An Efficient Algorithm for Computing the $\mathcal{H}_\infty$ Norm

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***Abstract*—This technical note addresses the computation of the $\mathcal{H}_\infty$ norm by directly computing the isolated common zeros of two bivariate polynomials, unlike the iteration algorithm that is currently used to find the $\mathcal{H}_\infty$ norm. The proposed method to $\mathcal{H}_\infty$ norm calculation is compared with the existing method [by Bruinsma and Steinbuch (1990)] using numerical experiments on random transfer functions of orders upto 240; the time taken is better by 15 to 30 times, in addition to improved accuracy. The proposed method uses techniques involving structured linearization of the Bezoutian matrix constructed from two bivariate polynomials.**

***Index Terms*—Behaviors, Bezoutian matrix, dissipativity, $\mathcal{H}_\infty$-norm computation, roots of bivariate polynomials, structured linearization of polynomial matrices.**

I. Introduction

The most widely used method currently for computing the $\mathcal{H}_\infty$ norm of a rational matrix is based on the procedure described in Bruinsma and Steinbuch [5]: the implementations in both Matlab and Scilab are based on the iterative algorithm of [5], for instance. In this technical note we propose a method that calculates the norm directly. Since (generalized) eigenvalues of a constant matrix have to be computed only once in the proposed method, unlike within each iteration as in the current method, the proposed method performs much faster. (See Figs. 1 and 2 for comparison of the two methods for randomly generated transfer functions.)

The calculation of the $\mathcal{H}_\infty$ norm can be transformed into the calculation of common zeros of a bivariate polynomial and one of its partial derivatives: this key finding allows the use of classical methods, like the Bezoutian of two polynomials, to directly calculate the largest common real root, instead of iteratively converging to the $\mathcal{H}_\infty$ norm. (For a special case of the relation between the $\mathcal{H}_\infty$-norm calculation and the common root of two bivariate polynomials, see Caponetto *et al.* [6].)

The notation we use is standard: $\mathbb{R}$ and $\mathbb{C}$ respectively denote the fields of real numbers and complex numbers. $\mathbb{R}^{\mathtt{w}}$ stands for the vector space of $\mathtt{w}$-tuples over the field $\mathbb{R}$, and $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ for the space of $\mathbb{R}^{\mathtt{w}}$ valued functions from $\mathbb{R}$ having continuous derivatives of all orders. We sometimes need to include the point ∞ into the set we are considering. $\bar{\mathbb{R}}$ and $\bar{\mathbb{C}}$ correspond respectively to the sets of real and complex numbers, each including the point(s) at infinity. The sets $\mathbb{R}[s]$ and $\mathbb{R}(s)$ stand for the ring of polynomials with real coefficients and the field of rationals over this ring. Matrices with entries from these sets are defined the obvious way: $\mathbb{R}^{\mathtt{p}\times\mathtt{m}}(s)$, etc.

The technical note is organized as follows. The next section formulates the $\mathcal{H}_\infty$-norm calculation problem into that of finding the largest real common root of two bivariate polynomials. This uses basic theory of dissipative systems (see Willems & Trentelman [13]). Section III contains key results that allow the application of classical methods for computation of isolated roots of bivariate polynomials to the problem at hand. Section IV states methods to calculate the zeros of a one-variable polynomial matrix, in particular row and column compression and the method of linearization of structured polynomial matrices (see Mackey, *et al.* [9]). Section V contains comparison of our proposed method with the current method using various numerical experiments (see Figs. 1 and 2) and a brief discussion about the reasons why time is saved and accuracy is better.

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II. Problem Formulation

Let $G \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}(s)$ be a real, proper, rational transfer matrix with no poles in the closed right half complex plane. Its $\mathcal{H}_\infty$-norm is defined as the supremum over the closed right half complex plane of the maximum singular value of $G(s)$

$$\|G\|_{\mathcal{H}_\infty} = \sup_{\mathrm{Re}(s)\geqslant 0} \sigma_{\max}\left(G(s)\right). \tag{1}$$

It is well-known that, since we assumed G is stable, it is enough to consider the L_∞-norm, i.e., $\|G\|_{\mathcal{H}_\infty} = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(G(i\omega))$. Let G be given by a right-coprime factorization $M_2M_1^{-1}$ with $M_1, M_2 \in \mathbb{R}^{\bullet\times\mathtt{m}}[s]$.

