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Fast modes in the set of minimal dissipation trajectories ☆

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ABSTRACT

In this paper, we study the set of trajectories satisfying both a given LTI system's laws and also laws of the corresponding 'adjoint' system: in other words, trajectories in the *intersection* of the system's behavior and that of the adjoint system. This intersection has important system theoretic significance: for example, it is known that the trajectories in this intersection are the ones with minimal 'dissipation'. Underlying the notion of adjoint is that of a power supply: it is with respect to this supply rate that the trajectories in the intersection are known to be 'stationary'. In this paper, we deal with *half-line* solutions to the differential equations governing both the system and its adjoint. Analysis of half-line solutions plays a central role for example in initial value problems and in well-posedness studies of an interconnection. We interpret the set of half-line trajectories allowed by a system and its adjoint as an *interconnection* of these two systems, and thus address issues about well-posedness/ill-posedness of the interconnection. We formulate necessary and sufficient conditions for this intersection to be autonomous. For the case of an ill-posed interconnection and resulting autonomous system, we derive conditions for existence of initial conditions that lead to impulsive solutions in the states of the system. We link our conditions with the strongly reachable and weakly unobservable subspaces of a state space system. We show that absence of impulsive initial conditions is equivalent to the well-known subspace iteration algorithms for these subspaces converging in one step.

1. Introduction

For an LTI system, the intersection of the sets of trajectories allowed by the system and its 'adjoint' (dual) system has significance in various areas: as 'stationary' trajectories in the context of LQ control (see [1,2]), as Hamiltonian systems (see [3]), and as trajectories of minimal dissipation (see [4]). Under suitable regularity assumptions, this intersection exhibits desirable properties—like, *autonomy*, having McMillan degree equal to *twice* the McMillan degree of the original system. One or both of these properties are lost when the regularity assumptions are relaxed. Consequently, under non-satisfaction of the regularity assumptions, the usage of the interconnection of the system and its adjoint, in control problems, becomes subject to major modifications. For example, in singular LQ control, the intersection of the system and its adjoint may or may not contain impulsive optimal solutions: see [5,1,6] for a related exposition. In this paper, we go beyond the intersection and view the same as an 'interconnection'. While the interconnection point of view does not provide, for the regular case, any significant leverage over that of the intersection, the former point of view can handle the singular case better than the latter; this is because the singular case is nothing but an 'ill-posed' interconnection of the system and its adjoint.

Following the tradition of the study of ill-posedness in the interconnection paradigm, in this paper, we study *half-line* solutions of the interconnection of the system and its adjoint. Further, we investigate the issue of whether this interconnection, when ill-posed, contains *impulsive* modes. For the purpose of this paper: 'impulsive' modes are those trajectories that contain one or more derivatives of the Dirac delta δ. 'Fast' modes include impulsive modes and jumps.

Without dwelling on the essential preliminaries (which are elaborated below in Section 2), we first list the main questions we address in this paper. Let $\mathfrak{B}$ be the *behavior* of the system, that is, the collection of all the allowable trajectories under the system's dynamical equations. Further, let Σ, a constant real symmetric

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matrix, induce the quadratic supply rate $w^T \Sigma w$ on trajectories $w \in \mathfrak{B}$. Let $\mathfrak{B}^{\perp_\Sigma}$ denote the adjoint of $\mathfrak{B}$ with respect to the supply rate $w^T \Sigma w$. We address the following issues:

1. Given Σ and controllable/observable state space representations of a system $\mathfrak{B}$ and its adjoint system $\mathfrak{B}^{\perp_\Sigma}$, when is the interconnection $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ an autonomous system?
2. Find conditions on $\mathfrak{B}$ under which the interconnection is an ill-posed interconnection.
3. If the interconnection is autonomous and ill-posed: find conditions under which there are no initial state-space conditions causing impulsive solutions.
4. Find conditions on the system $\mathfrak{B}$ under which the *external* system variables exhibit impulsive solutions: relate these conditions to those in Item 3 above.
5. Can there be situations under which one or more of the states of the interconnected system are impulsive, but the external system variables are not impulsive? Does 'impulse unobservability' or 'unobservability at infinity' resolve this?

In this paper we formulate necessary and sufficient conditions for resolving some of the above questions and we provide counter-examples for the unresolved ones. When studying the interconnection of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$, there are three important representations for the interconnected system: $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$:

1. the (possibly singular descriptor) state space system obtained from the minimal state space representations of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$,
2. the kernel representation of $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ obtained by using the kernel representations of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$, and
3. the latent variable representation $w = M(\frac{d}{dt})\ell$ and $M(-\frac{d}{dt})^T \Sigma M(\frac{d}{dt})\ell = 0$.

Note that, while the various representations listed above all lead to the same set of solutions for the case of well-posed interconnection between $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$, it is ill-posed interconnection that results in difference in the fast solution sets of the various representations: this paper focuses only on the fast modes. In this context, it turns out that even when the state space of the interconnected system has impulsive initial conditions, the external system variables do not necessarily have impulsive modes. See also [7]. In the later part of the paper, we describe numerical examples with these features, and further, investigate if impulse unobservability can explain why the system is 'impulse unobservable'.

A brief overview of the main results in this paper and the paper organization are as follows. The following section contains definitions pertaining to the behavioral approach, quadratic differential forms (QDFs), and preliminary results on well-posedness of interconnection and the notion of zeros at infinity of a polynomial matrix and its relation to inadmissible initial conditions, i.e. those initial conditions that cause impulsive solutions. In Section 3, we summarize the assumptions used in this paper and also their system-theoretic justifications. Section 4 contains new results on ill-posedness of interconnection of a system $\mathfrak{B}$ and its dual $\mathfrak{B}^{\perp_\Sigma}$, and conditions for the interconnection to be autonomous. Section 5 contains another main result of this paper: necessary and sufficient conditions for the ill-posed interconnection case under which the interconnected system has no impulsive initial conditions. Section 6 raises questions about how the presence/absence of impulsive solutions need not be the same for the case of state variables, manifest system variables and the latent variable used in an image representation. Section 7 contains some concluding remarks.

We use standard notation in this paper: $\mathbb{R}$ and $\mathbb{C}$ stand for the fields of real and complex numbers respectively. The ring of polynomials in the indeterminate ξ with coefficients from $\mathbb{R}$ is denoted as $\mathbb{R}[\xi]$, while matrices with entries from $\mathbb{R}[\xi]$ and having $\mathtt{p}$ rows and $\mathtt{m}$ columns are denoted by $\mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{m}}$, which for polynomials is also $\mathbb{R}^{\mathtt{p}\times\mathtt{m}}[\xi]$. The spaces $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ and $\mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ stand for the spaces of infinitely often differentiable functions and locally integrable functions each from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$. In this paper, we also need $\mathfrak{C}^\infty(\mathbb{R}_+, \mathbb{R}^{\mathtt{w}})$, where $\mathbb{R}_+$ stands for $(0, \infty)$. The set of those elements in $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ which have compact support is denoted by $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. When the co-domain is clear from the context, then we drop the co-domain and write $\mathfrak{C}^\infty(\mathbb{R}_+)$, for example. Further, when both domain and co-domain are clear, we write just $\mathfrak{C}^\infty$ or $\mathfrak{L}_1^{\text{loc}}$.

2. Preliminaries

This section deals with the preliminaries that are required for this paper. The following subsection reviews required results from the behavioral approach to dynamical systems.

