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Computing the radius of controllability for state space systems[☆]

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ABSTRACT

In this paper, we discuss the problem of computing the nearest uncontrollable system to a given control system represented by a matrix pair (A, B). In order to do so, we construct a sequence of structured matrices from given system matrices A and B. Controllability of the pair (A, B) is equivalent to a condition on the null-space dimension of an appropriate matrix in this sequence. We show that the dimension of the reachability space is also related to the above condition. Further, it is shown that the nearest Structured Low Rank Approximation (SLRA) of this structured matrix corresponds to a nearest uncontrollable system to the pair (A, B).

1. Introduction

Controllability of a system is a central concept of systems theory. Consider a system in state space representation as

$$\dot{x} = Ax + Bu \tag{1}$$

where $A \in \mathbb{R}^{g\times g}$, $B \in \mathbb{R}^{g\times p}$. The well known test for controllability is to check if the controllability matrix, $\mathcal{C}(A, B) = [B\ AB\ \cdots\ A^{g-1}B]$, is full rank. Further, the Hautus test for controllability states that the matrix $H = [\lambda I - A\ B]$ is required to be full rank for all $\lambda \in \mathbb{C}$ for the system to be controllable (for further details, see [1]). All these tests answer the question of whether the system is controllable. However, knowing whether the system is controllable is often not enough. This is illustrated in the following example.

Example 1.1. Let the system be described as in (1) where $A = \begin{bmatrix} 10 & \epsilon \\ 1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. For any $\epsilon \neq 0$, the system (A, b) is controllable. However note that for very small ϵ such that $|\epsilon| > 0$, a small perturbation in the matrix A can make this system uncontrollable.

From the example above, it is clear that though the system is controllable for all $\epsilon \neq 0$, the numerical robustness for controllability changes with ϵ. In order to overcome this difficulty, attempts have been made to define a continuous metric for checking controllability as opposed to the yes/no kind of discrete metric (see [2–5]). In [4], the distance between the given pair (A, B) and the set of all uncontrollable systems is shown to be same as

$$\min_{\lambda\in\mathbb{C}} \sigma_{\min}\, [\lambda I - A\ B] \tag{2}$$

where $\sigma_{\min}$ denotes the smallest singular value of the matrix. A relation between the positive definite solution of a certain Riccati equation and the nearness to uncontrollability is proved in [6]. From this result, upper and lower bounds are derived for the distance to uncontrollability. The distance between the given system and the nearest uncontrollable system in real as well as complex cases is discussed in [7] along with some properties of the reachability Gramian. Various numerical algorithms have been considered to compute the lower and upper bounds for the distance to uncontrollability. See for instance [8–10]. A bisection method is developed in [11] to compute this distance. In [12], the algorithm proposed in [11] is improved with respect to computational cost. The improved algorithm is based on a trisection algorithm and a novel algorithm to extract eigenvalues of a matrix with Kronecker structure. The algorithm proposed in [11] has complexity $O(n^6)$ while the improved version in [12] has complexity $O(n^4)$ on average.

The real[1] radius of controllability or radius of controllability is defined as follows:

$$r_c = \min_{\Delta A\in\mathbb{R}^{g\times g},\, \Delta B\in\mathbb{R}^{g\times p}} \{\|[\Delta A\ \Delta B]\|_F \mid \text{the pair } (A+\Delta A, B+\Delta B) \text{ is uncontrollable}\}. \tag{3}$$

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[1] The word *real* in the definition of real radius of controllability is emphasized to indicate that the perturbations that are allowed in the system matrices are real matrices. When the complex perturbations are allowed, the term is defined as the complex radius of controllability.

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In [13], the real as well as complex radius of controllability, r_c, is defined with the induced matrix 2-norm as opposed to the Frobenius norm that we intend to use. Note that the complex radius of controllability is equivalent with respect to the 2-norm and the Frobenius norm, but the real stability radius is different with respect to these norms. Further a numerical algorithm is also discussed to compute r_c in [13]. This algorithm obtains upper and lower bounds on the value of r_c instead of performing global optimization. Then using this upper bound as a starting point, nonlinear programming methods are used to compute the optimal value of r_c.

Another formulation of the radius of controllability can be found in [14]. The structured distance to uncontrollability is also discussed in [14] where the class of perturbations is restricted to the special class of matrices namely, symmetric or Hermitian.

In this paper, we give an efficient numerical algorithm to compute the radius of controllability for any given pair (A, B). Our approach involves constructing a sequence of structured matrices from the system matrices A and B. We show that null-spaces of these structured matrices are related to the controllability of the system. Further we show that the distance of some structured matrix in this sequence to its nearest Structured Low Rank Approximation (SLRA) is related to the controllability radius of the pair (A, B). A minor contribution of this paper is an equivalent criterion for testing controllability of the system in terms of a certain structured matrix which further helps in computing the nearest uncontrollable system.

The paper is organized as follows. In Section 2, we prove basic results required to define the concept of real radius of controllability of a given system. In Section 3, we discuss the SLRA problem and an algorithm to compute the nearest SLRA. We formulate the problem of computing the radius of controllability as an SLRA problem and present an algorithm to find the radius of controllability. Further, numerical examples are considered in Section 4 and the results obtained are compared with those in the literature. We conclude in Section 5.

