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Uncontrollable dissipative systems: observability and embeddability

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The theory of dissipativity is well developed for controllable systems. A more appropriate definition of dissipativity in the context of uncontrollable systems is in terms of the existence of a storage function, namely a function such that, along every system trajectory, its rate of change at each time instant is at most the power supplied to the system at that time. However, even when the supplied power is expressible in terms of just the external variables, the dissipativity property for uncontrollable systems crucially hinges on whether or not the storage function depends on variables unobservable/hidden from the external variables: this paper investigates the key aspects of both cases, and also proposes another intuitive definition of dissipativity. These three definitions are compared: we show that drawbacks of one definition are addressed by another.

Dealing first with *observable* storage functions, under the conditions that no two uncontrollable poles add to zero and that dissipativity is strict as frequency tends to infinity, we prove that the dissipativities of a system and its controllable part are equivalent. We use the behavioural approach for formalising key notions: a system *behaviour* is the set of all system trajectories. We prove that storage functions *have* to be unobservable for 'lossless' uncontrollable systems. It is known, however, that unobservable storage functions result in certain 'fallacious' examples of lossless systems. We propose an intuitive definition of dissipativity: a system/behaviour is called dissipative if it can be *embedded* in a controllable dissipative *superbehaviour*. We prove embeddability results and use them to resolve the fallacy in the example termed 'lossless' due to unobservable storage functions. We next show that, quite unreasonably, the embeddability definition admits behaviours that are both strictly dissipative and strictly antidissipative. Drawbacks of the embeddability definition in the context of RLC circuits are finally related to the inability to realise/synthesise the special one-port electrical network, called the nullator, using only passive electrical components.

Keywords: algebraic Riccati equation; indefinite linear algebra; Lyapunov equation; storage function; Hamiltonian matrix; behavioural approach

1. Introduction

The theory of dissipativity for linear dynamical systems helps in the analysis and design of control systems for several control problems, for example, linear quadratic regulator/linear quadratic Gaussian control, $\mathcal{H}_\infty$ control, synthesis of passive systems and optimal estimation problems. When dealing with linear time-invariant (LTI) controllable systems, it is easy to define dissipativity since controllability ensures that, loosely speaking, the compactly supported system trajectories are 'dense' in the set of all allowed trajectories. However, this is not the case for uncontrollable systems; alternative dissipativity definitions and their interrelationships are the central focus of this paper.

In this paper, we first investigate a less-often-used definition of dissipativity for systems, possibly uncontrollable, and generalise key results using techniques from indefinite linear algebra (see Gohberg, Lancaster, & Rodman, 2005) for solving algebraic Riccati equalities. Like in Çamlıbel, Willems, and Belur (2003) and Pal and Belur (2008), we first define a system as dissipative if there exists a storage function that satisfies the dissipation inequality for all system trajectories. The inequality states that the rate of increase of the stored energy cannot exceed the power supplied to the system. The power supplied, called the supply rate, is a function of only the external variables of the system. Important here is the issue whether the storage function is allowed to depend only on the external variables and their derivatives, or also on 'unobservable'/hidden variables (see Willems, 2004, for a detailed exposition). We investigate both the definitions: observable storage function and unobservable storage function. We next propose another intuitive definition of dissipativity: existence of a controllable dissipative superbehaviour. This brings us to embeddability of a possibly uncontrollable behaviour in a dissipative controllable behaviour.

A summary of the main results and the organisation of the paper is as follows. Section 2 contains preliminaries of behavioural systems theory, quadratic differential forms and indefinite linear algebra. Section 2 also contains the definition of a storage function, with a distinction between

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observable storage functions, unobservable storage functions and state functions: this distinction is central to this paper. Section 3 contains our first main result (Theorem 3.1): sufficient conditions for the existence of an observable storage function. Auxiliary results and the proof of Theorem 3.1 are in Sections 3 and 4. Section 5 studies inevitability of unobservable storage functions when some uncontrollable poles are on the imaginary axis. We study orthogonality of systems in Section 6 where we propose the embeddability definition for orthogonality/losslessness and dissipativity. This section contains key results on the smallest controllable 'superbehaviour' and a count on the number of inputs of the superbehaviour: here we resolve how certain (intuitively fallacious) systems that are orthogonal by the existence of an unobservable storage function definition are not orthogonal by the embeddability definition. The results on smallest controllable superbehaviours of Section 6 are used in Section 7 where we show that, when using the embeddability definition of dissipativity, there exist systems that are both strictly dissipative and strictly antidissipative. A summary of main results and an examination of the inter-relations of the three definitions of dissipativity finally follow in Section 8.

2. Preliminaries

In this section, we include various definitions about the behavioural framework for studying dynamical systems (Section 2.1) and then introduce quadratic differential forms (QDFs) (Section 2.2). Section 2.3 reviews the definition of a state of a system and Section 2.4 contains a brief notation about indefinite linear algebra from Gohberg et al. (2005). Section 2.5 contains the definition of a storage function.

We use $\mathbb{R}$ for the set of all real numbers and $\mathbb{R}[\xi]$ for the set of polynomials in the indeterminate ξ and real coefficients; matrices and polynomial matrices are denoted the standard way. When unnecessary or when it follows from the context, we use $\bullet$ to leave the row dimension unspecified, for example, $\mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$. The space of infinitely often differentiable functions from $\mathbb{R}$ to, say, $\mathbb{R}^{\mathtt{n}}$ is denoted by $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$, and $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ denotes the set of all compactly supported functions within $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$. The number of components in a vector w is denoted by $\mathtt{w}$, for example, $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ or $w(t) \in \mathbb{R}^{\mathtt{w}}$: this helps to avoid too many letters when denoting dimensions.

2.1 The behavioural approach

When dealing with linear differential systems, it is convenient to use polynomial matrices for describing the differential equations. Suppose $R_0, R_1, \ldots, R_N \in \mathbb{R}^{\bullet\times\mathtt{w}}$ are constant matrices of the same size such that

$$R_0 w + R_1 \tfrac{\mathrm{d}}{\mathrm{d}t} w + R_2 \tfrac{\mathrm{d}^2}{\mathrm{d}t^2} w + \cdots + R_N \tfrac{\mathrm{d}^N}{\mathrm{d}t^N} w = 0$$

is a system of linear, constant-coefficient ordinary differential equations in the variable w. Define the polynomial matrix $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ by $R(\xi) : R_0 + R_1\xi + \cdots + R_N\xi^N$, and represent the above differential equation conveniently as $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$.

A linear differential behaviour, denoted by $\mathfrak{B}$, is defined as the set of all infinitely often differentiable trajectories that satisfy such a system of differential equations, i.e. for $R(\xi) \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$,

$$\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0\},$$

and $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is called a *kernel representation* for $\mathfrak{B}$. We also write $\mathfrak{B} = \ker R(\frac{\mathrm{d}}{\mathrm{d}t})$. We denote the set of all such linear differential behaviours, with $\mathtt{w}$ being the number of dependent variables, by $\mathfrak{L}^{\mathtt{w}}$. (The independent variable is time t for the trajectories.) The matrix R is not unique and one can use elementary row operations to modify R and this does not change the set of solutions $\mathfrak{B}$: this thus allows assuming, without loss of generality, that R has full row rank (see Polderman & Willems, 1998, Section 2.5). Out of the $\mathtt{w}$ variables in w, some of the variables (say, inputs) can be chosen as arbitrary $\mathfrak{C}^\infty$ functions and the system equations then determine the rest of the variables (say, outputs): this defines an input/output (i/o) partition of the variables w. Though there are several choices of i/o classifications, the maximum *number* of inputs depends only on $\mathfrak{B}$: we call this number the *input cardinality*, and denote it by $\mathtt{m}(\mathfrak{B})$. The number of outputs, the *output cardinality* of $\mathfrak{B}$, is denoted by $\mathtt{p}(\mathfrak{B})$: this is equal to $\mathtt{w} - \mathtt{m}(\mathfrak{B})$. The number of outputs is the number of independent system equations, i.e. $\mathtt{p}(\mathfrak{B}) = \operatorname{rank}(R)$, where $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is any kernel representation of $\mathfrak{B}$. See also Note 1.

A fundamental concept is controllability of a system. A behaviour $\mathfrak{B} = \ker R(\frac{\mathrm{d}}{\mathrm{d}t})$ is said to be *controllable*, if for every w_1 and $w_2 \in \mathfrak{B}$, there exist $w_3 \in \mathfrak{B}$ and $\tau > 0$ such that

$$w_3(t) = \begin{cases} w_1(t) & \text{for all } t \leqslant 0, \\ w_2(t) & \text{for all } t \geqslant \tau. \end{cases}$$

The set of all *controllable* behaviours with $\mathtt{w}$ variables is denoted by $\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. It is shown in Polderman and Willems (1998, Section 5.2.1) that this patchability definition of controllability yields the traditional Kalman state-space definition of controllability for the state-space case. Further, it is shown there that $\mathfrak{B} = \ker R(\frac{\mathrm{d}}{\mathrm{d}t})$ is controllable if and only if $R(\lambda)$ has constant rank for all $\lambda \in \mathbb{C}$. Another important equivalence of controllability of a behaviour $\mathfrak{B}$ is that, for some $M(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$,

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{there exists } \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}}) \text{ such that } w = M(\tfrac{\mathrm{d}}{\mathrm{d}t})\ell\ \}.$$

This representation of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is known as an *image representation* and we also write $\mathfrak{B} = \operatorname{im} M(\frac{\mathrm{d}}{\mathrm{d}t})$. The variable ℓ is called a *latent* variable: these are auxiliary variables used to describe the behaviour; we distinguish the variable w as the *manifest variable*, the variable of interest. In such situations where we deal with the manifest variables and some more variables, say, ℓ, we often need to consider the *full behaviour* $\mathfrak{B}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\ell}$: this is the behaviour with variables (w, ℓ), all those that arise in the equations. Thus, a general form of a *latent variable representation* of a linear differential behaviour is $R(\frac{\mathrm{d}}{\mathrm{d}t})w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$, with R and M polynomial matrices of appropriate sizes. The *manifest behaviour* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ defined through such a latent variable representation is defined as

$$\mathfrak{B} := \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) | \text{ there exists } \ell \text{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}} \}.$$

Consider again the image representation $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$: it is known that $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ always allows an image representation with $M(\xi)$ such that $M(\lambda)$ has full column rank for every $\lambda \in \mathbb{C}$. This kind of image representation is known as an *observable* image representation. In this paper, unless otherwise stated explicitly, we assume[1] the image representations are observable. The use of the term 'observable' is motivated by the fact that the variable ℓ is *observable* from the variable w. This notion is defined as follows. For a behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}+\ell}$ with variables w and ℓ, the variable ℓ is said to be observable from w in $\mathfrak{B}$ if whenever (w, ℓ_1) and (w, ℓ_2) both are in $\mathfrak{B}$, we have $\ell_1 = \ell_2$. Observability of ℓ from w in a behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}+\ell}$ is equivalent to the existence[2] of a polynomial matrix $F(\xi)$ such that $\ell = F(\frac{\mathrm{d}}{\mathrm{d}t})w$ for all (w, ℓ) in $\mathfrak{B}$.

We next define relevant notions in the context of uncontrollable behaviours. For a behaviour $\mathfrak{B}$, possibly uncontrollable, the largest controllable behaviour contained in $\mathfrak{B}$ is called the *controllable* part of $\mathfrak{B}$, and denoted by $\mathfrak{B}_{\text{cont}}$. The controllable part of $\mathfrak{B}$ satisfies $\mathtt{m}(\mathfrak{B}_{\text{cont}}) = \mathtt{m}(\mathfrak{B})$. The set of complex numbers λ for which $R(\lambda)$ loses rank is called the set of *uncontrollable poles* and is denoted by Λ_{un}. The notion of uncontrollable poles is motivated by the poles of an *autonomous* behaviour. A behaviour $\mathfrak{B}$ is called autonomous if $\mathtt{m}(\mathfrak{B}) = 0$. Thus, any minimal kernel representation $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ of an autonomous $\mathfrak{B}$ is such that R is square and $\det R \neq 0$. The roots of $\det R$ depend only on $\mathfrak{B}$ and not on R: the poles of $\mathfrak{B}$ are defined as the roots of $\det R$ counted with multiplicity. For a recent exposition on behaviours, controllability and observability, we refer the reader to Willems (2007).

2.2 Quadratic differential forms

The concept of quadratic differential forms (QDFs) is central to this paper. See Willems and Trentelman (1998) for a detailed exposition. Consider a two-variable polynomial matrix with real coefficients, $\Phi(\zeta, \eta) := \sum_{j,k} \Phi_{jk}\zeta^j\eta^k \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, where $\Phi_{jk} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. The QDF Q_Φ induced by Φ is a map $Q_\Phi : \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \longrightarrow \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})$ defined as

$$Q_\Phi(w) := \sum_{j,k} \left(\frac{\mathrm{d}^j}{\mathrm{d}t^j} w\right)^T \Phi_{jk} \frac{\mathrm{d}^k}{\mathrm{d}t^k} w. \tag{1}$$

When dealing with quadratic forms in w and its derivatives, we can assume without loss of generality that $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^T$: such a Φ is called a symmetric two-variable polynomial matrix. A quadratic form induced by a real symmetric constant matrix $S \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ is a special QDF. We frequently need the number of positive and negative eigenvalues (counted with multiplicity) of a non-singular and symmetric matrix S: they are denoted by $\sigma_+(S)$ and $\sigma_-(S)$, respectively. For a symmetric and positive-definite matrix S, we define $S^{1/2}$ to be the unique symmetric positive-definite matrix P that satisfies $P^2 = S$. See Note 11 for *our* definition of $S^{1/2}$ when S is indefinite.

2.3 States

The state variable formalises the intuitive requirement of concatenability. A variable x is said to satisfy the *property of state* if the following holds: whenever $(w_1, x_1), (w_2, x_2) \in \mathfrak{B}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{x}}$ and $x_1(0) = x_2(0)$, then the new trajectory (w, x) formed by concatenating (w_1, x_1) to (w_2, x_2) at $t = 0$, defined as follows:

$$(w, x)(t) = \begin{cases} (w_1, x_1)(t) & \text{for all} \quad t \leqslant 0, \\ (w_2, x_2)(t) & \text{for all} \quad t > 0, \end{cases}$$

also satisfies the system equations of $\mathfrak{B}_{\text{full}}$ in a distributional[3] sense. In the context when we are given a behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and we *seek* a latent variable representation in which the latent variable x satisfies the state property and also x has the least number of components, one can ensure that x is observable from w and also that (w, x) satisfies the familiar *input/state/output* (i/s/o) representation below.[4] The i/o partition (after a permutation of components, if necessary) $w = (w_i, w_o)$ then results in properness of the transfer matrix from input w_i to output w_o (see Willems & Trentelman, 1998, p. 1706):

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw_i, \quad w_o = Cx + Dw_i, \quad \text{with } (C, A) \text{ observable.} \tag{2}$$

In this case, behavioural controllability of the system $\mathfrak{B}$ is equivalent to the conventional controllability of the pair (A, B).