2.1. The behavioral approach

A linear differential behavior $\mathfrak{B}$ is defined as the subspace of $\mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ consisting of all solutions to a set of ordinary linear differential[1] equations with constant coefficients; i.e. for $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$

$$\mathfrak{B} := \left\{ w \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{d}{dt}\right) w = 0 \right\}. \tag{1}$$

This representation is called a *kernel representation* of $\mathfrak{B}$ and w is called the *manifest variable*. We assume a kernel representation matrix $R(\xi)$ to be of full row rank without loss of generality (see [8]); such a full row rank kernel representation is called a *minimal* kernel representation. The set of subsets of $\mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ that can be described by an equation of the form in Eq. (1) is defined as $\mathfrak{L}^{\mathtt{w}}$.

The familiar steerability aspect of the state controllability definition has been extended for behaviors and the PBH rank test generalizes [8, Chapter 5] as follows. For a behavior $\mathfrak{B}$ described by a minimal kernel representation $R(\frac{d}{dt})w = 0$, $\mathfrak{B}$ is controllable if and only if $R(\lambda)$ has full row rank for every $\lambda \in \mathbb{C}$. The set of controllable behaviors in $\mathtt{w}$ variables is denoted as $\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. It is also known that a behavior $\mathfrak{B}$ is controllable if and only if there exists a polynomial matrix $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[\xi]$ such that

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{ there exists } \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}}) \text{ such that } w = M\left(\frac{d}{dt}\right)\ell \right\}. \tag{2}$$

This representation of $\mathfrak{B}$ is called an image representation. It turns out (see [8]) that for an image representation, without loss of generality, one can assume $M(\xi)$ to be such that $M(\lambda)$ has full column rank for each $\lambda \in \mathbb{C}$: we call this an *observable* image representation. A special case when a polynomial matrix U has for each $\lambda \in \mathbb{C}$ both full row rank and full column rank is when its determinant is a nonzero constant: such polynomial matrices are called unimodular.

2.2. Dissipativity

In this subsection we review the essential notions of dissipativity theory: see [9] for a thorough treatment. Consider $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$,

[1] The differential equations are required to be satisfied in only a weak sense, i.e. in the distributional sense.

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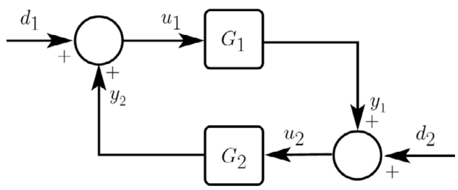

Fig. 1. Interconnection $\mathfrak{B}_1 \wedge \mathfrak{B}_2$.

assumed symmetric and nonsingular without loss of generality. A controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is said to be Σ-dissipative if

$$\int_{-\infty}^{\infty} w^T \Sigma w \, \mathrm{d}t \geqslant 0 \quad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

Suppose $w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$ is an observable image representation of the behavior. Then $\mathfrak{B}$ is dissipative if and only if $M(-j\omega)^T \Sigma M(j\omega)$ is non-negative definite for every $\omega \in \mathbb{R}$.

2.3. Well-posed/ill-posed interconnections, impulsive solutions

This section contains the definition of well-posedness of an interconnection and the link with existence of impulsive initial conditions. For the purpose of this paper, we deal with interconnection when an input/output partition of the system variables is specified. For the system $\mathfrak{B} \subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, a partition of the system variable w into $w = (u, y)$ is said to be an input/output partition with respect to $\mathfrak{C}^\infty$, with u as input and y as output, if for every u in $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$ there exists a $y \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{p}})$ and, no further component in y can be chosen arbitrarily in $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$. Conforming to this partition of w, rewrite the minimal kernel representation $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ as

$$Q\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) u + P\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) y = 0.$$

Then $w = (u, y)$ is an input/output partition with respect to $\mathfrak{C}^\infty$ if and only if P is square and has nonzero determinant. With respect to this input/output partition, we speak about the transfer matrix $G(s) := -P(s)^{-1}Q(s)$. Important in this paper is $\mathfrak{L}_1^{\text{loc}}$ properties of the variables, instead of $\mathfrak{C}^\infty$: the partition $w = (u, y)$ is an input/output partition with respect to $\mathfrak{L}_1^{\text{loc}}$ if and only if, in addition to P being square and nonsingular, $G(s) = -P(s)^{-1}Q(s)$ is *proper*.

A natural question that arises when dealing with two behaviors $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$ is whether upon 'interconnecting' the systems, properness-type conditions are lost. This brings us to the definition of a well-posed interconnection.

Definition 2.1. Consider behaviors $\mathfrak{B}_1$ and $\mathfrak{B}_2$, both with the system variable w, and suppose the partitions $w = (u_1, y_1)$ and $w = (y_2, u_2)$ are input/output partitions with respect to $\mathfrak{L}_1^{\text{loc}}$ for $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively. Assume the number of components in u_1 equals that in y_2. The *interconnection* $\mathfrak{B}_1 \wedge \mathfrak{B}_2$ of the systems $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is defined as the system with variables w and (d_1, d_2), and laws being those of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, together with

$$u_1 = y_2 + d_1 \quad \text{and} \quad u_2 = y_1 + d_2. \tag{3}$$

Further, the interconnection of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is said to be *well-posed* if for any $d_1, d_2 \in \mathfrak{L}_1^{\text{loc}}$, there exist unique $u_1, y_1, u_2, y_2 \in \mathfrak{L}_1^{\text{loc}}$ such that the laws of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ and Eq. (3) are satisfied. In other words, the interconnection is said to be well-posed if $(d_1, d_2, u_1, y_1, u_2, y_2) = ((d_1, d_2), (u_1, y_1, u_2, y_2))$ is an input/output partition with respect to $\mathfrak{L}_1^{\text{loc}}$ for $\mathfrak{B}_1 \wedge \mathfrak{B}_2$.

Fig. 1 illustrates Definition 2.1. The interconnection $\mathfrak{B}_1 \wedge \mathfrak{B}_2$ is said to be *ill-posed* if it is not well-posed. Note that well-posedness can be checked by merely considering the system $\mathfrak{B}_1 \wedge \mathfrak{B}_2$ with $d_1 = 0$ and $d_2 = 0$: see [10, Theorem 2.1] and [11, Theorem 7.1]. In the case when $d_1 = 0$ and $d_2 = 0$, the $\mathfrak{C}^\infty$-trajectories in the interconnection $\mathfrak{B}_1 \wedge \mathfrak{B}_2$ can be uniquely identified with those in $\mathfrak{B}_1 \cap \mathfrak{B}_2$: the identification being $u_1 = y_2$ and $u_2 = y_1$. In view of this, for the rest of the paper, the system laws of $\mathfrak{B}_1 \wedge \mathfrak{B}_2$ will include $d_1 = 0$ and $d_2 = 0$, and hence $\mathfrak{B}_1 \wedge \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}$. Crucially, for the well-posed case, the trajectories in $\mathfrak{B}_1 \wedge \mathfrak{B}_2$ are exactly those in $\mathfrak{B}_1 \cap \mathfrak{B}_2$. Then, a kernel representation of $\mathfrak{B}_1 \cap \mathfrak{B}_2$ (and hence $\mathfrak{B}_1 \wedge \mathfrak{B}_2$) is $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and $R_2(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, where $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ and $R_2(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ are respectively minimal kernel representations of $\mathfrak{B}_1$ and $\mathfrak{B}_2$. In this paper, we focus on the case when trajectories are $\mathfrak{L}_1^{\text{loc}}$.

It follows from Definition 2.1 that when an interconnection is ill-posed, then one or more of the following is the case.

- Non-autonomous interconnected system: for some d_1 and $d_2 \in \mathfrak{L}_1^{\text{loc}}$ there do not exist u_1, u_2, y_1 and y_2 satisfying the system laws.
- Autonomous interconnected system, but impulse causing initial conditions (singular descriptor interconnected system): there exist d_1 and $d_2 \in \mathfrak{L}_1^{\text{loc}}$ such that one or more of u_1, u_2, y_1 and y_2 are impulsive.

