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Dissipative systems synthesis: A linear algebraic approach

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Abstract

In this paper we consider the problem of synthesis of dissipative systems for the case that first and higher order derivatives of the concerned variables also appear in the weighting function. The problem is formulated and solved using the behavioral approach to systems and control. We relate the problem of weighted $\mathcal{H}_\infty$ control as a special case of this synthesis problem. The synthesis problem and its solution can be systematically understood when one notices that it is similar to finding a non-negative subspace (non-negative with respect to a given constant matrix) within a finite dimensional vector space satisfying certain inclusion and dimension constraints.
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Keywords: Dissipative systems; Behaviors; Weighted *H*-infinity control; Supply rate containing dynamics

1. Introduction and preliminaries

$\mathcal{H}_\infty$-control has been a subject of intensive research for almost three decades now for two main reasons: proven performance in practical applications and the elegance of the theory. Weighted $\mathcal{H}_\infty$-control has been studied in several contexts because of its equally wide range of applications.

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In this paper we consider a more general formulation of this problem, called the dissipativity synthesis problem (DSP), and show how the behavioral approach allows us to solve, as a special case, the weighted $\mathscr{H}_\infty$-control problem in a more straightforward fashion. Formulating the DSP in the behavioral framework turns out to reveal an immediate connection of the DSP to a simpler problem concerning subspaces of a finite dimensional real vector space.

The paper is structured as follows. The notation and other basic definitions form the remainder of this section. The next section (Section 2) contains the definition of dissipativity and some concepts that are essential for the formulation of the DSP. The relation of this problem to the weighted $\mathscr{H}_\infty$ control problem is the content of Section 3, followed by Section 4, which contains the main result. We then move on to Section 5 to study a similar problem concerning subspaces within a finite dimensional vector space. As mentioned above, the DSP has a parallel problem in this context and this problem can be of interest in its own right. This is covered in Section 5. This section also contains the analogous main result of this paper and its proof is in Section 6. Section 7 contains a simpler proof of the DSP, but under some additional conditions, like strictness of the given dissipativity. A few remarks about this paper are finally summarised in Section 8.

The notation we use is standard. $\mathbb{R}$ stands for the field of real numbers and $\mathbb{R}^n$ for the *n*-dimensional real vector space. $\mathbb{R}[\xi]$ is the ring of polynomials in one indeterminate, ξ, with real coefficients. We also consider polynomial matrices in one and two indeterminates: $\mathbb{R}^{n \times m}[\xi]$ and $\mathbb{R}^{n \times m}[\zeta, \eta]$ are the sets of polynomial matrices in the corresponding indeterminates, each matrix having *n* rows and *m* columns. We use $\bullet$ when it is unnecessary to specify the number of rows, for example, $\mathbb{R}^{\bullet \times m}[\xi]$, etc. $\mathbb{Z}_+$ stands for the set of non-negative integers.

In order to keep track of the number of components in a vector v, we use the same variable v (in a different font) to indicate the dimension. Let $v \in \mathbb{R}^v$, and let $\Sigma \in \mathbb{R}^{v \times v}$ be a symmetric matrix. Then $|v|^2_\Sigma$ denotes $v^T \Sigma v$, and when $\Sigma = I$, the identity matrix, we skip the I in $|v|^2_I$, and write $|v|^2$.

2. Behaviors and dissipativity

A linear differential controllable behavior $\mathscr{B}$ is the set of those trajectories $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w)$ that are in the image of some matrix differential operator $M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$, where $M(\xi)$ is a polynomial matrix with w rows.[3] More precisely,

$$\mathscr{B} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w) \mid \text{ there exists } \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\ell) \text{ such that } w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell \right\}. \quad (1)$$

The set of such controllable behaviors with w components is denoted by $\mathfrak{L}^w_{\text{cont}}$. The description $w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell$ in Eq. (1) above is called an *image representation* of $\mathscr{B}$. For the purpose of this paper, the easiest way to define the input cardinality of a behavior $\mathscr{B} \in \mathfrak{L}^w_{\text{cont}}$ is the rank of the polynomial matrix $M(\xi)$ in an image representation. We denote the input cardinality of $\mathscr{B}$ by $\mathtt{m}(\mathscr{B})$. This integer invariant indicates the number of input variables of the system. The remaining number of components $(\mathtt{w}-\mathtt{m}(\mathscr{B}))$ is the number of outputs of the system; it is called the output cardinality, and is denoted by $\mathtt{p}(\mathscr{B})$. We refer the reader to Polderman and Willems [7] for a good exposition on the behavioral approach to systems and control.

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[3] We use this definition for this paper. The equivalence of this with the 'patchability' property of the behavior and relation to the Kalman state space definition of controllability is explained in Polderman and Willems [7].

We also deal with bilinear and quadratic forms on the elements of a behavior. In this context we deal with polynomial matrices in two variables. Induced by $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$, we have the *bilinear differential form* $L_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\mathtt{w}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\mathtt{w}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined as follows. Let $\Phi(\zeta, \eta)$ be written as a (finite) sum $\Phi(\zeta, \eta) = \Sigma_{k,\ell \in \mathbb{Z}_+} \Phi_{k\ell} \zeta^k \eta^\ell$ with $\Phi_{k\ell} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$. Define $L_\Phi(w, v)$ by

$$L_\Phi(w, v) := \sum_{k,\ell \in \mathbb{Z}_+} \left(\frac{\mathrm{d}^k}{\mathrm{d}t^k} w \right)^{\mathrm{T}} \Phi_{k\ell} \left(\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} v \right).$$

Using the bilinear differential form L_Φ induced by $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$, we define the quadratic differential form $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\mathtt{w}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ by

$$Q_\Phi(w) := L_\Phi(w, w) = \sum_{k,\ell \in \mathbb{Z}_+} \left(\frac{\mathrm{d}^k}{\mathrm{d}t^k} w \right)^{\mathrm{T}} \Phi_{k\ell} \left(\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} w \right).$$

In this paper we also consider integrals of quadratic differential forms. Given $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$, $\mathscr{B} \in \mathfrak{L}_{\mathrm{cont}}^\mathtt{w}$ is said to be *Φ-dissipative* if $\int_\mathbb{R} Q_\Phi(w)\mathrm{d}t \geqslant 0$ for all $w \in \mathscr{B} \cap \mathfrak{D}$. ($\mathscr{B} \cap \mathfrak{D}$ is the subspace of those trajectories in $\mathscr{B}$ which are compactly supported.) In this context of dissipativity, Φ is said to induce the *supply rate Q_Φ*. $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ is called symmetric if $\Phi(\zeta, \eta) = \Phi^T(\eta, \zeta)$. Notice that when considering the quadratic differential form induced by Φ, we can assume that Φ is symmetric without loss of generality. Let $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$. Define $\partial \Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ by $\partial \Phi(\xi) = \Phi(-\xi, \xi)$. It is easy to see that if Φ is symmetric then the the complex matrix $\partial \Phi(i\omega)$ is Hermitian for all $\omega \in \mathbb{R}$.

In this paper we consider the problem of synthesis of dissipative behaviors. In this context we require the relation between the input cardinality of the behavior and the *signature* of the polynomial matrix that induces the supply rate. (The signature of a non-singular symmetric constant matrix M, denoted by $\mathrm{sign}(M)$, is defined as $(\sigma_-(M), \sigma_+(M))$, where $\sigma_-(M)$ is the number of negative eigenvalues of M, and $\sigma_+(M)$ is the number of positive eigenvalues of M.) In order to make an analogous definition for the signature of a two variable polynomial matrix $\Phi(\zeta, \eta)$, we make certain assumptions on Φ, and these assumptions remain for the rest of this paper.

Assumption 1. Let $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ be symmetric. Assume that $\partial \Phi$ is non-singular and that $\partial \Phi$ admits a J-spectral factorization, i.e. $\partial \Phi(\xi) = F^{\mathrm{T}}(-\xi)J F(\xi)$ for some $F \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ and $J \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ of the form

$$J = \begin{bmatrix} I_+ & 0 \\ 0 & -I_- \end{bmatrix}.$$

Under these ssumptions, we define $(\sigma_-(\partial \Phi), \sigma_+(\partial \Phi)) = \mathrm{sign}(\partial \Phi) := \mathrm{sign}(J)$.

It is well known that J-spectral factorizability of $\partial \Phi$ is equivalent to $\partial \Phi(i\omega)$ having constant signature for almost all $\omega \in \mathbb{R}$ (see [8]). We are now ready to state the dissipativity synthesis problem (DSP).

Dissipativity synthesis problem (DSP): Assume $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ satisfies Assumption 1 and let $\mathscr{N}, \mathscr{P} \in \mathfrak{L}_{\mathrm{cont}}^\mathtt{w}$ be two controllable behaviors satisfying $\mathscr{N} \subseteq \mathscr{P}$. Find conditions under which there exists a behavior $\mathscr{K} \in \mathfrak{L}_{\mathrm{cont}}^\mathtt{w}$ satisfying

(1) $\mathscr{N} \subseteq \mathscr{K} \subseteq \mathscr{P}$,
(2) $\mathscr{K}$ is Φ-dissipative, and
(3) $\mathtt{m}(\mathscr{K}) = \sigma_+(\partial \Phi)$.

$\mathcal{K}$ is called the controlled behavior while $\mathcal{N}$ and $\mathcal{P}$ are called the hidden and the plant behaviors respectively. Each of the three conditions above have important implications in systems theory, and more on this can be found in [14,10]. In [14,10], however, the problem was solved only for the case that Φ is a constant matrix. (This means that the supply rate does not depend on *derivatives* of the concerned variables.) Another important difference between the above problem and the one studied in [10,14] is that the dissipativity there was required to hold on the *half-line*. Half-line dissipativity is a concept stronger than just dissipativity as defined above. The relation between half-line dissipativity and internal stability of the controlled behavior $\mathcal{K}$ has been brought out in [10]. In this paper we do not require $\mathcal{K}$ to be half-line dissipative and hence we do not go more into the notion of half-line dissipativity. However, the difficulty in extending the main results of this paper to the half-line dissipativity case is addressed in Remark 15 in the last section. We note here that, as far as the (weighted) $\mathcal{H}_\infty$-control problem is concerned, the above DSP implies that the closed loop behavior is not required to be internally stable, i.e. the corresponding closed loop transfer function is allowed to have poles in the open right half plane also. (Dissipativity and the input cardinality condition, together, rule out poles on the imaginary axis.)