2.4 Indefinite linear algebra

In this paper, we use certain properties of matrices that are self-adjoint with respect to an indefinite inner product. We briefly review self-adjoint matrices and neutral subspaces (see Gohberg et al., 2005, Section 2.1). For a complex number $\lambda \in \mathbb{C}$, its complex conjugate is denoted by $\bar{\lambda}$, while x^* and A^*, respectively, denote the complex-conjugate transpose of the vector $x \in \mathbb{C}^{\mathtt{n}}$ and complex matrix A. Let $P \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ be a non-singular and Hermitian matrix. The matrix P defines an *indefinite* inner product on $\mathbb{C}^{\mathtt{n}}$ by $[x, y]_P$: x^*Py. The term 'indefinite' is motivated by the possible indefiniteness of the Hermitian matrix P. If all the eigenvalues of P are non-negative or non-positive, P is said to be sign-definite.

Consider matrices A and $P \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ with P invertible and Hermitian. The *P-adjoint* of the matrix A, denoted by $A^{[*]}$, is defined as $A^{[*]}$: $=P^{-1}A^*P$. A matrix $A \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ is said to be *P-self-adjoint* if $A = A^{[*]}$, i.e. $A = P^{-1}A^*P$.

When a non-singular matrix P is not sign-definite, depending on the vector x, the value of $[x, x]_P$ can be zero, positive or negative. A subspace $\mathcal{M} \subseteq \mathbb{C}^{\mathtt{n}}$ is said to be *P-neutral* if the indefinite inner product $[x, x]_P = 0$ for all $x \in \mathcal{M}$.

The set of all eigenvalues (counted with multiplicity, for this paper) of a matrix A is denoted by $\sigma(A)$. The *partial multiplicities* of an eigenvalue λ are the sizes of the Jordan blocks corresponding to λ in the Jordan canonical form of A (see Gohberg et al., 2005, p. 326). Thus, the *number* of integers in the partial multiplicities of λ is exactly the geometric multiplicity of λ and these integers themselves, all positive, add up to the algebraic multiplicity of λ. When all the partial multiplicities of λ are one, i.e. when algebraic and geometric multiplicities of λ are equal, then λ is said to be a *semisimple* eigenvalue. When the algebraic multiplicity is one, then λ is called a simple eigenvalue. Since the Jordan blocks of a matrix A are unique only up to their order, the partial multiplicities too require to be ordered, say, ascending, while comparing partial multiplicities of a common eigenvalue λ of two matrices, say, A and B.

2.5 Storage functions: observable and unobservable

In this paper, we deal with systems that satisfy the dissipativity property, i.e. energy is *absorbed* by the system along every system trajectory, perhaps not necessarily at every time instant, but 'totally' when integrated over time. The total aspect of the energy involves an integral, thus bringing in the initial and final conditions of the concerned trajectory. The convenience of considering just those trajectories that start-from-rest and end-at-rest applies to only controllable systems since the compactly supported trajectories in a behaviour are dense in the behaviour only for controllable behaviours (see Pillai & Shankar, 1998, p. 398). The notion of a storage function helps in formulating the dissipation property as an inequality to be satisfied *at each time-instant*. A central issue in this paper is what variables should the storage function be allowed to depend on. We explore the dependence on the following:

- On a latent variable ℓ [the most general, since ℓ could include[5] w, with $(w, \ell) \in \mathfrak{B}^{\ell}_{\text{full}}$ described by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$], or
- On a state variable x (very intuitive, since storage requires 'memory'), or
- On the manifest variable w and its derivatives (observability from the manifest variables w, since the supplied power is expressed in w, see Note 2).

This dependence is explicitly indicated in the storage function subscript within the following definition. See also Willems and Trentelman (1998, Remark 5.9) and Willems (2004, 2007) for earlier remarks about the use of the storage function's existence as a definition of dissipativity and the link with controllability/observability.

Definition 2.1: *Storage function:* Let $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be a non-singular and symmetric matrix that induces the supply rate $w^T\Sigma w$. Consider a behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ with a manifest variable w and latent variable ℓ, with the corresponding full behaviour $\mathfrak{B}^{\ell}_{\text{full}}$, and ℓ possibly unobservable from w. For the behaviour $\mathfrak{B}$, let x be a state variable with the corresponding full behaviour $\mathfrak{B}^{x}_{\text{full}}$, and x possibly unobservable from w. With respect to the supply rate Σ and the behaviour $\mathfrak{B}$, the QDF Q_{Ψ_ℓ} is said to be a *storage function* if the *dissipation inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{\Psi_\ell}(\ell) \leqslant w^T\Sigma w \qquad (3)$$

is satisfied for all $(w, \ell) \in \mathfrak{B}^{\ell}_{\text{full}}$.

(i) A storage function Q_{Ψ_ℓ} is said to be an *observable* storage function if the latent variable ℓ is observable from the manifest variable w. In this case (see Note 2), there exists a storage function Q_{Ψ_w} such that $\frac{\mathrm{d}}{\mathrm{d}t} Q_{\Psi_w}(w) \leqslant w^T\Sigma w$ for all $w \in \mathfrak{B}$.

(ii) A storage function $Q_{\Psi_x}(x)$ is said to be a *state function* if $Q_{\Psi_x}(x)$ is equal to x^TKx for some constant matrix K.

(iii) A storage function x^TKx, with $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ a constant matrix, is called *observable* if the state x is observable from w.

As mentioned above, the question of whether the storage function is observable from w is relevant since the instantaneous power, i.e. the supply rate, is specified using the manifest variable w by $w^T\Sigma w$. We adopt the convention that the power $w^T\Sigma w$ is *absorbed* by the system when $w^T\Sigma w$ is positive. Observability of the storage function [Property (i) above] from the manifest variable w is motivated[6] by the fact that the power is expressed in terms of w. While at certain instants along a system trajectory, power absorbed can be negative, this can happen only due to a *decrease* of internal stored energy: this 'instantaneous energy auditing' is the meaning of the dissipation inequality (3). Since energy *storage* intuitively appears to be linked with 'memory', Property (ii) above relates the storage function to a *static-state* function (see Willems and Trentelman, 1998, Theorem 5.5). This paper studies situations when the storage function is a state function, but unobservable. The question whether there are situations that force storage functions to depend on *derivatives* of the state remains unaddressed.

This paper deals with uncontrollable dissipative LTI systems, and investigates observability properties and state-function properties of the storage function. The controllable case was resolved in Willems and Trentelman (1998, Theorem 5.5). We state this below for easy reference.

Proposition 2.1: *Consider $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and let $w = M(\frac{d}{dt})\ell$ be an observable image representation. Let x be a state variable for $\mathfrak{B}$ such that x is observable from w. Suppose $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ is symmetric and non-singular. Then, the following are equivalent:*

(1) $\int_{\mathbb{R}} w^T \Sigma w dt \geqslant 0$ *for all $w \in \mathfrak{B} \cap \mathfrak{D}$, the compactly supported trajectories in $\mathfrak{B}$.*
(2) $M(-j\omega)^T \Sigma M(j\omega) \geq 0$ *for all $\omega \in \mathbb{R}$.*
(3) *There exists a QDF Q_{Ψ_w} such that $\frac{d}{dt} Q_\Psi(w) \leqslant w^T \Sigma w$ for all $w \in \mathfrak{B}$.*
(4) *There exists a constant and symmetric matrix K such that $x^T K x$ is a storage function.*

The significance of the above theorem is that, for controllable systems, it is possible to verify dissipativity, in principle, by checking non-negativity of the integral $\int_{\mathbb{R}} w^T \Sigma w \, dt$ over all compactly supported trajectories: the compact support signifying that we calculate the 'net power' transferred when the system 'starts from rest' and 'ends at rest'. The term 'rest' is suggestive that all variables and their derivatives are zero and hence the internal/stored energy is also zero: this crucially helps in ruling out the storage function from playing a role in the dissipativity definition. In fact, the integral inequality property [Statement (1)] is used as the definition of dissipativity for controllable systems. The same cannot be done for uncontrollable systems due to the compactly supported trajectories not[7] being 'dense' in the behaviour (see Pillai & Shankar, 1998, p. 398, and Willems and Trentelman, 2002, p. 55).

As noted in Willems and Trentelman (1998, Remark 5.9), for uncontrollable systems, the existence of a storage function is the most natural definition of a dissipative system. Notwithstanding the *existential* aspect of the storage function, we show in this paper that whether or not the storage function should be assumed to be observable and/or a state function crucially affects the conclusion of dissipativity of a system. We propose a new definition of dissipativity: embeddability in a controllable dissipative behaviour, which resolves this drawback. This is dealt with in Section 6.

In order to focus on the issues raised above, and not on the variety[8] of notions of power, for the rest of this paper, we consider a coordinate transformation that simplifies all the main results and proofs, and further without loss of generality. When studying dissipativity with respect to a constant, non-singular and symmetric $S \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, one can transform the coordinates of the variable w by using a non-singular matrix V such that $V^T \Sigma V = S$ with Σ a diagonal matrix consisting of only $+1$ and -1 along the diagonal. We hence assume throughout this paper that the supply rate is $w^T \Sigma w$, with Σ given in Equation (4). It is known (see Willems & Trentelman, 1998, Remark 5.11, and Willems and Trentelman, 2002, p. 56) that Σ-dissipativity of a behaviour $\mathfrak{B}$ implies that $\mathtt{m}(\mathfrak{B})$, the input cardinality of $\mathfrak{B}$, cannot exceed $\sigma_+(\Sigma)$, i.e. $\mathtt{m}(\mathfrak{B}) \leqslant \sigma_+(\Sigma)$. Moreover, there exists an i/o partition such that all the inputs correspond to $+1$ only and such that the transfer function matrix from these inputs to all other remaining variables is *proper* (see Willems & Trentelman, 1998, Remark 5.11). In view of these facts and the inequality $\mathtt{m}(\mathfrak{B}) \leqslant \sigma_+(\Sigma)$, we assume without loss of generality

$$\Sigma = \begin{bmatrix} I_{\mathtt{m}} & 0 & 0 \\ 0 & I_{\mathtt{q}} & 0 \\ 0 & 0 & -I_{\mathtt{r}} \end{bmatrix} \text{ and also define } J_{\mathtt{q},\mathtt{r}} := \begin{bmatrix} I_{\mathtt{q}} & 0 \\ 0 & -I_{\mathtt{r}} \end{bmatrix}. \tag{4}$$

Recall that $\mathtt{p}(\mathfrak{B})$ denotes the number of outputs, i.e. $\mathtt{w} - \mathtt{m}(\mathfrak{B})$. The above partition is made noting that $\mathtt{m}(\mathfrak{B}) \leqslant \sigma_+(\Sigma)$, and using $\mathtt{r} := \sigma_-(\Sigma)$, which gives $\mathtt{q} = \mathtt{p} - \sigma_-(\Sigma)$ and also $\mathtt{q} = \sigma_+(\Sigma) - \mathtt{m}(\mathfrak{B})$. The partition of w corresponds to an i/o partition of $\mathfrak{B}$: the first $\mathtt{m}$ components of w are inputs, while the remaining $\mathtt{p} = (\mathtt{q} + \mathtt{r})$ components are outputs, and further with the transfer function matrix from the input to the output being proper. The case when $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$ plays a key role in the results of this paper: we call this the *maximum input cardinality condition*. In this case $J_{\mathtt{q},\mathtt{r}}$ is negative definite and of size $\mathtt{r}$. The matrices Σ and $J_{\mathtt{q},\mathtt{r}}$ have also been called 'signature matrices' in the literature.

3. Observable storage functions: main results

In this section, we state our first main result: assuming the so-called unmixing condition on the uncontrollable poles and assuming a strictness of dissipativity asymptotically as frequency tends to ∞, we show that the existence of an observable storage function for a given uncontrollable behaviour is equivalent to the controllable subsystem's dissipativity. We use the existence of an observable storage function as the definition of dissipativity in this section.

We call the assumption that no pair of uncontrollable poles are symmetric with respect to the imaginary axis as the *'unmixing assumption'* on the uncontrollable poles[9] of the behaviour. This is the same as no two uncontrollable poles adding to zero. Stabilisability of the system, for example, ensures this condition: see Polderman and Willems (1998, Section 5.2).

Theorem 3.1: *Consider a behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ as in Equation* (4)*. Assume that the uncontrollable poles Λ_{un} are such that $\Lambda_{un} \cap -\Lambda_{un} = \emptyset$. Let $\mathfrak{B}_{\text{cont}}$ have an observable image representation $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ and consider an i/o partition of $\mathfrak{B}_{\text{cont}}$ such that the corresponding partition of M(s) into*

$$M(s) = \begin{bmatrix} W_1(s) \\ W_2(s) \end{bmatrix}, \quad W_1 \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[s],\ W_2 \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}[s], \quad (5)$$

results in a proper $W_2(s)W_1(s)^{-1}$. Define $G(s) : W_2(s)W_1(s)^{-1}$ and $D : \lim_{s\to\infty} G(s)$. Assume $\mathfrak{B}_{\text{cont}}$ such that $(I_{\mathtt{m}} + D^T J_{\mathtt{q,r}} D) > 0$. Then, the following are equivalent:

(1) *$\mathfrak{B}$ is Σ-dissipative, i.e. there exists an observable storage function $Q_\Psi(w)$ satisfying $\frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) \leqslant w^T \Sigma w$ for all $w \in \mathfrak{B}$.*
(2) *The controllable part $\mathfrak{B}_{\text{cont}}$ is Σ-dissipative, i.e. $\int_{\mathbb{R}} w^T \Sigma w \mathrm{d}t \geqslant 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$, the compactly supported trajectories in $\mathfrak{B}$.*

Theorem 3.1 is proved in the next section. The above result gives conditions under which the autonomous part of a behaviour poses no hindrance to dissipativity of the entire behaviour once the controllable subsystem is dissipative. One of the conditions for this is that the uncontrollable poles are not 'mixed'. For autonomous LTI systems, this condition is a necessary and sufficient condition for solvability of the Lyapunov equation: see Rosenblum (1956). Also see Peeters and Rapisarda (2001) for an extension of the solvability condition for the *polynomial* Lyapunov equation. Of course, storage functions are just generalisations of Lyapunov functions to non-autonomous systems. The other condition: $(I_{\mathtt{m}} + D^T J_{\mathtt{q,r}} D) > 0$ on the controllable part is a kind of strictness of dissipativity 'at the infinity[10] frequency'. This is elaborated in the remark below.