These two issues are central for this paper. We review the notion of an inadmissible initial condition and that of zeros at infinity of a polynomial matrix. For this paper, this is defined only for an autonomous system, i.e. for systems which, loosely speaking, the system variables do not 'respond' to external inputs, in other words, all variables are outputs. A system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called *autonomous* if in any input/output partition, all system variables are outputs. Thus $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ with minimal kernel representation $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is autonomous if and only if $P(\xi)$ is square and nonsingular.

Suppose an autonomous system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ has a minimal kernel representation $P(\frac{\mathrm{d}}{\mathrm{d}t})w(t) = 0$ with $P(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ nonsingular. Let $\mathtt{N}$ be the highest degree of all the polynomial entries in $P(\xi)$. Let $w(0), w^{(1)}(0), \ldots, w^{(\mathtt{N}-1)}(0)$ be the values of w, $\frac{\mathrm{d}}{\mathrm{d}t}w, \ldots, \frac{\mathrm{d}^{\mathtt{N}-1}}{\mathrm{d}t^{\mathtt{N}-1}}w$ at time $t = 0^-$. Define $\overline{w}(0) = [w(0), w^{(1)}(0), \ldots, w^{(\mathtt{N}-1)}(0)]$. We call the vector $\overline{w}(0) \in \mathbb{R}^{\mathtt{N}\mathtt{w}}$ an *initial condition* vector. A vector $\overline{w}(0)$ is said to be an *inadmissible* initial condition vector if the corresponding solution $w(t)$ nontrivially contains a Dirac impulse $\delta(t)$ and/or its distributional derivatives. See [12–14] for a similar treatment.

There are various (equivalent) definitions of the notion of a zero at infinity of a polynomial/rational matrix. Loosely speaking, a matrix $P(s)$ has one or more zeros at infinity if the matrix $Q(\lambda) := P(1/\lambda)$ has one or more zeros at the origin $\lambda = 0$. We adopt a more direct definition: consider $P(s) \in \mathbb{R}^{\mathtt{q}\times\mathtt{w}}[s]$ of rank say r. The polynomial matrix $P(s)$ is said to have no zeros at infinity if all the following inequalities hold:

$$\nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_r,$$

where $\nu_i := \max_{s \in S_i}\{\deg \det(s)\}$ and S_i is the set of all $i \times i$ minors of $P(s)$. If $r = 1$, there is no inequality to be satisfied: in this case there are no zeros at infinity. Of course, the negative of each of the ν_i are the ith valuations at ∞: see [15]. Using the above definition, and the fact that a unimodular polynomial matrix $U(\xi)$ has a nonzero constant as its determinant, we note that any nonconstant unimodular $U(\xi)$ has zeros at infinity.

The relevance of zeros at infinity is due to Vardulakis [14, Theorem 4.32], which states that a necessary and sufficient condition for absence of inadmissible initial conditions for the autonomous system $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ with P square and nonsingular is that P has no zeros at infinity. See also [12].

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2.4. Orthogonal complement of a behavior and Hamiltonian systems

We review the definition of the Σ-orthogonal complement of the system $\mathfrak{B}$. This system is familiar in the control literature as dual or adjoint system. See also [16] where this dynamical system is termed the 'costate dynamics'.

Definition 2.2 (*See [9]*). Consider a controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and a symmetric, nonsingular matrix $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. *The Σ-orthogonal complement of* $\mathfrak{B}$, denoted by $\mathfrak{B}^{\perp\Sigma}$, is the set of all the trajectories $v \in \mathfrak{L}^{\text{loc}}_1(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ such that $\int_{-\infty}^{\infty} v^T \Sigma w \, dt = 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$.

Suppose $R(\frac{d}{dt})w = 0$ is a minimal kernel representation and $w = M\left(\frac{d}{dt}\right)\ell$ is an observable image representation of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. It is known that (see [9, Section 10], for example) a minimal kernel representation for $\mathfrak{B}^{\perp\Sigma}$ is $M\left(-\frac{d}{dt}\right)^T \Sigma v = 0$ and $v = \Sigma R(-\frac{d}{dt})^T \ell$ is an observable image representation for $\mathfrak{B}^{\perp\Sigma}$. In this paper we deal with three situations.

(a) $\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma}$ is non-autonomous,
(b) $\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma}$ is autonomous and the interconnection $\mathfrak{B} \wedge \mathfrak{B}^{\perp\Sigma}$ is ill-posed, and
(c) $\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma}$ is autonomous and the interconnection $\mathfrak{B} \wedge \mathfrak{B}^{\perp\Sigma}$ is well-posed.

For the ill-posed case, we characterize conditions for existence of initial conditions resulting in impulses in one or more variables.

We dwell further on different representations of $\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma}$: see also [1, equations (10), (12) and (18)] and [17, Proposition 4.1 and Theorem 3.4] for a similar treatment. We view the set $\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma}$ as those trajectories in the 'interconnected' or the 'closed-loop' system obtained by connecting the system $\mathfrak{B}$ with its adjoint system: when asking questions about well-posedness of the interconnection, we distinguish the *interconnection* $\mathfrak{B} \wedge \mathfrak{B}^{\perp\Sigma}$ from the intersection $\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma}$.

$$\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma} = \left\{ w \in \mathfrak{L}^{\text{loc}}_1(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{d}{dt}\right) w = 0 \right. \\ \left. \text{and } M\left(-\frac{d}{dt}\right)^T \Sigma w = 0 \right\}. \tag{4}$$

Another representation of $\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma}$ can be obtained by using the image representations of $\mathfrak{B}$ and $\mathfrak{B}^{\perp\Sigma}$:

$$\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma} = \left\{ w \in \mathfrak{L}^{\text{loc}}_1(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid w = M\left(\frac{d}{dt}\right)\ell \right. \\ \left. \text{with } \ell \text{ satisfying } M\left(-\frac{d}{dt}\right)^T \Sigma M\left(\frac{d}{dt}\right)\ell = 0 \right\}. \tag{5}$$

It is easy to see that $\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma}$ is autonomous if and only if the $\mathtt{w} \times \mathtt{w}$ polynomial matrix $R_{\text{Ham}}(\xi) := \begin{bmatrix} R(\xi) \\ M(-\xi)^T\Sigma \end{bmatrix}$, which plays a role in Eq. (4), is nonsingular. This nonsingularity is equivalent to that of $M(-\xi)^T \Sigma M(\xi)$. It is *less* easy to see that, while the two representations (i.e. Eqs. (4) and (5)) indeed both describe $\mathfrak{B}\cap\mathfrak{B}^{\perp\Sigma}$ as far as *slow* solutions are concerned: the fast modes need not be the same. More precisely, while $R_{\text{Ham}}(\xi)$ could have nontrivial impulsive half-line solutions, the corresponding $M(-\xi)^T \Sigma M(\xi)$ need not necessarily have: see Section 6 for concrete examples. A third important representation, a first order one, is what we need often.