Define $M \in \mathbb{R}^{(\mathtt{m}+\mathtt{p})\times\mathtt{m}}[s]$ by $M := \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$. The condition of G having no poles on the imaginary axis is equivalent to $M_1(s)$ having no zeros on $i\mathbb{R}$. Define $\Sigma_\gamma \in \mathbb{R}^{(\mathtt{m}+\mathtt{p})\times(\mathtt{m}+p)}$ by

$$\Sigma_\gamma := \begin{pmatrix} \gamma^2 I_{\mathtt{m}} & 0 \\ 0 & -I_{\mathtt{p}} \end{pmatrix}$$

where $I_{\mathtt{m}}$ and $I_{\mathtt{p}}$ are identity matrices of sizes corresponding to row-sizes of M_1 and M_2 respectively.

The following theorem yields alternative ways to describe the $\mathcal{H}_\infty$-norm. The proof, which essentially utilizes [13, Theorem 6.4], is pretty straightforward, and can be found in [2].

Theorem 1: With the above notation the following 3 statements are equivalent:

1) $\|G\|_{\mathcal{H}_\infty} \leqslant \gamma$;
2) $M^T(-i\omega)\Sigma_\gamma M(i\omega) \geqslant 0$ for all $\omega \in \mathbb{R}$;
3) For every $\omega \in \mathbb{R}$ & $v \in \mathbb{C}^{\mathtt{m}}$, $(\|M_2(i\omega)v\|_2/\|M_1(i\omega)v\|_2) \leqslant \gamma$.

Associated with the second statement of the above theorem, we define the matrix$H_\gamma(\omega) \in \mathbb{C}^{\mathtt{m}\times\mathtt{m}}[\gamma, \omega]$

$$\begin{aligned}\gamma(\omega) &:= \gamma^2 M_1^T(-i\omega)M_1(i\omega) - M_2^T(-i\omega)M_2(i\omega) \\ &= M^T(-i\omega)\Sigma_\gamma M(i\omega)\end{aligned} \tag{2}$$

and the function $\Gamma : \mathbb{R}\times\mathbb{C}_o^{\mathtt{m}} \to \mathbb{R}^+$

$$\Gamma(\omega, v) := \frac{\|M_2(i\omega)v\|_2}{\|M_1(i\omega)v\|_2} \tag{3}$$

associated with the third statement, where $\mathbb{C}_o^{\mathtt{m}} = \mathbb{C}^{\mathtt{m}} - \{0\}$ is the set of nonzero vectors in $\mathbb{C}^{\mathtt{m}}$. Since G is proper, the definition of Γ can be extended to $\bar{\mathbb{R}}\times\mathbb{C}_o^{\mathtt{m}}$ by taking the limit for $\omega \to \infty$.

According to Theorem 1, to find the $\mathcal{H}_\infty$-norm of G, it is sufficient to compute the minimal γ such that $H_\gamma(\omega) \geqslant 0$ for all $\omega \in \mathbb{R}$, or equivalently, $\Gamma(\omega, v) \leqslant \gamma$ for all $\omega \in \mathbb{R}$, and $v \in \mathbb{C}_o^{\mathtt{m}}$. Moreover, such a minimal γ is non-negative. We denote this minimum by $\bar{\gamma}$: it equals the $\mathcal{H}_\infty$-norm of the system by Theorem 1. Thus the problem of computation of the $\mathcal{H}_\infty$-norm is reformulated into a property of polynomial matrices: the further formulation to computation of isolated common roots of a pair of polynomials is done in the following section.

III. Isolated Roots of Bivariate Polynomials

As noted above, we need to find the minimal γ such that $H_\gamma(\omega)$ is non-negative for all $\omega \in \mathbb{R}$, or $\Gamma(\omega, v) \leqslant \gamma$ for all $\omega \in \mathbb{R}$, $v \in \mathbb{C}^{\mathbb{m}}_o$. Clearly $\Gamma(\omega, cv) = \Gamma(\omega, v)$ for all $c \neq 0$. So we can restrict v to $\mathbb{S}^{\mathbb{m}-1}$, the unit sphere in $\mathbb{C}^{\mathbb{m}}$. Then Γ is a continuous function on the compact set $\bar{\mathbb{R}} \times \mathbb{S}^{\mathbb{m}-1}$, and hence it attains its maximum $\bar\gamma$ at a (possibly non-unique) point $(\bar\omega, \bar v)$. In the sequel we distinguish between the following three cases.

i) $\bar\omega$ is arbitrary: For every $\omega \in \bar{\mathbb{R}}$ there exists a $v \in \mathbb{C}^{\mathbb{m}}_o$ such that $\Gamma(\omega, \bar v) = \bar\gamma$.
ii) $\bar\omega$ is unique and equals ∞.
iii) There exists an isolated $\bar\omega \in \mathbb{R}$.

