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Reduced-order controller synthesis with regional pole constraint

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ABSTRACT
A reduced order output feedback controller is designed for a linear time invariant system, which guarantees that the closed-loop poles are placed within some pre-specified stability region in the complex plane. A convex approximation of the non-convex constraints is used to pose a sequence of semi-definite programs, which provide the lowest order proper controller satisfying the approximate constraints. The proposed method is demonstrated on two practical controller design applications.

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Linear systems; output feedback; LMIs; controller reduction

1. Introduction

The problem of finding minimum order output feedback controllers for various control objectives has proved to be difficult due to the underlying non-convexity of the optimisations involved (Bernstein, 1992; Hammer, 1983; Karimi, Khatibi, & Longchamp, 2007). If all the closed-loop poles are specified for an n-th order linear time invariant (LTI) single-input single-output (SISO) system, then it is well known that the minimum order output feedback controller which achieves these pole locations is $(n - 1)$ (Qiu & Zhou, 2009; Wellstead, 1991). For the multi-input multi-output (MIMO) case, a minimum degree observer–controller configuration achieving arbitrary pole placement is given by the classic result due to Luenberger (1964). However, if there are no precise requirements on the closed-loop poles, but they are only required to belong to some pre-specified region in the complex plane, then these extra degrees of freedom can be used to further reduce the controller order (e.g. below $(n - 1)$ in the SISO case). This pole placement scenario is more relevant in practice (Datta & Chakraborty, 2013, 2014; Datta, Chakraborty, & Belur, 2012; Datta, Chakraborty, & Chaudhuri, 2012) since the performance specifications usually mention time domain characteristics like settling time/damping ratio. Hence, it would be enough if a designed controller guarantees that all the closed-loop poles are placed within some desirable region in the complex plane. Low-order controllers, on the other hand, are usually desirable due to reduced implementation/computational complexities and related costs.

Under such a pole placement paradigm, in this article, we propose convex formulations to design a reduced-order controller for general MIMO systems. The developed algorithms ensure that the resulting controller is a proper/strictly proper controller and the closed-loop poles are placed inside a pre-specified region in the complex plane. The regional pole placement requirements on the closed-loop poles are first translated into constraints in the coefficients of the corresponding polynomial matrices using the eliminant matrix (Antsaklis & Michel, 2006). Thereafter, this constraint set is convexified using a recent result in inner approximation of the polynomial matrix stability region (Henrion, Arzelier, & Peaucelle, 2003; Yang, Gani, & Henrion, 2007). Finally, we show that a sequence of semi-definite programs (SDPs) has to be solved to obtain a reduced-order proper controller for a strictly proper MIMO plant.

For SISO systems, we are able to pose and solve a slightly more general, partial pole placement problem. Frequently, one needs to exactly place a subset of the closed-loop poles (which we call the critical poles) while the remaining poles (non-critical) can be placed anywhere within some pre-specified region. For example, in large interconnected power systems, the inter-area oscillations caused by electro-mechanical modes are a cause of concern for power system engineers and a typical controller would like to precisely place the poles corresponding to these oscillation modes. The other poles in the system are already quite stable and they can be allowed to be placed anywhere within some pre-specified region corresponding to, e.g., some settling time

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requirements. Such a pole placement paradigm was proposed in Datta, Chakraborty, and Chaudhuri (2012) where simultaneously the state feedback controller norm was minimised. Here, we use the same constructions as in Datta, Chakraborty, and Chaudhuri (2012) and Datta, Chakraborty, and Belur (2012) along with a Sylvester parameterisation of output feedback controllers to optimise the controller order, while satisfying the separate requirements on the critical and non-critical poles. Similar to the MIMO case, we use an inner convex approximation of the polynomial stability region (Henrion, Sebek, & Kucera, 2003; Yang et al., 2007) to define a linear matrix inequality (LMI) on the coefficients of the polynomials associated with the output feedback controller. It is shown that a reduced-order proper output feedback controller, satisfying the approximated regional pole placement requirements, can be found by solving a sequence of SDPs.

The traditional approaches to find a reduced-order controller for a linear system are based on various model or controller order reduction techniques (see e.g., Obinata & Anderson, 2001, and the references therein). These methods provide no guarantee on the closed-loop specifications and hence reduced-order controller design with guaranteed closed-loop performance remains an important problem (Bernstein, 1992). In Mesbahi and Papavassilopoulos (1997) and Mesbahi (1998), a similar problem to the one treated in this article, is posed as a rank minimisation problem. It is shown that if the associated feasible set is a hyper-lattice, then it can be solved through an equivalent SDP. Similarly, in Wang and Chow (2000), a convex suboptimal problem, associated with obtaining a reduced-order controller, is solved by using the strictly positive real condition. In these approaches, convexification is achieved at the cost of optimality or some special system properties are assumed. In Keel and Bhattacharyya (1990), a reduced-order controller is designed with regional pole placement requirements. An approximated output feedback controller is obtained through a non-convex iterative algorithm. This algorithm requires initial guess of a pseudo-diagonal matrix consisting of eigenvalues taken from the predefined stability region and the controller gain matrix. Since the optimisation uses Sylvester equation, one has to check before the start of each iteration that the eigenvalues of the pseudo-diagonal matrix are not close to the open-loop poles. Furthermore, this approach fails to place a subset of the eigenvalues at specific locations. On the other hand, the algorithm proposed in this paper requires one to heuristically choose certain polynomials which in turn determines the conservativeness of the solution obtained. From the implementation perspective, one of the requirements while designing a reduced-order controller is that the computed controller should be proper or strictly proper. Typical approaches adopted in the literature are: (1) representing the dynamics of plant as well as controller in a state-space form (Han, Oliveira, & Skelton, 2006; Keel & Bhattacharyya, 1990; Mesbahi, 1998; Mesbahi & Papavassilopoulos, 1997), (2) expressing the controller as a proper transfer function where the highest degree coefficient of denominator polynomial is set to one (Wang & Chow, 2000) and (3) imposing extra constraints in the optimisations (Han et al., 2006). Another related problem is the design of fixed-order controllers, where Yang et al. (2007), Khatibi, Karimi, and Longchamp (2008) and Karimi et al. (2007) have focused on obtaining fixed-order controllers for plants with polytopic uncertainty.

The rest of this paper is organised as follows. First, we present some known definitions and results on polynomial matrices in Section 2. In Section 2.3, a procedure to construct the eliminant matrix associated with polynomial system matrices is reviewed. This matrix is used in Section 3 to synthesise a reduced-order controller satisfying regional pole placement requirements. The partial pole placement problem for SISO systems is formulated in Section 4. In Section 5, the reduced-order controller is obtained by solving at most n SDPs. Finally, case studies demonstrating the application of the proposed theory on a NASA F-8 DFBW aircraft and a four-machine, two-area power system are included in Section 6.

2. Regional pole placement for MIMO systems

We introduce some notations and definitions related to polynomial matrices before formulating the problem.

2.1 Preliminaries

Let us denote $\mathbb{R}[s]$ and $\mathbb{R}^{m\times p}[s]$ as the sets of all polynomials and $(m \times p)$ polynomial matrices, respectively. Consider a polynomial matrix $A(s) \in \mathbb{R}^{m\times p}[s]$ with its entries $a_{ij}(s) \in \mathbb{R}[s]$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, p$. Let r be the highest degree occurring among the degrees of the polynomial entries $a_{ij}(s)$. Then the polynomial matrix $A(s)$ can be represented as

$$A(s) = A_r s^r + A_{r-1}s^{r-1} + \cdots + A_1 s + A_0 \qquad (1)$$

where $A_k \in \mathbb{R}^{m\times p}$ for $k = 0, 1, 2, \ldots, r$ are the coefficient matrices of $A(s)$. Henceforth, we will denote A_k as the k-th coefficient of $A(s)$ and r as the *degree* of the polynomial matrix (Antsaklis & Michel, 2006; Wolovich, 1974). The maximum degree occurring among the degrees of all elements in the j-th column, $a_j(s)$ of polynomial matrix $A(s)$, is referred to as the *column degree* of $a_j(s)$ and denoted as $\delta_c(a_j(s))$. Likewise, the maximum degree occurring among the degrees of all elements in the i-th row, $a_i(s)$ of $A(s)$, is referred to as the *row degree* of $a_i(s)$ and denoted as $\delta_r(a_i(s))$. A polynomial matrix $A(s) \in \mathbb{R}^{m\times m}[s]$ is said to be $\mathbb{S}$-stable if all the *zeros* of $A(s)$ (i.e. roots of $det\,A(s) = 0$, where *det* denotes the determinant) belong to some stability region $\mathbb{S}$ in the complex plane. Following Henrion et al. (2003), we will define $\mathbb{S}$ as follows:

$$\mathbb{S} = \left\{ s \in \mathbb{C} : \begin{bmatrix} 1 & s^* \end{bmatrix} \underbrace{\begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}}_{S} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0 \right\} \tag{2}$$

where $s*$ denotes the complex conjugate of s and $S \in \mathbb{R}^{2\times 2}$. It has been shown that this region $\mathbb{S}$ can be used to represent some common stability regions in the complex plane (e.g. arbitrary half planes and discs (Henrion et al., 2003)).