3. Dissipativity synthesis and its applications

An example where we encounter dissipativity with respect to a supply rate induced by a non-constant Φ is in mechanical systems. Here, power can be expressed as force F times the derivative of the position x, i.e. power$= F \frac{\mathrm{d}}{\mathrm{d}t} x$. Thus, for $w = (F, x)$, the supply rate $Q_\Phi(w) = F \frac{\mathrm{d}}{\mathrm{d}t} x$ is induced by $\Phi \in \mathbb{R}^{2\times 2}[\zeta, \eta]$ defined as

$$\Phi(\zeta, \eta) := \frac{1}{2} \begin{bmatrix} 0 & \eta \\ \zeta & 0 \end{bmatrix}.$$

Hence the problem of synthesis of a passive mechanical system (i.e. the controlled mechanical system can only *absorb* net energy) can be formulated as a DSP, with the above Φ.

We now briefly describe how the weighted $\mathcal{H}_\infty$-control problem (or rather the weighted $\mathcal{L}_\infty$-control problem) can be viewed as a special case of the DSP. Consider the $\mathcal{H}_\infty$-disturbance attenuation control problem, in which the problem is to design a controller that ensures that the effect of exogenous disturbance e on the endogenous to-be-regulated output z is sufficiently small. Interpret $G(s)$ as the to-be-shaped transfer function from e to z. For several applications, it is useful to shape the frequency response of not $G(s)$ but of $W_1(s)G(s)W_2(s)$, with $W_1(s)$ and $W_2(s)$ suitably chosen weighting transfer matrices. Well-studied applications include high frequency roll-off and partial pole placement. More on this can be found in Kwakernaak [4] and Tsai et al. [11].

This problem can be rephrased as finding a controller such that the $\mathcal{L}_\infty$-norm of $W_1(s)G(s)$ $W_2(s)$ is not greater than a given positive γ. Assume for simplicity that $G(s)$, $W_1(s)$ and $W_2(s)$ are scalar transfer functions, in which case, due to commutativity, we can write just $W(s)$ instead $W_1(s)$ and $W_2(s)$. This problem can be reformulated into a DSP as follows. Define $w := (e, z)$ and choose $\Phi \in \mathbb{R}^{2\times 2}[\zeta, \eta]$ as

$$\Phi(\zeta, \eta) := \begin{bmatrix} \gamma^2 d(\zeta)d(\eta) & 0 \\ 0 & -n(\zeta)n(\eta) \end{bmatrix},$$

where $W(s) = \frac{n(s)}{d(s)}$ is a factorization of $W(s)$. Once the dissipativity synthesis problem is solved, we obtain a $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ that satisfies the requirements. Corresponding to this $\mathcal{K}$, we obtain the transfer function $G(s)$ from e to z. One can verify that requiring $\mathcal{K}$ to be Φ-dissipative is equivalent

to requiring the $\mathscr{L}_\infty$-norm of $W(s)G(s)$ to be not more than γ. (Notice that both $G(s)$ and $W(s)$ are allowed to have poles in the open right half plane also.) This illustrates how the weighted $\mathscr{L}_\infty$-control problem can be posed as a special case of the DSP.

4. Main results

Before we provide necessary and sufficient conditions for the existence of a $\mathscr{K}$ satisfying the requirements of the DSP, we state and prove the following lemma which shows how the input cardinality condition in the DSP is a maximality requirement.

Lemma 2. *Suppose* $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ *satisfies Assumption* 1. *Let* $\mathscr{K} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ *be* Φ-*dissipative. Then* $\mathtt{m}(\mathscr{K}) \leqslant \sigma_+(\partial \Phi)$.

Proof. Let $F \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ be such that $\partial \Phi(\xi) = F^T(-\xi)JF(\xi)$. We have

$$\int_{\mathbb{R}} Q_\Phi(w)\mathrm{d}t = \int_{\mathbb{R}} \left(F\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w \right)^{\mathrm{T}} JF\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w\mathrm{d}t \quad \text{for all } w \in \mathscr{K} \cap \mathfrak{D} \tag{2}$$

from which we infer that $\mathscr{K}$ is Φ-dissipative if and only if $\left(F\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \mathscr{K} \right)$ is J-dissipative. Let $M(\xi)$ be a full column rank polynomial matrix such that $\mathscr{K}$ is the image of $M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$. Define $\mathscr{K}' := F\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \mathscr{K}$. Since, F is non-singular, $\mathrm{rank}(F(\xi)M(\xi)) = \mathrm{rank}(M(\xi))$ and hence $\mathtt{m}(\mathscr{K}') = \mathtt{m}(\mathscr{K}) = \mathrm{rank}(FM)$. Corresponding to J, partition $FM =: M'$ into

$$M' = \begin{bmatrix} M'_1 \\ M'_2 \end{bmatrix}.$$

We will show that M'_1 is full column rank, and this would imply that $\mathtt{m}(\mathscr{K}) = \mathrm{rank}(M') \leqslant \sigma_+(\partial \Phi)$, since $\sigma_+(\partial \Phi)$ is the number of rows of M'_1, thus completing the proof.

Suppose M'_1 does not have full column rank, i.e. there exists a polynomial vector $x \neq 0$ such that $M'_1 x = 0$. However, since M' has full column rank, we have $M'_2 x \neq 0$. We now construct $w \in \mathscr{K}' \cap \mathfrak{D}$ by choosing any $f \in \mathbb{C}^\infty(\mathbb{R}, \mathbb{R}) \cap \mathfrak{D}$ (with $f \neq 0$), and then by defining $w = M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) x\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) f$. Since $x \neq 0$, and since f has compact support, $w \neq 0$. This w clearly contradicts J-dissipativity of $\mathscr{K}'$ and hence we proved that M'_1 has full column rank. $\quad\square$

The above lemma is an analogue of Proposition 2 of [14] which had a similar result for the case that Φ is a constant. We now need the notion of orthogonal complement of a behavior. Consider a behavior $\mathscr{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$, the orthogonal complement $\mathscr{B}^\perp$ of $\mathscr{B}$ is defined as follows

$$\mathscr{B}^\perp := \left\{ w \in \mathbb{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \Big| \int_{\mathbb{R}} w^{\mathrm{T}} v \mathrm{d}t = 0 \quad \text{for all } v \in \mathscr{B} \cap \mathfrak{D} \right\}.$$

The orthogonal complement $\mathscr{B}^\perp$ of a controllable behavior $\mathscr{B}$ turns out to be a controllable behavior too. These facts, together with other relations about orthogonality and their proofs, can be found in [2]. We also require the notion of orthogonal complement with respect to a non-singular $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$. The Φ-orthogonal complement $\mathscr{B}^{\perp_\Phi}$ of $\mathscr{B}$ is defined as $\left(\partial \Phi\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \mathscr{B} \right)^\perp$. One can show that $\mathscr{B}^{\perp_\Phi}$ is the largest controllable behavior such that

$$\int_{\mathbb{R}} L_{\Phi}(w, v)\mathrm{d}t = 0 \quad \text{for all } w \in \mathscr{B} \text{ and } v \in \mathscr{B}^{\perp_{\Phi}} \cap \mathfrak{D}.$$

Thus $\mathscr{B}^{\perp}$ is, in fact, $\mathscr{B}^{\perp_I}$, the orthogonal complement of $\mathscr{B}$ with respect to I. (I denotes identity matrix.) Using this notion of the Φ-orthogonal complement of a behavior we are ready to state the main result of this paper: necessary and sufficient conditions for the solvability of the DSP.

Theorem 3. *Suppose $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ satisfies Assumption 1 and let $\mathscr{N}, \mathscr{P} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ with $\mathscr{N} \subseteq \mathscr{P}$. There exists a behavior $\mathscr{K} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ satisfying*

(1) $\mathscr{N} \subseteq \mathscr{K} \subseteq \mathscr{P}$,
(2) *$\mathscr{K}$ is Φ-dissipative, and*
(3) $\mathtt{m}(\mathscr{K}) = \sigma_+(\partial \Phi)$.

if and only if

(1) *$\mathscr{N}$ is Φ-dissipative, and*
(2) *$\mathscr{P}^{\perp_{\Phi}}$ is $(-\Phi)$-dissipative.*

Notice the similarity in the conditions for the solvability of the DSP to those in the main result of [14, Theorem 5]. The conditions are similar except for the absence of a third condition that suitably couples the dissipativities of $\mathscr{N}$ and $\mathscr{P}^{\perp_{\Phi}}$. This coupling condition was an outcome of the half-line dissipativity requirement on $\mathscr{K}$. An important difference in our paper is that the proof does not resort to any state-space representations of the various behaviors. The proof of the above result will be done for an analogous problem that we consider in the following section. Remark 15 relates the result in [14, Theorem 5], the above theorem and the half-line dissipativity constraint.

5. An analogous DSP problem

In this section we deal with a problem concerning only finite dimensional spaces (unlike the previous sections where the behaviors are infinite dimensional subspaces of $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, except for very trivial cases). Consider a real vector space $\mathbb{V}$ of dimension, say, $\mathtt{v}$ and a symmetric non-singular matrix $\Sigma \in \mathbb{R}^{\mathtt{v} \times \mathtt{v}}$. A subspace $\mathbb{B}$ is said to be Σ-positive if $|v|^2_{\Sigma} > 0$ for all non-zero $v \in \mathbb{B}$, and subspace $\mathbb{B}$ is called Σ-neutral if $|v|^2_{\Sigma} = 0$ for all $v \in \mathbb{B}$. Similarly, we have the obvious definitions for Σ-non-negativity, Σ-non-positivity and Σ-negativity of a subspace.

We denote the dimension of a subspace $\mathbb{B}$ by $\mathtt{m}(\mathbb{B})$. The reason behind the choice of the notation for the dimension of a subspace $\mathbb{B}$, notwithstanding the obvious confusion with the input cardinality $\mathtt{m}(\mathscr{B})$ of a behavior $\mathscr{B}$, becomes clear after the following easy result whose proof is straightforward, and can be easily found. $\sigma_+(\Sigma)$ below stands for the number of positive eigenvalues of Σ and this definition of σ_+ is a special case of the definition in Assumption 1.

Proposition 4. *Assume $\Sigma \in \mathbb{R}^{\mathtt{v} \times \mathtt{v}}$ is symmetric and non-singular. Let $\mathbb{B}$ be a subspace of $\mathbb{R}^{\mathtt{v}}$. If $\mathbb{B}$ is Σ-non-negative, then $\mathtt{m}(\mathbb{B}) \leqslant \sigma_+(\Sigma)$.*

Remark 5. The similarities between the property of non-negativity of a subspace $\mathbb{B}$ of $\mathbb{V}$, dissipativity of a behavior $\mathscr{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$, and the related analogy between the input cardinality condition (Lemma 2) and the dimension condition (Proposition 4) become more obvious when we consider

the following argument. The number of elements in any basis for $\mathbb{B}$ is the dimension of $\mathbb{B}$, $\mathfrak{m}(\mathbb{B})$. Analogously, consider Eq. (1), and interpret the columns of the matrix $M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ as 'generating' the behavior $\mathscr{B}$. The number of 'independent' columns of $M(\xi)$ is the input cardinality of $\mathscr{B}$. The independence here is over the field of rational functions $\mathbb{R}(\xi)$. Keeping this analogy in mind, we define in this section the related notions like orthogonality for finite dimensional subspaces.