Remark 3.1: The matrix $I + D^T J_{\mathtt{q,r}} D$ being nonnegative is a necessary condition for dissipativity of $\mathfrak{B}_{\text{cont}}$, and the non-negativity denotes dissipativity at very high frequencies, i.e. as $\omega \to \infty$: see Statement (2) of Proposition 2.1. Positive definiteness of $I + D^T J_{\mathtt{q,r}} D$ is nothing but strict dissipativity at frequency tending to infinity. This assumption helps in the existence of a Hamiltonian matrix. Our proofs use the Hamiltonian matrix properties and techniques from indefinite linear algebra, in particular, from Gohberg et al. (2005). Positive definiteness of $I + D^T J_{\mathtt{q,r}} D$ is guaranteed, for example, by strict dissipativity of a behaviour (see Pal & Belur, 2008); on the other hand, the matrix $I + D^T J_{\mathtt{q,r}} D$ is zero for lossless controllable behaviours.

Though most of the main results in this paper are formulated in terms of the system behaviour, the proofs use many known results about algebraic Riccati equations (AREs), Hamiltonian matrices and indefinite linear algebra: the availability of these results in the literature and the intuition built using them make it relatively difficult and less worthwhile to prove using just kernel representations and/or general latent variable representations. We now review some existing results and formulate/prove new results about Hamiltonian matrices in the context of dissipativity of controllable and uncontrollable systems. The rest of this section contains results about ARE solvability for the case of an uncontrollable pair (A, B) when some of the uncontrollable eigenvalues are on the imaginary axis.

The proposition below relates the existence of a symmetric solution to the dissipation LMI and a storage function $x^T K x$ for Σ-dissipativity. The following result is well known (refer to Trentelman & Willlems, 1991). See also Boyd, Ghaoui, Feron, and Balakrishnan (1994) for an elaborate treatment on LMIs.

Proposition 3.1 *(Trentelman & Willlems, 1991, Theorem 8.4.2)***:** *Consider a behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ with an i/s/o representation and A, B, C, D as state space matrices: with (C, A) observable and (A, B) possibly uncontrollable. Suppose $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is symmetric. Then $x^T K x$ is a storage function for Σ-dissipativity of $\mathfrak{B}$ if and only if K solves the LMI*

$$\begin{bmatrix} (KA + A^T K - C^T J_{\mathtt{q,r}} C) & (KB - C^T J_{\mathtt{q,r}} D) \\ (KB - C^T J_{\mathtt{q,r}} D)^T & -(I_{\mathtt{m}} + D^T J_{\mathtt{q,r}} D) \end{bmatrix} \leqslant 0. \quad (6)$$

As $(I_{\mathtt{m}} + D^T J_{\mathtt{q,r}} D)$ is invertible, the Schur complement of $(I_{\mathtt{m}} + D^T J_{\mathtt{q,r}} D)$ in the above LMI gives[11] the algebraic Riccati inequality (ARI)

$$\begin{aligned} &K(A - B(I_{\mathtt{m}} + D^T J_{\mathtt{q,r}} D)^{-1} D^T J_{\mathtt{q,r}} C) \\ &\quad + (A - B(I_{\mathtt{m}} + D^T J_{\mathtt{q,r}} D)^{-1} D^T J_{\mathtt{q,r}} C)^T K \\ &\quad + KB(I_{\mathtt{m}} + D^T J_{\mathtt{q,r}} D)^{-1} B^T K \\ &\quad - C^T (J_{\mathtt{q,r}} + DD^T)^{-1} C \end{aligned} \leqslant 0. \quad (7)$$

The corresponding *equation* is an ARE and we use properties of this ARE in proving Theorem 3.1. Define $\tilde{A} := (A - B(I_{\mathtt{m}} + D^T J_{\mathtt{q,r}} D)^{-1} D^T J_{\mathtt{q,r}} C), \tilde{D} := B(I_{\mathtt{m}} + D^T J_{\mathtt{q,r}} D)^{-1} B^T$ and $\tilde{C} := C^T (J_{\mathtt{q,r}} + DD^T)^{-1} C$, and rewrite the ARE as

$$K\tilde{A} + \tilde{A}^T K + K\tilde{D}K - \tilde{C} = 0. \quad (8)$$

Corresponding to this ARE, define the Hamiltonian matrix H and also the matrices M, P and $\widehat{P}$ as done in

Gohberg et al. (2005, Section 14.5):

$$H := \begin{bmatrix} \tilde{A} & \tilde{D} \\ \tilde{C} & -\tilde{A}^* \end{bmatrix}, \quad M := jH, \quad P := \begin{bmatrix} -\tilde{C} & \tilde{A}^* \\ \tilde{A} & \tilde{D} \end{bmatrix}$$
$$\text{and} \quad \widehat{P} := j\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \tag{9}$$

It is well known that a symmetric solution to an ARE can be obtained from an n-dimensional, M-invariant and P-neutral subspace and we state this in the proposition below. (Non-singularity of the P above is not a concern due to Gohberg et al., 2005, Lemma 14.5.1.) Let $K \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ be a Hermitian matrix. The *graph subspace* $\mathcal{G}(K) \subset \mathbb{C}^{2\mathtt{n}}$ corresponding to the matrix K is defined as

$$\mathcal{G}(K) := \operatorname{im} \begin{bmatrix} I \\ K \end{bmatrix}.$$

The following result, a restatement combining Propositions 14.4.1 and 14.5.2 of Gohberg et al. (2005), relates solvability of the ARE in terms of P-neutrality with P defined in (9).

Proposition 3.2 *(Gohberg et al., 2005, Propositions 14.4.1 and 14.5.2)***:** *Consider the ARE (8) with $(\tilde{A}, \tilde{D})$ possibly uncontrollable. Then $K \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ is a Hermitian solution of the ARE if and only if $\mathcal{G}(K)$ is M-invariant and P-neutral.*

The following decomposition of the state space is standard (see e.g. Kailath, 1980, p. 133): this simplifies the matrices we deal with.

Proposition 3.3: *Consider the behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ with an i/s/o representation $\frac{d}{dt}x = Ax + Bw_1$, $w_2 = Cx + Dw_1$, where $w = (w_1, w_2)$. Then there exists a non-singular matrix $T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that*

$$T^{-1}AT = \begin{bmatrix} A_c & A_{cp} \\ 0 & A_u \end{bmatrix},\ T^{-1}B = \begin{bmatrix} B_c \\ 0 \end{bmatrix} \text{ and } CT = \begin{bmatrix} C_c & C_u \end{bmatrix}$$
$$\text{with the pair } (A_c, B_c) \text{ controllable.} \tag{10}$$

Further, an i/s/o representation for the controllable subsystem $\mathfrak{B}_{\text{cont}}$ is given by

$$\frac{d}{dt}z = A_c z + B_c w_1, \quad w_2 = C_c z + D w_1. \tag{11}$$

In the context of the controllable part $\mathfrak{B}_{\text{cont}}$ of an uncontrollable behaviour $\mathfrak{B}$, we need the H_c and M_c matrices analogous to those in Equation (9) for $\mathfrak{B}$. Let $\mathtt{n_c}$ be the size of A_c above: recall that we began with an *observable* state-space realisation in (2). Define $\tilde{A}_c := (A_c - B_c(I_{\mathtt{m}} + D^T J_{\mathtt{q,r}} D)^{-1} D^T J_{\mathtt{q,r}} C_c)$, $\tilde{D}_c := B_c(I_{\mathtt{m}} + D^T J_{\mathtt{q,r}} D)^{-1} B_c^T$ and $\tilde{C}_c := C_c^T (J_{\mathtt{q,r}} + DD^T)^{-1} C_c$. Use these to define $H_c \in \mathbb{R}^{2\mathtt{n_c}\times 2\mathtt{n_c}}$:

$$H_c := \begin{bmatrix} \tilde{A}_c & \tilde{D}_c \\ \tilde{C}_c & -\tilde{A}_c^* \end{bmatrix} \text{ and } M_c := jH_c. \tag{12}$$

Lemma 3.1: *Suppose $\mathfrak{B}_{\text{cont}} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ satisfies the assumption that $(I_{\mathtt{m}} + D^T J_{\mathtt{q,r}} D) > 0$. If $\mathfrak{B}_{\text{cont}}$ is Σ-dissipative, then the partial multiplicities corresponding to the real eigenvalues of M_c, if any, are all even.*

In order to prove Lemma 3.1, we use a result from Gohberg et al. (2005) concerning the partial multiplicities of real eigenvalues of M_c. We use $\mathcal{R}_{\lambda_0}(A)$ to denote the generalised[12] eigenspace corresponding to an eigenvalue λ_0 in the spectrum $\sigma(A)$. Denote the controllable subspace of the pair $(\tilde{A}, \tilde{D})$ by $\mathfrak{C}_{\tilde{A},\tilde{D}}$.

Proposition 3.4 (Gohberg et al., 2005, Theorem 14.7.2)**:** *Consider the state-space description as in Equation* (11) *with (A_c, B_c) controllable and (C_c, A_c) observable. Construct the Hamiltonian matrix $H_c \in \mathbb{R}^{2\mathtt{n_c}\times 2\mathtt{n_c}}$ as in Equation* (12)*. Assume $\tilde{D}_c \geqslant 0$, $\tilde{C}_c^* = \tilde{C}_c$ and there exists a Hermitian solution $K \in \mathbb{C}^{\mathtt{n_c}\times\mathtt{n_c}}$ to the ARE (8). Suppose every purely imaginary eigenvalue λ_0 of $(\tilde{A}_c + \tilde{D}_c K)$, if any exists on the imaginary axis, satisfies $\mathcal{R}_{\lambda_0}(\tilde{A}_c + \tilde{D}_c K) \subseteq \mathfrak{C}_{\tilde{A}_c,\tilde{D}_c}$. Then, the partial multiplicities of corresponding real eigenvalues of M_c are all even and are twice the partial multiplicities of the corresponding purely imaginary eigenvalues of $(\tilde{A}_c + \tilde{D}_c K)$.*

Proof of Lemma 3.1: As the controllable part is Σ-dissipative, due to Proposition 2.1, there exists a storage function $x^T K x$ such that K is real and symmetric. By Proposition 3.1, the matrix K is a solution of the ARE. Since the system is controllable, the controllability subspace $\mathfrak{C}_{A_c,B_c} = \mathbb{C}^{\mathtt{n_c}}$ and hence $\mathfrak{C}_{\tilde{A}_c,\tilde{D}_c} = \mathbb{C}^{\mathtt{n_c}}$ too.[13] This proves that $\mathcal{R}_{\lambda_0}(\tilde{A}_c + \tilde{D}_c K) \subseteq \mathfrak{C}_{\tilde{A}_c,\tilde{D}_c}$ for every purely imaginary eigenvalue λ_0 of $(\tilde{A}_c + \tilde{D}_c K)$, if any exists on the imaginary axis. Thus, using Proposition 3.4, the partial multiplicities of every real eigenvalue of M_c, if any, are all even. This completes the proof of Lemma 3.1. □

For a complex matrix $M \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$, we define[14] a set called the *c-set* as in Gohberg et al. (2005, Section 5.12). The existence of such a *c-set* guarantees the existence of a unique P-neutral, M-invariant subspace $\mathcal{N}$ under suitable conditions on the partial multiplicities assumption. This is made precise in the proposition given below. We use $\overline{\mathcal{C}}$ to denote the set of complex conjugates of the elements in $\mathcal{C} \subset \mathbb{C}$.

Definition 3.1 (Gohberg et al., 2005, Section 5.12)**:** Let $M \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$ and let $\mathcal{C}$ be a finite set of non-real complex numbers. The set $\mathcal{C}$ is called a *c-set* of M if $\mathcal{C}$ satisfies the following properties:

(1) $\mathcal{C} \cap \overline{\mathcal{C}} = \emptyset$

(2) $\mathcal{C} \cup \overline{\mathcal{C}} = \sigma(M) \setminus \mathbb{R}$, the set of all non-real eigenvalues of M.

We use this set in this and the following sections. We denote the restriction of a matrix M to a subspace $\mathcal{N}$ by $M|_{\mathcal{N}}$. When $\mathcal{N}$ is M-invariant, the eigenvalues of the restriction $M|_{\mathcal{N}}$ are denoted by $\sigma(M|_{\mathcal{N}})$.

Proposition 3.5 *(Gohberg et al., 2005, Theorem 5.12.3)***:** *Let $M \in \mathbb{C}^{2\mathtt{n}\times 2\mathtt{n}}$ be a P-self-adjoint matrix such that the sizes of the Jordan blocks of M, say, m_1, m_2, …, m_r, corresponding to all real eigenvalues of M, are even. Then for every c-set $\mathcal{C}$ there exists a unique P-neutral, M-invariant subspace $\mathcal{N}$ of dimension $\mathtt{n}$ and $\sigma(M|_{\mathcal{N}}) \setminus \mathbb{R} = \mathcal{C}$. Further, the sizes of the Jordan blocks of $M|_{\mathcal{N}}$ corresponding to the real eigenvalues are $\frac{1}{2}m_1, \frac{1}{2}m_2, \ldots, \frac{1}{2}m_r$, respectively.*

It is easy to verify that M and P defined in Equation (9) satisfy $PM = M^*P$, i.e. M is P-self-adjoint. This allows the use of Proposition 3.5 when we need P-neutral, M-invariant subspaces corresponding to a suitable c-set.

The following proposition, which is a reformulation and combination of Theorems A.6.1, A.6.2 and A.6.3 in Gohberg et al. (2005), states that the partial multiplicities of an eigenvalue are unaffected by pre-multiplying and/or post-multiplying by a non-singular matrix. More generally, the multiplicity of the roots of the invariant polynomials remains unaffected too. [The invariant polynomials of a polynomial matrix $P(s)$ are the diagonal entries of the Smith form of $P(s)$.] This result is used in the proof of Theorem 3.1.

Proposition 3.6 (Gohberg et al., 2005, Section A.6)**:** *Consider $S_1(\xi)$ and $S_2(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ and let $p_{j,1}(\xi)$ and $p_{j,2}(\xi) \in \mathbb{R}[\xi]$ for $j = 1, \ldots, \mathtt{w}$ be the invariant polynomials of $S_1(\xi)$ and $S_2(\xi)$, respectively. Suppose $S_1(\xi) = T_1 S_2(\xi) T_2$, for invertible $T_1, T_2 \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Let $\lambda \in \mathbb{C}$ and $\beta_{1,i}, \ldots, \beta_{\mathtt{w},i}$ be the maximum integers such that, for each of $i = 1$ and 2, the factor $(\xi - \lambda)^{\beta_{j,i}}$ divides $p_{j,i}(\xi)$ for all $j = 1, \ldots, \mathtt{w}$. Then, $\beta_{j,1} = \beta_{j,2}$ for $j = 1, \ldots, \mathtt{w}$. In particular, if*

$$S_1(\xi) = \begin{bmatrix} \xi I_1 - P_1 & 0 \\ 0 & I_2 \end{bmatrix}$$

$$\text{and} \qquad S_2(\xi) = \begin{bmatrix} \xi I_1 - P_2 & 0 \\ 0 & Q(\xi) \end{bmatrix}$$

with $\det Q(\lambda) \neq 0$ *and P_1 and P_2 square constant matrices of appropriate size, then the partial multiplicities of $S_1(\xi)$ and $S_2(\xi)$ corresponding to λ are equal.*

We state and prove another lemma that is useful for proving Theorem 3.1. The following lemma relates the partial multiplicities of purely imaginary eigenvalues of the Hamiltonian matrix corresponding to the controllable part and those of the uncontrollable behaviour.