Let us assume that the column degrees of a polynomial matrix $A(s) \in \mathbb{R}^{m\times m}[s]$ are $\delta_c(a_j(s)) = \mu_j$ for $j = 1, 2, \ldots, m$. Then $A(s)$ can always be written as $A(s) = A_h P(s) + A_l(s)$ where $P(s) = \mathrm{diag}\,\{s^{\mu_1}; s^{\mu_2}; \cdots; s^{\mu_m}\}$, (where diag{•} denotes the diagonal matrix) $A_h \in \mathbb{R}^{m\times m}$ is the *highest column degree coefficient matrix* of $A(s)$ and $A_l(s)$ is the polynomial matrix consisting of remaining lower degree terms of $A(s)$. We say that $A(s)$ is *column reduced* if $det\,A_h \neq 0$ (Antsaklis & Michel, 2006; Kailath, 1980; Wolovich, 1974). Likewise, let $\delta_r(x_i(s)) = \nu_i$ for $i = 1, 2, \ldots, m$ are the row degrees of a polynomial matrix $X(s) \in \mathbb{R}^{m\times m}[s]$. Then $X(s)$ can always be written as $X(s) = P(s)X_h + X_l(s)$, where $P(s) = \mathrm{diag}\,\{s^{\nu_1}; s^{\nu_2}; \cdots; s^{\nu_m}\}$. We say that $X(s)$ is *row reduced* if $det\,X_h \neq 0$.

2.2 Problem formulation

Let $H(s)$ be the transfer function matrix associated with a controllable and observable MIMO system with m inputs and p outputs. Then, it is well known (Antsaklis & Michel, 2006; Kailath, 1980; Wolovich, 1974) that $H(s)$ can be represented by the following co-prime factorisation:

$$H(s) = B(s)A(s)^{-1} \tag{3}$$

where $B(s) \in \mathbb{R}^{p\times m}[s]$ and $A(s) \in \mathbb{R}^{m\times m}[s]$ is column reduced. Let $\delta_c(a_j(s)) = \mu_j$ for $j = 1, 2, \ldots, m$ and assume that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. If this is not the case, then one has to perform suitable column operations on $A(s)$ to make $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. Note that, the same operations need to be performed on $B(s)$ to keep the transfer function matrix $H(s)$ the same. Since $A(s)$ is column reduced, we have (Antsaklis & Michel, 2006)

$$det\,A(s) = det\,A_h s^{\sum \mu_j} + \text{lower degree terms}$$

and, hence the *order* of the plant is $n = \sum_{j=1}^{m} \mu_j$. Assume that the plant is strictly proper, that is, $\delta_c(b_j(s)) < \mu_j$ for $j = 1, 2, \ldots, m$. Consider a controller $C(s)$, represented by the following factorisation:

$$C(s) = X(s)^{-1}Y(s) \tag{4}$$

where $Y(s) \in \mathbb{R}^{m\times p}[s]$ and $X(s) \in \mathbb{R}^{m\times m}[s]$ is row reduced. By denoting $\delta_r(x_i(s)) = \nu_i$, we define the *order* of the controller as $\kappa := \sum_{i=1}^{m} \nu_i$.

It is well known that if the plant (3) and controller (4) are interconnected, then the closed-loop poles are the zeros of the polynomial matrix

$$D(s) = X(s)A(s) + Y(s)B(s). \tag{5}$$

Then, the problem of interest can precisely be written as follows.

Problem 2.1: Find a minimum order ($\kappa \leq n$) proper/strictly proper controller $C(s)$ such that all the closed-loop poles, that is, zeros of $D(s)$, are placed anywhere in the stability region $\mathbb{S}$.

In the following section, we first introduce the eliminant matrix and then we show how that can be used to design a reduced-order controller.

2.3 Eliminant matrix

For any fixed integer $\nu > 0$, let us define a polynomial matrix $W^\nu(s) \in \mathbb{R}^{(n+m\nu)\times m}[s]$ as follows:

$$W^\nu(s) := \begin{bmatrix} 1 & & & \\ s & & & \\ \vdots & & & \\ s^{\mu_1+\nu-1} & & & \\ & 1 & & \\ & s & & \\ & \vdots & & \\ & s^{\mu_2+\nu-1} & & \\ & & \ddots & \\ & & & 1 \\ & & & s \\ & & & \vdots \\ & & & s^{\mu_m+\nu-1} \end{bmatrix}. \tag{6}$$

Then, corresponding to the polynomial matrices $A(s)$ and $B(s)$ of (3), for some integer $\nu > 0$, we can write

$$\begin{bmatrix} B(s) \\ sB(s) \\ \vdots \\ s^{\nu-1}B(s) \\ A(s) \\ sA(s) \\ \vdots \\ s^{\nu-1}A(s) \end{bmatrix} = M_\nu W^\nu(s). \tag{7}$$

In (7), we say the matrix $M_\nu \in \mathbb{R}^{\nu(p+m)\times(n+m\nu)}$ as the *eliminant matrix* associated with the polynomial matrices $A(s)$ and $B(s)$ (Antsaklis & Michel, 2006; Wolovich, 1974).

Next we assume that the degree of $X(s)$ and $Y(s)$ are both $\nu - 1$. This is reasonable since we do not know the relative degree of the controller a priori. Denote $X_k \in \mathbb{R}^{m\times m}$ and $Y_k \in \mathbb{R}^{m\times p}$ for $k = 0, 1, \ldots, \nu - 1$ as the coefficient matrices associated with $X(s)$ and $Y(s)$, respectively. Then, by defining a controller coefficient matrix

$$\mathcal{K}_{\nu-1} := \left[Y_0\ Y_1 \cdots Y_{\nu-1}\ X_0\ X_1 \cdots X_{\nu-1} \right], \tag{8}$$

we can write

$$X(s)A(s) + Y(s)B(s) = \mathcal{K}_{\nu-1} \begin{bmatrix} B(s) \\ sB(s) \\ \vdots \\ s^{\nu-1}B(s) \\ A(s) \\ sA(s) \\ \vdots \\ s^{\nu-1}A(s) \end{bmatrix} = \mathcal{K}_{\nu-1} M_\nu W^\nu(s). \tag{9}$$

Since, $A(s)$ and $B(s)$ are right co-prime, it is known (Antsaklis & Michel, 2006, Chapter 7, Theorem 2.13) that there exists some $\nu > 0$ such that the eliminant matrix is full column rank, that is, $rank(M_\nu) = n + m\nu$ where $n = \sum_{j=1}^{m} \mu_j$. Hence, for any arbitrary choice of $D(s) \in \mathbb{R}^{m\times m}[s]$ such that $\delta_c(d_j(s)) \le \mu_j + \nu - 1$, we can solve for $X(s)$ and $Y(s)$ which satisfy the Diophantine equation (5). Furthermore, since $\delta_c(d_j(s)) \le \mu_j + \nu - 1$, we have

$$D(s) = \mathbf{D}_t W^\nu(s) \tag{10}$$

where $\mathbf{D}_t \in \mathbb{R}^{m\times(n+m\nu)}$ (recall that column degrees of $W^\nu(s)$ are $\mu_j + \nu - 1$). Then, using (9) and (10), the Diophantine equation (5) will be satisfied if and only if following relation

$$\mathcal{K}_{\nu-1} M_\nu = \mathbf{D}_t \tag{11}$$

holds (Antsaklis & Michel, 2006). In the following section, we use (11) to design a reduced-order controller.