Suppose $\mathfrak{B}$ has the following minimal input/state/output (i/s/o) representation

$$\dot{x} = Ax + Bw_1 \quad \text{and} \quad w_2 = Cx + Dw_1 \tag{6}$$

where x is the state vector, w_1 is the input vector, w_2 is the output vector. Consider $\Sigma = \begin{bmatrix} I_{\mathtt{m}} & 0 \\ 0 & -I_{\mathtt{p}} \end{bmatrix}$, then a state space representation of $\mathfrak{B}^{\perp\Sigma}$ is given by

$$\dot{z} = -A^T z - C^T v_1 \quad \text{and} \quad v_2 = B^T z + D^T v_1. \tag{7}$$

Under the interconnection $w_2 = v_1$ and $w_1 = v_2$, a first order representation of the interconnected system simplifies to

$$\begin{bmatrix} \dot{x} \\ \dot{z} \\ 0 \end{bmatrix} = \begin{bmatrix} A & BB^T & BD^T \\ 0 & -A^T & -C^T \\ -C & -DB^T & I_{\mathtt{p}} - DD^T \end{bmatrix} \begin{bmatrix} x \\ z \\ v_1 \end{bmatrix}. \tag{8}$$

It is well-known (see [18, Lemma 5.1], for example) that $\mathfrak{B} \wedge \mathfrak{B}^{\perp\Sigma}$ is well-posed if and only if $(I_{\mathtt{p}} - DD^T)$ is nonsingular. In that case, the last line in the matrix–vector equation (8) can be rewritten as $v_1 = (I_{\mathtt{p}} - DD^T)^{-1}Cx + (I_{\mathtt{p}} - DD^T)^{-1}DB^T z$. Substituting this in Eq. (8) to eliminate v_1 we get $\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = H \begin{bmatrix} x \\ z \end{bmatrix}$ where

$$H := \begin{bmatrix} A + BD^T(I_{\mathtt{p}} - DD^T)^{-1}C & BB^T + BD^T(I_{\mathtt{p}} - DD^T)^{-1}DB^T \\ -C^T(I_{\mathtt{p}} - DD^T)^{-1}C & -(A^T + C^T(I_{\mathtt{p}} - DD^T)^{-1}DB^T) \end{bmatrix}.$$

The $(2\mathtt{n} \times 2\mathtt{n})$ matrix H above is a *Hamiltonian* matrix, i.e. H is similar to $-H^T$. Note, however, that the above derivation fails when $(I_{\mathtt{p}} - DD^T)$ is singular. The nonsingularity of $(I_{\mathtt{p}} - DD^T)$ remains a standing assumption in various applications: optimal control through algebraic Riccati equations and to Hamiltonian matrices, see [19,20], for example. In this paper, we focus on the case when $(I_{\mathtt{p}} - DD^T)$ is singular, and analyze the interconnection $\mathfrak{B} \wedge \mathfrak{B}^{\perp\Sigma}$ when it is not well-posed.

We note here that the set $\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma}$ has received attention in various contexts. For example, stationarity properties of this set have been noted in [1] in the context of LQ control, and the link with Euler–Lagrange equation has been brought out there. Similar studies have been pursued later in [3,2]. In the context of interpolation at spectral zeros, the matrix in Eq. (8) is the one in [21], although for the 'passivity supply rate' instead of the bounded real supply rate Σ (Eq. (9)) of this paper. The set $\mathfrak{B} \cap \mathfrak{B}^{\perp\Sigma}$ has been shown to be the set of trajectories of 'minimal dissipation' in [4]. See also [22–24] for use of the notion of adjoint system for system identification and construction of canonical state maps, for example. We mention here that, unlike smooth modes, impulsive solutions in a system behavior do get affected by equation manipulation of the kind arising from (nonconstant) unimodular matrices. However, the similarity transformations on the states, being constant matrices, do not affect impulsive modes in the system: hence their usage in [25], for example.

3. Assumptions and justifications

In this section we list the assumptions we make throughout this paper. We also give system theoretic justification for the assumptions.

3.1. Maximum input cardinality

Consider the supply rate induced by the matrix Σ used for defining dissipativity above. By considering a suitable coordinate transformation in the w variable, Σ can be assumed without loss of generality to be equal to

$$\Sigma = \begin{bmatrix} I_+ & 0 \\ 0 & -I_- \end{bmatrix}. \tag{9}$$

Denote the positive and negative signatures of Σ, (the number of positive and negative eigenvalues of the matrix Σ respectively), by $\sigma_+(\Sigma)$ and $\sigma_-(\Sigma)$. Suppose the system is described by a minimal kernel representation $R(\frac{d}{dt})w = 0$ and observable image representation $w = M(\frac{d}{dt})\ell$ with $R(\xi) \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ and $M(\xi) \in \mathbb{R}^{\mathtt{w}\times(\mathtt{w}-\mathtt{p})}[\xi]$ and define $\mathtt{m} := \text{rank}\,(M) = \mathtt{w} - \mathtt{p}$. It is known that [9, Remark 5.11] $\mathtt{m} \leqslant \sigma_+(\Sigma)$ is necessary for Σ-dissipativity of $\mathfrak{B}$. This paper deals with the special case $\mathtt{m} = \sigma_+(\Sigma)$: we call this the *maximum input cardinality condition* and hence use

$$\Sigma = \begin{bmatrix} I_{\mathtt{m}} & 0 \\ 0 & -I_{\mathtt{p}} \end{bmatrix}. \tag{10}$$

Corresponding to the partition of Σ, partition $w = (u, y)$. It is known that Σ-dissipativity ensures that this partition is, in fact, an $\mathfrak{L}_1^{\text{loc}}$-input/output partition. In other words, partitioning R of the kernel representation $R(\frac{d}{dt})w = 0$ above into $R = [Q \;\; P]$ with respect to $w = (u, y)$, Σ-dissipativity results in the transfer matrix $G(s) := -P(s)^{-1}Q(s)$ to be proper, and $\|G(s)\|_{\mathcal{L}_\infty} \leqslant 1$. Further, since $G(s)$ is proper, one can obtain a minimal state-space representation of $G(s)$, for which, up to a similarity transformation, the state-space representation is unique.

3.2. Feedthrough term $D = I$

Assume a controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ satisfying assumptions in Section 3.1 is Σ-dissipative, and suppose $\frac{d}{dt}x = Ax + Bu$ and $y = Cx + Du$ is a minimal state space representation. Then $I_{\mathtt{p}} - DD^T$ is non-negative definite. The case when $I_{\mathtt{p}} - DD^T$ is positive definite is well-understood and results in a well-posed interconnection. This paper focuses on the case when $I - DD^T$ is *singular*. In order to avoid pursuing with a decomposition of the (u, y) variables into nullspace and image of $I_{\mathtt{p}} - DD^T$, we assume that $(I_{\mathtt{p}} - DD^T)$ is zero; further, we also assume that $D = I$ and $\Sigma = \text{diag}\,(I_{\mathtt{m}}, -I_{\mathtt{m}})$. The reasons are elaborated below.

Clearly, for ill-posedness, $I - DD^T$ is singular, i.e. one or more of the singular values of D are equal to one. Further, when the system is Σ-dissipative, i.e. when the system transfer matrix has $\mathcal{L}_\infty$-norm at most one, the remaining singular values are strictly less than one. The singular values of D that are strictly less than one do not cause ill-posedness of the interconnection and hence a state space similarity transformation combined with a coordinate transformation in u and y variables (see [26, equation (A.3)]) results in a modified $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ such that $\tilde{D}$ is diagonal with all diagonal entries being either zero or one. The diagonal entries being zero are as good as the corresponding transfer matrix being *strictly proper*. Since the strictly proper case and the situation when $I - DD^T > 0$, both result in the well-understood regular case, this particular aspect in the more general singular $(I - DD^T)$ case can be handled by a corresponding regular part in the final singular descriptor state space system. In order to analyze the situation due to singularity, we thus focus on the extreme case of ill-posedness, namely, when D is the identity matrix $I_{\mathtt{m}}$. As a special case, for a SISO system, assuming $\Sigma = \text{diag}\,(1, -1)$, ill-posedness of the interconnection is *equivalent* to $D = 1$.

3.3. Full column rank condition on input matrix B

For the rest of this paper, we assume the input matrix B is full column rank. By a dual argument, we also assume C is full row rank. This is elaborated in this subsection. We first state a necessary condition on state space representations of $\mathfrak{B}$ under which $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ is autonomous. The rest of the paper deals with autonomy of the interconnected system.