We now formulate the problem analogous to the DSP. We call this new problem DSP2.

(DSP2): Let $\Sigma \in \mathbb{R}^{v \times v}$ be symmetric and non-singular satisfying $\Sigma = \Sigma^{\mathrm{T}}$ and assume $\mathbb{N}$, $\mathbb{P}$ are subspaces of $\mathbb{V}$ such that $\mathbb{N} \subseteq \mathbb{P}$. Find conditions under which there exists a subspace $\mathbb{K}$ of $\mathbb{V}$ satisfying

(1) $\mathbb{N} \subseteq \mathbb{K} \subseteq \mathbb{P}$,
(2) $\mathbb{K}$ is Σ-non-negative, and
(3) $\mathfrak{m}(\mathbb{K}) = \sigma_+(\Sigma)$.

Let $\Sigma \in \mathbb{R}^{v \times v}$ be symmetric and non-singular. (This is a standing assumption throughout this paper, and is the assumption analogous to Assumption 1.) Subspaces $\mathbb{B}_1$ and $\mathbb{B}_2$ of $\mathbb{V}$ are called orthogonal with respect to Σ (or Σ-orthogonal) if $v_1^{\mathrm{T}} \Sigma v_2 = 0$ for all $(v_1, v_2) \in \mathbb{B}_1 \times \mathbb{B}_2$. Given a subspace $\mathbb{B}$, we define the Σ-orthogonal complement of $\mathbb{B}^{\perp_\Sigma}$ as follows

$$\mathbb{B}^{\perp_\Sigma} := \{v \in \mathbb{V} \mid v^{\mathrm{T}} \Sigma w = 0 \quad \text{for all } w \in \mathbb{B}\}. \tag{3}$$

Obviously, $\mathbb{B}^{\perp_\Sigma}$ is also a subspace of $\mathbb{V}$. In the languague of indefinite metric spaces, Σ in said to induce an indefinite metric. Σ-invariance properties of positive/negative subspaces have been studied in, for example, [1]. When the orthogonal complement is taken with respect to $\Sigma = I$, the identity matrix, then we skip the Σ and write simply $\mathbb{B}^\perp$ to denote the orthogonal complement. Notice that $\mathbb{B}^{\perp_\Sigma} = (\Sigma\mathbb{B})^\perp = \Sigma^{-1}\mathbb{B}^\perp$ and that $\mathbb{B}^{\perp_\Sigma}$ is Σ-non-negative (positive) if and only if $\mathbb{B}^\perp$ is Σ^{-1} non-negative (positive, respectively). Moreover, $(\mathbb{B}^{\perp_\Sigma})^{\perp_\Sigma} = \mathbb{B}$. We have the following analogous and main result.

Theorem 6. *Suppose $\Sigma \in \mathbb{R}^{v \times v}$ is symmetric and non-singular. Let $\mathbb{N}$, $\mathbb{P}$ be subspaces of a finite dimensional vector space $\mathbb{V}$ such that $\mathbb{N} \subseteq \mathbb{P}$. There exists a $\mathbb{K}$ satisfying*

(1) $\mathbb{N} \subseteq \mathbb{K} \subseteq \mathbb{P}$,
(2) $\mathbb{K}$ *is Σ-non-negative, and*
(3) $\mathfrak{m}(\mathbb{K}) = \sigma_+(\Sigma)$,

if and only if the following two conditions are satisfied:

(1) $\mathbb{N}$ *is Σ-non-negative, and*
(2) $\mathbb{P}^{\perp_\Sigma}$ *is Σ-non-positive.*

The motivation behind studying the above linear algebra problem is clear from the new problem formulation and its solution. We prove only the above result (in the following section), and the proof of Theorem 3 follows exactly along the same lines. The important difference is explained in the following remark. The rest of this section consists of some properties of $\mathbb{B}$ and its Σ-orthogonal complement, which we shall use in the proof of the main result.

There is a close similarity between the following two issues:

- A subspace $\mathbb{B} \subseteq \mathbb{V}$ and its non-negativity with respect to a symmetric non-singular $\Sigma \in \mathbb{R}^{\mathtt{v} \times \mathtt{v}}$, and
- A controllable behavior $\mathscr{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ ($\mathbb{B} \subseteq \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$) and its dissipativity with respect to a $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ that satisfies Assumption 1.

The intersection of two finite dimensional subspaces $\mathbb{B}_1$ and $\mathbb{B}_2$ of $\mathbb{V}$ is a subspace, the dimensions satisfying

$$\mathfrak{m}(\mathbb{B}_1 + \mathbb{B}_2) + \mathfrak{m}(\mathbb{B}_1 \cap \mathbb{B}_2) = \mathfrak{m}(\mathbb{B}_1) + \mathfrak{m}(\mathbb{B}_2).$$

For the case of behaviors, the intersection of two controllable behaviors, $\mathscr{B}_1$ and $\mathscr{B}_2$, is a behavior, however, it may not be controllable. With the suitable generalization of the definition of input cardinality for behaviors that are not controllable (see [12]), we have

$$\mathtt{m}(\mathscr{B}_1 + \mathscr{B}_2) + \mathtt{m}(\mathscr{B}_1 \cap \mathscr{B}_2) = \mathtt{m}(\mathscr{B}_1) + \mathtt{m}(\mathscr{B}_2). \tag{4}$$

Moreover, as far as the proof of the main result (Theorems 3/6) is concerned, when we encounter an uncontrollable behavior $\mathscr{B}$ (due to intersection of two (un)controllable behaviors, for example), we continue with the 'controllable part' of $\mathscr{B}$, which is defined as the largest controllable behavior contained in $\mathscr{B}$, and is denoted by $\mathscr{B}_{\mathrm{cont}}$. Moreover, $\mathscr{B}$ and $\mathscr{B}_{\mathrm{cont}}$ have the same input cardinality. Another important characterization of $\mathscr{B}_{\mathrm{cont}}$ is that it is the smallest behavior containing all the compactly supported trajectories in $\mathscr{B}$. This property is of use when we define the lossless part of a dissipative behavior. (See Lemma 11 below.) Detailed exposition on this together with proofs about these claims can be found in [2, Chapter 2].

With these important similarities in the two dissipativity synthesis problems, it is easier to prove the result for DSP1 by first proving Theorem 6, the case for finite dimensional subspaces. In this and the following sections, we prove explicitly the finite dimensional synthesis problem, and we skip the simpler details about the synthesis problem for the case of behaviors.

The following relation is easily verified.

$$(\mathbb{B} \cap \mathbb{B}^{\perp_\Sigma})^{\perp_\Sigma} = (\mathbb{B}^{\perp_\Sigma}) + (\mathbb{B}^{\perp_\Sigma})^{\perp_\Sigma} = \mathbb{B} + \mathbb{B}^{\perp_\Sigma}. \tag{5}$$

The equation above implies that $(\mathbb{B} \cap \mathbb{B}^{\perp_\Sigma}) \perp_\Sigma (\mathbb{B} + \mathbb{B}^{\perp_\Sigma})$. A second fact that is easily proved is $\mathfrak{m}(\mathbb{B}^{\perp_\Sigma}) + \mathfrak{m}(\mathbb{B}) = \mathtt{v}$. This is true even when $\mathbb{B}^{\perp_\Sigma}$ and $\mathbb{B}$ intersect non-trivially.

Let $\mathbb{B}$ be a Σ-non-negative subspace. Define $\mathbb{B}_L$ as the set of all elements in $\mathbb{B}$ that are Σ-neutral. The following lemma brings out a few properties about $\mathbb{B}_L$ that are essential for the proof of the main result of this paper.

Lemma 7. *Let $\Sigma \in \mathbb{R}^{\mathtt{v} \times \mathtt{v}}$ and suppose $\mathbb{B}$ is a Σ-non-negative subspace of $\mathbb{V}$. Then,*

(1) $\mathbb{B}_L$ *is the largest subspace within $\mathbb{B}$ which is Σ-neutral,*
(2) $\mathbb{B}_L = \mathbb{B} \cap \mathbb{B}^{\perp_\Sigma}$,
(3) $\mathbb{B}_L = 0 \Leftrightarrow \mathbb{B}$ *is Σ-positive,*
(4) $\Sigma\mathbb{B}_L \oplus (\mathbb{B} + \mathbb{B}^{\perp_\Sigma}) = \mathbb{V}$,
(5) $\Sigma\mathbb{B}_L \subseteq \mathbb{B}^{\perp}$, *and*
(6) $\Sigma\mathbb{B}_L$ *is Σ^{-1} neutral.*

Proof

(1) We first show that $\mathbb{B}_L$ is a subspace. Let $w_1, w_2 \in \mathbb{B}_L$. It easily follows that $\alpha w_1 \in \mathbb{B}_L$ for any $\alpha \in \mathbb{R}$. Now consider

$$|w_1 + w_2|_\Sigma^2 = |w_1|_\Sigma^2 + w_1^{\mathrm{T}} \Sigma w_2 + |w_2|_\Sigma^2.$$

We need only to show that $w_1^{\mathrm{T}} \Sigma w_2 = 0$. Suppose it is non-zero, and assume without loss of generality that it is negative (otherwise, consider $-w_1$ instead of w_1). Then, clearly, we arrive at a contradiction to Σ-non-negativity of $\mathbb{B}$ by considering the element $(w_1 + w_2)$. This thus shows that $w_1^{\mathrm{T}} \Sigma w_2 = 0$ and hence $(w_1 + w_2) \in \mathbb{B}_L$. This proves that $\mathbb{B}_L$ is a subspace. By definition, it is the largest set within $\mathbb{B}$ that is Σ-neutral. So, it follows that $\mathbb{B}_L$ is the largest subspace within $\mathbb{B}$ satisfying Σ-neutrality.