3. Reduced -order controller design

Recall that μ_1 was assumed to be the largest among all μ_js. Then, according to the definition (6), $W^\nu(s)$ can be written in the following form:

$$W^\nu(s) = W_0 + W_1 s + W_2 s^2 + \cdots + W_{\mu_1+\nu-1} s^{\mu_1+\nu-1}.$$

Next, using M_ν, $\mathbf{D}_t$ and $W^\nu(s)$, we construct two new matrices L_ν and $\mathbf{D}$ as follows:

$$\begin{aligned} &L_\nu = \left[L_0\ L_1\ \cdots\ L_{\mu_1+\nu-1} \right] \quad \text{where} \\ &L_0 = M_\nu W_0,\ L_1 = M_\nu W_1, \cdots, L_{\mu_1+\nu-1} = M_\nu W_{\mu_1+\nu-1} \\ &\mathbf{D} = \left[D_0\ D_1\ \cdots\ D_{\mu_1+\nu-1} \right] \quad \text{where} \\ &D_0 = \mathbf{D}_t W_0,\ D_1 = \mathbf{D}_t W_1, \cdots, D_{\mu_1+\nu-1} = \mathbf{D}_t W_{\mu_1+\nu-1} \end{aligned} \tag{12}$$

with $L_k \in \mathbb{R}^{\nu(p+m)\times m}$ and $D_k \in \mathbb{R}^{m\times m}$ for $k = 1, 2, \ldots, \mu_1 + \nu - 1$. In the previous section, we saw that the Diophantine equation (5) can be written as $\mathcal{K}_{\nu-1} M_\nu W^\nu(s) = \mathbf{D}_t W^\nu(s)$, and hence by equating the coefficients of both sides, we can write

$$\mathcal{K}_{\nu-1} L_\nu = \mathbf{D}. \tag{13}$$

The elements D_ks in $\mathbf{D}$ are the coefficients of the polynomial matrix $D(s)$. Since there exists some $\nu > 0$ such that (11) is solvable, relation (13) also has a solution.

Recall that we are interested in designing a minimum order proper controller $C(s)$ such that the zeros of polynomial matrix $D(s)$ belong to some stability region $\mathbb{S}$. For this purpose, let us define a set

$$\mathcal{N}_s := \left\{ D(s) \in \mathbb{R}^{m\times m}[s] \text{ of degree } \mu_1 + \nu - 1 : \text{the zeros of } D(s) \in \mathbb{S} \right\}.$$

Let $q := \mu_1 + \nu - 1$. Hence, Problem 2.1 can be posed as follows: find a minimum order proper controller $C(s)$ such that $D(s) \in \mathcal{N}_s$. However, it was shown (Henrion et al., 2003) that the set $\mathcal{N}_s$ is a non-convex set. Hence, to convexify the optimisation, we use a result by Henrion et al. (2003) which is described briefly in the following section.

3.1 LMI stability region

Let us define following two matrices corresponding to the polynomial matrices $D(s)$ and an arbitrary but fixed $\bar{D}(s)$ (of degree q), respectively:

$$\mathbf{D} := [D_0 \quad D_1 \quad \cdots \quad D_{q-1} \quad D_q] \in \mathbb{R}^{m\times(q+1)m};$$
$$\bar{\mathbf{D}} := [\bar{D}_0 \quad \bar{D}_1 \quad \cdots \quad \bar{D}_{q-1} \quad \bar{D}_q] \in \mathbb{R}^{m\times(q+1)m}.$$

Then, for a fixed $\bar{D}(s) \in \mathcal{N}_s$, define the set:

$$\mathcal{M}_s := \{D(s) \in \mathbb{R}^{m\times m}[s] \,:\, \bar{\mathbf{D}}^T\mathbf{D} + \mathbf{D}^T\bar{\mathbf{D}} - \Pi^T(S\otimes T)\Pi \succ 0, \text{ for some } T = T^T \in \mathbb{R}^{qm\times qm}\} \quad (14)$$

where $\otimes$ refers to the Kronecker product, $\succ 0$ implies a positive definite matrix, S as defined in (2) and $\Pi \in \mathbb{R}^{2qm\times(q+1)m}$ denotes a projection matrix given by

$$\Pi = \begin{bmatrix} I_m & & & 0 & \cdots & 0 \\ & \ddots & & I_m & & \\ & & I_m & & \ddots & \\ 0 & \cdots & 0 & & & I_m \end{bmatrix}^T. \quad (15)$$

It was shown (Henrion et al., 2003, Lemma 1) that for any given $\mathbb{S}$-stable polynomial matrix $\bar{D}(s) \in \mathcal{N}_s$, the polynomial matrix $D(s) \in \mathcal{N}_s$ if there exists a symmetric matrix $T \in \mathbb{R}^{qm\times qm}$ satisfying the matrix inequality $\bar{\mathbf{D}}^T\mathbf{D} + \mathbf{D}^T\bar{\mathbf{D}} - \Pi^T(S\otimes T)\Pi \succ 0$. Hence, for every fixed $\bar{D}(s)$, this result characterises a subset of the stable polynomial matrices and hence the set $\mathcal{M}_s \subseteq \mathcal{N}_s$. Then, by replacing the set $\mathcal{N}_s$ with an approximated set $\mathcal{M}_s$, we can pose a convexified but suboptimal version of Problem 2.1 as follows:

Problem 3.1: Find a minimum order ($\kappa \le n$) proper controller $C(s)$ such that $D(s) \in \mathcal{M}_s$.

Note that, since we use a convex inner approximated set of the stable polynomial matrices, the optimisation might not produce the minimum order controller, and hence we refer to the resulting controller as *reduced-order controller*. However, the order of the resulting controller is minimum with respect to the approximated stability region.

3.2 LMI formulation for controller design

Recall (13), i.e. $\mathcal{K}_{\nu-1}L_\nu = \mathbf{D}$. Then, the matrix inequality $\bar{\mathbf{D}}^T\mathbf{D} + \mathbf{D}^T\bar{\mathbf{D}} - \Pi^T(S\otimes T)\Pi \succ 0$, used to describe the set $\mathcal{M}_s$, would be

$$\bar{\mathbf{D}}^T\mathcal{K}_{\nu-1}L_\nu + L_\nu^T\mathcal{K}_{\nu-1}^T\bar{\mathbf{D}} + \Pi^T(S\otimes T)\Pi \succ 0. \quad (16)$$

For a given $\bar{\mathbf{D}}$, the inequality in (16) is linear in variables $\mathcal{K}_{\nu-1}$ and T. Hence, we can use this as a constraint in the optimisation problem. However, the solution of the LMI in (16) might not produce a row-reduced polynomial matrix $X(s)$, and hence the resulting controller may not be a proper/strictly proper controller. Next, we propose a methodology to overcome this difficulty. Before proceeding further, let us introduce some more notations. Denote $\mathbf{x}_i^T (i = 1, 2, \ldots, m)$ as the i-th row of the matrix $X_{\nu-1} \in \mathbb{R}^{m\times m}$ which was defined as the highest degree coefficient of polynomial matrix $X(s)$. Then, construct a vector $\tilde{\mathbf{x}}_i^T \in \mathbb{R}^{m-1}$ by taking all the elements of $\mathbf{x}_i^T$ excluding x_{ii} for $i = 1, 2, \ldots, m$. Denote the elements of $\tilde{\mathbf{x}}_i^T$ as $\tilde{x}_{ik}$ for $k = 1, 2, \ldots, m$ and $i \neq k$. Then, the following result holds.

Theorem 3.2: *For a fixed polynomial matrix $\bar{D}(s) \in \mathcal{N}_s$, if $\mathcal{K}_{\nu-1}$ and a symmetric matrix T, satisfy the following conditions:*

$$(i)\quad \bar{\mathbf{D}}^T\mathcal{K}_{\nu-1}L_\nu + L_\nu^T\mathcal{K}_{\nu-1}^T\bar{\mathbf{D}} + \Pi^T(S\otimes T)\Pi \succ 0$$
$$(ii)\quad \begin{bmatrix} \frac{1}{(m-1)}x_{ii}I_{m-1} & \tilde{\mathbf{x}}_i \\ \tilde{\mathbf{x}}_i^T & x_{ii} \end{bmatrix} \succ 0 \text{ for } i = 1, 2, \ldots, m,$$

then all the closed-loop poles are placed within the stability region $\mathbb{S}$. Furthermore, the resulting controller would be either proper or strictly proper and the order of the controller $\kappa = \sum_{i=1}^m \delta_r(x_i(s))$.