Remark 2: It is important to note what the above three cases correspond to. In this remark we describe examples for each of the three cases and also provide general reasons for an example to belong to a case. Consider the all-pass filter $G(s) = (s-1)/(s+1)$, with $\|G\|_{\mathcal{H}_\infty} = 1$. For this case, $\Gamma(\omega, v)$ is independent of ω (and also of v, since G is SISO): in fact, $\Gamma(\omega, v) = 1$. Further, the polynomial $H_{\bar\gamma}(\omega)$ is identically zero for this case. The Bode magnitude plot for such a transfer function is flat and hence the maximum is attained at all values of ω. On the other hand, case (ii) captures the situation when the supremum in (1) is attained as $\omega \to \infty$: this is standard, and Lemma 3 deals with this situation. An example for case (ii) is $(s+1)/(s+2)$. The third case is the one that is more commonly encountered, and the problem is computationally most intensive for this case, since the precision of the value of the computed ω where the supremum in (1) is achieved decides the precision of the computed $\|G\|_{\mathcal{H}_\infty}$; the peak value in the Bode magnitude plot is attained at a finite and nonzero value of ω. An example about this case is $1/(s^2+s+1)$. In any of these three cases, it is possible that the determinant of the the one-variable polynomial matrix $B(\gamma)$ (defined below after (5)) is identically zero. The computational problem due to this aspect is elaborated in Remark 9 below.

The following points about $H_\gamma(\omega)$ defined in (2) and its determinant[1] $p \in \mathbb{R}[\gamma, \omega]$ play a key role in the computation of $\bar\gamma$.

1) We say that p has a zero at (a, ∞) if the leading coefficient of p with respect to ω (this leading coefficient is a polynomial in γ) has a zero at a. (Stated differently, if $p_a(\omega)$ has a lower degree than degree of ω in p.) A similar definition is used for p having a zero (∞, b).
2) The leading coefficient of p as polynomial in ω is, in fact, the characteristic polynomial of $G^T(\infty)G(\infty)$, see lemma 3 below.

The situation when the infimum, and hence the $\mathcal{L}_\infty$ norm, is attained in the limit as ω approaches ∞ is well-known to be related to the feedthrough matrix D of any state space realization of $G(s)$. See Boyd, *et al.* [4], for example. The following lemma is analogous to this situation.

Lemma 3: Let $p_\gamma(\omega) \in \mathbb{R}[\gamma, \omega]$ be the determinant of $H_\gamma(\omega)$ as defined above in (2). Suppose $a_{2n}(\gamma)$ is the leading coefficient of $p_\gamma(\omega)$. Assume D is the feedthrough term in a state space realization of $G(s)$, i.e. $D := \lim_{s\to\infty} G(s)$. Then $a_{2n}(\gamma) = \det(\gamma I_{\mathbb{m}} - D^T D)$.

Proof: Notice that $(M_1(-i\omega)^T)^{-1} H_\gamma(\omega) M_1(i\omega)^{-1} = \gamma I_{\mathbb{m}} - G^T(-i\omega)G(i\omega)$. Further, since G is proper, we can write this as follows:

$$\gamma I_{\mathbb{m}} - G^T(-i\omega)G(i\omega) = \gamma I_{\mathbb{m}} - D^T D + \frac{1}{\omega} P(\omega)$$

with $P(\omega) \in \mathbb{C}^{\mathbb{m}\times\mathbb{m}}(\omega)$ a proper complex rational matrix. This implies that

$$\det\left(\gamma I_{\mathrm{m}} - G^T(-i\omega)G(i\omega)\right) = \det(\gamma I_{\mathrm{m}} - D^T D) + \frac{1}{\omega} h_\gamma(\omega)$$

with $h_\gamma(\omega)$ such that it is polynomial in γ and proper rational in ω.

Further

$$\begin{aligned}\det&\left[\left(M_1(-i\omega)^T\right)^{-1} H_\gamma(\omega) M_1(i\omega)^{-1}\right] \\ &= \left(\det M_1(-i\omega)^T\right)^{-1} (\det H_\gamma(\omega)) (\det M_1(i\omega))^{-1}. \\ &= \frac{p_\gamma(\omega)}{\det\left(M_1(-i\omega)^T M_1(i\omega)\right)}\end{aligned}$$

This implies that

$$p_\gamma(\omega) = \left(\det\left(M_1^T(-i\omega)M_1(i\omega)\right)\det(\gamma I_{\mathbb{m}} - D^T D)\right) + \ldots$$

the dots representing terms with degree in ω strictly less than $2n$.