2. Radius of controllability: theory

The radius of controllability of a system is defined as the distance of this system to the nearest uncontrollable system. In order to define this concept, we require some theoretical preliminaries which we prove in the sequel. Consider the system as described in (1) with $A \in \mathbb{R}^{g\times g}$ and $B \in \mathbb{R}^{g\times p}$. We construct a sequence of structured matrices from A and B as follows: construct $X_0 = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$ where I is a $g \times g$ identity matrix and $0 \in \mathbb{R}^{g\times p}$ is the zero matrix. Thus $X_0 \in \mathbb{R}^{2g\times(g+p)}$. Further construct

$$X_1 = \left[\begin{array}{c|c} X_0 & 0 \\ \hline 0 & X_0 \end{array}\right] \qquad X_2 = \left[\begin{array}{c|c|c} X_0 & 0 & 0 \\ & & 0 \\ \hline 0 & X_0 & \\ \hline 0 & 0 & X_0 \end{array}\right], \ldots \tag{4}$$

where 0 in the above equation is the zero matrix of size $g \times (g+p)$. For any $i \in \mathbb{N}$, $X_i \in \mathbb{R}^{(i+2)g\times(i+1)(g+p)}$. Let $\mathcal{K}_i$ be the null-space of matrix X_i and let $d_i = \dim(\mathcal{K}_i)$. We now prove some properties of this sequence $\{d_i\}$.

Lemma 2.1. *Let $\{X_i\}_{i=0,1,2,\ldots}$ be the sequence of structured matrices as defined in (4). Let $\mathcal{K}_i$ be the null-space of X_i and $d_i = \dim(\mathcal{K}_i)$ for $i = 0, 1, 2, \ldots$. Then the sequence $\{d_i\}_{i=0,1,2,\ldots}$ is a nondecreasing sequence of integers.*

Proof. Note that for any $i \in \mathbb{N}$, $X_i \in \mathbb{R}^{(i+2)g\times(i+1)(g+p)}$. Let $n_0 \in \mathbb{N}$ be the smallest positive integer such that $d_{n_0} > 0$. Let $0 \neq y \in \mathbb{R}^{(n_0+1)(g+p)}$ be such that $y \in \mathcal{K}_{n_0}$. Then, from the structure of matrices X_i, it is clear that for $0 \in \mathbb{R}^{g+p}$, both $\begin{bmatrix} y \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \in \mathcal{K}_{n_0+1}$. Thus $d_{n_0+1} \geqslant 2d_{n_0}$. In particular $d_{n_0+1} > d_{n_0}$. Let $d_{n_0+1} = 2d_{n_0} + \alpha_1$. Then using similar argument we can show that $d_{n_0+2} = 3d_{n_0} + 2\alpha_1 + \alpha_2 = d_{n_0+1} + d_{n_0} + \alpha_1 + \alpha_2 > d_{n_0+1}$. Generalizing this argument it follows that

$$\begin{aligned} d_{n_0+j} &= (j+1)d_0 + \sum_{k=1}^{j}(j-k+1)\alpha_k \\ &= d_{n_0+j-1} + \left(d_0 + \sum_{k=1}^{j}\alpha_k\right) > d_{n_0+j-1} \end{aligned}$$

for $j = 1, 2, \ldots$. Note that the first n_0 terms of the sequence are 0. This proves that $\{d_i\}_{i=0,1,2,\ldots}$ is a nondecreasing sequence of integers. $\square$

Remark 2.2. In the notation of Lemma 2.1, the first n_0 terms of the sequence $\{d_i\}_{i=0,1,2,\ldots}$ are 0. Further from the n_0th term onwards, the sequence $\{d_i\}$ is, in fact, a strictly increasing sequence.

We now show that the sequence $\{d_i\}_{i=0,1,\ldots}$ is related to the controllability of the pair (A, B).

Theorem 2.3. *Let the system be represented as in (1) where $A \in \mathbb{R}^{g\times g}$ and $B \in \mathbb{R}^{g\times p}$. Construct the sequence of structured matrices $\{X_i\}_{i=0,1,2,\ldots}$ as in (4). Let $\mathcal{K}_i$ be the null-space of X_i and $d_i = \dim(\mathcal{K}_i)$ for $i = 0, 1, 2, \ldots$. Let $\mathcal{C}(A, B) = \begin{bmatrix} B\; AB\; \cdots\; A^{g-1}B \end{bmatrix}$ be the controllability matrix. Then* $\dim(\mathrm{null}(\mathcal{C}(A, B))) = d_{g-1}$.

Proof. For $i \in \mathbb{N}$ such that $i \leqslant g-1$, we construct the matrix X_i as follows:

$$\begin{aligned} X_i &= \left[\begin{array}{c|c} X_0 & 0 \\ \hline 0 & X_{i-1} \end{array}\right] \\ &= \begin{bmatrix} A & B & & & & & & & \\ I & 0 & A & B & & & & & \\ & & I & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & A & B & & \\ & & & & & I & 0 & A & B \\ & & & & & & & I & 0 \end{bmatrix}. \end{aligned} \tag{5}$$

In order to determine the $\dim(\mathcal{K}_i)$, we compute the rank of X_i by reducing it to a special form using Gaussian elimination. We use I blocks to eliminate A. We start the elimination procedure from the last row of the matrix. Then we have,

$$X_i \sim \begin{bmatrix} 0 & B & & -AB & & & & (-1)^i A^i B \\ I & 0 & 0 & B & & & & (-1)^{i-1}A^{i-1}B \\ & & & & \ddots & & & \\ & & & & & -AB & & A^2B \\ & & & & 0 & B & & -AB \\ & & & & I & 0 & 0 & B \\ & & & & & & I & 0 \end{bmatrix}. \tag{6}$$