Proof: For a fixed $\bar{\mathbf{D}}$, since the matrices $\mathcal{K}_{\nu-1}$ and T are satisfying condition (i), we have $\bar{\mathbf{D}}^T\mathbf{D} + \mathbf{D}^T\bar{\mathbf{D}} - \Pi^T(S\otimes T)\Pi \succ 0$. Hence, the polynomial matrix $D(s) \in \mathcal{M}_s$. However, following the previous discussion, we have $\mathcal{M}_s \subseteq \mathcal{N}_s$ and hence all the closed-loop poles belong to the stability region $\mathbb{S}$. According to the Schur complement relation, the condition (ii) is equivalent to

$$x_{ii} > 0 \quad\text{and}\quad x_{ii}^2 - (m-1)\tilde{\mathbf{x}}_i^T\tilde{\mathbf{x}}_i > 0,$$

and hence we can write

$$x_{ii}^2 > (m-1)\tilde{\mathbf{x}}_i^T\tilde{\mathbf{x}}_i \quad \text{for } i = 1, 2, \ldots, m$$
$$= (m-1)\sum_{k=1}^m |\tilde{x}_{ik}|^2 \quad \text{for } i = 1, 2, \ldots, m \quad \text{and } i \neq k$$
$$= \sum_{k=1}^m |\tilde{x}_{ik}|^2 + (m-2)\left(\sum_{k=1}^m |\tilde{x}_{ik}|^2\right)$$

$$\geq \sum_{k=1}^{m} |\tilde{x}_{ik}|^2 + 2\left(\sum_{l=1}^{m}\sum_{k=1}^{m} |x_{ik}||x_{il}|\right) \quad \text{for } i \neq k \neq l \quad (\star)$$

$$= \left(\sum_{k=1}^{m} |\tilde{x}_{ik}|\right)^2$$

$$\Rightarrow |x_{ii}| > \sum_{k=1}^{m} |\tilde{x}_{ik}| \text{ for } i \neq k. \tag{17}$$

In the above equation, the inequality at step ($\star$) follows from the Young's inequality.[1] According to (17), the matrix $X_{\nu-1}$ is strictly diagonally dominant matrix and hence is nonsingular (Horn & Johnson, 1985). In addition, none of the diagonal entries are zero. Hence, according to the definition, the polynomial matrix $X(s)$ is row reduced with $\delta_r(x_i(s)) = \nu - 1$. Hence, we can write (Antsaklis & Michel, 2006)

$$det\, X(s) = det\, X_{\nu-1}s^{m\nu-m} + \text{lower degree terms.}$$

This leads to the conclusion that inverse of the polynomial matrix $X(s)$ exists. In addition, following the construction of the controller coefficient matrix $\mathcal{K}_{\nu-1}$ (see (8)), we have $\delta_r(x_i(s)) \geq \delta_r(y_i(s))$. Hence, the resulting controller $C(s) = X(s)^{-1}Y(s)$ is either proper or strictly proper. The polynomial matrix $X(s)$ being row reduced leads to the fact that the order of the controller is equal to $\sum_{i=1}^{m} \delta_r(x_i(s))$. This completes the proof. □

3.3 Synthesis procedure

Note that according to Theorem 3.2, if conditions (*i*) and (*ii*) are satisfied, then we achieve our objectives: regional pole placement requirement with proper/strictly proper controller. Since we are interested in designing a minimum order proper controller $C(s)$, we can check the satisfiability of conditions (*i*) and (*ii*) starting with a zeroth-order controller, that is, $\nu = 1$. If there is no feasible solution, then the value can be sequentially increased until a feasible solution is reached. The satisfiability conditions can be checked by formulating the following LMI optimisation problem:

Problem 3.3: Find $\max_{\mathcal{K}_{\nu-1},T,\gamma} \gamma$ subject to

$$(i) \quad \bar{\mathbf{D}}^T\mathcal{K}_{\nu-1}L_\nu + L_\nu^T\mathcal{K}_{\nu-1}^T\bar{\mathbf{D}} + \Pi^T(S\otimes T)\Pi - \gamma I \succ 0$$

$$(ii) \quad \begin{bmatrix} \frac{1}{(m-1)}x_{ii}I & \tilde{\mathbf{x}}_i \\ \tilde{\mathbf{x}}_i^T & x_{ii} \end{bmatrix} \succ 0 \quad \text{for} \quad i = 1, 2, \ldots, m$$

where γ is a positive scalar and I denotes the identity matrix of appropriate dimension.

Since $A(s)$ and $B(s)$ are co-prime, it is guaranteed that there exists some ν such that constraint (*i*) of the above problem is satisfiable. The role of constraint (ii) is explained below. Note that Problem 3.3 is an LMI optimisation and hence can be solved by standard LMI solvers like *SeDuMi*. However, to solve Problem 3.3, we need to first choose a central polynomial matrix $\bar{D}(s) \in \mathcal{N}_s$. At the current state of research, the polynomial matrix $\bar{D}(s)$ has to be chosen heuristically. In the numerical example below, we choose a diagonal $\bar{D}(s)$.

Remark 1: Note that constraint (*ii*) in Problem 3.3 is required to guarantee that the resulting $X(s)$ is row reduced. However, according to the definition of row-reduced polynomial matrix, the highest row degree coefficient matrix X_h should be nonsingular. Since it is difficult to write that as an LMI, a sufficient condition, i.e., $X_{\nu-1}$ should be nonsingular, is imposed in Problem 3.3. From our experience with numerical examples, it seems that the solution of $\max_{\mathcal{K}_{\nu-1},T,\gamma} \gamma$ with only constraint (i) of Problem 3.3, usually results in a nonsingular X_h. Hence, it is preferable that Problem 3.3 should be first solved without the possibly conservative constraint (*ii*). If a feasible solution does not result in a nonsingular X_h, then we need to impose constraint (*ii*) in the optimisation.

Note that if we impose constraint (*ii*) in Problem 3.3, then the highest row degree coefficient matrix $X_h = X_{\nu-1}$, and as a result, the order of the controller would be $\sum_{i=1}^{m} \delta_r(x_i(s)) = m(\nu-1)$. On the other hand, if the optimisation is solved without considering constraint (*ii*) and results in a non-singular X_h, then X_h might not be equal to $X_{\nu-1}$, and hence the controller order could be $\sum_{i=1}^{m} \delta_r(x_i(s)) \leq m(\nu-1)$.

Summarising the above, we propose following design procedure to obtain a reduced-order controller:

Design steps

(1) Start with $\nu = 1$.
(2) Following the procedure proposed in Section 2.3, compute the eliminant matrix M_ν and construct L_ν.
(3) Choose a stability region $\mathbb{S}$ in the complex plane. Design a central polynomial matrix $\bar{\mathbf{D}}(s)$ such that all the zeros of $\bar{\mathbf{D}}(s)$ are within $\mathbb{S}$. Some trial and error iterations are required at this stage. Then, solve Problem 3.3 without considering constraint (*ii*). If the feasible solution results in a row-reduced $X(s)$, then stop; otherwise, solve Problem 3.3. If there is no feasible solution to Problem 3.3 (without considering constraint (*ii*)), then go to Step 4.
(4) Increase the value of ν by one and follow the procedure from Step 2. The increment of ν should be continued until a feasible solution is achieved.

Remark 2: Although there is no unique strategy for the choice of $\mathbb{S}$-stable polynomial matrix $\bar{D}(s)$, we have observed through several practical examples that a diagonal polynomial matrix, where the diagonal elements are constructed by considering all the well-damped/stable open-loop poles (which are inside the stability region $\mathbb{S}$), produces satisfactory results. Since in the applications, most of the open-loop poles are stable and well damped, it is preferable to consider them while constructing the diagonal polynomials of $\bar{D}(s)$. However, if there are not enough well-damped/stable open-loop poles, then we need to choose, heuristically, complex numbers (with conjugate) which are inside the stability region $\mathbb{S}$ in the complex plane to form diagonal polynomials of $\bar{D}(s)$. We have experienced from the numerical examples that the choice of complex numbers near to the boundary of $\mathbb{S}$ produces satisfactory results. More details on designing $\bar{D}(s)$ are included in the numerical examples.

4. Partial pole placement for SISO systems

In this part, we specialise the above controller order reduction technique for SISO system, so as to address the partial pole placement paradigm (Datta & Chakraborty, 2013; Datta, Chakraborty, & Chaudhuri, 2012) described in Section 1. In particular, we consider the following constraints: (1) an arbitrary subset (critical poles) of the closed-loop poles are to be placed at precise pre-defined locations in the complex plane, and (ii) the remaining (non-critical) poles are to be placed within a pre-defined region in the complex plane.

Consider an LTI SISO system represented by the following transfer function.

$$P(s) = \frac{b(s)}{a(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{18}$$

where the polynomials $a(s)$ and $b(s)$ are co-prime. Assume that the plant $P(s)$ is strictly proper. Let us consider an output feedback controller of the following form:

$$C(s) = \frac{y(s)}{x(s)} = \frac{y_m s^m + y_{n-1}s^{n-1} + \cdots + y_1 s + y_0}{x_m s^m + x_{m-1}s^{m-1} + \cdots + x_1 s + x_0} \tag{19}$$

with $x_m \neq 0$ and $m \leq (n-1)$. The closed loop, comprising of plant $P(s)$ and controller $C(s)$, would be

$$G(s) = \frac{P(s)}{1 + P(s)C(s)} = \frac{b(s)x(s)}{a(s)x(s) + b(s)y(s)}, \tag{20}$$

and hence the corresponding characteristic polynomial is

$$\sigma(s) = a(s)x(s) + b(s)y(s) \tag{21}$$

with degree $(n + m)$.