(2) It is easy to see that the subspace $\mathbb{B} \cap \mathbb{B}^{\perp_\Sigma}$ is Σ-neutral, and hence that $\mathbb{B} \cap \mathbb{B}^{\perp_\Sigma} \subseteq \mathbb{B}_L$. We now show that $\mathbb{B}_L \subseteq \mathbb{B} \cap \mathbb{B}^{\perp_\Sigma}$. Let $w_0 \in \mathbb{B}_L$. We prove that $w_0^{\mathrm{T}} \Sigma v = 0$ for all $v \in \mathbb{B}$, thus proving that $w_0 \in \mathbb{B}^{\perp_\Sigma}$ also.
Suppose $w_0^{\mathrm{T}} \Sigma v \neq 0$ for some $v \in \mathbb{B}$. Assume again without loss of generality that $w_0^{\mathrm{T}} \Sigma v > 0$. (Otherwise, choose $-w_0$ instead of w_0.) Now consider, for some $\alpha \in \mathbb{R}$,

$$(\alpha w_0 + v)^{\mathrm{T}} \Sigma (\alpha w_0 + v) = 2\alpha v^{\mathrm{T}} \Sigma w_0 + v^{\mathrm{T}} \Sigma v.$$

We know $v^{\mathrm{T}} \Sigma v \geqslant 0$, since $\mathbb{B}$ is Σ-non-negative. We now take $\alpha < \frac{-|v|_\Sigma^2}{2 v^{\mathrm{T}} \Sigma w_0}$, to obtain a contradiction to Σ-non-negativity of $\mathbb{B}$. This proves that $w_0^{\mathrm{T}} \Sigma v = 0$ for all $w_0 \in \mathbb{B}_L$ and all $v \in \mathbb{B}$, or in other words, $\mathbb{B}_L \perp_\Sigma \mathbb{B}$. This, in turn, implies that $\mathbb{B}_L \subseteq \mathbb{B}^{\perp_\Sigma}$. Thus we have shown that $\mathbb{B}_L \subseteq \mathbb{B} \cap \mathbb{B}^{\perp_\Sigma}$.

(3) If $\mathbb{B}_L = 0$, then it follows that 0 is the only element in $\mathbb{B}$ which is Σ-neutral, and hence $\mathbb{B}$ is Σ-positive. The converse is equally easy.

(4) We show that $\Sigma \mathbb{B}_L \cap (\mathbb{B} + \mathbb{B}^{\perp_\Sigma}) = 0$. Notice that, from Eq. (5), $\mathbb{B}_L \perp_\Sigma (\mathbb{B} + \mathbb{B}^{\perp_\Sigma})$ is equivalent to $\Sigma \mathbb{B}_L \perp (\mathbb{B} + \mathbb{B}^{\perp_\Sigma})$. If $w \in \Sigma \mathbb{B}_L \cap (\mathbb{B} + \mathbb{B}^{\perp_\Sigma})$, then $|w|^2 = 0$, thus implying $w = 0$. Further,

$$\dim(\mathbb{B} + \mathbb{B}^{\perp_\Sigma}) + \dim(\mathbb{B} \cap \mathbb{B}^{\perp_\Sigma}) = \dim(\mathbb{B}) + \dim(\mathbb{B}^{\perp_\Sigma}), \text{ i.e.}$$
$$\dim(\mathbb{B} + \mathbb{B}^{\perp_\Sigma}) + \dim(\Sigma \mathbb{B}_L) = \dim(\mathbb{V})$$

Since the dimensions of $\Sigma \mathbb{B}_L$ and $(\mathbb{B} + \mathbb{B}^{\perp_\Sigma})$ add up to $\dim(\mathbb{V})$, and since they intersect trivially, we conclude that $\Sigma \mathbb{B}_L \oplus (\mathbb{B} + \mathbb{B}^{\perp_\Sigma}) = \mathbb{V}$.

(5) We know that $\mathbb{B}_L \perp_\Sigma \mathbb{B}$, which means that $\Sigma \mathbb{B}_L \perp \mathbb{B}$. In other words, $\Sigma \mathbb{B}_L \subseteq \mathbb{B}^\perp$.

(6) Suppose $v \in \Sigma \mathbb{B}_L$. $|v|_{\Sigma^{-1}}^2 = |\Sigma^{-1} v|_\Sigma^2 = 0$, since $\Sigma^{-1} v \in \mathbb{B}_L$, and since we know $\mathbb{B}_L$ is Σ-neutral. $\quad\square$

In this context, notice that once we have $\mathbb{B}_L$, the Σ-neutral part of $\mathbb{B}$ (supposing $\mathbb{B}$ is Σ non-negative), there exists a (non-unique, in general) subspace $\mathbb{B}_+$ within $\mathbb{B}$ such that $\mathbb{B}_L \oplus \mathbb{B}_+ = \mathbb{B}$. Clearly, $\mathbb{B}_+$ is Σ-positive. In this paper, we often need to construct $\mathbb{B}_+$ explicitly; we define $\mathbb{B}_+ := \mathbb{B}_L^\perp \cap \mathbb{B}$.

Further, an important fact is that $\mathbb{B}_L \oplus \Sigma \mathbb{B}_L$ is Σ-indefinite, except under trivial conditions. Addressing this issue turns out to be central in proving the main result. Another issue that turns out to be crucial is as follows. Suppose $\mathbb{N}$ is Σ-non-negative. $\mathbb{N} \subseteq \mathbb{P}$ implies that $\mathbb{N}_L \subseteq \mathbb{P}$. However, in general $\Sigma \mathbb{N}_L \not\subseteq \mathbb{P}$ and this makes it necessary to decompose $\Sigma \mathbb{N}_L$ into the part contained in $\mathbb{P}$ and a complement (defined as $\mathbb{N}_1$ and $\mathbb{N}_2$, respectively; see the table of definitions of all these

subspaces). Definition of $\mathbb{K}$ can be done only after the intersection of various subspaces within $\mathbb{P}$ and their dimensions are found. We now prove Theorem 6.

6. Proof of the main results

This section contains the proof of Theorem 6, and this proof requires certain auxiliary results that we formulate and prove as and when we need them within this section.

Proof of 'only if part' of Theorem 6. Suppose there exists $\mathbb{K}$ satisfying the conditions of the DSP2. Since $\mathbb{N} \subseteq \mathbb{K}$, we have that $\mathbb{N}$ is Σ-non-negative. In order to show that $\mathbb{P}^{\perp_\Sigma}$ is Σ-non-positive, we require Lemma A-3 from [13], which is stated below as Proposition 8.

Proposition 8. *Let $\mathbb{L}$ be a linear subspace of $\mathbb{R}^n$. Consider the quadratic form $x^{\mathrm{T}}Qx$ on $\mathbb{R}^n$ with $Q = Q^{\mathrm{T}}$ non-singular. Assume that $\sigma_+(Q) = \mathfrak{m}(\mathbb{L})$. Then*

- *$a^{\mathrm{T}}Qa > 0$ for all non-zero $a \in \mathbb{L}$ if and only if $b^{\mathrm{T}}Q^{-1}b < 0$ for all non-zero $b \in \mathbb{L}^\perp$, and*
- *$a^{\mathrm{T}}Qa \geqslant 0$ for all $a \in \mathbb{L}$ if and only if $b^{\mathrm{T}}Q^{-1}b \leqslant 0$ for all $b \in \mathbb{L}^\perp$.*

As mentioned, we use the above proposition to show that $\mathbb{P}^{\perp_\Sigma}$ is Σ-non-positive. Since $\mathfrak{m}(\mathbb{K}) = \sigma_+(\Sigma)$, we have $\mathbb{K}^{\perp_\Sigma}$ is Σ-non-positive. $\mathbb{K} \subseteq \mathbb{P}$ results in $\mathbb{P}^{\perp_\Sigma} \subseteq \mathbb{K}^{\perp_\Sigma}$ and this means that $\mathbb{P}^{\perp_\Sigma}$ is also Σ-non-positive. This completes the 'only-if' part of Theorem 6.

Proof of 'if part' of Theorem 6. Let $\mathbb{N}$ and $\mathbb{P}$ be subspaces of $\mathbb{V}$ satisfying $\mathbb{N} \subseteq \mathbb{P}$. and let $\mathbb{N}$ be Σ-non-negative and let $\mathbb{P}^{\perp_\Sigma}$ be Σ-non-positive. Define $\mathbb{N}_L := \mathbb{N} \cap \mathbb{N}^{\perp_\Sigma}$ and $\mathbb{P}_L := \mathbb{P} \cap \mathbb{P}^{\perp_\Sigma}$. Then, using statement (4) of Lemma 7, we have $(\mathbb{N} + \mathbb{N}^{\perp_\Sigma}) \oplus \Sigma\mathbb{N}_L = \mathbb{V}$ and $(\mathbb{P} + \mathbb{P}^{\perp_\Sigma}) \oplus \Sigma\mathbb{P}_L = \mathbb{V}$. We now study some properties interlacing these behaviors. The following lemma is one such property.

Lemma 9. $\Sigma\mathbb{N}_L \cap \mathbb{P}_L = 0$.

Proof. $\mathbb{N} \subseteq \mathbb{P}$ is equivalent to $\mathbb{P}^{\perp_\Sigma} \subseteq \mathbb{N}^{\perp_\Sigma}$. Now, since $\Sigma\mathbb{N}_L \cap \mathbb{N}^{\perp_\Sigma} = 0$, we have $\Sigma\mathbb{N}_L \cap \mathbb{P}^{\perp_\Sigma} = 0$ too, and since $\mathbb{P}_L \subseteq \mathbb{P}^{\perp_\Sigma}$, we infer that $\Sigma\mathbb{N}_L \cap \mathbb{P}_L = 0$. $\quad\square$

We now continue with the proof of Theorem 6. Let $\mathbb{L}_{\mathbb{NP}} := \mathbb{N}_L \cap \mathbb{P}_L$. Notice that $\mathbb{P} + \mathbb{N}^{\perp_\Sigma} = (\mathbb{L}_{\mathbb{NP}})^{\perp_\Sigma}$. This can be seen as follows.

$$\mathbb{P} + \mathbb{P}^{\perp_\Sigma} = (\mathbb{P}_L)^{\perp_\Sigma}, \qquad \mathbb{N} + \mathbb{N}^{\perp_\Sigma} = (\mathbb{N}_L)^{\perp_\Sigma}, \text{ and hence,}$$
$$\mathbb{P} + \mathbb{P}^{\perp_\Sigma} + \mathbb{N} + \mathbb{N}^{\perp_\Sigma} = \mathbb{P}_L^{\perp_\Sigma} + \mathbb{N}_L^{\perp_\Sigma} = (\mathbb{P}_L \cap \mathbb{N}_L)^{\perp_\Sigma} = (\mathbb{L}_{\mathbb{NP}})^{\perp_\Sigma}.$$

Now, since $\mathbb{N} \subseteq \mathbb{P}$, we have $\mathbb{P}^{\perp_\Sigma} \subseteq \mathbb{N}^{\perp_\Sigma}$, and this simplifies the left-hand-side above to give $\mathbb{P} + \mathbb{N}^{\perp_\Sigma} = (\mathbb{L}_{\mathbb{NP}})^{\perp_\Sigma}$.