Lemma 3.2: *Consider the behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ with the set of uncontrollable poles Λ_{un} satisfying $\Lambda_{un} \cap -\Lambda_{un} = \emptyset$. Suppose $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bw_1$, $w_2 = Cx + Dw_1$, induced by $w = (w_1, w_2)$, is an observable i/s/o representation of $\mathfrak{B}$. Assume $\mathfrak{B}$ satisfies strict dissipativity at infinity, i.e. $(I_{\mathtt{m}} + D^T J_{\mathtt{q},\mathtt{r}} D) > 0$. Construct the Hamiltonian matrix H as in (9). Let $\mathfrak{B}_{\mathrm{cont}} = \operatorname{im} M(\frac{\mathrm{d}}{\mathrm{d}t})$ be an observable image representation. Then, the following hold:*

(1) *$\sigma(H) = \Lambda_{\mathrm{un}} \cup -\Lambda_{\mathrm{un}} \cup \mathit{roots} : \det M(-\xi)^T \Sigma M(\xi)$, counted with multiplicity.*
(2) *If the controllable part $\mathfrak{B}_{\mathrm{cont}}$ is Σ-dissipative, then the partial multiplicities corresponding to the purely imaginary eigenvalues of H, if any, are all even.*

Proof of Lemma 3.2: *Statement 1*: See Pal and Belur (2008, Theorem 5.4).

Statement 2: Consider the following i/s/o representation for $\mathfrak{B}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_c \\ x_u \end{bmatrix} = \begin{bmatrix} \hat{A}_c & \hat{A}_{cp} \\ 0 & \hat{A}_u \end{bmatrix}\begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} \hat{B}_c \\ 0 \end{bmatrix} w_1 \text{ and } w_2 = \begin{bmatrix} \hat{C}_c & \hat{C}_u \end{bmatrix}\begin{bmatrix} x_c \\ x_u \end{bmatrix} + \hat{D}w_1 \tag{13}$$

with $(\hat{A}_c, \hat{B}_c)$ controllable. Then, the Hamiltonian matrix gets the form

$$H = \begin{bmatrix} A_c & A_{cp} & B_c B_c^T & 0 \\ 0 & A_u & 0 & 0 \\ C_c^T C_c & C_c^T C_u & -A_c^T & 0 \\ C_u^T C_c & C_u^T C_u & -A_{cp}^T & -A_u^T \end{bmatrix}$$

$$\text{with} \quad \begin{aligned} A_u &:= \hat{A}_u, \\ A_c &:= \hat{A}_c - \hat{B}_c(I_{\mathtt{m}} + \hat{D}^T J_{\mathtt{q},\mathtt{r}}\hat{D})^{-1}\hat{D}^T J_{\mathtt{q},\mathtt{r}}\hat{C}_c, \\ A_{cp} &:= \hat{A}_{cp} - \hat{B}_c(I_{\mathtt{m}} + \hat{D}^T J_{\mathtt{q},\mathtt{r}}\hat{D})^{-1}\hat{D}^T J_{\mathtt{q},\mathtt{r}}\hat{C}_u, \\ B_c &:= \hat{B}_c(I_{\mathtt{m}} + \hat{D}^T J_{\mathtt{q},\mathtt{r}}\hat{D})^{-\frac{1}{2}}, \\ C_c &:= (J_{\mathtt{q},\mathtt{r}} + \hat{D}\hat{D}^T)^{-\frac{1}{2}}\hat{C}_c, \\ C_u &:= (J_{\mathtt{q},\mathtt{r}} + \hat{D}\hat{D}^T)^{-\frac{1}{2}}\hat{C}_u. \end{aligned} \tag{14}$$

The Hamiltonian matrix H_c for $\mathfrak{B}_{\mathrm{cont}}$ is

$$H_c = \begin{bmatrix} A_c & B_c B_c^T \\ C_c^T C_c & -A_c^T \end{bmatrix}.$$

In order to deduce Statement (2), we use Proposition 3.6 to relate the partial multiplicities of the $j\mathbb{R}$ eigenvalues of H and H_c. To achieve this, we multiply the polynomial matrix $\xi I_{2n} - H$ to its left and right by constant matrices E_i defined below to obtain the modified $\xi I_{2n} - H$ in a convenient form.

Consider

$$\xi I_{2n} - H = \begin{bmatrix} \xi I_{n_c} - A_c & -A_{cp} & -B_c B_c^T & 0 \\ 0 & \xi I_{n_u} - A_u & 0 & 0 \\ -C_c^T C_c & -C_c^T C_u & \xi I_{n_c} + A_c^T & 0 \\ -C_u^T C_c & -C_u^T C_u & A_{cp}^T & \xi I_{n_u} + A_u^T \end{bmatrix}.$$

Pre- and post-multiplication by E_1 of $\xi I_{2n} - H$ gives $E_1(\xi I_{2n} - H)E_1 =$

$$\begin{bmatrix} \xi I_{n_c} - A_c & -B_c B_c^T & -A_{cp} & 0 \\ -C_c^T C_c & \xi I_{n_c} + A_c^T & -C_c^T C_u & 0 \\ 0 & 0 & \xi I_{n_u} - A_u & 0 \\ -C_u^T C_c & A_{cp}^T & -C_u^T C_u & \xi I_{n_u} + A_u^T \end{bmatrix}$$

$$\text{with } E_1 := \begin{bmatrix} I_{n_c} & 0 & 0 & 0 \\ 0 & 0 & I_{n_c} & 0 \\ 0 & I_{n_u} & 0 & 0 \\ 0 & 0 & 0 & I_{n_u} \end{bmatrix}.$$

Since $\Lambda_{\text{un}} \cap j\mathbb{R} = \emptyset$, for $\lambda \in \sigma(H_c) \cap j\mathbb{R}$, the matrix blocks $\lambda I_{n_u} - A_u$ and $\lambda I_{n_u} + A_u^T$ are invertible. Pre-multiplying $E_1(\xi I_{2n} - H)E_1$ by E_2 and post-multiplying by E_3, we get

$$E_2 E_1(\xi I_{2n} - H)E_1 E_3 = \begin{bmatrix} \xi I_{n_c} - A_c & -B_c B_c^T & 0 & 0 \\ -C_c^T C_c & \xi I_{n_c} + A_c^T & 0 & 0 \\ 0 & 0 & \xi I_{n_u} - A_u & 0 \\ 0 & 0 & -C_u^T C_u & \xi I_{n_u} + A_u^T \end{bmatrix} \tag{15}$$

with

$$E_2 := \begin{bmatrix} I_{n_c} & 0 & -T_3 T_1^{-1} & 0 \\ 0 & I_{n_c} & -T_4 T_1^{-1} & 0 \\ 0 & 0 & I_{n_u} & 0 \\ 0 & 0 & 0 & I_{n_u} \end{bmatrix},$$

$$E_3 := \begin{bmatrix} I_{n_c} & 0 & 0 & 0 \\ 0 & I_{n_c} & 0 & 0 \\ 0 & 0 & I_{n_u} & 0 \\ -T_2^{-1} T_4^T & T_2^{-1} T_3^T & 0 & I_{n_u} \end{bmatrix}$$

and $T_1 := \lambda I_{n_u} - A_u$, $T_2 := \lambda I_{n_u} + A_u^T$, $T_3 := -A_{cp}$ and $T_4 := -C_c^T C_u$. Thus,

$$E_2 E_1(\xi I_{2n} - H)E_1 E_2 = \begin{bmatrix} \xi I_{2n_c} - H_c & 0 \\ 0 & Q_u \end{bmatrix},$$

$$\text{where } Q_u = \begin{bmatrix} T_1 & 0 \\ -C_u^T C_u & T_2 \end{bmatrix}.$$

Recall that $\Lambda_{\text{un}} \cap j\mathbb{R} = \emptyset$ and hence Q_u is invertible for all $\lambda \in j\mathbb{R}$. Finally, use Proposition 3.6 to conclude that the partial multiplicities of purely imaginary eigenvalues of H and H_c are both even. This completes the proof of Lemma 3.2. □

Of course, notwithstanding the manipulations involved, Statement (2) of Lemma 3.2 is fairly expected: since there are no uncontrollable poles on the imaginary axis, the property that the partial multiplicities of $j\mathbb{R}$ eigenvalues are even holds for H_c and hence for H too.

4. Observable storage function: proof of Theorem 3.1 and examples

In this section, we prove our first main result, namely, assuming no two uncontrollable poles of a behaviour $\mathfrak{B}$ add to zero, and assuming strict dissipativity on the controllable part $\mathfrak{B}_{\text{cont}}$ at the infinity frequency, the dissipativity of $\mathfrak{B}_{\text{cont}}$ is equivalent to the existence of an *observable* storage function for the whole behaviour $\mathfrak{B}$. This section also has a corollary showing that the storage function constructed within the proof is, in fact, a state function. We then apply these results to two examples.

4.1 Proof of Theorem 3.1

Proof of Theorem 3.1: (2)⇒(1): Assume that the controllable part $\mathfrak{B}_{\text{cont}}$ is Σ-dissipative and the assumptions in the theorem are satisfied. The proof outline is as follows. By Propositions 3.1 and 3.2, in order to prove that the behaviour $\mathfrak{B}$ is dissipative, it suffices to show the existence of a matrix $K \in \mathbb{C}^{n\times n}$ such that the corresponding graph subspace is an n-dimensional, M-invariant, P-neutral subspace. To show the existence, we use Proposition 3.5 to construct a c-set such that the corresponding n-dimensional M-invariant, P-neutral subspace is a graph subspace. This gives a matrix K which is a solution to the ARE, and a storage function for the whole behaviour would then be defined as $x^T Kx$, thus completing the proof.

Since the unmixing assumption on uncontrollable poles Λ_{un} holds, $\lambda \in j\mathbb{R} \cap \sigma(H)$ means that $\lambda \notin \Lambda_{\text{un}}$ and hence $\lambda \in \sigma(H_c)$. Due to the assumption that the controllable part is Σ-dissipative, from Lemma 3.2, the partial multiplicities of real eigenvalues of $M(:= iH)$ are all even. Using this fact, Proposition 3.5 can be used to infer that there exists a unique n-dimensional, M-invariant, P-neutral subspace for every c-set. It remains to show the existence of a c-set such that the corresponding n-dimensional, M-invariant, P-neutral subspace is also a graph subspace.

Choose a c-set $\mathcal{C}$ for M. Let $\mathcal{L}$ be the n-dimensional, P-neutral, M-invariant subspace of $\mathbb{C}^{2n}$ corresponding to $\mathcal{C}$

and suppose

$$\mathcal{L} = \operatorname{im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

for matrices X_1 and $X_2 \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$. In order to prove that $\mathcal{L}$ is a graph subspace, it is required to show that X_1 is invertible. This is proved by contradiction: we assume X_1 is singular and show that we get a contradiction to the unmixing assumption on Λ_{un}. This constitutes the rest of the proof of the '(2)⇒(1) part'. We prove this exactly along the lines of Gohberg et al. (2005, Proof of Lemma 14.6.1) and Pal and Belur (2008, Proof of Theorem 5.5).

M-invariance of $\mathcal{L}$ implies that

$$j\begin{bmatrix} \tilde{A} & \tilde{D} \\ \tilde{C} & -\tilde{A}^* \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} T$$

for some $T \in \mathbb{C}^{\mathtt{n}\times\mathtt{n}}$. In other words,

$$j(\tilde{A}X_1 + \tilde{D}X_2) = X_1 T \text{ and } j(\tilde{C}X_1 - \tilde{A}^* X_2) = X_2 T. \tag{16}$$

Since $\mathcal{L}$ is also P-neutral,

$$\begin{bmatrix} X_1^* & X_2^* \end{bmatrix}\begin{bmatrix} -\tilde{C} & \tilde{A}^* \\ \tilde{A} & \tilde{D} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = X_2^*\tilde{D}X_2 + X_1^*\tilde{A}^*X_2 + X_2^*\tilde{A}X_1 - X_1^*\tilde{C}X_1 = 0. \tag{17}$$

Suppose X_1 is singular. Define $\mathcal{K} := \ker X_1 \neq \{0\}$. From (17), for every $x \in \mathcal{K}$ we have

$$x^* X_2^*\tilde{D}X_2 x + x^* X_1^*\tilde{A}^* X_2 x + x^* X_2^*\tilde{A}X_1 x - x^* X_1^*\tilde{C}X_1 x = 0, \tag{18}$$

which implies $x^* X_2^* \tilde{D} X_2 x = 0$. Since $\tilde{D} \geqslant 0$, $X_2 x \in \ker \tilde{D}$, i.e. $X_2\mathcal{K} \subseteq \ker\tilde{D}$. Now, for every $x \in \mathcal{K}$, from the first equation in (16) we have $X_1 T x = j\tilde{A}X_1 x + j\tilde{D}X_2 x = 0$, that is, $T\mathcal{K} \subseteq \mathcal{K}$. This implies $\mathcal{K}$ is T-invariant. Due to $\mathcal{K} \neq \{0\}$, there exists an eigenvector v of T with $v \in \mathcal{K}$ corresponding to an eigenvalue, say, λ, of T and of M. We first claim and prove that λ cannot be a real eigenvalue.

Post-multiplying the second equation of (16) by v, we get

$$j\tilde{C}X_1 v - j\tilde{A}^* X_2 v = X_2 T v \quad \text{and hence } -j\tilde{A}^* X_2 v = \lambda X_2 v. \tag{19}$$

Since $X_1 v = 0$, and due to $\mathcal{L}$ being n-dimensional, $X_2 v \neq 0$. Equation (19) implies that $X_2 v$ is a left eigenvector of $\tilde{A}$ with eigenvalue $-j\,\bar{\lambda}$. We also had shown (immediately following Equation (18)) that $X_2 v \in \ker\tilde{D}$: this implies $B^T X_2 v = 0$. This means that $-j\,\bar{\lambda}$ is an uncontrollable eigenvalue of $\tilde{A}$ and $-j\,\bar{\lambda} \in \Lambda_{\text{un}}$, i.e. $\bar{\lambda} \in j\Lambda_{\text{un}}$. Now, if λ were real, then $-j\,\bar{\lambda}$ and $j\,\bar{\lambda}$ both belong to Λ_{un} and thus contradicts the unmixing assumption $\Lambda_{\text{un}} \cap \Lambda_{\text{un}} = \emptyset$. This proves the claim that λ cannot be real. It remains to show the contradiction on the unmixing assumption Λ_{un} when λ is non-real.