Lemma 3.1. *Consider $\Sigma = \text{diag}\,(I_{\mathtt{m}}, -I_{\mathtt{m}})$ and suppose $\mathfrak{B} \in \mathfrak{L}^{2\mathtt{m}}_{\text{cont}}$ has variable w partitioned into $w = (u, y)$ with respect to which the transfer matrix G from u to y has a minimal state space realization $(A, B, C, I_{\mathtt{m}})$. Suppose $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ is autonomous. Then B is full column rank.*

The proof can be found in [2, Remark 4.3]. Further, this is related to Hautus and Silverman [27, Theorem 3.26] and Heemels et al., [28, Lemma 3.3], and our assumption that $D = I$. Consider the assumption of B being full column rank. Under the situation that D is the identity matrix, it can be proved that if B is not full column rank, then the inputs corresponding to the null-space of B result in a non-autonomous all-pass subsystem in the interconnection of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$. We outline this proof here. Suppose v is a constant nonzero vector such that $Bv = 0$. Then $u = v\ell$ for any nonzero compactly supported function ℓ has the corresponding output $y = v\ell$, assuming initial condition is zero. Clearly, (u, y) is an element of both $\mathfrak{B}$ and of $\mathfrak{B}^{\perp_\Sigma}$ and is a nonzero compactly supported function. This proves that the intersection is non-autonomous. Thus the assumption that B is full column rank is a necessary condition for the interconnected system to be autonomous.

For the situation addressed in the above result, we interpret the above lemma, loosely speaking, as a nontrivial kernel of B resulting in a non-autonomous and all-pass subsystem in the interconnection of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$.

By a similar argument, we also assume the full row rank condition on C: this is also a necessary condition for $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ to be autonomous. In summary, the following assumptions hold for the rest of this paper.

(a) The dissipativity being considered is with respect to $u^Tu - y^Ty$: dissipativity ensures the input/output partition is unique, and the corresponding transfer matrix is proper.
(b) With respect to the above input/output partition, the feedthrough term $D = I$, i.e. a state space system of $\mathfrak{B}$ is $\frac{d}{dt}x = Ax + Bu$ and $y = Cx + u$.
(c) B is full column rank and C is full row rank.

4. Ill-posed interconnection $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$

In this section, we obtain a state space representation of $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ for the case that the interconnection is not well-posed, i.e. $(I_{\mathtt{p}} - DD^T)$ is singular.

The main result needs the notions of the weakly unobservable subspace V and the strongly reachable subspace W as proposed in [27]. The weakly unobservable subspace V is defined as the set of all initial conditions x_0 for which there exists an input $u \in \mathfrak{C}^\infty(\mathbb{R}_+)$ such that the corresponding output $y(t)$ is identically zero on $[0, \infty)$. The strongly reachable subspace $W \subseteq \mathbb{R}^{\mathtt{n}}$ is defined as the set of all states reachable by an impulsive input without the output y being impulsive. The sets V and W can be computed by the following subspace iteration algorithms [27, equations (3.20) and (3.22)], each of which has been shown there to converge in at most $\mathtt{n}$ steps.

Consider the state space system $\dot{x} = Kx + Lu, y = Mx + Nu$ with $K \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $N \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$, and L and M of corresponding sizes.

$$V_0 := \mathbb{R}^{\mathtt{n}},$$
$$V_{i+1} := \begin{bmatrix} K \\ M \end{bmatrix}^{-1} \left\{ (V_i \oplus 0) + \text{im} \begin{bmatrix} L \\ N \end{bmatrix} \right\}, \qquad V_n =: V \tag{11}$$

$$W_0 := \{0\},$$
$$W_{i+1} := [K,\; L]\; \{(W_i \oplus \mathbb{R}^{\mathtt{m}}) \cap \ker [M,\; N]\} \qquad W_n =: W. \tag{12}$$

It is known that the above subspace iteration algorithms converge in at most $\mathtt{n}$ steps. The set W_1 has a special significance in this paper: it is the set of states reachable by an input containing δ but no derivatives of δ. Further, together with autonomy of the system, $W = W_1$ is equivalent to no initial condition resulting in an impulsive solution. We use V_i and W_i in Theorems 4.1 and 5.1.

Suppose $\mathfrak{B}$ has a minimal state space representation $\frac{d}{dt}x = Ax + Bu$ and $y = Cx + u$: the assumptions listed in the previous section

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hold for the rest of the paper. Recall the state space representation for $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ described in Eq. (8):

$$\begin{bmatrix} \dot{x} \\ \dot{z} \\ 0 \end{bmatrix} = \begin{bmatrix} A & BB^T & B \\ 0 & -A^T & -C^T \\ -C & -B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ v_1 \end{bmatrix}. \tag{13}$$

Define

$$\widetilde{A} := \begin{bmatrix} A & BB^T \\ 0 & -A^T \end{bmatrix}, \qquad \widetilde{B} := \begin{bmatrix} B \\ -C^T \end{bmatrix}, \\ \widetilde{C} := \begin{bmatrix} C & B^T \end{bmatrix}. \tag{14}$$

The above matrices are used to characterize the situation when the interconnection is autonomous. See Remark 5.2 for the relation with all-pass behavior of the system.

Theorem 4.1. *Consider the interconnection of the behaviors $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$ with Σ as defined in Eq. (9). Let (A, B, C, D) be a minimal i/s/o representation with $D = I$, B being full column rank, and C being full row rank. Then statements 1, 2 and 3 are equivalent.*

1. *The interconnected system is autonomous.*
2. *The $\mathtt{m} \times \mathtt{m}$ matrix $\widetilde{C}(sI - \widetilde{A})^{-1}\widetilde{B}$ is invertible as a rational matrix.*
3. *$V \oplus W = \mathbb{R}^{2\mathtt{n}}$.*

Further, suppose any one of the above is true. Then we have the following:

4. $\ker(\widetilde{C}\widetilde{B}) \cap \ker(\widetilde{C}\widetilde{A}\widetilde{B}) \cap \cdots \cap \ker(\widetilde{C}\widetilde{A}^{2\mathtt{n}-1}\widetilde{B}) = \{0\}$.

Proof. The proof is organized as follows. We prove 1 ⇔ 2, 1 ⇔ 3, and then 2 ⇒ 4.
(**1 ⇔ 2**) Define

$$P(s) := \begin{bmatrix} sI_{\mathtt{n}} - A & -BB^T & -B \\ 0 & sI_{\mathtt{n}} + A^T & C^T \\ C & B^T & 0 \end{bmatrix}.$$

