_{\text{full}}$ is a behavior $\mathcal{P}_{\text{full}}^{\text{ext}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{d}}$ (with $\mathtt{d}$ an arbitrary positive integer), with variables (w, c, d), such that

2) d is free in $\mathcal{P}_{\text{full}}^{\text{ext}}$;
3) $\mathcal{P}_{\text{full}} = \{(w, c) \mid \text{such that } (w, c, 0) \in \mathcal{P}_{\text{full}}^{\text{ext}}\}$.

Thus, $\mathcal{P}_{\text{full}}^{\text{ext}}$ being an extension of $\mathcal{P}_{\text{full}}$ formalizes that $\mathcal{P}_{\text{full}}$ has exactly those signals (w, c) that are compatible with the disturbance $d = 0$ in $\mathcal{P}_{\text{full}}^{\text{ext}}$. Of course, a given full behavior $\mathcal{P}_{\text{full}}$ has many extensions.

For a given extension $\mathcal{P}_{\text{full}}^{\text{ext}}$ and a given controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$, we define the extended controlled behavior by

$$\mathcal{K}_{\text{full}}^{\text{ext}} = \left\{(w, c, d) \mid (w, c, d) \in \mathcal{P}_{\text{full}}^{\text{ext}} \text{ and } c \in \mathcal{C}\right\}.$$

A controller $\mathcal{C}$ shall be acceptable only if the disturbance d remains free in $\mathcal{K}_{\text{full}}^{\text{ext}}$, *for any possible extension* $\mathcal{P}_{\text{full}}^{\text{ext}}$. It turns out that this is guaranteed exactly, by the regularity of the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$!

Theorem 12: The following two conditions are equivalent:

2) the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular;
3) for any extension $\mathcal{P}_{\text{full}}^{\text{ext}}$ of $\mathcal{P}_{\text{full}}$, d is free in $\mathcal{K}_{\text{full}}^{\text{ext}}$.

Proof: ((1)⇒(2)) Suppose $\mathcal{P}_{\text{full}}^{\text{ext}}$ is represented minimally by $R_1w + R_2c + R_3d = 0$. Then $\mathcal{P}_{\text{full}}$ is represented by $R_1w + R_2c = 0$.

We first claim that $[R_1\ R_2]$ also has full-row rank. Indeed, assume this matrix did not have full-row rank. Then after premultiplication by a unimodular matrix, $\mathcal{P}_{\text{full}}^{\text{ext}}$ is represented minimally by

$$\begin{bmatrix} R_1' & R_2' & R_3' \\ 0 & 0 & R_3'' \end{bmatrix} \begin{bmatrix} w \\ c \\ d \end{bmatrix} = 0, \tag{14}$$

with $R_3'' \neq 0$. Equation (14) has $R_3''d = 0$, and this means that d is not free (against our assumption). Thus $[R_1\ R_2]$ has full-row rank, as claimed.

Assume $Cc = 0$ is a minimal kernel representation of the controller $\mathcal{C}$. Since $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ are interconnected regularly, $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ also has full-row rank.

Consider the following minimal kernel representation of the extended controlled behavior $\mathcal{K}_{\text{full}}^{\text{ext}}$:

$$\begin{bmatrix} R_1 & R_2 & R_3 \\ 0 & C & 0 \end{bmatrix} \begin{bmatrix} w \\ c \\ d \end{bmatrix} = 0, \text{ or}$$
$$\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = \begin{bmatrix} -R_3 \\ 0 \end{bmatrix} d. \tag{15}$$

Because of the full-row rank condition on $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$, by the theorem on input–output partition (see [7, Th. 3.3.22]), d is free in the C^∞ sense, in $\mathcal{K}_{\text{full}}^{\text{ext}}$ also.

((2)⇒(1)) Let $R_1w + R_2c = 0$ be a minimal representation of $\mathcal{P}_{\text{full}}$. One of the $\mathcal{P}_{\text{full}}^{\text{ext}}$ that yields $\mathcal{P}_{\text{full}}$, is represented by $R_1w + R_2c + d = 0$. Let $\mathcal{C}$ be given by the minimal kernel representation $Cc = 0$. Then, we have that d is free in

$$\begin{aligned} R_1w + R_2c + d &= 0, \\ Cc &= 0. \end{aligned} \tag{16}$$

We now show that $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ has full-row rank. Suppose this matrix did not have full-row rank. Then, there exists a polynomial row vector $[p_1 \quad p_2] \neq [0 \quad 0]$, such that $[p_1 \quad p_2]\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix} = [0 \quad 0]$. Now we claim that $p_1 \neq 0$. For, otherwise, we get $p_2 C = 0$, and this means $p_2 = 0$ too, since C has full-row rank. Hence, as claimed, $p_1 \neq 0$. From (16), we get that for all (w, c, d) in $\mathcal{K}_{\text{full}}^{\text{ext}}$, we have $[p_1 \quad p_2]\begin{bmatrix} R_1 & R_2 & I \\ 0 & C & 0 \end{bmatrix}\begin{bmatrix} w \\ c \\ d \end{bmatrix} = 0$. This results in $p_1 d = 0$ which would mean that d is not free in $\mathcal{K}_{\text{full}}$. Hence $[p_1 \quad p_2] \neq [0 \quad 0]$ leads to a contradiction. This means that $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ has full-row rank and we have shown that the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is indeed regular. □

VIII. Conclusion

In this paper, we have studied control by interconnection in a behavioral framework. In particular, for linear differential systems with two types of variables, to-be-controlled variables and control variables, we have established necessary and sufficient conditions for regular implementability of a given sub-behavior of the manifest plant behavior. We have formulated the pole placement problem and the stabilization problem as problems of finding suitable, regularly implementable sub-behaviors. These formulations were completely representation-free. Using our characterization of regular implementability, we have obtained necessary and sufficient conditions for pole placement and stabilization. Again, these conditions were expressed in terms of properties of the plant behavior itself, and not as properties of a particular representation of it. As an illustration, we have studied the case that the plant is given in an input–state–output representation. We have proven that the controlled behaviors obtained in the pole placement problem and the stabilization problem can, in fact, be implemented by means of (singular) feedback. In fact, if for the plant to be controlled an actuator–sensor structure is specified in advance, then a feedback controller can be found that respects this actuator–sensor structure. Finally, we have established the connection between freedom of disturbances in the controlled system, and regularity of interconnections.

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