Performing column permutations on the matrix in the above equation and multiplying the columns by -1 whenever necessary, we obtain,

$$X_i \sim \begin{bmatrix} B & AB & \cdots & A^iB & & & & \\ & -B & & -A^{i-1}B & I & & & \\ & & & A^{i-2}B & & I & & \\ & & & \vdots & & & \ddots & \\ & & & -B & & & & I \\ & & & & & & & & I \end{bmatrix}. \tag{7}$$

From (7), we note that

$$\operatorname{rank}\left(\left[B\,AB\,\cdots\,A^iB\right]\right) + \dim\left(\operatorname{null}\left(\left[B\,AB\,\cdots\,A^iB\right]\right)\right)$$
$$= (i+1)\mathrm{p} \tag{8}$$

$$\operatorname{rank}\left(\left[B\,AB\,\cdots\,A^iB\right]\right) + (i+1)\mathrm{g} = \operatorname{rank}(X_i) \tag{9}$$

$$\operatorname{rank}(X_i) + d_i = (i+1)(\mathrm{g}+\mathrm{p}). \tag{10}$$

From (8) to (10), we get

$$d_i = \dim\left(\operatorname{null}\left(\left[B\,AB\,\cdots\,A^iB\right]\right)\right). \tag{11}$$

Thus for $i = \mathrm{g}-1$, we have $d_{\mathrm{g}-1} = \dim(\operatorname{null}(\mathcal{C}(A,B)))$. $\square$

Before we proceed further, there are some observations in order.

Remark 2.4. We can use the matrix $X_{\mathrm{g}-1}$ to check controllability of the pair (A, B) instead of the controllability matrix $\mathcal{C}(A, B)$. In order to check controllability using the controllability matrix, we have to compute the controllability matrix first and then determine its rank. For $A \in \mathbb{R}^{\mathrm{g}\times\mathrm{g}}$ and $B \in \mathbb{R}^{\mathrm{g}\times\mathrm{p}}$, the floating point operations (flops) required to compute $\mathcal{C}(A, B)$ are $(\mathrm{g}^2\mathrm{p})(\mathrm{g}-1)$ which is $O(\mathrm{g}^3\mathrm{p})$. Further to determine the rank of $\mathcal{C}(A, B)$, we have to compute the QR decomposition of $\mathcal{C}(A, B)$ which is $O(\mathrm{g}^3)$. Note that $\mathcal{C}(A, B)$ is usually extremely ill-conditioned (see [15, Chapter 5]). The special structure of the matrix $X_{\mathrm{g}-1}$ allows us to determine its rank with $O(\mathrm{g}+1)^3$ flops without requiring to compute $\mathcal{C}(A, B)$.

Remark 2.5. The reachability subspace is the range space of the controllability matrix $\mathcal{C}(A, B)$. By Theorem 2.3, we can determine the dimension of the reachability subspace from the sequence $\{d_i\}_{i=0,1,\ldots}$. The dimension of the reachability space is given by $\mathrm{gp}-d_{\mathrm{g}-1}$, where gp is the number of columns in the controllability matrix $\mathcal{C}(A, B)$. Thus if the system is controllable, then from the above theorem, $d_{\mathrm{g}-1} = \mathrm{gp}-\mathrm{g}$ and the dimension of the reachability space is g.

From the above remark, it is clear that the dimension of the reachability space is related to the dimension of the null-space of the structured matrix $X_{\mathrm{g}-1}$. In fact dimensions of null-spaces of structured matrices X_i are related to Kronecker indices of the matrix pencil $[sI - A\ B]$ and the controllability indices of the pair (A, B); see for instance [16,17]. We now give an explicit formula to compute controllability indices[2] from the sequence $\{d_i\}_{i=0,1,\ldots}$. Let $\mathcal{B}$ denote the range space of $B \in \mathbb{R}^{\mathrm{g}\times\mathrm{p}}$, the input matrix. Let us assume that the pair (A, B) is controllable. Define the spaces $\mathcal{S}_j = \mathcal{B} + A\mathcal{B} + \cdots + A^j\mathcal{B}$ for $j = 0, 1, \ldots, \mathrm{g}-1$. Then define

$$\rho_0 = \dim \mathcal{S}_0, \tag{12a}$$

$$\rho_j = \dim\left(\mathcal{S}_j/\mathcal{S}_{j-1}\right) \quad \text{for } j = 1, 2, \ldots, \mathrm{g}-1. \tag{12b}$$

[2] The definitions of controllability indices in the literature have slight variations; see for instance [1]. We use the formulation as given in [18].

Note that $\rho_0 \geqslant \rho_1 \geqslant \rho_2 \cdots \geqslant \rho_{\mathrm{g}-1} \geqslant 0$ and $\sum_{j=0}^{\mathrm{g}-1}\rho_j = \mathrm{g}$. Then the controllability indices $\kappa_j,\ j = 1, 2, \ldots, \mathrm{p}$ are defined as follows:

$$\kappa_j = \text{number of integers in the set}$$
$$\{\rho_0, \rho_1, \ldots, \rho_{\mathrm{g}-1}\} \quad \text{which are } \geqslant j. \tag{13}$$

Controllability indices satisfy $\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_{\mathrm{p}}$ and $\sum_{j=1}^{\mathrm{p}}\kappa_j = \mathrm{g}$. (see [18, Section 5.7])

The above construction of ρ_i's and κ_i's is independent of controllability of the pair (A, B). Therefore given any pair (A, B) we can construct ρ_i's and κ_i's. Technically controllability indices are defined only for a controllable pair (A, B). However, the above construction can be done for any pair (A, B). For this paper, by slight abuse of notation, we call κ_i's controllability indices even for the case when the pair (A, B) is not controllable.