Evaluating the determinant of $P(s)$ by using the Schur complement with respect to the top-left $2\mathtt{n} \times 2\mathtt{n}$ block: $\begin{bmatrix} sI_{\mathtt{n}} - A & -BB^T \\ 0 & sI_{\mathtt{n}} + A^T \end{bmatrix}$ we get

$$\det(P(s)) = \chi_A(-s)\chi_A(s)\det(\widetilde{C}(sI_{2\mathtt{n}} - \widetilde{A})^{-1}\widetilde{B}), \tag{15}$$

where $\chi_A(s)$ is the characteristic polynomial of A. Since autonomy of the interconnected system is equivalent to nonsingularity of $P(s)$, we infer the equivalence of 1 and 2.
(**1 ⇔ 3**) This has been shown in [28, Lemma 3.3]. See also [27].
(**2 ⇒ 4**) Suppose the condition within statement 4 does not hold, that is, $\ker(\widetilde{C}\widetilde{B}) \cap \ker(\widetilde{C}\widetilde{A}\widetilde{B}) \cap \cdots \cap \ker(\widetilde{C}\widetilde{A}^{2\mathtt{n}-1}\widetilde{B}) \neq \{0\}$. We show that, this implies $\widetilde{C}(sI - \widetilde{A})^{-1}\widetilde{B}$ cannot be invertible as a rational function matrix. Let $v \in \mathbb{R}^{2\mathtt{n}} \setminus \{0\}$ be in $\ker \widetilde{C}\widetilde{A}^i\widetilde{B}$ for all $0 \leqslant i \leqslant 2\mathtt{n} - 1$. Note that using the Cayley–Hamilton theorem,

$$\widetilde{C}e^{\widetilde{A}t}\widetilde{B} = \left[\alpha_0(t)\widetilde{C}\widetilde{B} + \alpha_1(t)\widetilde{C}\widetilde{A}\widetilde{B} + \cdots + \alpha_{2\mathtt{n}-1}(t)\widetilde{C}\widetilde{A}^{2\mathtt{n}-1}\widetilde{B}\right]$$

for suitable analytic functions $\alpha_i(t)$. Since $v \in \ker \widetilde{C}\widetilde{A}^i\widetilde{B}$ for all $0 \leqslant i \leqslant 2\mathtt{n} - 1$, it follows from the last equation that $\widetilde{C}e^{\widetilde{A}t}\widetilde{B}v = 0$ for all $t \geqslant 0$. Taking Laplace transform of $\widetilde{C}e^{\widetilde{A}t}\widetilde{B}v$, we get that

$$\widetilde{C}(sI - \widetilde{A})^{-1}\widetilde{B}v = 0, \quad \text{and} \quad v \neq 0,$$

which means $\widetilde{C}(sI - \widetilde{A})^{-1}\widetilde{B}$ is not invertible as a rational function matrix. □

5. Impulsive initial conditions

In this section we formulate necessary and sufficient conditions for the interconnected system $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ to have inadmissible initial conditions, i.e. initial conditions that cause impulsive solutions.

The following result is one of the main results of this paper: necessary and sufficient conditions on $\mathfrak{B}$ for the interconnection $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ to have no inadmissible initial conditions. The relation of the conditions with all-pass characteristics of a MIMO system is elaborated in Remark 5.2. Also compare the corresponding equivalent statements in Theorem 4.1 where we characterized just autonomy.

Theorem 5.1. *Consider the state space representation of the systems $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$ as in Eqs. (6) and (7) and their interconnection $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$. Assume the resulting Hamiltonian system given by Eq. (13) is autonomous. Consider $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ as defined in Eq. (14) and use them to construct V_i and W_i as described in Eqs. (11) and (12). Then the following are equivalent:*

1. *The singular Hamiltonian system has no inadmissible initial conditions.*
2. $\ker(\widetilde{C}\widetilde{B}) = \{0\}$.
3. $\det(CB - (CB)^T) \neq 0$.
4. *$\widetilde{C}e^{\widetilde{A}t}\widetilde{B}$ is nonsingular at $t = 0$.*
5. *$W = W_1$ and $V = V_1$.*

Proof. The proof is organized as follows: 1 ⇒ 2 ⇔ 3 ⇔ 4 ⇒ 5 ⇒ 1.
(**1 ⇒ 2**) Suppose statement 2 is not true, we show that this implies statement 1 also is not true. Take $\ell \in \ker \widetilde{C}\widetilde{B}$; because 2 has been assumed to be false, we have $\ell \neq 0$. Now, consider solving Eq. (13) with initial condition

$$\begin{bmatrix} x_0 \\ z_0 \end{bmatrix} := \widetilde{A}\widetilde{B}\ell,$$

and distributional input

$$v_1(t) := -\ell\delta'(t),$$

where $\delta'(t)$ denotes the distributional derivative of the Dirac delta distribution, $\delta(\cdot)$, supported at $t = 0$. The resulting state trajectory is given by:

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{cases} e^{\widetilde{A}t}\begin{bmatrix} x_0 \\ z_0 \end{bmatrix} - \widetilde{A}e^{\widetilde{A}t}\widetilde{B}\ell - \widetilde{B}\ell\delta(t) & \text{for } t \geqslant 0 \\ 0 & \text{for } t < 0. \end{cases}$$

Since $\begin{bmatrix} x_0 \\ z_0 \end{bmatrix} := \widetilde{A}\widetilde{B}\ell$, and $\ell \in \ker \widetilde{C}\widetilde{B}$ it follows that

$$\widetilde{C}\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \equiv 0.$$

(Here, we have made use of the fact that $e^{\widetilde{A}t}$ commutes with $\widetilde{A}$.) Therefore, $\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$ as above and $v_1(t) = \ell\delta'(t)$ solves Eq. (13). Since, the Hamiltonian system has been assumed to be autonomous, $v_1(t) = -\ell\delta'(t)$ is the unique distribution that makes $\widetilde{C}\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \equiv 0$ for the chosen initial condition, $\begin{bmatrix} x_0 \\ z_0 \end{bmatrix} := \widetilde{A}\widetilde{B}\ell$. However, $\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$ clearly contains an impulse. Therefore, $\begin{bmatrix} x_0 \\ z_0 \end{bmatrix} := \widetilde{A}\widetilde{B}\ell$ is an inadmissible initial condition, which is contrary to statement 1.
(**2 ⇔ 3**) This follows by using the definition of $\widetilde{B}$ and $\widetilde{C}$ from Eq. (14).
(**3 ⇔ 4**) This is seen by noting that $\widetilde{C}e^{\widetilde{A}t}\widetilde{B}|_{t=0} = \widetilde{C}\widetilde{B}$.
(**4 ⇒ 5**) We assume that $\widetilde{C}e^{\widetilde{A}t}\widetilde{B}$ is non-singular at $t = 0$, and want to show that $W = W_1$ and $V = V_1$. First, the equivalence

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of statements 2, 3 and 4 implies that $\widetilde{C}e^{\widetilde{A}t}\widetilde{B}$ being non-singular at $t = 0$ is equivalent to $\ker \widetilde{C}\widetilde{B} = \{0\}$. It follows that $\ker \widetilde{C} \cap \operatorname{im} \widetilde{B} = \{0\}$. Due to the full column rank assumption on B and C^T, we obtain the full column rank assumption on $\widetilde{B}$ and $\widetilde{C}^T$, which results in $\dim\left(\ker \widetilde{C}\right) = 2\mathtt{n} - \mathtt{m}$ and $\dim\left(\operatorname{im} \widetilde{B}\right) = \mathtt{m}$. These two facts together imply that

$$\ker \widetilde{C} \oplus \operatorname{im} \widetilde{B} = \mathbb{R}^{2\mathtt{n}}. \tag{16}$$

Using [29, Lemma 4.2], we note that a necessary and sufficient condition for $\ker \widetilde{C}$ to be $(\widetilde{A}, \widetilde{B})$-invariant is: $\widetilde{A}\left(\ker \widetilde{C}\right) \subseteq \ker \widetilde{C} + \operatorname{im} \widetilde{B}$. Therefore Eq. (16) implies that $\ker \widetilde{C}$ is $(\widetilde{A}, \widetilde{B})$-*invariant*, i.e., there exists a state-feedback matrix $F \in \mathbb{R}^{\mathtt{m}\times 2\mathtt{n}}$ such that $\ker \widetilde{C}$ is $(\widetilde{A} + \widetilde{B}F)$-invariant. It then follows that the consistent subspace V, for the case at hand, is nothing but $\ker \widetilde{C}$ (because, in this particular case with $D = 0$, the consistent subspace V is the largest $(\widetilde{A}, \widetilde{B})$-invariant subspace contained in $\ker \widetilde{C}$ [27, Theorem 3.10]). However, using the iteration Eq. (11), we have $V_1 = \ker \widetilde{C}$. Therefore, $V = V_1$. Now, since the system has been assumed to be autonomous, by Heemels et al. [28, Lemma 3.3] we get that

$$V \oplus W = \mathbb{R}^{2\mathtt{n}},$$

where W is the jump space of the system. It then follows that $\dim W = 2\mathtt{n} - (2\mathtt{n} - \mathtt{m}) = \mathtt{m}$ because $V = \ker \widetilde{C}$ and $\dim\left(\ker \widetilde{C}\right) = 2\mathtt{n} - \mathtt{m}$. However, note that $W \supseteq \operatorname{im} \widetilde{B}$ and $\dim\left(\operatorname{im} \widetilde{B}\right) = \mathtt{m}$. Hence, $W = \operatorname{im} \widetilde{B} = W_1$.