A similar argument using Lemma 7 (statement 4) results in $(\mathbb{P} + \mathbb{N}^{\perp_\Sigma}) \oplus (\Sigma\mathbb{L}_{\mathbb{NP}}) = \mathbb{V}$. The idea behind the rest of the proof is to obtain a direct sum decomposition of $\mathbb{P} + \mathbb{N}^{\perp_\Sigma}$, and in turn of $\mathbb{V}$, and then to carefully choose the right subspaces to construct $\mathbb{K}$.

Define $\mathbb{N}_{L_r}$ to be the subspace defined by $\mathbb{N}_{L_r} := \mathbb{N}_L \cap (\mathbb{L}_{\mathbb{NP}})^\perp$. We thus have $\mathbb{N}_L = \mathbb{L}_{\mathbb{NP}} \oplus \mathbb{N}_{L_r}$. (In other words, $\mathbb{N}_{L_r}$ complements $\mathbb{L}_{\mathbb{NP}}$ in $\mathbb{N}_L$.) Similarly, define $\mathbb{P}_{L_r} := \mathbb{P}_L \cap (\mathbb{L}_{\mathbb{NP}})^\perp$. We correspondingly have $\mathbb{P}_L = \mathbb{L}_{\mathbb{NP}} \oplus \mathbb{P}_{L_r}$.

Further, define $\mathbb{N}_d := \mathbb{N} \cap (\mathbb{N}_L)^\perp$, i.e. $\mathbb{N} = \mathbb{N}_L \oplus \mathbb{N}_d$. It can be seen that $\mathbb{N}_d$ is Σ-positive. Similarly, $\mathbb{P}_d := \mathbb{P}^{\perp_\Sigma} \cap (\mathbb{P}_L)^\perp$ resulting in $\mathbb{P}^{\perp_\Sigma} = \mathbb{P}_L \oplus \mathbb{P}_d$, with $\mathbb{P}_d$ being Σ-negative.

We need to decompose $\Sigma\mathbb{N}_{L_r}$ into the part within $\mathbb{P}$ and the rest of it. Let $\Sigma\mathbb{N}_{L_r} \cap \mathbb{P} =: \mathbb{N}_1$. Define $\mathbb{N}_2 := \Sigma\mathbb{N}_{L_r} \cap \mathbb{N}_1^\perp$. We thus have $\Sigma\mathbb{N}_{L_r} = \mathbb{N}_1 \oplus \mathbb{N}_2$. This also implies that $\mathbb{N}_2 \cap \mathbb{P} = 0$.

In the construction of a Σ-non-negative $\mathbb{K}$, in addition to $\mathbb{N}$, we need to take a suitable part from $\mathbb{P} \cap \mathbb{N}^{\perp_\Sigma}$. However, $\mathbb{P}_L$ and $\mathbb{N}_L$ will be contained in $\mathbb{P} \cap \mathbb{N}^{\perp_\Sigma}$, which we will take into $\mathbb{K}$ anyway. The Σ-non-negative part in $\mathbb{P} \cap \mathbb{N}^{\perp_\Sigma}$ outside $\mathbb{N}_L$ and $\mathbb{P}_L$ is what remains to be found and taken into $\mathbb{K}$. Define $\mathbb{F} := (\mathbb{P} \cap \mathbb{N}^{\perp_\Sigma}) \cap \mathbb{N}_L^\perp \cap \mathbb{P}_L^\perp$.

We restrict Σ to $\mathbb{F}$ and decompose $\mathbb{F}$ into subspaces $\mathbb{F}_+$ and $\mathbb{F}_-$ such that $\mathbb{F}_+$ is Σ-positive, and $\mathbb{F}_-$ is Σ-negative; this can be done as follows. Let $\mathtt{f}$ be the dimension of $\mathbb{F}$ and suppose $F \in \mathbb{R}^{\mathtt{v}\times\mathtt{f}}$ is a matrix with full column rank whose image is $\mathbb{F}$, i.e. $v = Fu$ is an image representation of $\mathbb{F}$. Construct $F^\mathrm{T}\Sigma F$. Notice that, by construction, $F^\mathrm{T}\Sigma F$ is symmetric and non-singular. Let $\mathtt{f}_+$ and $\mathtt{f}_-$ denote the positive and negative signatures, $\sigma_+(F^\mathrm{T}\Sigma F)$ and $\sigma_-(F^\mathrm{T}\Sigma F)$, of $F^\mathrm{T}\Sigma F$, respectively. There exists a non-singular matrix $U \in \mathbb{R}^{\mathtt{f}\times\mathtt{f}}$ partitioned suitably into

$$U = \begin{bmatrix} U_+ \\ U_- \end{bmatrix} \quad \text{such that} \quad F^\mathrm{T}\Sigma F = \begin{bmatrix} U_+^\mathrm{T} & U_-^\mathrm{T} \end{bmatrix} \begin{bmatrix} I_{\mathtt{f}_+} & 0 \\ 0 & -I_{\mathtt{f}_-} \end{bmatrix} \begin{bmatrix} U_+ \\ U_- \end{bmatrix}. \tag{6}$$

Now define $\mathbb{F}_+ \subseteq \mathbb{F}$ by $\mathbb{F}_+ := F(\mathrm{kernel}(U_-))$, i.e.

$$\mathbb{F}_+ := \{v \in \mathbb{V} \mid \text{there exists } u \in \mathbb{R}^{\mathtt{f}} \text{ such that } v = Fu \text{ and } U_- u = 0\}. \tag{7}$$

$\mathbb{F}_-$ is defined analogously as $\mathbb{F}_- := F(\mathrm{kernel}(U_+))$.

We now show that $\mathbb{F}_+$ is Σ non-negative, and that $\mathbb{F}_+$ and $\mathbb{F}_-$ are Σ-orthogonal to each other. Consider $v_1 \in \mathbb{F}_+$ and suppose $u_1 \in \mathbb{R}^{\mathtt{f}}$ is such that $v_1 = Fu_1$. We have $|v_1|_\Sigma^2 = |U_+u_1|^2 - |U_-u_1|^2 = |U_+u_1|^2$, since $U_-u_1 = 0$ by definition of $\mathbb{F}_+$. This shows that $\mathbb{F}_+$ is Σ non-negative. Using the non-singularity of U, one further shows that $U_+u \neq 0$ when $u \neq 0$, and hence $\mathbb{F}_+$ is Σ positive. Similarly, one also proves that $\mathbb{F}_-$ is Σ-negative. We now prove that $\mathbb{F}_+$ and $\mathbb{F}_-$ are Σ-orthogonal. Let $v^+ \in \mathbb{F}_+$ and $v^- \in \mathbb{F}_-$, and let u^+ and u^- be such that $v^+ = Fu^+$ and $v^- = Fu^-$. Using the definitions of $\mathbb{F}_+$ and $\mathbb{F}_-$, together with definition of U_+ and U_-, it is easily verified that $(v^+)^\mathrm{T}\Sigma v^- = 0$. This proves the Σ-orthogonality of $\mathbb{F}_+$ and $\mathbb{F}_-$.

Before we proceed further to construct a suitable $\mathbb{K}$, we recapitulate the definitions of the various subspaces so far:

Subspace definition and description	Dimension	Whether $\subseteq \mathbb{P}$ or $\cap \mathbb{P} = 0$?
$\mathbb{N}_L$: the Σ-neutral part of $\mathbb{N}$	$m_2 + m_3$	$\subseteq \mathbb{P}$
$\mathbb{P}_L$: the Σ-neutral part of $\mathbb{P}^{\perp_\Sigma}$	$m_3 + m_8$	$\subseteq \mathbb{P}$
$\mathbb{L}_{\mathbb{NP}} := \mathbb{N}_L \cap \mathbb{P}_L$	m_3	$\subseteq \mathbb{P}$
$\Sigma\mathbb{L}_{\mathbb{NP}}$	m_3	$\cap \mathbb{P} = 0$
$\mathbb{P}_{L_r} := \mathbb{P}_L \cap \mathbb{L}_{\mathbb{NP}}^\perp$ (the part of $\mathbb{P}_L$ outside $\mathbb{L}_{\mathbb{NP}}$)	m_6	$\subseteq \mathbb{P}$
$\Sigma\mathbb{P}_{L_r}$	m_6	$\cap \mathbb{P} = 0$
$\mathbb{N}_{L_r} := \mathbb{N}_L \cap \mathbb{L}_{\mathbb{NP}}^\perp$ (the part of $\mathbb{N}_L$ outside $\mathbb{L}_{\mathbb{NP}}$)	m_2	$\subseteq \mathbb{P}$
$\mathbb{N}_d := \mathbb{N} \cap \mathbb{N}_L^\perp$ (the Σ-positive part of $\mathbb{N}$)	m_1	$\subseteq \mathbb{P}$
$\Sigma\mathbb{N}_{L_r}$ (to be decomposed into $\mathbb{N}_1$ and $\mathbb{N}_2$ below)	m_2	
$\mathbb{N}_1 := \Sigma\mathbb{N}_{L_r} \cap \mathbb{P}$	m_{21}	$\subseteq \mathbb{P}$
$\mathbb{N}_2 := \Sigma\mathbb{N}_{L_r} \cap \mathbb{N}_1^\perp$ (the part in $\Sigma\mathbb{N}_{L_r}$ outside $\mathbb{N}_1$)	$m_2 - m_{21}$	$\cap \mathbb{P} = 0$
$\mathbb{P}_d := \mathbb{P}^{\perp_\Sigma} \cap \mathbb{P}_L^\perp$ (the Σ-negative part of $\mathbb{P}^{\perp_\Sigma}$)	m_7	$\cap \mathbb{P} = 0$
$\mathbb{N}_r := \mathbb{N}^{\perp_\Sigma} \cap \mathbb{N}_L^\perp$: (the part in $\mathbb{N}^{\perp_\Sigma}$ outside $\mathbb{N}_L$: used below)		
$\mathbb{F} := \mathbb{P} \cap \mathbb{N}^{\perp_\Sigma} \cap \mathbb{P}_L^\perp \cap \mathbb{N}_L^\perp$ (the part in $\mathbb{P}$ and $\mathbb{N}_r$, but not in $\mathbb{P}_L$)	$m_4 + m_5$	$\subseteq \mathbb{P}$
$\mathbb{F}_+$: the Σ-positive part of $\mathbb{F}$	m_4	$\subseteq \mathbb{P}$
$\mathbb{F}_-$: the Σ-negative part of $\mathbb{F}$	m_5	$\subseteq \mathbb{P}$