Lemma 2.6. *Consider a pair (A, B) with $A \in \mathbb{R}^{\mathrm{g}\times\mathrm{g}}$, $B \in \mathbb{R}^{\mathrm{g}\times\mathrm{p}}$. Construct the sequence of structured matrices as in (4). Let $d_i = \dim(\ker X_i)$. Then*

$$\rho_0 = \mathrm{p} - d_0 \tag{14a}$$

$$\rho_j = \mathrm{p} - (d_{j+1} - d_j) \quad \text{for } j \in \{1, 2, \ldots, \mathrm{g}-1\} \tag{14b}$$

Proof. From Theorem 2.3, we see that

$$\operatorname{rank}[B\,AB\,\cdots\,A^jB] + d_j = (j+1)\mathrm{p}$$
$$\Rightarrow \dim \mathcal{S}_j = (j+1)\mathrm{p} - d_j \tag{15}$$

for $j = 0, 1, \ldots$. Substituting $j = 0$ in (15), we get, $\operatorname{rank}(B) + d_0 = \mathrm{p}$ and hence $\rho_0 = \dim \mathcal{S}_0 = \mathrm{p} - d_0$. For $k \in \{1, 2, \ldots, \mathrm{g}-1\}$, using (15) we get

$$\dim \mathcal{S}_k - \dim \mathcal{S}_{k-1} = (k+1)\mathrm{p} - d_k - (k\mathrm{p} - d_{k-1})$$
$$\Rightarrow \rho_k = \mathrm{p} - (d_k - d_{k-1}). \quad \square \tag{16}$$

We compute controllability indices κ_j's from ρ_j's as in (13), where ρ_j's are calculated using Lemma 2.6. We illustrate this in the following example.

Example 2.7. Let the given controllable pair (A, B), where $A \in \mathbb{R}^{5\times 5}$ and $B \in \mathbb{R}^{5\times 2}$, be given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then we compute $\rho_0 = 2$, $\rho_1 = 2$, $\rho_2 = 1$ and subsequently $\rho_i = 0$ for $i = 3, 4$. The controllability indices are therefore $\kappa_1 = 3$ and $\kappa_2 = 2$. On constructing the sequence of structured matrices $\{X_i\}_{i=0,1,\ldots}$, we obtain the sequence $\{d_i\}_{i=0,1,\ldots}$ as $0, 0, 1, 3, 5, \ldots$. Then from (16), we compute $\rho_0 = 2$, $\rho_1 = \mathrm{p} - (d_1 - d_0) = 2$, $\rho_2 = \mathrm{p} - (d_2 - d_1) = 2 - 1 = 1$ and $\rho_3 = \rho_4 = 0$. Hence the controllability indices by our method are $\kappa_1 = 3,\ \kappa_2 = 2$.

Controllability of a given pair (A, B) is equivalent to the dimension of the reachability space being equal to that of the state space, in our case g. Based on this characterization of controllability, we now define the real radius of controllability of order k as follows.

Definition 2.8. Let (A, B) be a given pair describing the system (1) and $A \in \mathbb{R}^{\mathrm{g}\times\mathrm{g}}$ and $B \in \mathbb{R}^{\mathrm{g}\times\mathrm{p}}$. Let k be any positive integer such that $\mathrm{k} \leqslant \mathrm{g}-1$. Then define

$$r_c(\mathrm{k}) = \min_{\Delta A, \Delta B}\{\|[\Delta A\ \Delta B]\|_F \mid \operatorname{rank}(\mathcal{C}(A+\Delta A, B+\Delta B))$$
$$\leqslant \mathrm{g}-\mathrm{k}\}. \tag{17}$$

$r_c(\mathrm{k})$ is called the *real radius of controllability of order* k.

Remark 2.9. Another way to understand Definition 2.8 is as follows: given a pair (A, B), there exists a ball of radius r with center at (A, B) where all pairs $(\tilde{A}, \tilde{B})$ in this ball have rank of $\mathcal{C}(\tilde{A}, \tilde{B}) > \mathrm{g} - \mathrm{k}$. The supremum of such r is $r_c(\mathrm{k})$.

The real radius of controllability as defined in (3) (see [13]) is indeed the real radius of controllability of order 1, denoted as $r_c(1)$.

Using Theorem 2.3 we now give a procedure to compute the nearest uncontrollable pair to the given pair (A, B). If the pair (A, B) is controllable, then $d_{\mathrm{g}-1} = \mathrm{gp} - \mathrm{g}$ and hence the dimension of the reachability space is g. Fixing some matrix norm, if one finds the nearest matrix to $X_{\mathrm{g}-1}$ that preserves the same structure as that of $X_{\mathrm{g}-1}$ (namely the block Toeplitz structure as in (5)) but has lower rank than $X_{\mathrm{g}-1}$, then one would have effectively found the closest uncontrollable system to the given system. Such problems of finding a lower rank matrix that preserves a certain structure have been investigated in the literature and goes under the name of Structured Low Rank Approximation, abbreviated as SLRA (see for instance [19,20]). Thus we have the following theorem.