(**5** $\Rightarrow$ **1**) Since the Hamiltonian system has been assumed to be autonomous, by Theorem 4.1, we must have

$$V \oplus W = \mathbb{R}^{2\mathtt{n}}.$$

This means every initial condition $\begin{bmatrix} x_0 \\ z_0 \end{bmatrix} \in \mathbb{R}^{2\mathtt{n}}$ can be decomposed as

$$\begin{bmatrix} x_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x_0^W \\ z_0^W \end{bmatrix} + \begin{bmatrix} x_0^V \\ z_0^V \end{bmatrix},$$

where $\begin{bmatrix} x_0^W \\ z_0^W \end{bmatrix} \in W$ and $\begin{bmatrix} x_0^V \\ z_0^V \end{bmatrix} \in V$. Two key observations will help complete the proof.

Observation 1. Note that $W = W_1 = \operatorname{im} \widetilde{B}$ and $V = V_1 = \ker \widetilde{C}$ imply that

$$\begin{bmatrix} x_0^W \\ z_0^W \end{bmatrix} \in \operatorname{im} \widetilde{B}, \quad \text{and} \quad \begin{bmatrix} x_0^V \\ z_0^V \end{bmatrix} \in \ker \widetilde{C}. \tag{17}$$

It then follows that there exists $\ell \in \mathbb{R}^{\mathtt{m}}$ such that

$$\begin{bmatrix} x_0^W \\ z_0^W \end{bmatrix} = \widetilde{B}\ell.$$

Consider the solution of Eq. (13) with $\begin{bmatrix} x_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x_0^W \\ z_0^W \end{bmatrix}$ and $v_1(t) = -\ell\delta(t)$. Clearly, using $\begin{bmatrix} x_0^W \\ z_0^W \end{bmatrix} = B\ell$, the state trajectory corresponding to this v_1 evaluates to

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = e^{\widetilde{A}t}\begin{bmatrix} x_0^W \\ z_0^W \end{bmatrix} - e^{\widetilde{A}t}\widetilde{B}\ell \equiv 0 \quad \text{for all } t > 0.$$

Observation 2. Since $V = V_1 = \ker \widetilde{C}$, there exists a state feedback matrix $F \in \mathbb{R}^{\mathtt{m}\times 2\mathtt{n}}$ such that $\ker \widetilde{C}$ is $(\widetilde{A} + \widetilde{B}F)$-invariant. Thus,

$$\widetilde{C}e^{(\widetilde{A}+\widetilde{B}F)t}\begin{bmatrix} x_0^V \\ z_0^V \end{bmatrix} = 0 \quad \text{for all } t \geqslant 0,$$

because $\begin{bmatrix} x_0^V \\ z_0^V \end{bmatrix} \in \ker \widetilde{C}$ and $\ker \widetilde{C}$ is $(\widetilde{A} + \widetilde{B}F)$-invariant.

Observations 1 and 2 together imply the following. Suppose for an initial condition $\begin{bmatrix} x_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x_0^W \\ z_0^W \end{bmatrix} + \begin{bmatrix} x_0^V \\ z_0^V \end{bmatrix}$, we have

$$v_1(t) = \begin{cases} Fe^{(\widetilde{A}+\widetilde{B}Ft)}\begin{bmatrix} x_0^V \\ z_0^V \end{bmatrix} - \ell\delta(t) & \text{for } t \geqslant 0 \\ 0 & \text{for } t < 0, \end{cases}$$

then the resulting trajectory $\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$ has the property that

$$\widetilde{C}\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \equiv 0,$$

in the distributional sense. This means, $\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}$, with the above-mentioned $v_1(t)$ solves Eq. (13). However, notice that

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} & \text{for } t = 0, \\ e^{(\widetilde{A}+\widetilde{B}F)t}\begin{bmatrix} x_0^V \\ z_0^V \end{bmatrix} & \text{for all } t > 0, \end{cases}$$

which is clearly impulse-free, but perhaps has jumps. At $t = 0$, we have $\begin{bmatrix} x_0 \\ z_0 \end{bmatrix}$, which was chosen arbitrarily, and hence it follows that the Hamiltonian system has no inadmissible initial conditions. This proves 1 and thus completes the proof of Theorem 5.1. □

Note that the minimal state space representation (A, B, C, D) for $\mathfrak{B}$ was arbitrary, except for $D = I$. In our analysis, we used the state space representation in Eq. (7) for $\mathfrak{B}^{\perp\Sigma}$. If one begins with a different minimal state space representation for $\mathfrak{B}^{\perp\Sigma}$, then there must exist a similarity transformation S such that the new representation is $\frac{d}{dt}z = -S^{-1}A^TSz - S^{-1}C^Ty$ and $u = B^TSz + y$ and it can be verified that each of the statements in the above theorem is unchanged due to the matrix S. In this way, our results are not dependent on any specific state space representation for $\mathfrak{B}$ or $\mathfrak{B}^{\perp\Sigma}$. In the context of Statement 5 of the above theorem, as mentioned before Theorem 4.1: together with autonomy of the system, $W = W_1$ is known to be equivalent to no initial condition resulting in an impulsive solution. The following remark relates condition 2 of the above Theorem with an all-pass MIMO transfer matrix.

Remark 5.2. Loosely speaking, Condition 2 of the above theorem is opposite to the condition required for a transfer function $G(s)$ to be all-pass.[2] More precisely, consider a square MIMO transfer function $G(s) \in \mathbb{R}(s)^{\mathtt{m}\times\mathtt{m}}$ which is all-pass, i.e. $I - G(-s)^TG(s) = 0$ for every $s \in j\mathbb{R}$. The feed-through term D of such a transfer matrix can be assumed to be I by considering a change of coordinates in either the u or the y variables. With this assumption on D, equating each of the Markov parameters of $I - G(-s)^TG(s)$ to zero, the all-pass condition on G results in the following conditions on matrices A, B and C of its state space realization:

$$\begin{aligned} &CB - B^TC^T = 0, \\ &CAB + (CAB)^T - B^TC^TCB = 0, \cdots. \end{aligned} \tag{18}$$

Notice that CB is nothing but the first moment of $G(s)$ about $s = \infty$. Thus a necessary condition on the first moment for G to be all-pass is that the skew-symmetric part of CB is zero. On the other hand,

[2] This remark is relevant for the case that the supply rate corresponds to $u^Tu - y^Ty$, for which 'lossless' corresponds to all-pass characteristics. When dealing with the supply rate u^Ty, relevant in passivity analysis, it is singularity of $(D + D^T)$ matrix that plays a role for the results of this paper; $G(s) + G(-s)^T$ then replaces $I - G(-s)^TG(s)$ for the statements made in this remark.

condition 3 of the above theorem requires the skew-symmetric part to be nonsingular. In this sense, the necessary and sufficient condition on $G(s)$ for the singular Hamiltonian system to not have any inadmissible initial conditions is opposite to the requirement that $G(s)$ is all-pass.