It is well known (Qiu & Zhou, 2009; Wellstead, 1991) that if all the $(n + m)$ poles of the closed-loop system are specified, then the minimum order of the required controller is $m = n - 1$. However, here, only a subset of the closed-loop poles are specified while remaining are free. Assume q out of $(n + m)$ closed-loop poles to be non-critical and hence not associated with any desired closed-loop location. The remaining $(n + m - q)$ poles are critical and are required to be placed at self-conjugate locations $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n+m-q}\}$ in closed loop. Further assume that the q free poles are required to be located inside a (stable) subset $\mathbb{S}$ (see (2)) of the complex plane $\mathbb{C}$. Then, consider the following problem:

Problem 4.1: Find a minimum order $(m \leq (n - 1))$ proper controller $C(s)$ such that the closed-loop poles have the following properties:

(1) $(n + m - q)$ out of the total $(n + m)$ poles are placed at $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n+m-q}\}$ and
(2) remaining q poles are placed anywhere in $\mathbb{S}$.

Denote $\{-\mu_1, -\mu_2, \ldots, -\mu_q\}$ as the unspecified q closed-loop poles of the system (20). Hence, the characteristic equation of the closed-loop system will be

$$\sigma(s) = \underbrace{\left[\prod_{j=1}^{q}(s + \mu_j)\right]}_{\alpha(s)} \underbrace{\left[\prod_{i=1}^{n+m-q}(s + \lambda_i)\right]}_{\beta(s)} \tag{22}$$

where

$$\alpha(s) := s^q + \alpha_{q-1}s^{q-1} + \cdots + \alpha_1 s + \alpha_0;$$
$$\beta(s) := s^{n+m-q} + \beta_{n+m-q-1}s^{n+m-q-1} + \cdots + \beta_1 s + \beta_0.$$

Note that in (22), $\beta(s)$ is a monic polynomial of known coefficients (completely defined from the problem specifications) while $\alpha(s)$ is a monic polynomial of unknown coefficients. According to Problem 4.1, the only requirement on $\alpha(s)$ is that the roots should be located in a pre-specified region $\mathbb{S} \subset \mathbb{C}$ defined in (2). Next, denote the set of all q-th degree monic polynomials with real coefficients as $\mathbb{R}[s]$, and define the set $C_s := \{\alpha(s) \in \mathbb{R}[s] :$ roots of $\alpha(s) \in \mathbb{S}\}$. Then, the second constraint of Problem 4.1 can be restated as $\alpha(s) \in C_s$. As pointed out in the MIMO case, the set $C_s \subset \mathbb{R}[s]$ is not a convex set for $q \geq 3$ (see Ackermann, 1980; Henrion et al., 2003) which leads to a non-convex optimisation problem. We convexify this constraint using a SISO version of the technique described in Section 3.1 (Henrion et al., 2003; Yang et al., 2007).

Assume that $\widehat{\alpha}(s)$ is a polynomial in the stability region C_s. Define the coefficient vectors corresponding to $\widehat{\alpha}(s)$ and $\alpha(s)$ (defined in (22)) as follows: $\widehat{\alpha} := \left[\widehat{\alpha}_0\, \widehat{\alpha}_1 \ldots \widehat{\alpha}_{q-1}\right]^T \in \mathbb{R}^q$ and $\alpha := \left[\alpha_0\, \alpha_1 \ldots \alpha_{q-1}\right]^T \in \mathbb{R}^q$, respectively. Furthermore, let $\alpha_e := \left[\alpha^T\ 1\right]^T \in \mathbb{R}^{q+1}$ and $\widehat{\alpha}_e := \left[\widehat{\alpha}^T\ 1\right]^T \in \mathbb{R}^{q+1}$. A restatement of (14) follows: for any given stable polynomial $\widehat{\alpha}(s) \in C_s$, the polynomial $\alpha(s)$ is also in C_s, provided there exists a symmetric matrix $P \in \mathbb{R}^{q\times q}$ satisfying the matrix inequality

$$\alpha_e\widehat{\alpha}_e^T + \widehat{\alpha}_e\alpha_e^T - \Pi^T(S\otimes P)\Pi \succeq 0. \tag{23}$$

where Π is defined in (15) with $m = 1$. This helps us to convexify Problem 4.1 by replacing constraint (2) with (23), leading to a suboptimal (but convex) version.

However, as in the matrix case, to compute (23) explicitly, we still need a priori the central polynomial $\widehat{\alpha}(s) \in C_s$. In Henrion et al. (2003) and Yang et al. (2007), various domain-dependent heuristics are provided for design choices for $\widehat{\alpha}(s)$. In our case, $\widehat{\alpha}(s)$ can be chosen to be any q-th degree polynomial with roots in the stability region $\mathbb{S}$. Note that the accuracy of the approximation is sensitive to the choice of the central polynomial $\widehat{\alpha}(s)$ (see Henrion et al., 2003; Yang et al., 2007), and hence some conservativeness is introduced in to the proposed methodology due to this dependence.

5. Controller synthesis for SISO systems

In this section, we show that a sub-optimal solution to Problem 4.1 can be obtained from an SDP. Define the following Toeplitz matrices corresponding to the polynomials $a(s)$ and $b(s)$ in (18):

$$T(a) := \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \\ 1 & a_{n-1} & \cdots & a_1 \\ 0 & 1 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad T(b) := \begin{bmatrix} b_0 & 0 & \cdots & 0 \\ b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-2} & b_{n-3} & \cdots & b_0 \\ b_{n-1} & b_{n-2} & \cdots & b_1 \\ 0 & b_{n-1} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{n-1} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{24}$$

The Sylvester's resultant matrix associated with $T(a) \in \mathbb{R}^{(n+m+1)\times(m+1)}$ and $T(b) \in \mathbb{R}^{(n+m+1)\times(m+1)}$ can be defined as follows:

$$\mathcal{R}(a, b, 2(m+1)) := \left[\, T(a)\ T(b)\, \right]_{(n+m+1)\times 2(m+1)} \tag{25}$$

Let us define the vector $\sigma =: \left[\sigma_0\ \sigma_1\ \cdots\ \sigma_{n+m-1}\ \sigma_{n+m}\right]^T \in \mathbb{R}^{n+m+1}$ associated with the closed-loop characteristic polynomial $\sigma(s) := \sigma_{n+m}s^{n+m} + \cdots + \sigma_1 s + \sigma_0$. Furthermore, define following vectors

$$\begin{aligned} x &:= \left[x_0\ x_1\ \cdots\ x_m\right]^T \in \mathbb{R}^{(m+1)} \quad \text{and} \\ y &:= \left[y_0\ y_1\ \cdots\ y_m\right]^T \in \mathbb{R}^{(m+1)} \end{aligned} \tag{26}$$

corresponding to the polynomials $x(s)$ and $y(s)$ defined in (19). The controller coefficient vector can then be defined as follows: $k := \left[x\ y\right]^T \in \mathbb{R}^{2(m+1)}$.

Arbitrary pole placement with the controller $C(s)$ can be achieved (e.g. see Qiu & Zhou, 2009; Wellstead, 1991) from the following relation:

$$[\mathcal{R}(a, b, 2(m+1))]\, k = \sigma \tag{27}$$

From (27), it can be verified that when $m = n - 1$, the matrix $\mathcal{R}(a, b, 2(m+1))$ is square and also non-singular ($a(s)$ and $b(s)$ are co-prime). Hence, there is a unique controller coefficient vector k corresponding to the specified σ. However, in our case, only a subset of the closed-loop poles are specified, which leads to the following result:

Lemma 5.1: *For a given set of closed-loop poles* $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n+m-q}\}$, (27) *defines the following linear equations*

$$\alpha = Fk + g \quad \text{and} \quad \tilde{E}k + \tilde{h} = \mathbf{0}. \tag{28}$$

for some $F \in \mathbb{R}^{q\times(2m+2)}$, $g \in \mathbb{R}^q$, $\tilde{E} \in \mathbb{R}^{(n+m+1-q)\times(2m+2)}$, $\tilde{h} \in \mathbb{R}^{(n+m+1-q)}$ *and* $\mathbf{0}$ *is a zero vector of appropriate dimension.*

Proof: Following Datta, Chakraborty, and Chaudhuri (2012), the next $(n + m + 1)$ linear equations can be derived from (27) and (22)

$$\begin{aligned} a_0x_0 + b_0y_0 &= \beta_0\alpha_0 \\ a_1x_0 + a_0x_1 + b_1y_0 + b_0y_1 &= \beta_0\alpha_1 + \beta_1\alpha_0 \\ &\vdots \\ x_{m-1} + a_{n-1}x_m + b_{n-1}y_m &= \beta_{n+m-q-1} + \alpha_{q-1} \\ x_m &= 1 \end{aligned} \tag{29}$$

From (29), it is possible to express α_j ($j = 0, 1, \ldots, q-1$) in terms of variables x_i's and y_i's

($i = 0, 1, \ldots, m$). Compactly, this can be written as $\alpha = Fk + g$, where $F \in \mathbb{R}^{q\times(2m+2)}$ and $g \in \mathbb{R}^q$.