Since we assumed that $M_1(s)$ is such that $\det M_1(s)$ is monic, we obtain that $a_{2n}(\gamma)$ is indeed equal to $\det(\gamma I_{\mathbb{m}} - D^T D)$. This completes the proof. ∎

As a consequence of this lemma, the γ-value corresponding to Cases (i) and (ii) turn out to be roots of $a_{2n}(\gamma)$; see Remark 9 below for further discussion. Returning to the observation at the beginning of this section we know that there exists an $\bar\omega \in \bar{\mathbb{R}}$ such that $p(\bar\gamma, \bar\omega) = 0$. This leads to the formulation of the norm computation problem as a special case of determination of isolated common zeros of a pair of bivariate polynomials: the Appendix contains a definition of isolated common zeros. Write p as a polynomial in ω with coefficients from $\mathbb{R}[\gamma]$:

$$p_\gamma(\omega) = a_0(\gamma) + a_2(\gamma)\omega^2 \cdots + a_{2n-2}(\gamma)\omega^{2n-2} + a_{2n}(\gamma)\omega^{2n}. \quad (4)$$

We need to find to the largest positive γ for which p_γ has a (finite or infinite) real root $\bar\omega$.

Considering the graph of p_γ for real values of ω (i.e., $p_\gamma(\omega)$ plotted against ω), it is evident that positive definiteness of H for large real γ implies that the graph of p_γ as a function of real ω lies strictly above the (horizontal) ω-axis for all $\gamma > \bar\gamma$ and touches the horizontal axis precisely for $\gamma = \bar\gamma$ for the situation that $\bar\omega$ is finite. The following theorem makes this precise.

Theorem 4: Let $H, \bar\gamma, \bar\omega$ and p be as above. Then $\bar\omega$ is a common root of $p_{\bar\gamma}$ and its derivative, $q_{\bar\gamma} := \partial p_{\bar\gamma}/\partial\omega$.

Proof: In case (*i*) $p_{\bar\gamma} = 0$ we have nothing to prove, so suppose $p_{\bar\gamma} \neq 0$. Hence q_γ, the derivative of p_γ with respect to ω, also satisfies $q_{\bar\gamma} \neq 0$.

Next assume that we are in case (*iii*): $\bar\omega \neq \infty$ and assume that $q_{\bar\gamma}(\bar\omega) \neq 0$. Then there exists an ω close to $\bar\omega$ such that $p_{\bar\gamma}(\omega) < 0$. Since $H(\gamma, \omega)$ is positive definite for all $\gamma > \bar\gamma$, we know that $p_\gamma(\omega) > 0$, for all $\gamma > \bar\gamma$. The continuity of p with respect to γ now yields a contradiction. So $q_{\bar\gamma}(\bar\omega) = 0$.

Finally, assume that $\bar\omega = \infty$ (case (*ii*)). Then p has a root at $(\bar\gamma, \infty)$. By definition, this implies that $\bar\gamma$ is a root of the leading term of p_γ. Since the leading term of q_γ is a nonzero multiple of the leading coefficient of p_γ, $(\bar\gamma, \infty)$ is also a root of q. This completes the proof. ∎

The question arises whether $(\bar\gamma, \bar\omega)$ is an *isolated* common zero of p and q. Unfortunately, this need not be true. There is the possibility that $p_{\bar\gamma}$ is identically zero. We now show that this is the only thing that can go wrong: we restate Theorem 4 in a stronger form:

Theorem 5: Let $H, \bar\gamma, \bar\omega$ and p be as above. Then either $p_{\bar\gamma} = 0$ or $(\bar\gamma, \bar\omega)$ is an isolated common root of p and q, the partial derivative of p with respect to ω.

Proof: Let $p_{\bar\gamma} \neq 0$. Let r be the gcd of p and q. Note that r_γ has no real zeros for any $\gamma > \bar\gamma$, since r divides p. Hence if $r_{\bar\gamma}(\bar\omega) = 0$, then $r_{\bar\gamma}$ has a root of even multiplicity at $\bar\omega$. Since $p_{\bar\gamma}$ also has a root of even multiplicity at $\bar\omega$, $q_{\bar\gamma}$ has a root of odd multiplicity there, and hence the root of $r_{\bar\gamma}$ at $\bar\omega$ is of strictly lower multiplicity than the roots of p and q, showing that $(\bar\gamma, \bar\omega)$ is an isolated common root of p and q. ∎

[1]The fact that the determinant of $H_\gamma(\omega)$ is a *real* polynomial in γ and ω follows due to $H_\gamma(\omega)$ being Hermitian for each real γ and ω.