Consider the following two direct sum decompositions of $\mathbb{V}$

$$\mathbb{V} = \mathbb{N}_L \oplus \mathbb{N}_d \oplus (\mathbb{N}^{\perp_\Sigma} \cap \mathbb{N}_L^\perp) \oplus \Sigma\mathbb{N}_L, \text{ and}$$
$$\mathbb{V} = \mathbb{P}_L \oplus \mathbb{P}_d \oplus (\mathbb{P} \cap \mathbb{P}_L^\perp) \oplus \Sigma\mathbb{P}_L.$$

Using the definitions of the various subspaces and the mutual intersections among them, we have

$$\mathbb{V} = \mathbb{L}_{\mathbb{N}\mathbb{P}} \oplus \mathbb{N}_{L_r} \oplus \mathbb{N}_d \oplus \mathbb{P}_{L_r} \oplus (\mathbb{P} \cap \mathbb{N}^{\perp_\Sigma} \cap \mathbb{N}_L^\perp \cap \mathbb{P}_L^\perp) \oplus \mathbb{P}_d \oplus \Sigma\mathbb{L}_{\mathbb{N}\mathbb{P}} \oplus \Sigma\mathbb{N}_{L_r} \oplus \Sigma\mathbb{P}_{L_r}.$$

Notice that except $\Sigma\mathbb{L}_{\mathbb{N}\mathbb{P}}$ above, all the subspaces belong to $\mathbb{P} + \mathbb{N}^{\perp_\Sigma}$. Using the above definition of $\mathbb{F}$, and also writing $\Sigma\mathbb{N}_{L_r} = \mathbb{N}_1 \oplus \mathbb{N}_2$, we decompose $\mathbb{V}$ as a direct sum of the following subspaces

$$\mathbb{V} = (\mathbb{L}_{\mathbb{N}\mathbb{P}} \oplus \mathbb{N}_{L_r} \oplus \mathbb{N}_1 \oplus \mathbb{N}_d \oplus \mathbb{F} \oplus \mathbb{P}_{L_r}) \oplus (\mathbb{N}_2 \oplus \Sigma\mathbb{L}_{\mathbb{N}\mathbb{P}} \oplus \Sigma\mathbb{P}_{L_r} \oplus \mathbb{P}_d). \tag{8}$$

Subspaces in the first group belong to $\mathbb{P}$, while those in the second group intersect trivially with $\mathbb{P}$ (i.e., $\cap\mathbb{P} = 0$). Moreover, by their definitions, subspaces in the first group form a direct sum decomposition of $\mathbb{P}$. Using $\mathbb{F} = \mathbb{F}_+ \oplus \mathbb{F}_-$, we have

$$\mathbb{P} = \mathbb{L}_{\mathbb{N}\mathbb{P}} \oplus \mathbb{N}_{L_r} \oplus \mathbb{N}_1 \oplus \mathbb{N}_d \oplus (\mathbb{F}_+ \oplus \mathbb{F}_-) \oplus \mathbb{P}_{L_r}. \tag{9}$$

With this decomposition of $\mathbb{P}$, it is straightforward to decide as to which of the subspaces ought to be taken into $\mathbb{K}$; define $\mathbb{K}$ as follows

$$\mathbb{K} := \mathbb{L}_{\mathbb{N}\mathbb{P}} \oplus \mathbb{N}_{L_r} \oplus \mathbb{N}_d \oplus \mathbb{F}_+ \oplus \mathbb{P}_{L_r}. \tag{10}$$

The subspaces that have been used to form $\mathbb{K}$ are mutually Σ-orthogonal and each of them are either Σ-neutral or Σ-positive. We use the following lemma to conclude that $\mathbb{K}$ is Σ-non-negative.

Lemma 10. *Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be Σ-non-negative subspaces of $\mathbb{V}$ satisfying $\mathbb{B}_1 \perp_\Sigma \mathbb{B}_2$. Then, $\mathbb{B}_1 + \mathbb{B}_2$ is also Σ-non-negative.*

Proof. Let $w_1 \in \mathbb{B}_1$ and $w_2 \in \mathbb{B}_2$. Then

$$|w_1 + w_2|_\Sigma^2 = |w_1|_\Sigma^2 + |w_2|_\Sigma^2,$$

due to Σ-orthogonality of $\mathbb{B}_1$ and $\mathbb{B}_2$. Since each of the two terms on the right hand side are non-negative, we have that $\mathbb{B}_1 + \mathbb{B}_2$ is Σ-non-negative. $\quad\square$

We continue with the proof of showing that above $\mathbb{K}$ meets the requirements. By construction we have $\mathbb{N} \subseteq \mathbb{K} \subseteq \mathbb{P}$. The definition of $\mathbb{K}$ in Eq. (10) above shows that $\mathbb{K}$ is nothing but $\mathbb{N} + \mathbb{P}_L + \mathbb{F}_+$. However, in general $\mathbb{N}$ and $\mathbb{P}_L$ intersect non-trivially, and it remains to show that the dimension of $\mathbb{K}$ is $\sigma_+(\Sigma)$, as claimed in the theorem. Consider again the decomposition of $\mathbb{V}$ as in Eq. (8), with a reordering of the subspaces as below

$$\mathbb{V} = \mathbb{N}_d \oplus \mathbb{N}_{L_r} \oplus \mathbb{N}_1 \oplus \mathbb{N}_2 \oplus \mathbb{L}_{\mathbb{N}\mathbb{P}} \oplus \Sigma\mathbb{L}_{\mathbb{N}\mathbb{P}} \oplus \mathbb{F}_+ \oplus \mathbb{F}_- \oplus \mathbb{P}_{L_r} \oplus \Sigma\mathbb{P}_{L_r} \oplus \mathbb{P}_d.$$

Using the Σ-positivity, Σ-negativity or the Σ-neutrality of the various concerned spaces, and from the method of their construction, one can show that there exists a basis for each of these subspaces such that with respect to this basis for $\mathbb{V}$, Σ acquires[4] the following matrix representation

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[4] More precisely, if R is the matrix that takes the new basis to the old basis, then the matrix of $R^T \Sigma R$ acquires the given form.

	$\mathbb{N}_d$	$\mathbb{N}_{L_r}$	$(\mathbb{N}_1 \oplus \mathbb{N}_2)$	$\mathbb{L}_{\mathbb{NP}}$	$\Sigma\mathbb{L}_{\mathbb{NP}}$	$\mathbb{F}_+$	$\mathbb{F}_-$	$\mathbb{P}_{L_r}$	$\Sigma\mathbb{P}_{L_r}$	$\mathbb{P}_d$
dimension:	m_1	m_2	m_2	m_3	m_3	m_4	m_5	m_6	m_6	m_7
whether contained in $\mathbb{P}$?	√	√	√	×	√	×	√	√	√	×
whether to be taken in $\mathbb{K}$?	√	√	×	√	×	√	×	√	×	×

$$
\begin{array}{l}
\mathbb{N}_d \\
\mathbb{N}_{L_r} \\
\mathbb{N}_1 \oplus \mathbb{N}_2 = \Sigma\mathbb{N}_{L_r} \\
\mathbb{L}_{\mathbb{NP}} \\
\Sigma\mathbb{L}_{\mathbb{NP}} \\
\mathbb{F}_+ \\
\mathbb{F}_- \\
\mathbb{P}_{L_r} \\
\Sigma\mathbb{P}_{L_r} \\
\mathbb{P}_d
\end{array}
\begin{bmatrix}
I_1 & & * & & * & & & & * & \\
& 0_2 & I_2 & & * & & & & * & \\
* & I_2 & 0_2 & * & * & * & * & * & * & * \\
& & * & 0_3 & I_3 & & & & * & \\
* & * & * & I_3 & 0_3 & * & * & * & * & * \\
& & * & & * & I_4 & & & * & \\
& & * & & * & & -I_5 & & * & \\
& & * & & * & & & 0_6 & I_6 & \\
* & * & * & * & * & * & * & I_6 & 0_6 & * \\
& & * & & * & & & & * & -I_7
\end{bmatrix},
$$

where ∗ indicates a possibly non-zero matrix of suitable size between subspaces that might possibly not be Σ-orthogonal, and the rest of the matrix (with entries empty) is zero. From the above matrix representation we conclude that the positive and negative signatures of Σ are related to the dimensions of the various subspaces as follows

$$\sigma_+(\Sigma) = m_1 + m_2 + m_3 + m_4 + m_6 \quad \text{and} \quad \sigma_-(\Sigma) = m_2 + m_3 + m_5 + m_6 + m_7.$$

Recalling the definition of $\mathbb{K}$ (Eq. (10)), $\mathbb{K} := \mathbb{N}_d \oplus \mathbb{N}_{L_r} \oplus \mathbb{L}_{\mathbb{NP}} \oplus \mathbb{F}_+ \oplus \mathbb{P}_{L_r}$, we have thus shown that $\mathbb{K}$ indeed has dimension $\sigma_+(\Sigma)$. This ends the proof of Theorem 6. $\square$

The above proof for finite dimensional vector space $\mathbb{V}$ works in a similar fashion for the case of linear differential behaviors, which are subspaces of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. We now state and prove the corresponding results for the case of behaviors. These results will form an outline of the proof of Theorem 3, the main result. We first need some definitions about losslessness.