Now, excluding $x_m = 1$, the coefficients $\alpha_0, \ldots, \alpha_{q-1}$ can be back-substituted in the set of $(n + m)$ Equations (29) to get $(n + m - q)$ linear equations in x_i's and y_i's (for $i = 0, 1, \ldots, m$). These equations can be written in the form: $Ek + h = \mathbf{0}$ where $E \in \mathbb{R}^{(n+m-q)\times(2m+2)}$, $h \in \mathbb{R}^{(n+m-q)}$ and $\mathbf{0}$ is a zero vector of appropriate dimension. Including the equation $x_m - 1 = 0$ to the above set of equations, $Ek + h = \mathbf{0}$ can be written as $\tilde{E}k + \tilde{h} = \mathbf{0}$. □

Corresponding to the relation $\alpha = Fk + g$, of Lemma 5.1, let us define α_e as

$$\alpha_e = \tilde{F}k + \tilde{g} \quad \text{where} \quad \tilde{F} = \begin{bmatrix} F_{q\times(2m+2)} \\ \mathbf{0}_{1\times(2m+2)} \end{bmatrix} \quad \text{and} \quad \tilde{g} = \begin{bmatrix} g \\ 1 \end{bmatrix} \tag{30}$$

Using (30), the LMI (23) becomes

$$\tilde{F}k\widehat{\alpha}_e^T + \widehat{\alpha}_e k^T \tilde{F}^T + \tilde{g}\widehat{\alpha}_e^T + \widehat{\alpha}_e \tilde{g}^T - \Pi^T(S \otimes P)\Pi \succeq 0 \tag{31}$$

Then the following result holds:

Theorem 5.2: *For any fixed $\widehat{\alpha}(s) \in C_s$, if for some $k \in \mathbb{R}^{2m+2}$ and for some $P = P^T \in \mathbb{R}^{q\times q}$, the relations (31) and $\tilde{E}k + \tilde{h} = \mathbf{0}$ hold, then the closed-loop poles (roots of the polynomial defined in (21)) satisfy the following properties:*

(1) *$(n + m - q)$ out of the total $(n + m)$ poles are $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n+m-q}\}$.*

(2) *the remaining q poles $-\mu_i \in \mathbb{S}$ for $i = 1, \ldots, q$.*

Furthermore, the resulting controller will be an m-th order proper or strictly proper controller.

Proof: Fix $\widehat{\alpha}(s) \in C_s$. Assume that some $k \in \mathbb{R}^{(2m+2)}$ and $P = P^T \in \mathbb{R}^{q\times q}$ satisfy (31). Then

$$\alpha_e\widehat{\alpha}_e^T + \widehat{\alpha}_e\alpha_e^T - \Pi^T(S \otimes P)\Pi \succeq 0.$$

Hence, the roots of $\alpha(s)$ lie in $\mathbb{S}$. The $(n + m - q)$ equations $\tilde{E}k + \tilde{h} = \mathbf{0}$ imply that the $(n + m - q)$ roots of polynomial $\beta(s)$ (see (22)) are placed at $\{-\lambda_1, \ldots, -\lambda_{n+m-q}\}$.

Since $x_m = 1$ (see (29)), the corresponding coefficient vector associated with polynomial $x(s)$ would be $x = \begin{bmatrix} x_0 & x_1 & \cdots & x_{m-1} & 1 \end{bmatrix}^T$. Hence, the denominator polynomial $x(s)$ of the controller $C(s)$ is a monic polynomial of degree m. The polynomial $y(s)$, on the other hand, is of degree not more than m, since there are only $m + 1$ entries in vector k corresponding to the polynomial $y(s)$. Hence, the resulting controller will either be a proper or strictly proper controller. □

Remark 3: We have assumed that the plant is strictly proper. Also, the resulting controller is proper or strictly proper. Since (see, e.g. Qiu & Zhou, 2009, Chapter 3, Theorem 3.26) all the closed-loop poles are in the stable region of complex plane, the feedback inter-connection is internally stable.

Note that, the corresponding Sylvester resultant matrix $\mathcal{R}(a, b, 2(m + 1))$ for $m < (n - 1)$ is a tall matrix and hence there may not exist a k which will satisfy (27) for a specified σ. However, since σ is not completely fixed in our case, (27) may be satisfied for some vector k. According to Theorem 5.2, the controller vector k satisfying the relations (31) and $\tilde{E}k + \tilde{h} = \mathbf{0}$ will guarantee that the pole placement requirements are achieved and (27) is satisfied. The conditions of Theorem 5.2 can be checked by solving the following SDP for increasing values of m.

Problem 5.3: Find $\max_{P,k,\gamma} \gamma$ subject to

$$(i) \quad \tilde{E}k + \tilde{h} = \mathbf{0}$$

$$(ii) \quad \Pi^T(S \otimes P)\Pi - \tilde{F}k\widehat{\alpha}_e^T - \widehat{\alpha}_e k^T\tilde{F}^T - \tilde{g}\widehat{\alpha}_e^T - \widehat{\alpha}_e\tilde{g}^T + \gamma I_{q+1} \preceq 0$$

where I_{q+1} is an identity matrix with dimension $q + 1$.

To obtain a minimum order controller, we have to start with a zero-order controller ($m = 0$) and check whether the solution γ to Problem 5.3 satisfies $\gamma > 0$. If this condition is not satisfied, then we should increase the order of the controller by one and recheck the satisfiability condition. At the stage of $m = (n - 1)$, it is guaranteed that the above problem has a feasible solution and hence to obtain the lowest order controller achievable through this method, we need to solve at most n SDPs. The above problem can be solved by using solvers like *SeDuMi* in MATLAB environment (Sed, 2010; Sturm, 2008).

Remark 4: Often in practical applications, only strictly proper controllers are allowed. The proposed algorithm is applicable in such a situation with some modifications. This can be done by enforcing the m-th component (y_m) of the vector y given in (26) to zero. Hence, the controller coefficient vector k would be of the dimension $(2m + 1)$. For this, the last column of the Sylvester resultant matrix $\mathcal{R}(a, b, 2(m + 1))$ need to be deleted from (27). Finally, the design procedure, discussed above, has to be followed after calculating the matrices F, E and vectors g, h as shown in Theorem 5.1. Note that, in this case, the design procedure has to be started with a first-order controller ($m = 1$).

Remark 5: The proposed approach can also be used to find a (single) reduced-order controller for a nonlinear plant operating at multiple operating points. Nonlinear plants, operating at different points, are often controlled via linear controllers. For example, in gain-scheduled controller designs (Shamma & Athans, 1990), the nonlinear plant is required to be linearised at various operating points and (different) linear controllers are designed for each of the resulting linearised plants. While such a design does not necessarily guarantee the stability of the overall nonlinear (switched) system, they are often necessary and extremely successful in practice. A relevant example is aircraft autopilot design, where the linearised airplane model changes frequently depending on atmospheric, altitude, and flight mode conditions (Ackermann, 1984). As a second example, consider a power system switching to different operating points in response to sudden faults (Pal & Chaudhuri, 2005) in the power network. A priori linear models for different fault situations are available and a single controller is supposed to stabilise any of the multiple fault models that the nominal system might change to arbitrarily.

In general, assume that there are w operating points of a nonlinear plant and the linearised models at those points are represented by: $P_l(s) = \frac{b_l(s)}{a_l(s)}$, for $l = 1, 2, \dots, w$, where $a_l(s)$ and $b_l(s)$ are co-primes and are in the form of (18). This leads to the following LMI satisfiability problem:

Problem 5.4: Find $\max_{P, k, \gamma} \gamma$ subject to

$$
\begin{aligned}
(i) \quad & \tilde{E}_l k + \tilde{h}_l = \mathbf{0} \\
(ii) \quad & \Pi^T (S \otimes P) \Pi - \tilde{F}_l k \widehat{\alpha}_e^T - \widehat{\alpha}_e k^T \tilde{F}_l^T - \tilde{g}_l \widehat{\alpha}_e^T \\
& - \widehat{\alpha}_e \tilde{g}_l^T + \gamma I_{q+1} \preceq 0
\end{aligned}
$$

for $l = 1, 2, \dots, w$.

Then, a reduced-order controller can be computed using the design steps described above. If there is a solution, then the resulting (single) controller should simultaneously satisfy the transient response requirements at all the operating points.