By definition of a c-set and Proposition 3.5, it follows that $\sigma(T) \setminus \mathbb{R} = \mathcal{C}$ and hence $\lambda \in \mathcal{C}$. We already showed that $\bar{\lambda} \in j\Lambda_{\text{un}}$ and we had started with $j\Lambda_{\text{un}} \subset \mathcal{C}$. This gives $\bar{\lambda} \in \mathcal{C}$ and hence the contradiction to $\mathcal{C}$ being a c-set. This contradiction proves that X_1 has to be non-singular. Define $K := X_2 X_1^{-1}$ and this K solves the LMI and hence $x^T K x$ is a storage function. This completes the proof of (2)⇒(1).

(1)⇒(2): Assume $\mathfrak{B}$ is Σ-dissipative. Then, there exists a storage function $Q_\psi(w)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_\psi(w) \leqslant w^T \Sigma w \qquad \text{for all } w \in \mathfrak{B}. \tag{20}$$

Integrating both sides, we get that for every $w \in \mathfrak{B} \cap \mathfrak{D}$, $\int_{\mathbb{R}} w^T \Sigma w \mathrm{d}t \geqslant 0$. This implies that $\mathfrak{B}_{\text{cont}}$ is Σ-dissipative. This completes the proof of Theorem 3.1. □

The above proof is constructive in the sense that if a behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ satisfies the following three conditions:

(1) Uncontrollable poles are unmixed, i.e. no two of them add to zero;
(2) The controllable part $\mathfrak{B}_{\text{cont}}$ is Σ-dissipative;
(3) The controllable part $\mathfrak{B}_{\text{cont}}$ is strictly dissipative at infinity, i.e. $(I_{\mathtt{m}} + D^T J_{\mathtt{q},\mathtt{r}} D) > 0$, where D is the feed-through term of the transfer function matrix for any choice of i/o partition that results in a proper transfer matrix;

then we construct a storage function $x^T K x$ that satisfies the dissipation inequality for the whole behaviour $\mathfrak{B}$. Further, the state x was assumed to be observable. These facts lead to the following corollary.

Corollary 4.1: *Consider Σ as in Equation* (4) *and let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be an uncontrollable behaviour that satisfies assumptions* (1), (2) *and* (3) *above. Then, the following are equivalent:*

(1) *$\mathfrak{B}_{\text{cont}}$ is dissipative.*
(2) *There exists a $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ such that $Q_\Psi(w)$ is a storage function, i.e. $\frac{\mathrm{d}}{\mathrm{d}t} Q_\psi(w) \leqslant w^T \Sigma w$ for all $w \in \mathfrak{B}$.*
(3) *There exists a matrix $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and an observable state variable x such that $\frac{\mathrm{d}}{\mathrm{d}t} x^T K x \leqslant w^T \Sigma w$ for all $w \in \mathfrak{B}$.*

Statement (3) formalises that the storage of energy requires no more memory of past evolution of trajectories than that required for arbitrary concatenation of any two system trajectories. See Willems and Trentelman (1998, Theorem 5.5) for the controllable case and related discussion.

$$F(\xi) = \left[\left(R\xi^3 + \left(\frac{1}{C_1} + \frac{2R^2}{C_1 L}\right)\xi^2 + \left(\frac{2R}{C_1 L} + \frac{R}{C_1 L C_2}\right)\xi + \frac{1}{C_1 L C_2}\right) - \left(\xi^3 + \frac{2R}{C_1 L}\xi^2 + \frac{1}{C_1 L C_2}\xi\right)\right].$$

4.2 Examples

In this section, we discuss two examples of uncontrollable systems that are dissipative. The AREs encountered in these cases are solved using the methods proposed in this paper; we also give explicit solutions.

The first example is of an uncontrollable system with uncontrollable poles satisfying the unmixing assumption, i.e. no two of the uncontrollable poles add to zero. However, the Hamiltonian matrix has eigenvalues on the imaginary axis.

Example 4.2: Consider the behaviour $\mathfrak{B}$ whose i/s/o representation is given by the following A, B, C and D matrices:

$$A = \begin{bmatrix} 0 & -0.5 \\ 1 & -1.5 \end{bmatrix}, \quad B = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & -0.5 \end{bmatrix}, \quad D = 0.5$$

with $\sigma(A) = \{-\frac{1}{2}, -1\}$. Here $\Lambda_{\text{un}} = \{-1\}$ which satisfies the unmixing assumption. A kernel representation of the behaviour is given by

$$\left[\left(\frac{d^2}{dt^2} + 2\frac{d}{dt} + 1\right) \; -\left(2\frac{d^2}{dt^2} + 3\frac{d}{dt} + 1\right)\right]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0.$$

In this case $\Sigma = \text{diag}(1, -1)$ and hence $\sigma_+(\Sigma) = \mathtt{m}(\mathfrak{B})$, it can be checked that the controllable part $\mathfrak{B}_{\text{cont}} = \ker\left[(\frac{d}{dt} + 1) \; -(2\frac{d}{dt} + 1)\right]$ is Σ-dissipative. Further $(I_{\mathtt{m}} - D^T D) = 0.75 > 0$. Thus, from Theorem 3.1, $\mathfrak{B}$ is Σ-dissipative. A two-dimensional, M-invariant, P-neutral subspace, say, $\mathcal{L}$, which gives a solution K, and the corresponding real symmetric matrix K that induces a storage function are

$$\mathcal{L} = \text{im}\begin{bmatrix} 1 & 2 \\ 1 & 8 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad K = \frac{1}{6}\begin{bmatrix} 7 & -1 \\ -1 & 1 \end{bmatrix}.$$

Example 4.3: Consider the RLC circuit in Fig. 1, for which the supply rate is vi, and due to only passive elements, we expect dissipativity. Assume $R_L = R_C =: R$. One can check that the system becomes uncontrollable when $L = R^2 C_2$. A kernel representation of the uncontrollable system is given by $F(\frac{d}{dt})\begin{bmatrix} i \\ v \end{bmatrix} = 0$ with

Let $R = 0.5$, $C_1 = C_2 = 1$ and $L = 0.25$. The uncontrollable pole is at -2 with multiplicity 2. In order to use our main results, we need the supply rate to be $u^2 - y^2$ instead of the supply rate vi: that this is always possible is explained in the text before Equation (4). We use the coordinate transformation $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ on the variables (v, i) to obtain the transformed system variables (u, y), and for the transformed system, we obtain an observable state-space realisation:

$$A = \begin{bmatrix} -\frac{14}{3} & 1 & 0 \\ -\frac{20}{3} & 0 & 1 \\ -\frac{8}{3} & 0 & 0 \end{bmatrix}, \quad B = \frac{1}{3}\begin{bmatrix} 2 \\ 8 \\ 8 \end{bmatrix},$$
$$C = \begin{bmatrix} -\frac{4}{3} & 0 & 0 \end{bmatrix}, \quad D = \frac{1}{3}. \tag{21}$$

Poles of the system are -2, -2 and $-\frac{2}{3}$ and two of them, -2 and -2, are uncontrollable. (The uncontrollable poles remain unchanged by the manifest variable coordinate transformation.) The transfer function G from u to y is $G(s) = \frac{s-2}{3s+2}$; the controllable part is not strictly dissipative with respect to $u^2 - y^2$, since the $\mathcal{H}_\infty$ norm[15] of $G(s)$ equals 1. Further $(I_{\mathtt{m}} - D^T D) > 0$, i.e. the controllable part is strictly dissipative at frequency tending to infinity: though the $\mathcal{H}_\infty$ norm is equal to 1, the supremum is attained at $\omega = 0$ and not asymptotically as $\omega \to \infty$. Check that the uncontrollable poles are unmixed. Thus, from Theorem 3.1, the system is dissipative. A solution to the ARE is given by the following symmetric matrix:

$$K = \frac{1}{64}\begin{bmatrix} 16 & 0 & -4 \\ 0 & 4 & -4 \\ -4 & -4 & 13 \end{bmatrix}. \tag{22}$$

Note that the state x, with respect to which the above K gives the storage function $x^T K x$, does not relate physically to the capacitor voltages and inductor current: these physical states are unobservable from the port variables (v, i) for the uncontrollable case. Equation (21) is an observable realisation of the uncontrollable system, after the above-mentioned coordinate transformation in the port variables (v, i). We will revisit this example later in the context of the embeddability definition of dissipativity and conclude that, quite unreasonably, this RLC system cannot be dissipative by that definition!

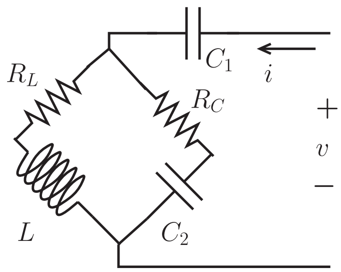

Fig. 1. An RLC circuit (Example 4.3).

5. Unobservable storage functions: lossless trajectories

One of the two key assumptions in the previous sections was that no two uncontrollable poles add to zero: the necessity of this is investigated in this section. We address the situation when the uncontrollable poles are on the imaginary axis. We first consider the case when the behaviour is autonomous and 'lossless' and later relax autonomy to some extent. In both cases, we show observable imaginary-axis poles rule out symmetric ARE solutions; with the *intuitive expectation* that storage functions are state functions. Thus, this section rules[16] out observable state functions as storage functions when there are some purely-imaginary-axis uncontrollable poles. We note that the case of imaginary-axis poles is just one converse to the unmixing assumption, the other – when two *non-zero real* poles add to zero – requires further investigation and is not pursued in this paper. We note, however, that Lyapunov equation solvability studies are closely linked with the unmixing assumption on the uncontrollable poles: the case of autonomous systems when two non-zero real poles add to zero has been related to 'the presence of time-reversible non-periodic solutions' in Willems (1998, p. 6). In a different context, it has been noted in Pal and Belur (2008, Theorem 8.3) that the single-output case, loosely speaking, couples the mixed uncontrollable poles due to which it has been shown there that unmixing of $\Lambda_{\rm un}$ is *necessary* for the existence of an ARE solution for the single-output case.

The following lemma shows that the existence of a storage function x^TKx which is a state function forces imaginary-axis eigenvalues to be unobservable. The link with losslessness is elaborated below in Remark 5.1.

Lemma 5.1: *Consider an autonomous behaviour $\mathfrak{B}_{\rm aut}$ described by $\frac{\rm d}{{\rm d}t}x = Ax$ and $w = Cx$, with $\sigma(A) \cap j\mathbb{R} \neq \emptyset$. Let the supply rate be $-w^Tw$. Suppose there exists a storage function x^TKx satisfying the inequality $\frac{\rm d}{{\rm d}t}x^TKx \leqslant -w^Tw$ for all $w \in \mathfrak{B}_{\rm aut}$. Then, every $\lambda \in \sigma(A) \cap j\mathbb{R}$ is unobservable.*

Proof: Suppose there exists a storage function x^TKx satisfying the dissipation LMI (6). In this case, the LMI boils down to the Lyapunov inequality $KA + A^*K + C^*C \leq 0$. Notice that every eigenvector x of A corresponding to eigenvalue $\lambda \in j\mathbb{R}$ gives $x^*(KA + A^*K + C^*C)x \leq 0$, and hence

$$\begin{aligned} &\lambda x^*Kx + \bar{\lambda}x^*Kx + x^*C^*Cx \leqslant 0 \\ &\text{which yields} \quad x^*C^*Cx \leqslant 0. \end{aligned} \tag{23}$$

However, since $x^*C^*Cx \geq 0$, we have $Cx = 0$ for every eigenvector of A corresponding to $\lambda \in \sigma(A) \cap j\mathbb{R}$. This implies that any $\lambda \in \sigma(A) \cap j\mathbb{R}$ is unobservable. This completes the proof.

The above result shows that for dissipativity of autonomous systems having eigenvalues on the imaginary axis, it is necessary to allow storage functions to depend on unobservable variables also. Note that this has been proved only for the special case when the supply rate satisfies $\sigma_+(\Sigma) = 0$, the maximum input cardinality condition for the case of an autonomous $\mathfrak{B}$. The condition $\sigma_+(\Sigma) = 0$ means that along every non-zero trajectory the power extracted out by the system is positive at every time instant.

Remark 5.1: It is reasonable to use the term 'lossless' for autonomous dynamical systems whose trajectories are periodic trajectories; more precisely, $\frac{\rm d}{{\rm d}t}x = Ax$ with $\sigma(A) \subset j\mathbb{R}$ and all eigenvalues semisimple. Such systems are just linear oscillatory systems without damping (see Rapisarda & Willems, 2005). In Sections 6 and 7, we touch upon 'losslessness' of non-autonomous systems, although in a different sense.

The autonomous aspect of the above result can be relaxed to some extent by allowing a 'static controllable part': the following state-space system makes this concrete. Consider $\frac{\rm d}{{\rm d}t}x = Ax$ and $y = Cx + Du$. Thus, all the states are uncontrollable. The lemma below states that imaginary-axis eigenvalues of A, together with observability, rule out real and symmetric solutions to the ARE, and hence rule out observable state functions as storage functions.

Lemma 5.2: *Consider a behaviour $\mathfrak{B}$ having a state-space representation $\frac{\rm d}{{\rm d}t}x = Ax$ and $y = Cx + Du$ with (C, A) observable. Consider the supply rate $u^Tu - y^Ty$. Suppose $I - D^TD > 0$ and, further, suppose $\Lambda_{\rm un} \subset j\mathbb{R}$. Then, there does not exist a symmetric solution to the corresponding ARE.*

The proof follows closely along the lines of that of Lemma 5.1; hence, we give only an outline. For this case too, the ARI boils down to a Lyapunov inequality $KA + A^*K + C^*(I - DD^T)^{-1}C \leq 0$. If a solution K exists, then the contradiction in (C, A) observability is obtained by noting that positive semi-definiteness of $C^*(I - DD^T)^{-1}C$ causes right eigenvectors of A corresponding to imaginary-axis eigenvalues of A to be in the nullspace of C. This proves that there does not exist an ARE/Lyapunov equation solution.

The assumptions in Lemma 5.2 can be understood as follows. Notice that the maximum input cardinality condition holds here and $J_{\mathtt{q},\mathtt{r}} = -I$. Further, relevant to this case, positive definiteness of $I - D^TD$ and $I - DD^T$ is equivalent. This helps to obtain the contradiction on the observability of (C, A) through Equation (23). When $D = 0$, the above lemma 'decouples' the input and output to yield Lemma 5.1.