IV. Calculation of the Largest Real Common Zero of Two Bivariate Polynomials

The implementation of our algorithm uses the Bezoutian matrix corresponding to two polynomials p and $q \in \mathbb{R}[s]$, and utilizes the symmetric structure of the Bezoutian matrix to find a symmetric pencil whose generalized eigenvalues are found to eventually yield the required common zeros of $p(\gamma_i, \omega)$ and $q(\gamma_i, \omega)$. This is described below.

Let p and $q \in \mathbb{R}[s]$ be polynomials of degree $(n+1)$ and $(m+1)$ respectively, and suppose $n \geqslant m$. The Bezoutian of p and q is defined as the two variable polynomial

$$b(s,t) := \frac{p(s)q(t) - q(s)p(t)}{s-t}.$$

This two variable polynomial $b(s,t)$ can be written as

$$b(s,t) = \begin{bmatrix} 1 \\ s \\ s^2 \\ \vdots \\ s^n \end{bmatrix}^T \begin{bmatrix} b_{00} & b_{01} & \cdots & b_{0n} \\ b_{10} & b_{11} & \cdots & b_{1n} \\ \vdots & \cdots & \ddots & \vdots \\ b_{n0} & b_{n1} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{bmatrix}. \quad (5)$$

Due to the symmetry in the two variable polynomial $b(s,t)$, it turns out that the matrix in the middle (called the Bezoutian matrix, say, B) is symmetric. The matrix B loses rank if and only if $p(s)$ and $q(s)$ have a common factor; moreover, the rank loss is equal to the degree of their gcd. Of course, we are dealing with the case when $p(s)$ and $q(s)$ have coefficients that are themselves polynomials in γ. In this situation, $B \in \mathbb{R}^{(n+1)\times(n+1)}[\gamma]$ and we are interested in those values of γ_i where $B(\gamma_i)$ loses its normal rank, (see Lev-Ari, *et al.* [8]). Note that this identifies also the zeros at infinity: the values of γ where the leading coefficient of p and q both vanish. In that case both the last column and the last row of B vanish, evidently lowering the normal rank.

The set of complex numbers where $B(\gamma)$ loses rank can be found using the following lemma. Its proof uses standard techniques relating to the Smith form of a polynomial matrix (see Polderman & Willems [11]); due to its straightforward nature, we omit the proof. The set of common zeros of polynomials p and q is denoted as $Z(p,q)$ (see Appendix below).

Lemma 6: Let $p, q \in \mathbb{R}[\gamma][\omega]$, and let U and V be unimodular[2] matrices with entries in $\mathbb{R}[\gamma]$ such that

$$B = U \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} V$$

where the 0's are blocks of zero matrices of appropriate sizes, and S is a square nonsingular matrix with entries from $\mathbb{R}[\gamma]$. Then, $\det S(\gamma) = 0$ if and only if there exists an ω such that $(\gamma, \omega) \in Z(p,q)$.

Using the above lemma, finding isolated common roots of p_γ and q_γ is easy: $\det(S)$ is a nonzero polynomial in γ, and its zeros $\gamma_1, \ldots, \gamma_n$ determine the values for which p_γ and q_γ have an isolated common root.

Though S is closely related to the Smith form of S, due to the numerical difficulties involved in Smith form computations, we use column and/or row compression routines to find U, V and S. This can be done, for instance, by `colred.m` in the polynomial toolbox of Matlab or by the algorithm described in Praagman, Trentelman and Zavala Yoé [12].

[2]A square polynomial matrix is called unimodular if its determinant is a nonzero constant.

We describe the algorithm for the special case here: $B(\gamma)$ is a symmetric real polynomial matrix.

Algorithm 7: Calculating isolated common roots:

Input: Symmetric polynomial matrix $B(s)$,

Output: zeros of $B(s)$

1) Form the Bezoutian matrix $B \in \mathbb{R}^{n\times n}[\gamma]$ of p and q as elements of $\mathbb{R}[\gamma][\omega]$. Here n is the maximum of the degrees of p and q in ω.
2) Find unimodular U such that $B^T U^T = (T\ 0)$ is column compressed.
3) Obtain the product $T^T U^T$ and partition it into $(S\ 0)$, with S square, nonsingular and a symmetric polynomial matrix.
4) Calculate $d = \det(S) \in \mathbb{R}[\gamma]$
5) Find the zeros $\{\gamma_1, \ldots, \gamma_n\}$ of d.
6) For each i, find the common zeros of $p(\gamma_i, \omega)$ and $q(\gamma_i, \omega)$.