6. Examples

Two numerical examples are presented in this section to illustrate the theory developed above.

6.1 NASA F-8 DFBW aircraft

In this example, we consider the linearised model of a NASA F-8 DFBW aircraft (Keel & Bhattacharyya, 1990). The lateral dynamics of the aircraft is represented by following matrices:

$$
A = \begin{bmatrix} -2.6 & 0.25 & -38 & 0 \\ -0.075 & -0.27 & 4.4 & 0 \\ 0.078 & -0.99 & -0.23 & 0.052 \\ 1 & 0.078 & 0 & 0 \end{bmatrix},
$$

$$
B = \begin{bmatrix} 17 & 7 \\ 0.82 & -3.2 \\ 0 & 0.046 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

The open-loop poles of the system are at -2.3965, -0.0249, $-0.3393 \pm 2.6235i$. To obtain a reduced-order controller for the above plant, we computed a right co-prime factorisation (refer to Structure theorem (Wolovich, 1974)) as follows:

$$
H(s) = B(s)A(s)^{-1} \quad \text{where}
$$
$$
A(s) = \begin{bmatrix} s^2 + 2.6062s + 0.7780 & 1.1081s + 5.6244 \\ -0.0533s + 0.8361 & 1.0181s^2 + 0.4938s + 6.5815 \end{bmatrix}
$$
$$
B(s) = \begin{bmatrix} 0.82s + 0.8113 & -3.2s - 0.2040 \\ 17.0640 & 6.7504 \end{bmatrix}.
$$

Note that $A(s)$ is column reduced. According to the design procedure, we first consider $\nu = 1$. Then, the polynomial matrix $W^1(s)$ would be

$$
W^1(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ 0 & 1 \\ 0 & s \\ 0 & s^2 \end{bmatrix}.
$$

Hence, following the procedure discussed in Section 2.3, the eliminant matrix would be as follows:

$$
M_1 = \begin{bmatrix} 0.8113 & 0.8200 & 0 & -0.2042 & -3.200 & 0 \\ 17.0640 & 0 & 0 & 6.7504 & 0 & 0 \\ 0.7780 & 2.6062 & 1 & 5.6244 & 1.0181 & 0 \\ 0.8361 & -0.0533 & 0 & 6.5815 & 0.4938 & 1 \end{bmatrix}.
$$

Corresponding to this M_1 the matrix L_1 would be as follows:

$$
L_1 = \begin{bmatrix} 0.8113 & -0.2042 & 0.8200 & -3.2000 & 0 & 0 \\ 17.0640 & 6.7504 & 0 & 0 & 0 & 0 \\ 0.7780 & 5.6244 & 2.6062 & 1.0181 & 1 & 0 \\ 0.8361 & 6.5815 & -0.0533 & 0.4938 & 0 & 1 \end{bmatrix}.
$$

It is required that all the poles of the closed-loop system, comprising the plant and a proper controller, should be placed left to a vertical line at -0.2 in the

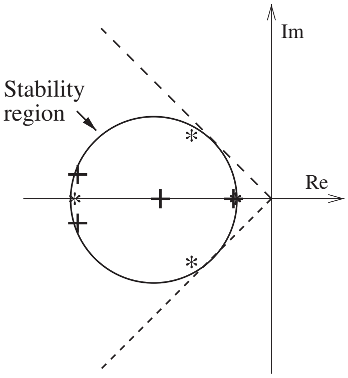

Figure 1. A disc having centre at −2.5 and radius 1.7 is chosen as stability region $\mathbb{S}$. The symbol $+$ shows the position of complex numbers which are used to form $\bar{D}(s)$ and $*$ shows the position of closed-loop poles.

complex plane. In addition, the damping ratio should be greater than or equal to 0.7. Corresponding to this stability region, we have chosen $\mathbb{S}$ as a disc having centre at −2.5 and radius 1.7 in the complex plane and hence the elements of S are $s_{11} = 3.36$, $s_{12} = s_{21} = 2.5$ and $s_{22} = 1$. To construct $\bar{D}(s)$, we choose the following set of complex numbers: −2.3965,−0.85 and $-4 \pm 0.5i$. Note that there is only one open-loop pole, that is, −2.3965 is inside $\mathbb{S}$ and hence we have chosen remaining three poles which are close to the boundary of $\mathbb{S}$, as shown in Figure 1. Corresponding to the above set of complex numbers, the polynomial matrix $\bar{D}(s)$ would be

$$\bar{D}(s) = \begin{bmatrix} s^2 + 3.2465s + 2.0370 & 0 \\ 0 & s^2 + 8s + 16.2500 \end{bmatrix}. \tag{32}$$

Then, by solving Problem 3.3 without considering constraint (*ii*), we have the following result:

$$X_0 = \begin{bmatrix} 4560.7716 & 2366.8460 \\ -436.5595 & 3144.9283 \end{bmatrix},$$

$$Y_0 = \begin{bmatrix} -4625.6015 & 112.7037 \\ -5071.1494 & -139.3457 \end{bmatrix}$$

with closed-loop poles at −4.0362, $-1.6639 \pm 1.4242i$ and −0.8338. The locations of the closed-loop poles are depicted in Figure 1. Hence, we have achieved our objective with a zeroth-order controller $C(s) = X_0^{-1}Y_0$. To show the effectiveness of the proposed design procedure for constructing $\bar{D}(s)$, we include the results corresponding to the choice of different $\bar{D}(s)$ in Table 1. Moreover, we obtain the following results by solving Problem 3.3 (with constraint (*ii*)), corresponding to the choice of $\bar{D}(s)$ as in (32):

$$X_0 = \begin{bmatrix} 6030.8688 & 3122.8716 \\ -814.2041 & 4054.4896 \end{bmatrix},$$

$$Y_0 = \begin{bmatrix} -6143.2737 & 147.7255 \\ -6501.7656 & -191.9227 \end{bmatrix}$$

with closed-loop poles at −4.0346, $-1.6654 \pm 1.4188i$ and −0.8359. Hence, we achieved our objective with a zeroth-order controller.

Note that, a zeroth-order controller is also computed in Keel and Bhattacharyya (1990) for the above example by solving non-convex optimisations iteratively. On the other hand, we obtained a zeroth-order controller by solving a convex optimisation corresponding to the choice of $\bar{D}(s)$. The optimisation proposed in Keel and

Table 1. Results corresponding to the choice of different $\bar{D}(s)$. The diagonal elements of $\bar{D}(s)$ are denoted as $\bar{d}_{11}(s)$ and $\bar{d}_{22}(s)$.

Complex numbers to form $\bar{D}(s)$	Diagonal elements of $\bar{D}(s)$	Resulting X_0 and Y_0	Closed-loop poles
−2.3965, −1,	$\bar{d}_{11}(s) = s^2 + 3.3965s + 2.3965$	$X_0 = \begin{bmatrix} 262.1306 & -423.1758 \\ 104.0616 & 494.2231 \end{bmatrix}$	$-3.8975, -1.6131 \pm 1.3061i$,
−0.85, −4	$\bar{d}_{22}(s) = s^2 + 4.8500s + 3.400$	$Y_0 = \begin{bmatrix} 633.7002 & 29.1730 \\ -780.3079 & -25.9751 \end{bmatrix}$	−0.8821
−2.3965, −0.9000,	$\bar{d}_{11}(s) = s^2 + 3.2965s + 2.1568$	$X_0 = \begin{bmatrix} 106.3675 & -169.0821 \\ 24.6396 & 178.2002 \end{bmatrix}$	$-3.1837, -1.7088 \pm 1.2993i$,
−0.8500, −3.5000	$\bar{d}_{22}(s) = s^2 + 4.3500s + 2.9750$	$Y_0 = \begin{bmatrix} 221.8113 & 13.5346 \\ -266.8398 & -11.9921 \end{bmatrix}$	−0.9851
−2.3965, −0.8500,	$\bar{d}_{11}(s) = s^2 + 3.2465s + 2.0370$	$X_0 = \begin{bmatrix} 2299.3968 & 910.6293 \\ -544.2107 & 2521.4922 \end{bmatrix}$	$-4.0378, -1.6760 \pm 1.4001i$,
$-3 \pm 1.5000i$	$\bar{d}_{22}(s) = s^2 + 6s + 11.2500$	$Y_0 = \begin{bmatrix} -1940.1561 & 66.5307 \\ -4057.3566 & -120.9291 \end{bmatrix}$	−0.8348
−2.3965, −1,	$\bar{d}_{11}(s) = s^2 + 3.3965s + 2.3965$	$X_0 = \begin{bmatrix} 6672.0534 & 2219.3619 \\ -1881.0828 & 13470.9057 \end{bmatrix}$	$-4.0002, -1.6267 \pm 1.4235i$,
$-2.500 \pm 1.600i$	$\bar{d}_{22}(s) = s^2 + 5s + 8.8100$	$Y_0 = \begin{bmatrix} -4548.3396 & 190.4769 \\ -21438.8574 & -677.5315 \end{bmatrix}$	−0.8849