6. Orthogonality and uncontrollable behaviours

In this section, we investigate the property of orthogonality of two behaviours in the absence of controllability. Note that orthogonality is a special case of losslessness (see Proposition 6.1 below); while we do not delve much into losslessness, the drawbacks we raise in the context of uncontrollable orthogonal behaviours are applicable to lossless behaviours too. We first review a result about orthogonality of controllable behaviours. The result is a combination of Willems and Trentelman (2002, Proposition 4, p. 57) and Willems and Trentelman (1998, Theorem 3.1).

Proposition 6.1 *(Willems & Trentelman, 2002, Proposition 4, Part I)***:** *Let $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be non-singular, and suppose $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. The following are equivalent:*

(1) $\int_{\mathbb{R}} w_1^T \Sigma w_2 dt = 0$ *for all* $w_1 \in \mathfrak{B}_1 \cap \mathfrak{D}$ *and for all* $w_2 \in \mathfrak{B}_2 \cap \mathfrak{D}$.
(2) $\mathfrak{B}_1 \times \mathfrak{B}_2$ *is lossless with respect to* $\begin{bmatrix} 0 & \Sigma \\ \Sigma^T & 0 \end{bmatrix}$.
(3) *There exists a bilinear*[17] *differential form* L_Ψ, *induced by* $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, *such that* $\frac{d}{dt} L_\Psi(w_1, w_2) = w_1^T \Sigma w_2$.

Statement (1) above is taken as the definition of orthogonality between two controllable behaviours $\mathfrak{B}_1$ and $\mathfrak{B}_2$ in Willems and Trentelman (2002, Part 1, Section V-B) (see also Willems & Trentelman, 1998, Section 10). The bilinear differential form (BDF) L_Ψ, called the *adapted* BDF in the situation of Statement (3), may be viewed as a variant of a storage function. Keeping in line with Definition 2.1 for dissipativity, Statement (3) could have been considered as the definition of orthogonality for behaviours not necessarily controllable. The drawback of using such a definition is elaborated below in this section (in Example 6.1). We pursue a different definition as follows. Notice that if $\mathfrak{B}_1$ and $\mathfrak{B}_2$ satisfy the integral condition in Statement (1), then this integral condition is satisfied for each respective subbehaviour $\mathfrak{B}_1'$ and $\mathfrak{B}_2'$ also. The following definition builds on this property.

Definition 6.1: Consider a non-singular $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and let $\mathfrak{B}_1$ and $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$. Behaviours $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are said to be Σ-orthogonal (and denoted by $\mathfrak{B}_1 \perp_\Sigma \mathfrak{B}_2$) if there exist $\mathfrak{B}_1^c$ and $\mathfrak{B}_2^c \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ such that

- $\int_{\mathbb{R}} w_1^T \Sigma w_2 \, dt = 0$ for all $w_1 \in \mathfrak{B}_1^c \cap \mathfrak{D}$ and for all $w_2 \in \mathfrak{B}_2^c \cap \mathfrak{D}$,
- $\mathfrak{B}_1 \subseteq \mathfrak{B}_1^c$ and $\mathfrak{B}_2 \subseteq \mathfrak{B}_2^c$.

When $\Sigma = I$, we denote $\mathfrak{B}_1 \perp_\Sigma \mathfrak{B}_2$ by $\mathfrak{B}_1 \perp \mathfrak{B}_2$. Instead of the definition being existential in the storage function (Section 3 and the results there), Definition 6.1 is existential in $\mathfrak{B}_1^c$ and $\mathfrak{B}_2^c$, raising new questions about how to check orthogonality. We show in the following subsection (in Theorem 6.1) that when $\mathfrak{B}$ is uncontrollable, any controllable $\mathfrak{B}^c \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ such that $\mathfrak{B} \subseteq \mathfrak{B}^c$ satisfies $\mathtt{m}(\mathfrak{B}) < \mathtt{m}(\mathfrak{B}^c)$ and we also obtain a precise count of the minimum difference $\mathtt{m}(\mathfrak{B}^c) - \mathtt{m}(\mathfrak{B})$.

6.1 Smallest controllable superbehaviour

Consider the following problem of 'embedding' a behaviour, possibly uncontrollable, in a controllable behaviour that is smallest in the sense of input cardinality.

Problem 6.1: Let $\mathfrak{B}_1 \in \mathfrak{L}^{\mathtt{w}}$. Find a controllable $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ such that $\mathfrak{B}_2 \supseteq \mathfrak{B}_1$ and $\mathfrak{B}_2$ has the smallest input cardinality amongst all controllable behaviours containing $\mathfrak{B}_1$.

Existence of such a smallest superbehaviour, its input cardinality and its uniqueness is addressed in the following result.

Theorem 6.1: *Let $R_1(\frac{d}{dt})w = 0$ be a minimal kernel representation of $\mathfrak{B}_1 \in \mathfrak{L}^{\mathtt{w}}$. The following statements are true:*

(1) *There exists $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ satisfying the requirements in Problem 6.1.*
(2) *Assume $\mathfrak{B}_1$ is uncontrollable. The input cardinality $\mathtt{m}(\mathfrak{B}_2)$ satisfies $\mathtt{m}(\mathfrak{B}_2) > \mathtt{m}(\mathfrak{B}_1)$. More precisely, $\mathtt{m}(\mathfrak{B}_2) = \mathtt{m}(\mathfrak{B}_1) + \mathtt{k}$ where $\mathtt{k} := \max_{\lambda\in\mathbb{C}} \operatorname{rank} R_1(\lambda) - \min_{\lambda\in\mathbb{C}} \operatorname{rank} R_1(\lambda)$.*
(3) *The behaviour $\mathfrak{B}_2$ is unique in and only in the following trivial cases:*
    (i) *$\mathfrak{B}_1$ is controllable, and then $\mathfrak{B}_1 = \mathfrak{B}_2$.*
    (ii) *$\mathfrak{B}_2 = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$: this is when $\min_{\lambda\in\mathbb{C}} \operatorname{rank} R_1(\lambda) = 0$.*

Proof: (1): This is shown by constructing a $\mathfrak{B}_2$ as required in Problem 6.1. Consider the minimal kernel representation $R_1(\frac{d}{dt})w = 0$ of $\mathfrak{B}_1$ and suppose $R_1 \in \mathbb{R}^{\mathtt{p}_1\times\mathtt{w}}[\xi]$. For any $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$, with kernel representation, say, $R_2(\frac{d}{dt})w = 0$ and $R_2 \in \mathbb{R}^{\mathtt{p}_2\times\mathtt{w}}[\xi]$, the behaviour $\mathfrak{B}_2$ satisfies $\mathfrak{B}_2 \supseteq \mathfrak{B}_1$ if and only if $FR_1 = R_2$ for some $F \in \mathbb{R}^{\mathtt{p}_2\times\mathtt{p}_1}[\xi]$. Thus, we need to find a suitable F such that the resulting $\mathfrak{B}_2$ satisfies the requirements in Problem 6.1. Without loss of generality, F can be assumed to be a full row rank polynomial matrix. Controllability of $\mathfrak{B}_2$ is equivalent to $F(\lambda)R_1(\lambda)$ being full row rank for every $\lambda \in \mathbb{C}$. The minimality of the input cardinality of $\mathfrak{B}_2$ is equivalent to requiring F to have the largest number of rows among all F such that $F(\lambda)R_1(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$.

Let Λ_{un} be the set of uncontrollable poles of $\mathfrak{B}_1$. Without loss[18] of generality, we assume R_1 to be in its Smith form: $R_1 = [\, S \; 0\,]$, and further, S partitioned as $S = \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}$ such that the identity matrix $I \in \mathbb{R}^{(\mathtt{p}_1-\mathtt{k})\times(\mathtt{p}_1-\mathtt{k})}$ and $D \in \mathbb{R}^{\mathtt{k}\times\mathtt{k}}[\xi]$ is a diagonal matrix with its non-constant diagonal entries $d_1, d_2, \ldots, d_{\mathtt{k}} \in \mathbb{R}[\xi]\backslash\mathbb{R}$ satisfying the divisibility property: $d_1|d_2$, $d_2|d_3$,... and $d_{\mathtt{k}-1}|d_{\mathtt{k}}$. Thus, $\mathtt{k}$ satisfies

$$\mathtt{k} = \max_{\lambda\in\mathbb{C}} \operatorname{rank}(R_1(\lambda)) - \min_{\lambda\in\mathbb{C}} \operatorname{rank}(R_1(\lambda)).$$

Partitioning $F =: [\, F_1 \;\; F_2\,]$ conforming to the partition of S, we have

$$R_2 = [\, F_1 \quad F_2\,] \begin{bmatrix} I & 0 & 0 \\ 0 & D & 0 \end{bmatrix} = [\, F_1 \quad F_2 D \quad 0\,]. \quad (24)$$

Notice that $F_2(\lambda)D(\lambda)$ loses rank for $\lambda \in \Lambda_{\text{un}}$. In fact, due to the divisibility property among the d_i, F_2D is the zero matrix when evaluated at the one or more roots of d_1, thus forcing F_1 also to be wide. The required minimality in Problem 6.1 results in F_1 to be not just a wide full row rank polynomial matrix, but in fact square and non-singular too. Any F_1 such that $R_2(\xi) := \left[\, F_1(\xi) \;\, F_2(\xi)D(\xi) \;\, 0 \,\right]$ is full row rank for every complex number results in $\mathfrak{B}_2$ satisfying the requirements in Problem 6.1: the identity matrix of size $(\mathtt{p}_1 - \mathtt{k})$ is a valid choice for F_1. For this choice of F_1, one can choose F_2 arbitrarily.

(2): The input cardinality of $\mathfrak{B}_2$ constructed above is given by

$$\mathtt{m}(\mathfrak{B}_2) = \mathtt{w} - \mathtt{p}_2 \quad = \mathtt{w} - (\mathtt{p}_1 - \mathtt{k}) \quad = \mathtt{m}(\mathfrak{B}_1) + \mathtt{k},$$

where, as calculated above, $\mathtt{k} := \max_{\lambda\in\mathbb{C}} \operatorname{rank}(R_1(\lambda)) - \min_{\lambda\in\mathbb{C}} \operatorname{rank}(R_1(\lambda))$. When $\mathfrak{B}_1$ is uncontrollable, $\mathtt{k} \geqslant 1$, and hence $\mathtt{m}(\mathfrak{B}_2) > \mathtt{m}(\mathfrak{B}_1)$.

(3): We first show that the two cases (i) and (ii) each results in a unique $\mathfrak{B}_2$. Suppose $\mathfrak{B}_1$ be controllable. Then, in the proof of Statement (1) above, $S = I$ and D has zero rows and columns. Thus, any kernel representation matrix of $\mathfrak{B}_2$ is given by $[F_1 \;\; 0]$ where F_1 has to be not just non-singular, but also unimodular. Since the kernel representations $\left[\, I \;\, 0\,\right]$ and $\left[\, F_1 \;\, 0\,\right]$ with F_1 unimodular correspond to the same behaviour, $\mathfrak{B}_2 = \mathfrak{B}_1$, thus making $\mathfrak{B}_2$ unique.

Consider case (ii): $\mathfrak{B}_2 = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is the same as $\mathtt{m}(\mathfrak{B}_2) = \mathtt{w}$, and this, together with the formula in Statement (2), results in $\min_{\lambda\in\mathbb{C}} \operatorname{rank} R_1(\lambda) = 0$. In this case *all* diagonal entries in the Smith form S are non-constant, and hence F_1 is absent. Since F_2D evaluates to the zero matrix at the one or more roots of d_1, the smallest $\mathfrak{B}_2$ is $\mathfrak{B}_2 = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$; in this case also, $\mathfrak{B}_2$ is unique.

We next show non-uniqueness of $\mathfrak{B}_2$ when $\mathfrak{B}_1$ be uncontrollable and when $\min_{\lambda\in\mathbb{C}} \operatorname{rank}(R_1(\lambda)) \geqslant 1$. In this case, the square and non-singular matrix D is of size at least 1 and at most $(\mathtt{p}_1 - 1)$, i.e. $D \in \mathbb{R}^{\mathtt{k}\times\mathtt{k}}[\xi]$ and $1 \leqslant \mathtt{k} \leqslant (\mathtt{p}_1 - 1)$. This allows a non-trivial choice of a square, non-singular $F_1 \in \mathbb{R}^{(\mathtt{p}_1-\mathtt{k})\times(\mathtt{p}_1-\mathtt{k})}[\xi]$ and $F_2 \in \mathbb{R}^{(\mathtt{p}_1-\mathtt{k})\times\mathtt{k}}[\xi]$ such that $[F_1(\lambda)F_2(\lambda)D(\lambda)]$ has full row rank for all $\lambda \in \mathbb{C}$. It is easy to see that any non-singular polynomial matrix $F_1(\xi)$ which is non-singular at each $\lambda \in \Lambda_{\text{un}}$ guarantees controllability of $\mathfrak{B}_2$ and since Λ_{un} is finite, the resulting $\mathfrak{B}_2$ can be ensured to be different by different choices of F_1. This completes the proof of Statement (3), and of Theorem 6.1. □

The above result has consequences on orthogonality of two uncontrollable behaviours: we see this in the following subsection.

6.2 Superbehaviours and orthogonality

We saw earlier in this section that orthogonality of two uncontrollable behaviours is defined by requiring these uncontrollable behaviours to be subbehaviours of two orthogonal *controllable* behaviours. Using the result on the existence of smallest controllable superbehaviours that are controllable, and their input cardinality, we reformulate the question of whether two uncontrollable behaviours are orthogonal as a question of finding a pair of smallest controllable superbehaviours that are mutually orthogonal. The requirement of them being smallest is motivated by the fact that orthogonality of two controllable behaviours imposes an upper bound on their input cardinalities: this is reviewed below for controllable behaviours and then formulated and proved for uncontrollable orthogonal behaviours.

For a behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and a non-singular matrix Σ, the set $\Sigma\mathfrak{B}$ is defined as

$$\Sigma\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{ there exists } v \in \mathfrak{B} \text{ such that } w = \Sigma v\}.$$

It is straightforward to see that $\Sigma\mathfrak{B}$ is also a behaviour, its controllability is equivalent to that of $\mathfrak{B}$, and the input cardinalities are equal. The statements and proofs of the following proposition can be found in Willems and Trentelman (2002, p. 57) and Belur, Pillai and Trentelman (2007, p. 752).

Proposition 6.2: *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and suppose $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ is non-singular. Then,*

(1) $\mathfrak{B}_1 \perp_\Sigma \mathfrak{B}_2 \Leftrightarrow \mathfrak{B}_1 \perp (\Sigma\mathfrak{B}_2)$,
(2) $\mathfrak{B}_1 \perp_\Sigma \mathfrak{B}_2 \Rightarrow \mathtt{m}(\mathfrak{B}_1) + \mathtt{m}(\mathfrak{B}_2) \leqslant \mathtt{w}$.