Of course, if generalized eigenvalues could be found using stable methods for irregular[3] matrix pencils (see [7]), then steps 2 and 3 would not be essential: these steps are relevant when the polynomial matrix B does not have full normal rank. We return to this point in Remark 9 below. Steps 4 and 5 can be replaced by a far more stable algorithm developed in [9]: we describe this briefly now. Noting that B, and hence S, is a symmetric polynomial matrix, their "linearization" can be achieved by symmetric constant matrices, thus resulting in improvement of the algorithm to compute generalized eigenvalues. We utilize[4] the algorithm for linearization of symmetric polynomial matrices, as developed and elaborated in Mackey, *et al.* [9]. Using this method, one obtains symmetric matrices E and A such that the set of zeros of $E\gamma - A$ is same as that of $B(\gamma)$. The Scilab codes achieving this are available at [1]. See Bora & Mehrmann [3] for a related treatment of the Hamiltonian matrix structure in the context of the current $\mathcal{H}_\infty$ norm computation algorithm.

We conclude this section by proposing an algorithm to calculate the $\mathcal{H}_\infty$-norm of a real rational, proper, and stable transfer function G, and then comparing the numerical efficiency of this algorithm with the current/existing method.

Algorithm 8: Calculation of the $\mathcal{H}_\infty$ norm:

Input: Proper real rational transfer function $G(s)$ with no poles on $i\mathbb{R}$,

Output: $\|G(s)\|_{\mathcal{L}_\infty}$

1) Obtain a right coprime factorization $G(s) = M_2(s)M_1^{-1}(s)$, with $M_1, M_2 \in \mathbb{R}^{\bullet\times\bullet}[s]$.
2) $p(\gamma, \omega) := \det(\gamma M_1^T(-i\omega)M_1(i\omega) - M_2^T(-i\omega)M_2(i\omega))$
3) $q(\gamma, \omega) := \partial p / \partial \omega$
4) Use p and q to construct the Bezoutian matrix B of (5).
5) Use Algorithm 7 to find the real $\gamma_1 > \ldots > \gamma_k > 0$ where B loses rank.
6) For $i = 1$ to k, check whether p_{γ_i} has a real root (including infinity) $\bar{\omega}$. If so, stop: set $\bar{\gamma} := \gamma_i$.
7) $\|G\|_{\mathcal{L}_\infty} := \sqrt{\bar{\gamma}}$.

As described in Section II, when G has no poles in the right half plane, then the $\mathcal{L}_\infty$ norm is, in fact, the $\mathcal{H}_\infty$ norm of G.

Remark 9: Our proposed method has a slight resemblance to a method proposed in Caponetto, *et al.* [6]. Differences are that the

[3]Irregular matrix pencils are pencils $sE - A$ satisfying $\det(sE - A) \equiv 0$.

[4]We thank Dr. Bibhas Adhikari for detailed discussions regarding structured linearization.

implementation of our method is through more recent methods of structured linearization developed in Mackey, *et al.* [9], but more importantly that our method is able to address the situations where p and q are not coprime, for instance[5]

$$G = \begin{bmatrix} \frac{1}{s^2+s+1} & 0 \\ 0 & \frac{1}{s^2+s+1} \end{bmatrix}. \tag{6}$$

Of course, for the sake of illustration, we have provided a very simple example, but this situation can happen with any of the three cases listed at the beginning of Section III. In situations like this, one utilizes row/column compression routines on polynomial matrix $B(\gamma)$ to obtain its finite zeros, or relies on methods for generalized eigenvalue computation of *irregular* matrix pencils, as pointed out after Algorithm 7. As is well known, both these problems are ill-conditioned. The numerical problems associated with polynomial matrix computations are avoided by introducing numerical problems regarding generalized eigenvalue computations. At this point it is hard to state which method is superior.

Another closely related work is that of Kanno & Smith [10], where Sturm chain tests are used for localizing to guaranteed accuracy. On the other hand, our method focuses on floating point arithmetic methods and compares very favorably with the current method (of [5]) in Matlab/Scilab: this comparison is elaborated in the following section.