A controllable behavior $\mathscr{B}$ is called Φ-lossless if for all $w \in \mathscr{B} \cap \mathfrak{D}$ we have $\int_{\mathbb{R}} Q_\Phi(w)\mathrm{d}t = 0$. For a given controllable Φ-dissipative behavior $\mathscr{B}$, we define the lossless part of $\mathscr{B}$, denoted by $\mathscr{B}_L$, as the smallest controllable behavior containing all elements $w \in \mathscr{B} \cap \mathfrak{D}$ that satisfy $\int_{\mathbb{R}} Q_\Phi(w)\mathrm{d}t = 0$. (This definition is well-defined; see remark after Eq. (4) about how the compactly supported trajectories used in the integral here define the controllable behavior $\mathscr{B}_L$.) Further, we need to distinguish dissipativity from a stricter dissipativity like the way we distinguish between Σ-non-negativity and Σ-positivity of a subspace. We call $\mathscr{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$

strictly Φ-dissipative[5] if for all non-zero $w \in \mathscr{B} \cap \mathfrak{D}$ we have $\int_{\mathbb{R}} Q_{\Phi}(w)\mathrm{d}t > 0$. We also need to define $\Phi^{-1} \in \mathbb{R}^{w \times w}[\zeta, \eta]$, which is done as follows. Suppose $\partial \Phi(\xi) = F^{\mathrm{T}}(-\xi)JF(\xi)$. Then, define Φ^{-1} as $\Phi^{-1}(\zeta, \eta) := L(-\zeta)JL^{\mathrm{T}}(-\eta)$, where $L(\xi)$ is the adjugate of $F(\xi)$, i.e. $L(\xi)F(\xi) = F(\xi)L(\xi) = \det(F(\xi))I_{w \times w}$. Though $F(\xi)$, and hence Φ^{-1}, is not unique, dissipativity properties of a behavior with respect to a Φ^{-1} is unaffected by which $F(\xi)$ is chosen in the J-spectral factorization of Φ. For proofs of these claims and for further details, see [5]. The following lemma analogous to Lemma 7 are easily proven. In the following lemma and in the rest of this paper, $\partial \Phi \mathscr{B}$ stands for $\partial \Phi \left(\frac{\mathrm{d}}{\mathrm{d}t} \right) \mathscr{B}$. An important difference between behaviors and finite dimensional subspaces of $\mathbb{V}$ is that the 'trivial' intersection of two behaviors is allowed to be non-zero: we only require the *controllable part* of the intersection to be zero. For example, we will use $\mathscr{B} \oplus \mathscr{B}^{\perp} = \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{w})$ when $\mathscr{B} + \mathscr{B}^{\perp} = \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{w})$ and $(\mathscr{B} \cap \mathscr{B}^{\perp})_{\mathrm{cont}} = 0$. Also note that $\mathscr{B}_{\mathrm{cont}} = 0$ is equivalent to $\mathtt{m}(\mathscr{B}) = 0$, which is same as $\mathscr{B}$ being 'autonomous'. This aspect about the trivial intersection of two behaviors being allowed to be non-zero and autonomous is relevant for the lemma below and for the rest of this paper.

Lemma 11. *Let* $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$ *satisfy Assumption* 1 *and let* $\mathscr{B} \in \mathfrak{L}^{w}_{\mathrm{cont}}$ *be a* Φ-*dissipative behavior. Then,*

(1) $\mathscr{B}_L$ *is the largest controllable* Φ-*lossless behavior within* $\mathscr{B}$,
(2) $\mathscr{B}_L = (\mathscr{B} \cap \mathscr{B}^{\perp_{\Phi}})_{\mathrm{cont}}$,
(3) $\mathscr{B}_L = 0 \Leftrightarrow \mathscr{B}$ *is strictly* Φ-*dissipative*,
(4) $\partial \Phi \mathscr{B}_L \oplus (\mathscr{B} + \mathscr{B}^{\perp_{\Phi}}) = \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{w})$,
(5) $\partial \Phi \mathscr{B}_L \subseteq \mathscr{B}^{\perp}$, *and*
(6) $\partial \Phi \mathscr{B}_L$ *is* Φ^{-1} *lossless*.

The following lemma is useful for the proof of Theorem 3; its proof is skipped, since it is straightforward and analogous to that of Lemma 9.

Lemma 12. $\mathtt{m}(\Sigma \mathscr{N}_L \cap \mathscr{P}_L) = 0$, *i.e.* $\Sigma \mathscr{N}_L \cap \mathscr{P}_L$ *is autonomous*.

Once the decomposition of $\mathscr{N}$ and $\mathscr{P}$ into their Φ dissipative and Φ lossless parts is made, we require an analogue of Lemma 10 to be able to form the required $\mathscr{K}$ from the various sub-behaviors within $\mathscr{P}$. This lemma helps in concluding that the sum of Φ-dissipative behaviors is Φ-dissipative, provided that they are Φ-orthogonal.

Lemma 13. *Let* $\mathscr{B}_1$ *and* $\mathscr{B}_2$ *be* Φ-*dissipative behaviors satisfying* $\mathscr{B}_1 \perp_{\Phi} \mathscr{B}_2$. *Then,* $\mathscr{B}_1 + \mathscr{B}_2$ *is also* Φ-*dissipative*.

Using these main lemmas above, we go by the constructive proof for the showing the existence of a $\mathscr{K}$ that meets the requirement of the DSP exactly like in the case of finite dimensional subspaces. We do not repeat the procedure here due to the close similarity, we only give the definition of $\mathscr{K}$ (the analogue of Eq. (10)). Define $\mathscr{K}$ as

$$\mathscr{K} := \mathscr{N} + \mathscr{P}_L + \mathscr{F}_+, \tag{11}$$

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[5] This definition of strict dissipativity meets the purpose of the proof in our paper, though it is different from previous definitions as appeared in [13,9].

where $\mathscr{F}_+$ is defined as explained below. Consider $\mathscr{F} := (\mathscr{P} \cap \mathscr{N}^{\perp_\Phi} \cap (\mathscr{P}_L)^\perp \cap (\mathscr{N}_L)^\perp)_{\mathrm{cont}}$. We decompose $\mathscr{F}$ into a Φ-dissipative sub-behavior $\mathscr{F}_+$ and a $-\Phi$-dissipative sub-behavior $\mathscr{F}_-$. This decomposition is done exactly as in Eq. (6) for the case of the finite dimensional subspace $\mathbb{F}$. We use an image representation for $\mathscr{F}$, and proceed with a J-spectral factorization instead of the factorization of constant symmetric matrix as in Eq. (6). The existence of such a J-spectral factorization follows due to the assumption on Φ (Assumption 1). This factorization, elaborated in the following section, is used to define $\mathscr{F}_+$ of the equation above and is similar to the definition of $\mathbb{F}_+$ in Eq. (7).

We remark here that the sum of sub-behaviors in Eq. (11) is not a direct sum, and hence proving the claim that $\mathscr{K}$ has input cardinality $\sigma_+(\partial\Phi)$ requires the decomposition of $\mathscr{N}$ and $\mathscr{P}^{\perp_\Phi}$ into their lossless parts and the intersection of these lossless parts. This part of the proof and the count of the input cardinalities again follow the same lines as in the proof for the finite dimensional case.

7. Special case: strict dissipativity

We now deal with a special case of the DSP, in which the dissipativities of $\mathscr{N}$ and $\mathscr{P}^{\perp_\Phi}$ are both strict in the sense that $\mathrm{m}(\mathscr{N}_L) = 0$ and $\mathrm{m}(\mathscr{P}_L) = 0$. In other words, there are no compactly supported non-trivial lossless trajectories in $\mathscr{N}$ and $\mathscr{P}^{\perp_\Phi}$. This special case is included here because of the simplicity of the proof of existence and construction of $\mathscr{K}$ that meets the requirements of the DSP.

Theorem 14. *Suppose $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ satisfies Assumption 1 and let $\mathscr{N}, \mathscr{P} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ with $\mathscr{N} \subseteq \mathscr{P}$. There exists a behavior $\mathscr{K} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ satisfying*

(1) $\mathscr{N} \subseteq \mathscr{K} \subseteq \mathscr{P}$,
(2) $\mathscr{K}$ *is Φ-dissipative*, *and*
(3) $\mathrm{m}(\mathscr{K}) = \sigma_+(\partial\Phi)$.

if

(1) $\mathscr{N}$ *is Φ-dissipative*,
(2) $\mathscr{P}^{\perp_\Phi}$ *is $(-\Phi)$-dissipative*, *and*
(3) $\mathrm{m}(\mathscr{N} \cap \mathscr{N}^{\perp_\Phi}) = 0 = \mathrm{m}(\mathscr{P} \cap \mathscr{P}^{\perp_\Phi})$.

Proof. We refer to condition 3 in the above theorem as a regularity assumption. We first show that this assumption implies that $\mathscr{N} \oplus (\mathscr{N}^{\perp_\Phi} \cap \mathscr{P}) \oplus \mathscr{P}^{\perp_\Phi} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. It is straightforward that the intersection of any two is trivial (i.e., their intersection has input cardinality zero). We show that their sum equals $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Using $\mathscr{N} \subseteq \mathscr{P}$ and $\mathscr{N} + \mathscr{N}^{\perp_\Phi} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ we have[6]

$$\mathscr{P} = (\mathscr{N} + \mathscr{N}^{\perp_\Phi}) \cap \mathscr{P} = \mathscr{N} + (\mathscr{N}^{\perp_\Phi} \cap \mathscr{P}). \tag{12}$$

Further, $\mathrm{m}(\mathscr{P} \cap \mathscr{P}^{\perp_\Phi}) = 0$ implies

$$\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) = \mathscr{P} + \mathscr{P}^{\perp_\Phi} = \mathscr{N} + (\mathscr{N}^{\perp_\Phi} \cap \mathscr{P}) + \mathscr{P}^{\perp_\Phi}.$$

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[6] Eq. (12) is also called the modular distributive rule, see [15, p. 4].

Now let $\mathscr{N}$, $(\mathscr{N}^{\perp_\Phi} \cap \mathscr{P})_{\mathrm{cont}}$ and $\mathscr{P}^{\perp_\Phi}$ have observable image representations $w_1 = M_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell_1$, $w_2 = M_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell_2$ and $w_3 = M_3\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell_3$, with input cardinalities ℓ_1, ℓ_2 and ℓ_3, respectively. Let $M := [M_1 \; M_2 \; M_3]$. The regularity assumption implies that $M \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ (i.e. $\mathtt{m}(\mathscr{N}) + \mathtt{m}(\mathscr{P} \cap \mathscr{N}^{\perp_\Phi}) + (\mathscr{P}^{\perp_\Phi}) = \mathtt{w}$). Moreover,

$$\mathscr{N} + \mathscr{P} \cap \mathscr{N}^{\perp_\Phi} + \mathscr{P}^{\perp_\Phi} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) = \mathrm{Image}\left(M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$$

implies that M is non-singular. Since $\partial\Phi$ admits a J-spectral factorization, $M^{\mathrm{T}}(-\xi)\partial\Phi(\xi)M(\xi)$ also admits one, and moreover, with the same J as that of $\partial\Phi$. In particular, this means that $M^{\mathrm{T}}(-\mathrm{i}\omega)\partial\Phi(\mathrm{i}\omega)M(\mathrm{i}\omega)$ has constant signature for almost all $\omega \in \mathbb{R}$. Further, $\mathscr{N}$, $(\mathscr{N}^{\perp_\Phi} \cap \mathscr{P})_{\mathrm{cont}}$ and $\mathscr{P}^{\perp_\Phi}$ are Φ-orthogonal to each other. This results in $M_i^{\mathrm{T}}(-\xi)\partial\Phi(\xi)M_j(\xi) = 0$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$. Consider

$$\begin{aligned}&M^{\mathrm{T}}(-\xi)\partial\Phi(\xi)M(\xi) \\ &= \begin{bmatrix} M_1^{\mathrm{T}}(-\xi)\partial\Phi(\xi)M_1(\xi) & 0 & 0 \\ 0 & M_2^{\mathrm{T}}(-\xi)\partial\Phi(\xi)M_2(\xi) & 0 \\ 0 & 0 & M_3^{\mathrm{T}}(-\xi)\partial\Phi(\xi)M_3(\xi) \end{bmatrix}.\end{aligned}$$