Theorem 2.10. *Let the given controllable system be represented as in (1) where $A \in \mathbb{R}^{\mathrm{g}\times\mathrm{g}}$ and $B \in \mathbb{R}^{\mathrm{g}\times\mathrm{p}}$. Construct the sequence of structured matrices $\{X_i\}_{i=0,1,2,\ldots}$ as in (4). Let $\tilde{X}_{\mathrm{g}-1}$ be the nearest SLRA of $X_{\mathrm{g}-1}$ in the Frobenius norm. Construct $\tilde{A}$ and $\tilde{B}$ from $\tilde{X}_{\mathrm{g}-1}$. Then*

$$r_c(1) = \left\| [A\, B] - \left[\tilde{A}\, \tilde{B}\right] \right\|_F, \tag{18}$$

where $r_c(1)$ is the radius of controllability of order 1.

Computation of $r_c(\mathrm{k})$ for $\mathrm{k} > 1$

For computing $r_c(\mathrm{k})$ for $\mathrm{k} > 1$ we require a generalization of Theorem 2.3 which we state below.

Lemma 2.11. *Consider a pair (A, B) with $A \in \mathbb{R}^{\mathrm{g}\times\mathrm{g}}$ and $B \in \mathbb{R}^{\mathrm{g}\times\mathrm{p}}$. For a given j define $\mathcal{C}_j(A, B) = \left[B\, AB\, \cdots\, A^{j-1}B\right]$. Construct the sequence of structured matrices X_i as in (4). Let $d_i = \dim(\mathrm{null}(X_i))$. Then $\dim(\mathrm{null}(\mathcal{C}_i(A, B))) = d_{i-1}$.*

Proof. See the proof of Theorem 2.3. □

In order to give a procedure to compute $r_c(\mathrm{k})$, we consider single-input and multi-input systems separately.

Single-input case

Let a controllable pair (A, B) be given where $A \in \mathbb{R}^{\mathrm{g}\times\mathrm{g}}$ and $B \in \mathbb{R}^{\mathrm{g}\times 1}$. We compute the nearest SLRA of $X_{\mathrm{g}-1}$ and by Theorem 2.10 we obtain the corresponding nearest uncontrollable pair $(\tilde{A}, \tilde{B})$. The dimension of the reachability subspace for the pair $(\tilde{A}, \tilde{B})$ turns out to be $\mathrm{g} - 1$. If we want to compute the nearest uncontrollable pair such that the dimension of the reachability space is at most $\mathrm{g} - \mathrm{k}$ where $1 \leqslant \mathrm{k} \leqslant \mathrm{g} - 1$, it is enough to compute the SLRA of the matrix $X_{\mathrm{g}-\mathrm{k}}$. By Lemma 2.11 the dimension of the subspace $\mathcal{C}_{\mathrm{g}-\mathrm{k}+1}(A, B)$ of the pair (A, B) is $\mathrm{g} - \mathrm{k} + 1$ (assuming (A, B) is a controllable pair). Therefore by finding the nearest SLRA of $X_{\mathrm{g}-\mathrm{k}}$, one effectively reduces the dimension of the subspace $\mathcal{C}_{\mathrm{g}-\mathrm{k}+1}(\tilde{A}, \tilde{B})$ to be strictly less than $\mathrm{g} - \mathrm{k} + 1$. As it is a single input system, this in fact forces the reachability space of $(\tilde{A}, \tilde{B})$ to be at most $\mathrm{g} - \mathrm{k}$ dimensional.

Multi-input case

For a single-input system there is only one controllability index whereas for a multi-input system there are several controllability indices. This complicates the computation of $r_c(\mathrm{k})$ for $\mathrm{k} > 1$. Consider a pair (A, B) with $A \in \mathbb{R}^{\mathrm{g}\times\mathrm{g}}$ and $B \in \mathbb{R}^{\mathrm{g}\times\mathrm{p}}$. Let $\kappa_1, \kappa_2, \ldots, \kappa_{\mathrm{p}}$ be the controllability indices as defined in (13). For a controllable pair (A, B), the dimension of $\mathcal{C}_i(A, B) = \mathrm{g}$ for all $i > \kappa_1$. Before giving a procedure to compute $r_c(\mathrm{k})$, we prove the following lemma which is useful.

Lemma 2.12. *Let (A, B) be given system with $A \in \mathbb{R}^{\mathrm{g}\times\mathrm{g}}$ and $B \in \mathbb{R}^{\mathrm{g}\times\mathrm{p}}$ such that the dimension of the reachability space of the pair (A, B) is $\mathrm{g} - \mathrm{k}$. Construct the sequence of structured matrices as in (4). Let $\kappa_1, \kappa_2, \ldots, \kappa_{\mathrm{p}}$ be the controllability indices as in (13). Then $d_i = (i + 1)\mathrm{p} - \mathrm{g} + \mathrm{k}$ for all $i > \kappa_1$.*

Proof. Since the dimension of the reachability space for the pair (A, B) is $\mathrm{g} - \mathrm{k}$, from Theorem 2.3 we get $d_{\mathrm{g}-1} = \mathrm{gp} - \mathrm{g} + \mathrm{k}$. By Lemma 2.11, $\dim(\mathrm{null}(\mathcal{C}_i(A, B))) = d_i$ for $i \geqslant 1$. Clearly $\dim(\mathrm{range}\,(\mathcal{C}_i(A, B))) = \mathrm{g} - \mathrm{k}$ for all $i > \kappa_1$. Therefore $\dim(\mathrm{null}(\mathcal{C}_i(A, B))) = d_i = (i + 1)\mathrm{p} - \mathrm{g} + \mathrm{k}$ for all $i > \kappa_1$. □