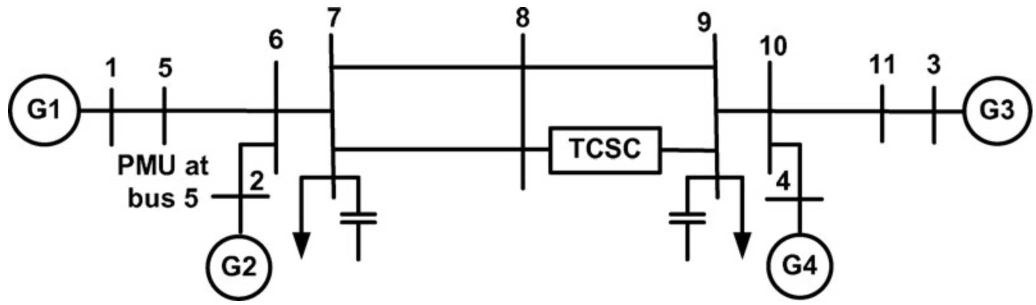

Figure 2. A four-machine two-area power system with a TCSC. The Phasor Measurement Unit (PMU), which can provide measurements of the states of monitoring buses, is installed at bus 5.

Bhattacharyya (1990) uses Sylvester equation for achieving the pole placement requirements. Hence, a priori verification of the eigenvalues of pseudo-diagonal matrix is required to ensure that they are not close to the open-loop poles; as a result, one can avoid non-existence of the solution of Sylvester equation. Such necessary actions are not required in the proposed approach. In fact, in the above example, we have considered the well-damped/stable open-loop poles (which are inside $\mathbb{S}$) to construct $\bar{D}(s)$. We will show in the next example that the proposed algorithm can place a subset of the closed-loop poles at specific locations which cannot be achieved by the algorithm developed in Keel and Bhattacharyya (1990).

6.2 Four-machine two-area power system

The performance of the proposed SISO controller design is validated through a case study on a simple power system. A single-line diagram of the test system is shown in Figure 2. It comprises four generators ($G1$–$G4$) spread over two geographical areas which are interconnected by two transmission lines. The loads are connected at bus 7 and 9. The details of study system can be found in Kundur (1994) and Pal and Chaudhuri (2005). A thyristor-controlled series capacitor (TCSC) (Hingorani & Gyugyi, 2000) is installed in the transmission corridor to facilitate power transfer between the two areas. Under normal condition, 400 MW power is transferred from *Area* 1 to *Area* 2 for which the TCSC is set to provide 10% compensation.

Linearised model of the above system about the nominal condition confirms the presence of one poorly damped electromechanical mode of oscillation (also known as inter-area oscillation) (Kundur, 1994) with about 0.6 Hz frequency. The open-loop poles are -42.5194, $-0.5701 \pm 6.9471i$, $-0.0467 \pm 3.9352i$, -1.9210, 0.0999, and $-0.7238 \pm 0.7318i$. The objective of this exercise is to improve the damping of this low-frequency mode through supplementary control of the TCSC (actuator). The design specification is to achieve a 10-second settling time for this critical mode with a minimum strictly proper controller. This implies shifting the eigenvalues corresponding to the inter-area mode from their open-loop position $-0.0467 \pm 3.9352i$ to $-0.4 \pm 3.9352i$ (corresponds to 10-second settling time) in closed loop while ensuring that the remaining closed-loop poles are restricted to the left of the vertical line at -0.5 in the complex plane. Phase angle difference between the voltages at bus 5 and 11 was chosen as the feedback signal due to its highest modal observability. A ninth-order linearised equivalent model is considered for the power system and a strictly proper reduced-order controller is designed following the proposed approach.

6.2.1 Controller design

As discussed above, the stability region for the free poles would be the closed left half of a vertical line at -0.5 in the complex plane. Hence, the stability region (2) will take the following form:

$$\mathbb{S} = \left\{ s \in \mathbb{C} : \begin{bmatrix} 1 & s^* \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0 \right\} \qquad (33)$$

According to the design steps, we first try with first-order controller and subsequently second-order controller to achieve our objectives. However, no feasible solutions exist for Problem 5.3 at these stages. Hence, the next step is to try with a third-order controller and it is observed that then Problem 5.3 does have a feasible solution.

Third-order controller: The order of the plant $P(s)$ is $n = 9$. The order of the controller $C(s)$ is 3. Hence, the number of closed-loop poles is 12. Among them, two poles (corresponding to the inter-area mode) are critical and hence already specified. The remaining 10 poles can take any positions in the stability region defined in (33). To form the central polynomial $\widehat{\alpha}(s)$, the following poles

Table 2. Pole locations table.

Closed-loop poles	ζ	t_s (seconds)
$-42.8375, -10.9750, -1.5865, -1.4234$	1	
$-0.5637 \pm 6.9493i$	0.08	7.09
$\mathbf{-0.4000 \pm 3.9352i}$	0.11	10
$-0.5332 \pm 0.8012i$	0.55	7.50
$-0.5794 \pm 0.0715i$	0.99	6.90

Note: Bold value to distinguish the electro-mechanical modes from the other modes.

Table 3. Comparison table.

Condition	Inter-area modes	f (Hz)	ζ	t_s i(seconds)
Open loop	$-0.0467 \pm 3.9352i$	0.62	0.01	86.65
Closed loop	$-0.4000 \pm 3.9352i$	0.62	0.11	10

are chosen inside the stability region $\mathbb{S}$:

$$\{-42.5194, -0.5701 \pm 6.9471i, -1.9210,\\ -0.7238 \pm 0.7318i, -0.55, -0.8, -0.6 \pm 1i\}$$

Note that the open-loop poles -42.5194, $-0.5701 \pm 6.9471i$, -1.9210, and $-0.7238 \pm 0.7318i$ are inside the stability region $\mathbb{S}$ and hence we consider them to form the central polynomial. The remaining poles to construct $\widehat{\alpha}(s)$ are chosen close to the boundary of $\mathbb{S}$.

The matrices F, E and vectors g, h are calculated following Section 5. Solving Problem 5.3, the resulting controller coefficient vector turns out to be

$$k = \begin{bmatrix} 39.1532 & 32.0201 & 13.7534 & 1 & 433.1596 & 174.5694 & 20.4935 \end{bmatrix}^T,$$

and hence the corresponding third-order controller would be:

$$C(s) = \frac{20.4945s^2 + 174.5694s + 433.1596}{s^3 + 13.7534s^2 + 32.0201s + 39.1532}.$$

The closed-loop poles are given in Table 2. It is clear that the poles corresponding to the inter-area mode are placed at the desired locations and their settling time should be less than 10 seconds. Furthermore, all the free poles have assumed positions in $\mathbb{S}$ as defined in (33). Hence, all the requirements on closed-loop poles are achieved with a third-order strictly proper controller.

The damping ratio (ζ), frequency of oscillation (f) and settling time (t_s) of inter-area modes for the open-loop and closed-loop plants are shown in Table 3.

7. Conclusion

In this article, we propose a design framework to obtain a reduced-order output feedback controller for linear systems while guaranteeing that the closed-loop poles are placed within some pre-specified region in the complex plane. In addition, the proposed method can achieve partial pole placement for SISO systems while optimising the controller order. This combination of objectives, though clearly important, has been rarely treated in the literature. Moreover, after suitable approximations, the proposed sub-optimal controller order minimisation algorithm is convex, corresponding to a fixed choice of stable polynomial matrix and solvable using standard semi-definite programming tools. Some of the conservativeness in the methodology developed here stems from (1) the inner approximation method used for convexifying the stability region in the coefficient space, and (2) the use of constraint *(ii)* of Theorem 3.2 to convexify the requirement of row-reduced $X(s)$. Furthermore, the optimal solution of Problem 3.3 with constraint *(ii)* produces a controller having an order equal to $m(\nu - 1)$, whereas without imposing it, the controller order could be less than or equal to $m(\nu - 1)$. Hence, we propose to solve Problem 3.3, first, without considering constraint (*ii*); and we have experienced through the numerical examples that the solution of Problem 3.3 without constraint (*ii*) usually results in a row-reduced polynomial matrix $X(s)$. While these problems are currently being investigated, the proposed algorithm seems to perform well in numerical examples.

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Note

1. Let x_1 and x_2 be two non-negative numbers. Then, according to Young's inequality, we have $2x_1x_2 \leq x_1^2 + x_2^2$.

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