V. Comparison of the Two Methods: Time Taken and Accuracy

The proposed algorithm has been implemented in Scilab for SISO and MIMO transfer function matrices, and the Scilab codes, together with a readme.txt file, are all available at http://www.ee.iitb.ac.in/~belur/hnorm/ The algorithm has been tested for randomly generated SISO transfer functions of varying orders and the time required to find the $\mathcal{H}_\infty$-norm using the proposed method has been compared with the existing method. Figs. 1 and 2 show plots whose vertical axis is the average computation time for 5 random examples of each order; the horizontal axis is the transfer function order. The figures for the strictly proper and biproper transfer functions are similar. The time taken in the proposed method is far better, and the improvement in time becomes more significant with increasing system orders: the improvement is about 10 to 25 times. The main reason for the time reduction is described below.

While a certain amount of iteration is involved in the current and proposed methods, it is noteworthy to distinguish between two kinds of iterations that are encountered:

- *An* '**inner**' *iteration to find eigenvalues for a given matrix*: present in both algorithms: current and the proposed.
- *An* '**outer**' *iteration over the γ value*: present in only the current method, and shown inessential by our proposed algorithm.

Thus the proposed algorithm has just the iteration that is inevitable in eigenvalue/root computation. On the other hand, the current method has nested iterations: an inner one inside an outer one. This is the primary reason for the improvement in the time taken by the proposed method.

The drastic reduction in time by the proposed method is, in fact, inspite of a significant improvement in the accuracy. The current method has an iteration to check and stop when the relative error between consecutive iterates is smaller than the user-defined relative error tolerance;

[5]In fact, any transfer matrix UGV, with G as in (6) and orthogonal matrices U & V, will result in the situation $\det B(\gamma) = 0$. In fact, for a given G, a class of *rational* matrices U, V can also be characterized that result in $\det B(\gamma) = 0$ for the transfer matrices UGV.

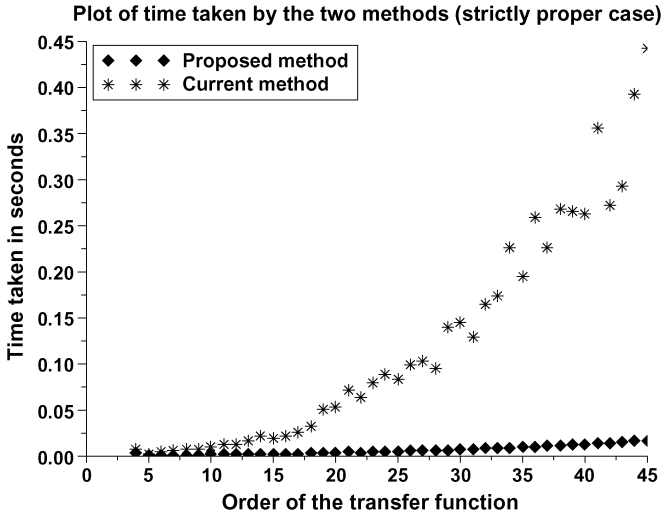

Fig. 1. Plot of time taken by the two methods for varying orders.

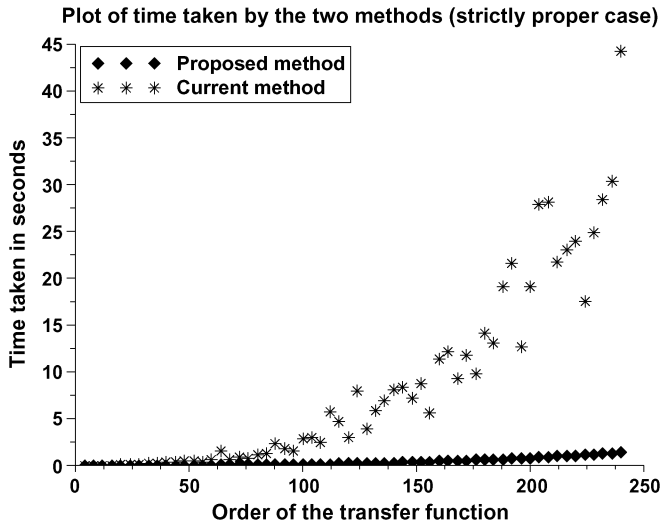

Fig. 2. Plot of time taken by the two methods for varying orders.

10^{-8} is the default relative error tolerance and the one for our comparison in the current method. On the other hand, absolute accuracy in the proposed method is limited by what LAPACK routines provide for generalized eigenvalue computation (using the QZ iteration): this default value is 10^{-16}, the machine precision. Of course, the accuracy limitation due to eigenvalue computation itself is faced by both methods.