In order to compute $r_c(\mathrm{k})$ when $\mathrm{k} = 1$, we compute the nearest SLRA to $X_{\mathrm{g}-1}$, say $\tilde{X}_{\mathrm{g}-1}$. Construct the pair $(\tilde{A}, \tilde{B})$ from $\tilde{X}_{\mathrm{g}-1}$. Then the pair $(\tilde{A}, \tilde{B})$ is the nearest uncontrollable pair to (A, B). For $r_c(\mathrm{k})$ with $\mathrm{k} > 1$, we cannot use an approach similar to that in the single-input case due to Lemma 2.12. We construct $X_{\mathrm{g}-1}$ and compute the nearest SLRA to $X_{\mathrm{g}-1}$, denoted as $\tilde{X}_{\mathrm{g}-1}$, that sets k singular values of $\tilde{X}_{\mathrm{g}-1}$ to zero. We construct the pair $(\tilde{A}, \tilde{B})$ from $\tilde{X}_{\mathrm{g}-1}$. Then from Lemma 2.12, we know that the dimension of the reachability space for the pair $(\tilde{A}, \tilde{B})$ is $\mathrm{g} - \mathrm{k}$.

3. SLRA and algorithm

In this section, we first state the problem of computing the nearest SLRA of a given affinely structured matrix. Then we formulate the problem of computing the nearest uncontrollable system as the SLRA problem. Finally we discuss a numerical algorithm to compute the nearest SLRA of a given matrix. Before we proceed further with the problem formulation we make an observation.

Remark 3.1. We obtain the sequence of structured matrices from the pair (A, B) as in (4). The set of structured matrices (to which X_i belongs for each i) of size $(i+2)\mathrm{g} \times (i+1)(\mathrm{g}+\mathrm{p})$ is an affine space of dimension $\mathrm{g}(\mathrm{g} + \mathrm{p})$. For example, when $\mathrm{g} = 3, \mathrm{p} = 1, i = 1$, the dimension of the affine space is 12.

SLRA formulation. For a given structure of matrices, let $\Omega \subset \mathbb{R}^{p\times q}$ denote the set of all structured matrices. Now we define the problem of computing the nearest SLRA as in [19].

Problem statement 3.2. Given $\Omega \subset \mathbb{R}^{p\times q}$, the set of matrices with the given structure, and $X \in \Omega$ such that $\mathrm{rank}(X) = k$ for $k \leqslant \min\{p, q\}$, find a matrix Y such that

$$\min_{Y\in\Omega,\,\mathrm{rank}(Y)=k-1} \|X - Y\|_F.$$

3.1. An algorithm to compute the nearest SLRA

The problem of computing the nearest SLRA of a given structured matrix is well studied in the literature (see [19,20]). Here we adopt the method discussed in [21] to the structured matrices with the structure as described in Section 2. We explain the Structured Total Least Norm (STLN) algorithm in this subsection.

We describe the algorithm for a general affine structure. Let $\Omega \subset \mathbb{R}^{p\times q}$ be the set of all structured matrices with a given affine structure. Thus Ω is an affine space in $\mathbb{R}^{p\times q}$. For a given $Z \in \Omega$ with rank r we need to compute the nearest $Y \in \Omega$ with rank $r - 1$. We partition $Z = [Z_1\, z]$, where $Z_1 \in \mathbb{R}^{p\times(q-1)}$ and $z \in \mathbb{R}^{p\times 1}$. Then the problem of computing the nearest SLRA can be formulated as

$$\min_{H,h,x} \|[H\, h]\|_F \quad \text{subject to } (Z_1 + H)x = (z + h) \tag{19}$$

where $H \in \mathbb{R}^{p\times(q-1)}$, $h \in \mathbb{R}^{p\times 1}$ are such that $[H\, h]$ belongs to the linear space corresponding to Ω and $x \in \mathbb{R}^{q-1}$. Note that this problem is similar to the Total Least Squares (TLS) problem with

Table 1
Comparison of radius of controllability for different values of $t \in \mathbb{R}$ (Example 4.1).

t	10	2	1.7	1.2	1.1	1	10^{-1}	10^{-3}	10^{-5}
r_c	0.2165	0.718	0.769	0.8596	0.8777	0.8954	9.127e − 2	9.129e − 4	9.129e − 6

an additional constraint on the structure of perturbation matrices H and h, hence the name Structured Total Least Squares (STLS) problem.

Let $\{B_1, B_2, \ldots, B_N\}$ be a basis for the linear space corresponding to Ω. Let $\Delta Z = [H\ h]$ be such that ΔZ belongs to the linear space corresponding to Ω. Let $\alpha = [\alpha_1\ \alpha_2\ \cdots\ \alpha_N]^T \in \mathbb{R}^N$ be coordinates of ΔZ with respect to the basis $B = \{B_1, B_2, \ldots, B_N\}$. Then we have $\Delta Z = \sum_{i=1}^{N} \alpha_i B_i$. We call α the representation of ΔZ. Let $P \in \mathbb{R}^{p\times N}$ be a matrix such that $h = P\alpha$. Then the structured minimization problem as in (19) can be stated as follows:

$$\min_{\alpha,x} \|D\alpha\|_2 \quad \text{subject to } \hat{r} = 0 \tag{20}$$

where the structured residual $\hat{r} = \hat{r}(\alpha, x) = z + P\alpha - (Z_1 + H)x$ and D is a positive definite weight matrix. If B is an orthonormal basis, the weight matrix can be chosen to be $D = I_N$. The above problem can be solved using the penalty method in the following way.