Since $\mathscr{N}$ is Φ-dissipative and $\mathscr{P}^{\perp_\Phi}$ is $(-\Phi)$-dissipative, we have

$$M_1^{\mathrm{T}}(-\mathrm{i}\omega)\partial\Phi(\mathrm{i}\omega)M_1(\mathrm{i}\omega) \geqslant 0 \quad \text{and} \quad M_3^{\mathrm{T}}(-\mathrm{i}\omega)\partial\Phi(\mathrm{i}\omega)M_3(\mathrm{i}\omega) \leqslant 0$$

for all $\omega \in \mathbb{R}$. The regularity assumption further ensures that the above inequalities are strict for almost all $\omega \in \mathbb{R}$. This results in $M_2^{\mathrm{T}}(-\mathrm{i}\omega)\partial\Phi(\mathrm{i}\omega)M_2(\mathrm{i}\omega)$ having constant signature for almost all $\omega \in \mathbb{R}$. By Theorem 5.1 of Ran and Rodman [8], this means that we can factorize $M_2^{\mathrm{T}}(-\xi)\partial\Phi(\xi)M_2(\xi)$ into

$$M_2^{\mathrm{T}}(-\xi)\partial\Phi(\xi)M_2(\xi) = [F_+^{\mathrm{T}}(-\xi) \; F_-^{\mathrm{T}}(-\xi)] \begin{bmatrix} I_+ & 0 \\ 0 & -I_- \end{bmatrix} \begin{bmatrix} F_+(\xi) \\ F_-(\xi) \end{bmatrix}. \tag{13}$$

Define $F \in \mathbb{R}^{\ell_2 \times \ell_2}[\xi]$ by $F := \begin{bmatrix} F_+ \\ F_- \end{bmatrix}$. Non-singularity of M and $\partial\Phi$ results in non-singularity of F. By the strict Φ-dissipativity assumption on $\mathscr{N}$, we have $\mathrm{rowdim}(F_+) = \sigma_+(\partial\Phi) - \mathtt{m}(\mathscr{N})$. Similarly, by strict $(-\Phi)$-dissipativity of $\mathscr{P}^{\perp_\Phi}$, we get

$$\mathrm{rowdim}(F_-) = \sigma_-(\partial\Phi) - \mathtt{m}(\mathscr{P}^{\perp_\Phi}).$$

In order to define $\mathscr{K}$, we have to look for a Φ-dissipative sub-behavior within $\mathscr{N}^{\perp_\Phi} \cap \mathscr{P}$. From the factorization in Eq. (13), it is easy to define this sub-behavior of $\mathscr{N}^{\perp_\Phi} \cap \mathscr{P}$; define the behavior $\mathscr{F}'_+ \in \mathfrak{L}^{\mathtt{w}}$ as the set[7] of all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ such that there exists an $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\ell)$ satisfying

$$\begin{bmatrix} w_2 \\ 0 \end{bmatrix} = \begin{bmatrix} M_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ F_-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix} \ell_2.$$

(Notice that $\widehat{\mathscr{F}}_+ := M_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left(\mathrm{kernel} F_-\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\right)$, and this is similar to the definition of $\mathbb{F}_+$ in Eq. (7).) Now define $\mathscr{F}_+ := \widehat{\mathscr{F}}_{+\mathrm{cont}}$, the controllable part of $\widehat{\mathscr{F}}_+$. It follows that $\mathscr{F}_+ \subseteq (\mathscr{N}^{\perp_\Phi} \cap$

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[7] In the behavioral literature, we call this definition a *latent variable representation of $\mathscr{F}'_+$*.

$\mathscr{P})_{\text{cont}}$. Moreover, $\mathscr{F}_+$ is Φ-dissipative. This can be seen as follows. We know $w_2 \in (\mathscr{N}^{\perp_\Phi} \cap \mathscr{P})_{\text{cont}}$ implies that there exists an ℓ_2 such that $w_2 = M_2 \left(\frac{d}{dt} \right) \ell_2$. Moreover, for this w_2,

$$\int_{\mathbb{R}} Q_\Phi(w_2) dt = \int_{\mathbb{R}} \left| F_+ \left(\frac{d}{dt} \right) \ell_2 \right|^2 - \left| F_- \left(\frac{d}{dt} \right) \ell_2 \right|^2 dt.$$

Further, for $w_2 \in \mathscr{F}_+$, we have $\int_{\mathbb{R}} Q_\Phi(w_2) dt = \int_{\mathbb{R}} |F_+ \left(\frac{d}{dt} \right) \ell_2|^2 dt \geqslant 0$. This proves Φ-dissipativity of $\mathscr{F}_+$. We now compute the input cardinality of $\mathscr{F}_+$. Using Lemma 8 of [3], (which gives an explicit formula for computing the output cardinality of a behavior described by a latent variable representation) we get:

$$\begin{aligned}
\mathrm{p}(\mathscr{F}_+) &= \mathrm{rank} \left(\begin{bmatrix} I_w & -M_2 \\ 0 & -F_- \end{bmatrix} \right) - \mathrm{rank} \left(\begin{bmatrix} -M_2 \\ -F_- \end{bmatrix} \right) \\
&= (\mathrm{w} + \mathrm{rowdim}(F_-)) - \mathrm{m}(\mathscr{N}^{\perp_\Phi} \cap \mathscr{P}) \\
&= \mathrm{w} + (\sigma_-(\partial \Phi) - \mathrm{m}(\mathscr{P}^{\perp_\Phi})) - (\mathrm{w} - \mathrm{m}(\mathscr{N}) - \mathrm{m}(\mathscr{P}^{\perp_\Phi})) \\
&= \sigma_-(\partial \Phi) + \mathrm{m}(\mathscr{N}).
\end{aligned}$$

Hence, $\mathrm{m}(\mathscr{F}_+) = \mathrm{w} - \mathrm{p}(\mathscr{F}_+) = \sigma_+(\partial \Phi) - \mathrm{m}(\mathscr{N})$. We used $\mathrm{w} = \sigma_-(\partial \Phi) + \sigma_+(\partial \Phi)$ and the full row rank condition on F_- (which follows from non-singularity of F). Since $\mathrm{m}(\mathscr{F}_+ \cap \mathscr{N}) = 0$ we get $\mathrm{m}(\mathscr{N} + \mathscr{F}_+) = \sigma_+(\partial \Phi)$. We define the required controlled behavior $\mathscr{K} := \mathscr{N} + \mathscr{F}_+$, and this $\mathscr{K}$ satisfies the requirements of the DSP. (Being the sum of two controllable behaviors, $\mathscr{K}$ is controllable; see [2, Lemma 2.10.6].) This completes the proof of Theorem 14. $\quad\square$

8. Conclusion

We have formulated and proved the dissipativity synthesis problem. The advantage of posing it in the behavioral framework is that the connection with an analogous linear algebra problem is immediate, and in turn, the solution is far more tractable. It has been pointed out by a reviewer that Phillips [6] has certain results that could be used to give an alternative proof to the main theorem of this paper. The weighted $\mathscr{L}_\infty$ control turns out to be an important special case, the difference with the weighted $\mathscr{H}_\infty$ control problem being that the stability of the controlled system (in a suitable sense) is not an objective in our paper. This absence of stability requirement has made the problem in this paper simpler; compare this proof to the one in [14], where the stability requirement on the final controlled system resulted in a good amount of usage of state space methods within the proofs. For the case that the necessary dissipativities in the main result are strict, the proof simplifies significantly. The proof of this special case was explicitly written for ease of readability of the proof of the main result.

Remark 15. We conclude this paper with a remark about the extension of the DSP from dissipativity to *half-line* dissipativity. $\mathscr{B} \in \mathfrak{L}^w_{\text{cont}}$ is called half-line dissipative on $\mathbb{R}_-$ if $\int_{-\infty}^0 Q_\Phi(w) dt \geqslant 0$ for all $w \in \mathscr{B} \cap \mathfrak{D}$. Similarly, we define half-line dissipativity on $\mathbb{R}_+$ by requiring $\int_\infty^0 Q_\Phi(w) dt \geqslant 0$ for all $w \in \mathscr{B} \cap \mathfrak{D}$. For the dissipativity synthesis problem in [14], one seeks a $\mathscr{K} \in \mathfrak{L}^w_{\text{cont}}$ that satisfies the three conditions as listed in Section 2 with dissipativity on $\mathbb{R}_-$. However, $\mathscr{K}$ is sought for the important special case of Φ being a constant. While the main result extends to the case of polynomial Φ perfectly for full-line dissipativity, the half-line dissipativity case fails to extend as

explained below. The proofs of the main results of this paper crucially depends on Proposition 8 above and on Proposition 12 of [14]; these results fail to be true for the half line dissipativity case for a $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ (using our definition of signature) as seen in the following example.

Let $\Phi \in \mathbb{R}^{2 \times 2}[\zeta, \eta]$ be defined by $\Phi(\zeta, \eta) = \begin{bmatrix} 1+(\zeta+\eta)a & 0 \\ 0 & -1+(\zeta+\eta)b \end{bmatrix}$ with $a, b \in \mathbb{R}$.

Define $\mathscr{K} \in \mathfrak{L}^2_{\mathrm{cont}}$ by $\mathscr{K} := \{(w, 0) \mid w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})\}$. Obviously, $\mathscr{K}^{\perp_\Phi} \in \mathfrak{L}^2_{\mathrm{cont}}$ is given by $\mathscr{K}^{\perp_\Phi} = \{(0, w) \mid w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})\}$. Further, it is easy to see that for all $a, b \in \mathbb{R}$, $\mathscr{K}$ is Φ-dissipative and $\mathscr{K}^{\perp_\Phi}$ is $-\Phi$-dissipative. However, $\mathscr{K}$ is Φ-dissipative on $\mathbb{R}_-$ if and only if $a \geqslant 0$, while $\mathscr{K}^{\perp_\Phi}$ is $-\Phi$-dissipative on $\mathbb{R}_+$ if and only if $b \geqslant 0$. Since a and b are arbitrary, we conclude that half-line dissipativities of $\mathscr{K}$ and $\mathscr{K}^{\perp_\Phi}$ (on $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively) are not equivalent.

This example shows that while our definition of signature (of $\partial\Phi$ on the imaginary axis) gives necessary and sufficient conditions exactly like in the case of constant Φ, the *half-line* dissipativity synthesis problem cannot yield the similar result, except for the constant Φ case addressed in [14].

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