Due to the above inequality constraint on the input cardinalities of orthogonal controllable behaviours, the uncontrollable behaviours too have a necessary condition to satisfy for mutual orthogonality.

Lemma 6.1: *Suppose* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$ *with at least one of them uncontrollable and let* $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ *be non-singular. Assume* $\mathfrak{B}_1 \perp_\Sigma \mathfrak{B}_2$ *(according to Definition 6.1). Then,* $\mathtt{m}(\mathfrak{B}_1) + \mathtt{m}(\mathfrak{B}_2) < \mathtt{w}$.

Proof: Since at least one of the two behaviours is uncontrollable, say, $\mathfrak{B}_1$, embedding $\mathfrak{B}_1$ in a smallest controllable superbehaviour $\mathfrak{B}_1^c$ results in a strict inequality in the input cardinalities of $\mathfrak{B}_1$ and $\mathfrak{B}_1^c$ [by Theorem 6.1, Statement (2)]. This combined with Proposition 6.2, Statement (2), gives the required inequality $\mathtt{m}(\mathfrak{B}_1) + \mathtt{m}(\mathfrak{B}_2) < \mathtt{w}$. □

The above result resolves an anomalous example of 'orthogonal' behaviours.

Example 6.1: Consider the pair of seemingly orthogonal behaviours $\mathfrak{B}_1$ and $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$ studied in Willems (2004, p. 360). Define an autonomous $\mathfrak{B}_1$ by choosing any $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $C \in \mathbb{R}^{\mathtt{w}\times\mathtt{n}}$, with $C \neq 0$, and let

$$\mathfrak{B}_1 := \{w_1 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{ there exists } x \text{ such that } \tfrac{\mathrm{d}}{\mathrm{d}t}x = Ax, \quad w_1 = Cx\}.$$

Define $\mathfrak{B}_2 := \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, with w_2 defined by, for example, $\frac{\mathrm{d}}{\mathrm{d}t}z = -A^T z + C^T w_2$ for some $z \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$. Consider the supply rate $w_1^T w_2$, i.e. $\Sigma = I$. It can be checked that the 'storage function' $x^T z$ (also called the adapted BDF, as mentioned after Proposition 6.1) satisfies $\frac{\mathrm{d}}{\mathrm{d}t}x^T z = w_1^T w_2$. In fact, the existence of such a storage function would be reasonable for a different definition of orthogonality of the two behaviours $\mathfrak{B}_1$ and $\mathfrak{B}_2$. (See Proposition 6.1, Statement (3), and the text that follows.) In other words, *any* autonomous behaviour $\mathfrak{B}_1 \in \mathfrak{L}^{\mathtt{w}}$ is then[19] 'orthogonal' to $\mathfrak{B}_2 = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$! However, the necessary condition of Lemma 6.1 is not satisfied for this pair. Hence, there does not exist a controllable behaviour $\mathfrak{B}_1^c$ such that $\mathfrak{B}_1^c \subseteq \mathfrak{B}_1$ and $\mathfrak{B}_1^c \perp_\Sigma \mathfrak{B}_2$ and thus $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are not orthogonal by Definition 6.1.

An obvious implication of Lemma 6.1 is as follows.

Corollary 6.1: *Suppose* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$ *and assume* $\mathfrak{B}_1 \perp \mathfrak{B}_2$ *according to Definition 6.1. Let* $\mathtt{m}(\mathfrak{B}_1) + \mathtt{m}(\mathfrak{B}_2) = \mathtt{w}$. *Then, both* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are controllable.*

As mentioned in the beginning of this section, Σ-orthogonality of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is equivalent to losslessness of $\mathfrak{B}_1 \times \mathfrak{B}_2$ with respect to the supply rate: $S := \begin{bmatrix} 0 & \Sigma \\ \Sigma^T & 0 \end{bmatrix}$. The matrix $S \in \mathbb{R}^{2\mathtt{w}\times 2\mathtt{w}}$ is special in the sense that, for any non-singular $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, $\sigma_+(S) = \sigma_-(S) = \mathtt{w}$. The condition $\mathtt{m}(\mathfrak{B}_1 \times \mathfrak{B}_2) = \mathtt{m}(\mathfrak{B}_1) + \mathtt{m}(\mathfrak{B}_2) = \mathtt{w}$ assumed in the above corollary is just the maximal input cardinality condition. Though this section focussed on embeddability in the context of *orthogonality*, one can define dissipativity (losslessness) as embeddability of a behaviour in a dissipative (lossless) controllable superbehaviour. In the context of losslessness, the question of how input cardinality being maximum rules out uncontrollability has been addressed and resolved in Rao (2012, Theorem 10): this turns out to follow from Corollary 6.1, albeit by a different definition.

Corollary 6.2: *Suppose* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$ *and consider* $\Sigma \in \mathbb{R}^{2\mathtt{w}\times 2\mathtt{w}}$ *with* $\Sigma := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. *Assume* $\mathfrak{B}_1 \times \mathfrak{B}_2$ *is* Σ-*lossless: there exists* $\mathfrak{B}_{12}^{\mathrm{cont}} \in \mathfrak{L}_{\mathrm{cont}}^{2\mathtt{w}}$ *such that* $\mathfrak{B}_{12}^{\mathrm{cont}}$ *is* Σ-*lossless and* $\mathfrak{B}_1 \times \mathfrak{B}_2 \subseteq \mathfrak{B}_{12}^{\mathrm{cont}}$. *Suppose* $\mathtt{m}(\mathfrak{B}_1) + \mathtt{m}(\mathfrak{B}_2) = \mathtt{w}$. *Then,* $\mathfrak{B}_1 \times \mathfrak{B}_2$ *is controllable, i.e. both* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are controllable.*

While the above corollary follows from embeddability arguments of Lemma 6.1 and Corollary 6.1, in Rao (2012) the existence of an observable storage function has been used as the definition of losslessness. Interestingly, at least as far as lossless and maximum input cardinality behaviours are concerned, Rao (2012, Theorem 10) and Corollary 6.2 here bring out the common aspects of the two definitions: embeddability in controllable superbehaviours and existence of observable storage functions. Also common to both results is the existential aspect of either the observable storage function or the controllable superbehaviour. In fact, the embeddability definition also yields the conclusion that for a behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ satisfying $\sigma_+(\Sigma) = \mathtt{m}(\mathfrak{B})$, if there exists a controllable Σ-dissipative behaviour $\mathfrak{B}^{\mathrm{cont}}$ containing $\mathfrak{B}$, then $\mathfrak{B}$ is controllable. See also Theorem 7.2: Statements (2) and (3).

7. Dissipative subbehaviours/superbehaviours

In this section, we look into the input cardinality condition for dissipative behaviours, and revisit the embeddability definition for orthogonal and dissipative behaviours. Recall that a behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ which is dissipative with respect to the supply rate Σ (a general constant, symmetric, non-singular matrix) satisfies the condition $\mathtt{m}(\mathfrak{B}) \leqslant \sigma_+(\Sigma)$. We now look into the possibility of embedding a behaviour $\mathfrak{B}$ in a controllable superbehaviour that is Σ-dissipative, and into a drawback of using this as a definition of dissipativity.

Problem 7.1: Given a non-singular, symmetric and indefinite $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, find conditions for the existence of a behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ such that

- There exist $\mathfrak{B}_+$ and $\mathfrak{B}_- \in \mathfrak{L}_{\mathrm{cont}}^{\mathtt{w}}$ with $\mathfrak{B} = \mathfrak{B}_+ \cap \mathfrak{B}_-$;
- $\mathfrak{B}_+$ is strictly Σ-dissipative and $\mathfrak{B}_-$ is strictly $-\Sigma$-dissipative.

The significance of the above problem is that if a nonzero behaviour $\mathfrak{B}$ satisfying the above conditions exists, then clearly such a behaviour would be both strictly Σ- and strictly $-\Sigma$-dissipative, raising concerns about whether embeddability in a dissipative controllable superbehaviour is a reasonable definition of dissipativity. The following theorem states that non-zero autonomous behaviours can indeed exist satisfying the above conditions.

Theorem 7.1: *Let $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be non-singular, symmetric and indefinite. Then, there exist non-zero behaviours $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ such that requirements in Problem 7.1 are satisfied. Further, any such $\mathfrak{B}$ is autonomous, i.e.* $\mathtt{m}(\mathfrak{B}) = 0$.

Proof: We first show that given any non-singular, symmetric and indefinite $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, a behaviour $\mathfrak{B}$ satisfying the above properties exists. Without loss of generality, let

$$\Sigma = \begin{bmatrix} I_+ & 0 \\ 0 & -I_- \end{bmatrix}$$

with sizes of the identity matrices I_+ and I_- equal to σ_+ and σ_-, respectively. Due to indefiniteness of Σ, $\sigma_+, \sigma_- \geqslant 1$ and, due to the non-singularity, they sum up to $\mathtt{w}$. Choose any[20] non-zero and non-constant polynomial matrix $M_+ \in \mathbb{R}^{\mathtt{w}\times\sigma_+}[\xi]$ such that for some $\epsilon_+ > 0$, $M_+(-j\omega)^T\Sigma M_+(j\omega) \geqslant \epsilon_+ I_{\mathtt{w}}$ for all $\omega \in \mathbb{R}$. Similarly, choose $M_- \in \mathbb{R}^{\mathtt{w}\times\sigma_-}[\xi]$ such that for some $\epsilon_- > 0$, $M_-(-j\omega)^T\Sigma M_-(j\omega) \leqslant -\epsilon_- I_{\mathtt{w}}$ for all $\omega \in \mathbb{R}$. Now define $\mathfrak{B}_+$ and $\mathfrak{B}_-$ by image representations $w = M_+(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ and $w = M_-(\frac{\mathrm{d}}{\mathrm{d}t})\ell$, respectively. Define $\mathfrak{B} := \mathfrak{B}_+ \cap \mathfrak{B}_-$, thus proving the existence of a non-zero[21] $\mathfrak{B}$ as stated in the theorem.

We now show that $\mathfrak{B}$ is autonomous. Let $R_+(\xi)$ and $R_-(\xi)$ induce the minimal kernel representations for $\mathfrak{B}_+$ and $\mathfrak{B}_-$, respectively. Then, $\mathfrak{B} = \mathfrak{B}_+ \cap \mathfrak{B}_-$ is described by the kernel representation matrix $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ with $R := \begin{bmatrix} R_+ \\ R_- \end{bmatrix}$. Clearly, $\operatorname{rank} R \leqslant \mathtt{w}$. Autonomy of $\mathfrak{B}$ is the same as $\operatorname{rank} R = \mathtt{w}$. This is proved by contradiction. Suppose $\operatorname{rank}(R) < \mathtt{w}$. Then, there exists $p \in \mathbb{R}^{\mathtt{w}}[\xi]$ and $p \neq 0$ such that $R_+p = 0$ and $R_-p = 0$. This implies $\operatorname{im} p(\frac{\mathrm{d}}{\mathrm{d}t}) \subseteq \mathfrak{B}_+$ and $\operatorname{im} p(\frac{\mathrm{d}}{\mathrm{d}t}) \subseteq \mathfrak{B}_-$. Defining $w := p(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ with $\ell \in \mathfrak{D}(\mathbb{R}, \mathbb{R})$ and $\ell \neq 0$, it follows that $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Further, $p \neq 0$, hence $w \neq 0$ because ℓ is non-zero and of compact support. Moreover, we have $\epsilon_+, \epsilon_- > 0$ such that

$$\int_{-\infty}^{\infty} w^T\Sigma w \,\mathrm{d}t \;\geqslant \epsilon_+\|w\|_{L_2}^2$$

$$\text{and} \qquad \int_{-\infty}^{\infty} w^T\Sigma w \,\mathrm{d}t \;\leqslant -\epsilon_-\|w\|_{L_2}^2.$$

Both the above conditions cannot be satisfied simultaneously for $w \neq 0$. Thus, $\operatorname{rank}(R) < \mathtt{w}$ gives a contradiction. This proves $\operatorname{rank}(R) = \mathtt{w}$ and hence autonomy of $\mathfrak{B}$.

We illustrate the above theorem using an example.

Example 7.1: Let $\Sigma = \operatorname{diag}(1, -1)$. Define $\mathfrak{B}_+$ and $\mathfrak{B}_-$ by image representations $w = M_+(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ and $w = M_-(\frac{\mathrm{d}}{\mathrm{d}t})\ell$, respectively, with

$$M_+(\xi) = \begin{bmatrix} \xi + 4 \\ 3 \end{bmatrix} \quad \text{and } M_-(\xi) = \begin{bmatrix} 2 \\ \xi + 5 \end{bmatrix}.$$

Strict dissipativities are easily verified. Calculating the kernel representations, we get a kernel representation for $\mathfrak{B} := \mathfrak{B}_+ \cap \mathfrak{B}_-$ as $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ with

$$R(\xi) = \begin{bmatrix} -3 & \xi + 4 \\ \xi + 5 & -2 \end{bmatrix}.$$

Clearly, R is non-singular and hence $\mathfrak{B}$ is autonomous. Moreover, $\det R$ has degree 2 and hence $\mathfrak{B}$ is non-zero. This is an example of a non-zero behaviour that can be embedded in a strictly Σ dissipative controllable behaviour $\mathfrak{B}_+$ and also in a strictly $-\Sigma$ dissipative controllable behaviour $\mathfrak{B}_-$.

For the non-strict dissipativity case, we have the following problem and the solution.

Problem 7.2: Given a non-singular, symmetric and indefinite $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, find conditions for the existence of a behaviour $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ such that:

- There exist $\mathfrak{B}_+$ and $\mathfrak{B}_- \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with $\mathfrak{B} = \mathfrak{B}_+ \cap \mathfrak{B}_-$,
- $\mathfrak{B}_+$ is Σ dissipative and $\mathfrak{B}_-$ is $-\Sigma$ dissipative.

Unlike Theorem 7.1, for the non-strict dissipativity case, non-autonomous behaviours can be embedded too: this is stated and proved below.