$$\min_{\alpha,x} \left\| \begin{bmatrix} \omega\hat{r}(\alpha, x) \\ \alpha \end{bmatrix} \right\|_2, \tag{21}$$

where ω is a very large positive constant. Typically in numerical simulations ω is taken in the range of 10^8–10^{10}. As proposed in [21], we linearize the structured residual as follows:

$$\begin{aligned} \hat{r}(\alpha + \Delta\alpha, x + \Delta x) &= z + P(\alpha + \Delta\alpha) \\ &\quad - (Z_1 + H + \Delta H)(x + \Delta x) \\ &\approx z + P\alpha + P\Delta\alpha - (Z_1 + H)x \\ &\quad - (Z_1 + H)\Delta x - \Delta H x. \end{aligned}$$

Let $S \in \mathbb{R}^{p\times N}$ be a matrix such that $S\Delta\alpha = \Delta H x$. The structure of S is similar to that of H. The entries in S depend on the entries of the vector x. Then (21) can be approximated by

$$\min_{\Delta\alpha, \Delta x} \left\| \begin{bmatrix} \omega(S - P) & \omega(Z_1 + H) \\ I_N & 0 \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta x \end{bmatrix} + \begin{bmatrix} -\omega\hat{r} \\ \alpha \end{bmatrix} \right\|_2. \tag{22}$$

We now summarize the algorithm.

Algorithm 3.3. STLN Algorithm

Input: Matrices Z_1, z and tolerance ϵ.

Output: Error matrix ΔX such that $\Delta X \in \Omega$, vector x and STLN error

Step 1: Choose a large number ω.

Step 2: Set $H = 0$, $h = 0$ and find x from $\min_x \|z - Z_1 x\|_2$ and S from x.

Step 3: Set $\hat{r} = z - Z_1 x$.

Step 4: Repeat
 (a) Solve the minimization problem in (22).
 (b) Set $x := x + \Delta x$, $\alpha := \alpha + \Delta\alpha$.
 (c) Construct $[H\ h]$ from α and S from x.
 (d) Compute $\hat{r} = (z + P\alpha) - (Z_1 + H)x$.

until $(\|\Delta x\|, \|\Delta\alpha\| \leqslant \epsilon)$.

Remark 3.4. The function we are trying to minimize is not a convex function and hence we do not guarantee the global minimizer. However this STLN formulation is equivalent to the TLS (Total Least Squares) formulation when there is no structure imposed on the residual. In this case, the solution obtained by the STLN algorithm is same as that obtained using TLS methods (see [21]).

The computational complexity of Algorithm 3.3 depends on the step (4a) where the least squares problem in (22) is solved. The matrix S is a specially structured matrix and it is shown in [21] that QR factorization of this structured matrix can be computed efficiently.

3.2. An algorithm to compute the nearest uncontrollable pair

In this subsection, we give an algorithm to compute the real radius of controllability for a given pair (A, B). The algorithm that we present is based on Theorems 2.3 and 2.10 and Algorithm 3.3.

Algorithm 3.5. Algorithm to compute the real radius of controllability $r_c(1)$

Input: Given a pair (A, B) such that $A \in \mathbb{R}^{\mathtt{g}\times\mathtt{g}}$, $B \in \mathbb{R}^{\mathtt{g}\times\mathtt{p}}$.

Output: The nearest pair $(\tilde{A}, \tilde{B})$ and the real radius of controllability $r_c(1)$.

Step 1: Construct $X_{\mathtt{g}-1}$ as in (4).

Step 2: Compute the nearest SLRA $\tilde{X}_{\mathtt{g}-1}$ of $X_{\mathtt{g}-1}$ using Algorithm 3.3.

Step 3: Construct $(\tilde{A}, \tilde{B})$ from $\tilde{X}_{\mathtt{g}-1}$.

Step 4: Compute $r_c(1) = \left\| \begin{bmatrix} \tilde{A}\ \tilde{B} \end{bmatrix} - [A\ B] \right\|_F$.

For the specific case of single-input systems, (i.e., $B \in \mathbb{R}^{\mathtt{g}\times 1}$), one can calculate $r_c(\mathtt{k})$ for any $\mathtt{k} \leqslant \mathtt{g} - 1$ by using the above algorithm. In order to do so, we construct $X_{\mathtt{g}-\mathtt{k}}$ in Step 1 of the algorithm and proceed with the subsequent steps of the algorithm with $X_{\mathtt{g}-\mathtt{k}}$ instead of $X_{\mathtt{g}-1}$. Then the pair $(\tilde{A}, \tilde{B})$ obtained from $\tilde{X}_{\mathtt{g}-\mathtt{k}}$ has the reachability space dimension equal to $\mathtt{g} - \mathtt{k}$. However, for multi-input systems, the above algorithm would not calculate $r_c(\mathtt{k})$ for $\mathtt{k} \geqslant 1$ (as discussed earlier).

Remark 3.6. The results presented in Section 2 hold true when the field under consideration is changed from $\mathbb{R}$ to $\mathbb{C}$. Further, the algorithm presented in Section 3 can be easily generalized for complex field (see [21]). Hence using our approach, we can easily compute the complex radius of controllability also.