The proposed method deals with polynomial matrix manipulation: this is known to be numerically less stable than constant matrix manipulation. However, the proposed method does eventually deal with constant matrix manipulation by reducing the problem to a generalized eigenvalue computation: only the 'outer' iteration is carefully made inessential. The polynomial aspect of our proposed algorithm is not encountered during the floating point operations. See Remark 9 above. Note that the size of the constant matrices whose generalized eigenvalues are to be computed (just once) is same as the order of the transfer function; this is after utilizing the property that $p_\gamma(\omega)$ is an even polynomial in ω. On the other hand, in the current method, the size of the Hamiltonian matrix (whose eigenvalues are repeatedly computed in order to converge to the $\mathcal{H}_\infty$ norm) is double the order of the transfer function.

A potential drawback of the proposed method is that *generalized* eigenvalue computation is numerically less stable than eigenvalue computation. For orders beyond 200, warnings about non-convergence in the QZ algorithm were encountered for many of the random transfer functions in the comparison experiments.

APPENDIX
ISOLATED COMMON ZEROS

For $p, q \in \mathbb{R}[\gamma][\omega]$, let $Z(p,q) := \{z \in \mathbb{C}^2 | p(z_1, z_2) = q(z_1, z_2) = 0\}$. This is an algebraic variety consisting of a finite number of irreducible components. Let r be the greatest common divisor of p and q, and let $p = r\tilde{p}$ and $q = r\tilde{q}$. If r is not a constant, its zero set is a one dimensional subvariety of $Z(p,q)$. The common zeros of $\tilde{p}$ and $\tilde{q}$ form a zero dimensional algebraic variety: a finite number of isolated points. We call these points the *isolated zeros* of p and q. Note that this definition allows isolated zeros to be contained in the zero set of r, as the following example shows.

Example 10: Let $p = (\gamma - \omega^2 - 1)^2$ and $q = \omega(\gamma - \omega^2 - 1)$. Then $r = \tilde{p} = \gamma - \omega^2 - 1$, $\tilde{q} = \omega$, so (1,0) is the only isolated zero of p and q, but is contained in the (one dimensional) zero set of r. □

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Convergence and Equivalence Results for the Jensen's Inequality—Application to Time-Delay and Sampled-Data Systems

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***Abstract*—The Jensen's inequality plays a crucial role in the analysis of time-delay and sampled-data systems. Its conservatism is studied through the use of the Grüss Inequality. It has been reported in the literature that fragmentation (or partitioning) schemes allow to empirically improve the results. We prove here that the Jensen's gap can be made arbitrarily small provided that the order of uniform fragmentation is chosen sufficiently large. Nonuniform fragmentation schemes are also shown to speed up the convergence in certain cases. Finally, a family of bounds is characterized and a comparison with other bounds of the literature is provided. It is shown that the other bounds are equivalent to Jensen's and that they exhibit interesting well-posedness and linearity properties which can be exploited to obtain better numerical results.**

***Index Terms*—Conservatism, fragmentation, Grüss inequality, Jensen's inequality, sampled-data systems, time-delay systems.**

I. INTRODUCTION

The Jensen's Inequality [1] has had a tremendous impact on many different fields; e.g., convex analysis, probability theory, information theory, statistics, control and systems theory [2]–[4]. It concerns the bounding of convex functions of integrals or sums:

Lemma 1.1: Let U be a given connected and compact set of $\mathbb{R}$, f a function measurable over U and ϕ a convex function measurable over $f(U)$. Then the inequality

$$\phi\left(\int_U f(s)d\mu(s)\right) \leq \mu(U)\int_U [\phi \circ f](s)d\mu(s) \tag{1}$$

holds where μ is a given nonnegative measure, e.g., the Lebesgue measure, and $\mu(U) = \int_U d\mu(s) < +\infty$ is the measure of the set U.

The discrete counterpart is given by:

Lemma 1.2: Let U be a given connected and compact set of $\mathbb{Z}$, f a function measurable over U and ϕ a convex function measurable over $f(U)$. Then the inequality

$$\phi\left(\mu(U)^{-1}\sum_{i\in U} f_i\mu_i\right) \leq \mu(U)^{-1}\sum_{i\in U}\phi(f_i)\mu_i \tag{2}$$

holds where $\{\mu_i\}_{i\in U}$ is a given nonnegative measure, e.g., the counting measure, and $\mu(U) = \sum_{i\in U}\mu_i < +\infty$ is the measure of the set U.

These inequalities have found applications in systems theory, for instance for the computation of an upper bound on the $\mathcal{L}_2$-gain of integral operators involved in time-delay systems analysis [2], [5], [6]. Another application in time-delay systems [7], [3], [8], [4] concerns the bounding of integral quadratic terms of the form

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