Theorem 7.2: *Let $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be non-singular, symmetric and indefinite. Then, the following hold:*

(1) *There exists $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ such that requirements in Problem 7.2 are satisfied.*
(2) *Any such $\mathfrak{B}$ satisfies* $\mathtt{m}(\mathfrak{B}) \leqslant \min(\sigma_+(\Sigma), \sigma_-(\Sigma))$.
(3) *In case $\mathfrak{B}$ is uncontrollable,* $\mathtt{m}(\mathfrak{B}) < \min(\sigma_+(\Sigma), \sigma_-(\Sigma))$.
(4) *If $\mathtt{m}(\mathfrak{B}) \geqslant 1$, then neither $\mathfrak{B}_+$ nor $\mathfrak{B}_-$ can be strictly dissipative.*

Proof: (**1**): Except for the strictness of the dissipativities, the proof proceeds in the same way as the proof for Theorem 7.1. Construct $\mathfrak{B}_+$ and $\mathfrak{B}_-$ as in the previous proof, but with ϵ_+ and ϵ_- equal to zero. We have

$$\int_{-\infty}^{\infty} w^T\Sigma w \,\mathrm{d}t \;\geqslant 0 \;\text{ for all }\; w \in \mathfrak{B}_+ \cap \mathfrak{D}$$

$$\text{and} \qquad \int_{-\infty}^{\infty} w^T\Sigma w \,\mathrm{d}t \;\leqslant 0 \;\text{ for all }\; w \in \mathfrak{B}_- \cap \mathfrak{D}.$$

The above two equations imply $\int_{-\infty}^{\infty} w^T\Sigma w \,\mathrm{d}t \; = 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$.

(**2**): Dissipativity with respect to Σ implies $\mathtt{m}(\mathfrak{B}) \leqslant \sigma_+(\Sigma)$. Similarly, dissipativity with respect to $-\Sigma$ implies $\mathtt{m}(\mathfrak{B}) \leqslant \sigma_-(\Sigma)$. This implies $\mathtt{m}(\mathfrak{B}) \leqslant \min(\sigma_+, \sigma_-)$.

(**3**): If $\mathfrak{B}$ is uncontrollable, then using Theorem 6.1, the input cardinality of $\mathfrak{B}$ has to be strictly less than that of both $\mathfrak{B}_+$ and $\mathfrak{B}_-$. This proves $\mathtt{m}(\mathfrak{B}) < \min(\sigma_+, \sigma_-)$.

(**4**): If $\mathtt{m}(\mathfrak{B}) \geqslant 1$, then from Theorem 7.1, $\mathfrak{B}_+$ and $\mathfrak{B}_-$ cannot be strictly dissipative with respect to Σ and $-\Sigma$, respectively. This completes the proof. □

As one of the consequences of the above theorem, if the maximum input cardinality condition is satisfied for an uncontrollable behaviour, i.e. $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$ or $\mathtt{m}(\mathfrak{B}) = \sigma_-(\Sigma)$, then such a behaviour can be embedded into neither a Σ-dissipative controllable behaviour nor a $-\Sigma$-dissipative controllable behaviour. However, an observable storage function for such a situation exists when the controllable part is strictly dissipative at ∞ and when the uncontrollable modes satisfy the unmixing condition (see Theorem 3.1). More concretely, the RLC circuit in Example 4.3 is a system which cannot be embedded in a controllable dissipative system.

On the other hand, dissipative behaviours satisfying $\mathtt{m}(\mathfrak{B}) < \sigma_+(\Sigma)$, controllable or uncontrollable, are not difficult to envision: these are subbehaviours of dissipative controllable behaviours. Do these exist physically, and, if so, can they be realised without using 'active' devices? The following remark deals with RLC realisability and the maximum input cardinality condition.

Remark 7.1: It is well known that a transfer function matrix being positive real is a necessary and sufficient condition for that transfer function matrix to be an impedance/admittance of a network comprising of only resistors, capacitors and inductors (see Brune, 1931, and also Bott & Duffin, 1949, for the case with transformers). Note that the 'transfer function matrix' captures only the controllable part of the behaviour, and the non-minimality aspect of the realisations has been well addressed too (see e.g. Willems, 2007, p. 150). Implicit to the definition of positive realness is the 'square' aspect of the transfer function matrix, i.e. the number of inputs is equal to the positive signature of the supply rate: $v^T i$, where v and i are the vectors of port voltages and port currents, respectively (in the load reference). Also see Rao (2012) for a discussion on this. We now consider an extreme example of the 'non-square' case.

Consider the single port network with $v = 0$ and $i = 0$. This port, called a nullator, behaves simultaneously as the open circuit and short circuit: see Belevitch (1968, p. 75) and Carlin (1964). Clearly, the nullator is a system that is controllable, dissipative, in fact, lossless, and further, it is a subbehaviour of every other behaviour, dissipative or otherwise. Further, the number of inputs (zero, here) is strictly less than the positive signature of the supply rate vi. While the nullator can be realised[22] using active elements, it is known that there cannot be a realisation using only passive elements (RLCT elements) (see Carlin, 1964, p. 68). This raises the more general question whether an externally dissipative system, say, $\mathfrak{B}$, satisfying $\mathtt{m}(\mathfrak{B}) < \sigma_+(\Sigma)$ is truly dissipative in the sense that they can be realised/synthesised physically without using active elements internally or does $\mathfrak{B}$ have to contain active internal devices.

8. Concluding remarks

We briefly review the main results in this paper. The definition of dissipativity involving the integral over all compactly supported trajectories was unsuitable for uncontrollable systems and hence we investigated three definitions of dissipativity regarding uncontrollable systems:

(1) Existence of an observable storage function;
(2) Existence of state functions that are storage functions but possibly unobservable; and
(3) Embeddability in a controllable dissipative superbehaviour.

We first used the existence of an observable storage function as the definition of a system's dissipativity and proved in Theorem 3.1 that, assuming the uncontrollable poles are unmixed and assuming the dissipativity at the infinity frequency is strict, the dissipativities of a behaviour and its controllable part are equivalent. This main result's proof used indefinite linear algebra techniques and also required formulation and proof of new results in the solvability of the ARE. These results also strengthened past results about the existence of observable storage functions. In this context, we also showed that the storage function is a static state function.

The need for unobservable storage functions was visible when relaxing the unmixing assumption by allowing imaginary-axis uncontrollable poles: we showed that for lossless autonomous dissipative systems, the storage function, if assumed a state function, cannot be observable, thus motivating the need for unobservability.

It has been noted in Willems (2004) about how by allowing unobservable storage functions in the definition of dissipativity/losslessness any autonomous system turns out to be 'orthogonal' to the whole space $\mathfrak{C}^\infty$ (see Example 6.1 above). This anomaly was resolved by our investigation into the third definition of dissipativity: embeddability of a behaviour into a controllable dissipative superbehaviour. We proved certain input cardinality necessary conditions: these conditions were not satisfied by Example 6.1. In the context of embeddability of an uncontrollable behaviour, we proved the existence and showed construction of a smallest controllable superbehaviour. However, the constraints on the number of inputs caused the situation that when an uncontrollable system has the number of inputs equal to the positive signature of the supply rate (the maximum input cardinality condition), no uncontrollable behaviour can be dissipative according to the embeddability definition. Note that the very physical and intuitively dissipative RLC circuit of Example 4.3 is precisely such a case. It is straightforward to construct many other RLC circuit examples that satisfy uncontrollability. We also saw Corollary 6.2 of how losslessness (by the embeddability definition) and maximum input cardinality rule out uncontrollability: a result from the literature that used the existence of an observable storage function for defining losslessness.

In the context of embeddability as a definition of dissipativity, we showed that one can construct non-zero behaviours that can be embedded in two controllable behaviours: one strictly dissipative and the other strictly 'antidissipative'. Key input cardinality constraints were proved in this embeddability result. This brought us to the question whether there exist physical dissipative systems in which the number of inputs is strictly less than the positive signature. We noted in Remark 7.1 that the one-port network called nullator is such a circuit example: its number of inputs is strictly less than the positive signature, and, moreover, it is not RLC realisable but requires active components for physical realisation.

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Notes

1. An observable image representation $w = M(\frac{d}{dt})\ell$ is helpful for an i/o partition and transfer matrix too: associated with any partition $M(\xi) = \begin{bmatrix} M_1(\xi) \\ M_2(\xi) \end{bmatrix}$, with $M_2(\xi)$ being square and non-singular, the corresponding partition of w into $w = (w_1, w_2)$ results in w_1 as the output and w_2 as the input. This is analogous to the partition of the matrix R (conformably to that of w) in a minimal kernel representation $R(\frac{d}{dt})w = 0$ into $R(\xi) = [P(\xi)Q(\xi)]$, with P being square and non-singular. For a controllable system, the transfer matrix G from w_2 to w_1 is $G = -P^{-1}Q = M_1M_2^{-1}$.
2. When dealing with a function, say, $Q(w, \ell)$ with $(w, \ell) \in \mathfrak{B}$, the map F helps crucially in expressing $Q(w, \ell)$ in terms of just w by $Q(w, F(\frac{d}{dt})w)$. We use this later when dealing with observable storage functions.
3. Note that elements of $\mathfrak{L}^{\mathtt{w}+\mathtt{x}}$ are subsets of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}+\mathtt{x}})$ and the concatenated trajectory is, in general, not $\mathfrak{C}^\infty$: see Rapisarda and Willems (1997, p. 1055) and Willems and Trentelman (2002, p. 59). Hence, we require satisfaction of the differential equations in only a *weak* sense.
4. One might have a state-space representation of the corresponding $\mathfrak{B}_{\text{full}}$: $\frac{d}{dt}x = Ax + Bu$, $y = Cx + Du$, with (C, A) unobservable. For example, at the parameter values causing uncontrollability, the capacitor voltages and the inductor currents in Example 4.3 are unobservable from the port variables (v, i). We address unobservability in Section 5.
5. The latent variable ℓ is not to be mistaken for that in an *image* representation $w = M(\frac{d}{dt})\ell$.
6. Though similarly motivated, the definition of observability of QDFs is slightly different in Willems and Trentelman (1998, Section 7).
7. In fact, a system is controllable if and only if the compactly supported trajectories are dense in the behaviour. While the 'denseness' aspect requires a more thorough elaboration of the topology used, it is not hard to expect that compactly supported trajectories are not 'representative' enough for the uncontrollable case. An extreme case is a non-trivial autonomous behaviour: while the zero trajectory is the only compactly supported trajectory, the behaviour consists of exponentials corresponding to the poles of $\mathfrak{B}$.
8. For a MIMO system with input u and output y, two key notions of power are u^Ty in passivity studies and $u^Tu - y^Ty$ in $\mathcal{H}_\infty$ control.
9. The term 'unmixing' has been used with almost the same meaning in Shayman (1983) and Scherer (1991). These papers also allow *imaginary-axis* uncontrollable poles and require non-symmetry of only the non-imaginary uncontrollable poles; our definition rules out imaginary-axis uncontrollable poles.
10. The feed-through term D is finite, since the transfer function is proper: see text before Equation (4). Once the transfer function is proper, then positive definiteness of $(I_{\mathtt{m}} + D^T J_{\mathtt{q,r}} D)$ is a property of just $\mathfrak{B}_{\text{cont}}$ and of neither $\mathfrak{B}$ nor the particular i/o partition. This independence is proved easily using arguments similar to those used in Willems and Trentelman (1998, p. 1739, proof of Theorem 5.7) while relating strict dissipativity and biproperness.
11. The constant term within the ARE, i.e. the term independent of K, simplifies by the use of the Matrix Inverse Lemma: $(J + DD^T)^{-1} = J - JD(I + D^TJD)^{-1}D^TJ$. Note that invertibilities of $I + D^TJD$ and $J + DD^T$ are equivalent. However, while $I + D^TJD$ is positive definite, $J + DD^T$ need not be sign-definite. In spite of this, in Equation (14), we use $(J + DD^T)^{1/2}$. With a slight abuse of the definition of $S^{1/2}$ of a symmetric and positive-definite matrix S (see Section 2.2 above), we define $J_{\mathtt{q,r}}^{1/2} := \begin{bmatrix} I_{\mathtt{q}} & 0 \\ 0 & jI_{\mathtt{r}} \end{bmatrix}$ for $J_{\mathtt{q,r}} = \begin{bmatrix} I_{\mathtt{q}} & 0 \\ 0 & -I_{\mathtt{r}} \end{bmatrix}$. For a symmetric non-singular S, we define $S^{1/2}$ as $V^T J_{\mathtt{q,r}}^{1/2} V$ using any factorisation $S = V^T J_{\mathtt{q,r}} V$, with V non-singular. As far as our use of $S^{1/2}$ for indefinite S (only in Equation (14)) is concerned, this definition is fine.
12. This is also called the *root subspace* of the matrix A corresponding to the eigenvalue λ (in, for example, Gohberg et al., 2005, Section 14.7), $\mathcal{R}_\lambda = \ker(A - \lambda I)^n$, where n is the size of the matrix A.
13. We use the property that the controllability subspaces of (A_1, B_1) and (A_2, B_2) are equal when $A_2 = A_1 + B_1K$ and $B_2 = B_1T$ for any K and any non-singular T of suitable dimensions. See also Gohberg et al. (2005, Proposition 14.1.8).
14. In our paper, the Hamiltonian matrix H of Equation (9) above can have $j\mathbb{R}$ eigenvalues. In the absence of such eigenvalues, the notion of c-set relates very closely with the co-primeness and unmixedness discussed in Trentelman and Rapisarda (2001, p. 984).
15. We define the $\mathcal{H}_\infty$ norm of a transfer function $G(s)$ with no poles in the closed right-half plane as $\|G\|_{\mathcal{H}_\infty} := \sup_{\omega\in\mathbb{R}} |G(j\omega)|$.
16. Strictly speaking, we rule out *ARE* solutions, while state functions which satisfy the dissipation inequality are *ARI* solutions. See Pal and Belur (2008, Remark 8.4) for more discussion on this.
17. A BDF is defined along the same lines as a QDF was defined in Equation (1). Consider $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ with $\Psi(\zeta, \eta) =$

$\sum_{j,k} \Psi_{jk} \zeta^j \eta^k$, where $\Psi_{jk} \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}$. The BDF L_Ψ induced by Ψ is a map $L_\Psi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{w}}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{w}}) \longrightarrow \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined as $L_\Psi(w, v) := \sum_{j,k} (\frac{\mathrm{d}^j}{\mathrm{d}t^j} w)^T \Psi_{jk} \frac{\mathrm{d}^k}{\mathrm{d}t^k} v$.

18. This amounts to a so-called module isomorphism on $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{w}})$ using a unimodular matrix, a kind of change of coordinates in $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{w}})$.
19. Though $\mathfrak{B}_1$ is, loosely speaking, a 'thin' set, notice that $0 \neq \mathfrak{B}_1 \subset \mathfrak{B}_2$: this is the anomalous aspect of the 'orthogonality' between $\mathfrak{B}_1$ and $\mathfrak{B}_2$.
20. There are ample such matrices due to the existence of many controllable strictly dissipative behaviours for every indefinite supply rate Σ. Non-constant here means at least one entry of M_+ has degree 1 or more.
21. It is not difficult to show that if M_+ and M_- are non-constant polynomial matrices, then $\mathfrak{B}$ is not the zero behaviour. Example 7.1 following the proof of Theorem 7.1 makes this easier to see.
22. As noted in Carlin (1964), the realisation of a nullator is combined with that of a so-called *norator*: a one-port network in which both the voltage and current have no laws to satisfy.

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