Remark 3.7. Note that Algorithm 3.5 depends on Algorithm 3.3 to compute the nearest SLRA. Algorithm 3.3 is known to converge to a local minimizer but the guarantee of convergence to a global minimizer is not assured. Therefore, we obtain an upper bound while computing the real radius of controllability. However through the numerical examples in the following section, we observe that the proposed approach estimates of the real radius of controllability are better than the ones available in the literature.

4. Numerical examples

In this section, we discuss some numerical examples where the proposed algorithm in the previous section is applied.

Example 4.1. The following pair (A, B) has been studied in [22]: $A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix}$ where $t \in \mathbb{R}$. Clearly, as $t \to 0$ the pair (A, B) gradually loses controllability. In Table 1,

Table 2
Radii of controllability for different dimensions of reachability spaces (Example 4.3).

Dimension (k) of the reachability space	3	2	1
$r_c(\mathtt{k})$	0.4607	0.5658	0.9996

Table 4
Radii of controllability for various sizes of the system (Example 4.5).

g	5	10	15	20
$r_c(1)$	0.4310	0.2281	0.1663	0.1312

we compare the values of real radius of controllability obtained for different values of $t \in \mathbb{R}$.

When the parameter $t = 10$, the perturbation matrices ΔA and ΔB such that the pair $(A + \Delta A, B + \Delta B)$ is uncontrollable are as given below. One can check that $[\Delta A\ \Delta B]$ is rank one and has 2-norm (and the Frobenius norm): 0.216487.

$$\Delta A = \begin{bmatrix} -9.39222e-11 & -1.86154e-10 & -1.774895e-11 \\ -9.008923e-3 & -1.785553e-2 & -1.702449e-3 \\ 9.448684e-2 & 1.872712e-1 & 1.785553e-2 \end{bmatrix}$$

$$\Delta B = \begin{bmatrix} 4.584664e-11 \\ 4.397604e-3 \\ -4.612268e-2 \end{bmatrix}.$$

Note that the 2-norm of the perturbation matrix reported in [22] is 0.219866. For rank 2 perturbation matrices $[\Delta A\ \Delta B]$, the Frobenius norm is larger than the 2-norm: this is reflected in the r_c values for smaller values of t when a pair of complex eigenvalues becomes uncontrollable in the perturbed system. See the following example for a similar situation and discussion about the choice of norm used in the definition.

Example 4.2. Consider the pair (A, B) that has been studied in [8,13] and [22, Example 4]

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0.1 & 3 & 5 \\ 0 & -1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0.1 \\ 0 \end{bmatrix}.$$

The real radius r_c of controllability obtained using our approach is 0.05734, and the perturbation matrices ΔA and ΔB are respectively

$$\begin{bmatrix} 5.8878e-4 & 4.966e-5 & -2.9287e-5 \\ -1.684774e-2 & -1.420985e-3 & 8.380330e-4 \\ -1.673060e-2 & -1.411105e-3 & 8.322061e-4 \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} -1.1427e-3 \\ -1.575356e-2 \\ -4.968473e-2 \end{bmatrix}.$$

In [8], where, like in this paper, the Frobenius norm was used for defining the radius of controllability, the same value 0.05734 was reported as the radius of controllability. In [13], the 2-norm of the perturbation required for uncontrollability was reported as 0.0492: the real radius defined there uses the 2-norm. The 2-norm of the above perturbation matrices obtained using our approach is 0.0564. On the other hand, the Frobenius norm of the perturbation matrix reported in [13] is 0.0696: this fact underlines the crucial (and quite unsurprising) dependence of the 'closest' uncontrollable pair on the choice of the matrix norm of the perturbation matrix in the definition of radius of controllability: this paper focuses on the Frobenius norm.

Example 4.3. Consider a pair

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly the dimension of the reachability space of this pair (A, B) is 4. In Table 2, we list the minimum norms of perturbation matrices that reduce the dimension of the corresponding perturbed system's reachability space. As expected, the norm of the perturbation required is larger for a lower dimensional reachability space.

Example 4.4. In this example, we illustrate the convergence properties of the SLRA algorithm. In order to do this, we have considered 1000 randomly generated test examples for various sizes g of the pair (A, B), with $A \in \mathbb{R}^{g\times g}$ and $B \in \mathbb{R}^{g\times 1}$. We note the average number of iterations required to compute the nearest SLRA and hence the radius controllability $r_c(1)$ for these cases (see Table 3).

Example 4.5. Let the pair (A, B) be given as

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & & 1 \\ 1 & 0 & \cdots & & 1 \end{bmatrix} \in \mathbb{R}^{g\times g}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{g\times 1}.$$

In Table 4, we list $r_c(1)$ for various values of g: the distance to uncontrollability is expectedly decreasing as the size grows.

5. Conclusions

In this paper, we have given a numerical algorithm to compute the radius of controllability of a specified order for a given pair (A, B). The approach discussed in this paper is different from the previous approaches to solve the problem. The problem of computing the radius of controllability is shown to be equivalent to the problem of computing the nearest SLRA of a certain structured matrix associated to the system described as in (1). We further solve the problem of computing the radius of controllability of order k, where k is the desired dimension of the reachability space of the perturbed system. The results proved here can be easily dualized to solve the corresponding problem of observability, namely computing the nearest unobservable system by considering the pair (A^T, C^T).

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Table 3
Average number of iterations for various sizes (Example 4.4).

g	5	7	10	12	15	20
Average number of iterations	4.055	4.800	3.257	3.222	6.158	7.0122

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