Another consequence of condition 3 in Theorem 5.1, under Lemma 3.1, is that square MIMO systems with an odd-number of inputs, in particular SISO systems, always has inadmissible initial conditions. This is seen in the following example.

Example 5.3. Consider $G(s) = \frac{s+1}{s+2}$ with input u and output y and consider its state space realization $(A, B, C, D) = (-2, 1, -1, 1)$. The adjoint system has transfer function $\frac{s-1}{s-2}$ and $(2, 1, 1, 1)$ is a state space description. The interconnected system (described in the variables: state x and output y of the system G and state z of its adjoint) turns out to be:

$$\frac{d}{dt}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ z\\ y\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ -1 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ z\\ y\end{bmatrix}.$$

It can be checked that the matrix pencil corresponding to the above first order differential equation has a zero at infinity. After elimination of the variable y too, the differential equation in just x and z turns out to contain inadmissible initial conditions. Theorem 5.1 can be used to obtain the same inference: since $CB - B^T C^T = 0$, we conclude that there exist inadmissible initial conditions. Of course, as noted in Remark 5.2, for SISO systems, ill-posed interconnection implies existence of inadmissible initial conditions.

6. Impulsive solutions in the manifest/system variables

The last section (in particular, Theorem 5.1) formulated conditions under which the *state-space* of the interconnected system (with the states being that of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$) has impulsive initial conditions. In this section we investigate further into the case when the *manifest* variables w have impulsive solutions in $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$.

Recall again from [14, Theorem 4.32] that a necessary and sufficient condition for absence of inadmissible initial conditions for an autonomous system $P(\frac{d}{dt})w = 0$ with P square and nonsingular is that P has no zeros at infinity. Using a kernel representation for $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ of Eq. (4), we note that there are no impulsive modes in the variable w if and only if $\begin{bmatrix}R(\xi)\\ M(-\xi)^T\Sigma\end{bmatrix}$ has no zeros at infinity. Alternatively, using the latent variable representation in Eq. (5):

$$\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma} = \Bigg\{ w \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid w = M\left(\frac{d}{dt}\right)\ell \\ \text{with } \ell \text{ satisfying } M\left(-\frac{d}{dt}\right)^T \Sigma M\left(\frac{d}{dt}\right)\ell = 0 \Bigg\}, \quad (19)$$

we see that w is impulsive if ℓ is impulsive, equivalently, in the presence of zeros at infinity in $M(-\xi)^T\Sigma M(\xi)$.

The natural question is whether these are equivalent. For single input systems, $M(-\xi)^T\Sigma M(\xi)$ is a scalar and hence never has zeros at infinity: but $\begin{bmatrix}R(\xi)\\ M(-\xi)^T\Sigma\end{bmatrix}$ can have: Example 5.3 is one such case. In this example, $G(s) = \frac{s+1}{s+2}$, with $w = (u, y)$, for which $R(\xi) := [(\xi+1) \ \ -(\xi+2)]$ and $M(\xi) := [(\xi+2) \ \ (\xi+1)]^T$. Verify that

$$R_{\text{Ham}}(\xi) := \begin{bmatrix}R(\xi)\\ M(-\xi)^T\Sigma\end{bmatrix} = \begin{bmatrix}(\xi+1) & -(\xi+2)\\ (\xi+2) & (\xi+1)\end{bmatrix}$$

while $M(-\xi)^T\Sigma M(\xi) = 1$.

Since $R_{\text{Ham}}(\xi)$ above is unimodular and nonconstant, $R_{\text{Ham}}(\xi)$ has zeros at infinity, which implies that one or both of the components in w have impulsive modes. We infer from this example that the latent variable representation in Eq. (19), which imposes restrictions on the latent variable ℓ by $M(-\frac{d}{dt})^T\Sigma M(\frac{d}{dt})\ell = 0$, need not be able to generate the impulsive modes in w (through $w = M(\frac{d}{dt})\ell$). In fact, for the case of single input, $M(-\xi)^T\Sigma M(\xi)$ cannot have zeros at infinity and is unable to reveal impulsive modes in w or the corresponding state space representation of $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$. A natural question is whether the impulsive modes as revealed by the state space representation of $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ and $R_{\text{Ham}}(\xi)$ agree with each other: the example below addresses this question.

Example 6.1. Consider $G(s) = \frac{(s+1)^2}{(s+2)^2}$, with $w = (u, y)$, for which define $R(s) = [(s+1)^2 \ -(s+2)^2]$ and $M(s) = [(s+2)^2 \ (s+1)^2]^T$. As far as the corresponding first order singular/descriptor state space representation is concerned, again like in Example 5.3, since the system is SISO, there are zeros at infinity in $sE - H$ and hence one or more of the states contain impulsive modes. While $M(-\xi)^T\Sigma M(\xi)$ evaluates to a scalar and hence causes no impulsive modes in ℓ, the 2×2 polynomial matrix $R_{\text{Ham}}(\xi)$ evaluates to

$$R_{\text{Ham}}(\xi) = \begin{bmatrix}R(\xi)\\ M(-\xi)^T\Sigma\end{bmatrix} = \begin{bmatrix}(\xi+1)^2 & -(\xi+2)^2\\ (\xi+2)^2 & (\xi+1)^2\end{bmatrix}$$

which has determinantal degree 2; using the definition of zeros at infinity, verify that there are no zeros at infinity. We infer from this example that while one or more of the states x and z have impulses, the kernel representation $R_{\text{Ham}}(\xi)w = 0$ is such that the manifest variable w contains no impulsive solutions.

A natural question that arises due to the absence of impulses in the external variables w in spite of x and z containing impulses is whether the system is *impulse unobservable*. We investigate this after a brief review of the notion of impulse observability, and the related notion: 'observability at ∞'.

Consider the definition of impulse observable from [25, page 1079]: a singular descriptor system $E\dot{x} = Ax$ and $y = Cx$ is said to be *impulse observable* (and also *observable at* ∞ in the sense of Verghese et al. [12]) if for every $\tau > 0$, knowledge of $y(\tau)$ is sufficient to determine $x(\tau)$. Closely related to impulse observable is the notion of 'observability at ∞' in the sense of Rosenbrock [30], in which knowledge of the distribution y over the duration $[0, \tau]$ for some $\tau > 0$ and $y(0^-)$ is sufficient to infer $x(0^-)$. Note that 'knowledge' of x and y in both definitions refers to in the distributional sense. We refer to Cobb [25] for a thorough comparison of the definitions/equivalent conditions about the two observabilities at ∞ as defined in [30,12,31]: we restrict ourselves to an equivalent condition each to check these observabilities. Observability at ∞ in the sense of Verghese is equivalent to $\begin{bmatrix}sE - A\\ C\end{bmatrix}$ having no zeros at ∞, while that in the sense of Rosenbrock is equivalent to $\begin{bmatrix}E - sA\\ C\end{bmatrix}$ having full column rank at $s = 0$: see [25, Theorems 9 and 10] and their proofs.

For Example 6.1, it can be checked as follows that the system is observable at ∞ in both senses. E, A and C matrices respectively evaluate to

$$\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}, \quad \begin{bmatrix}-4 & -4 & 1 & 0 & 1\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 4 & -1 & 2\\ 0 & 0 & 4 & 0 & 3\\ 2 & 3 & -1 & 0 & 0\end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix}0 & 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 1\end{bmatrix}.$$

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While $\begin{bmatrix} E \\ C \end{bmatrix}$ is obviously rank 5, absence of zeros at infinity of the matrix $\begin{bmatrix} sE - A \\ C \end{bmatrix}$ follows from noting that the degree of the determinant of the 5×5 matrix comprising of the first 4 rows and the last row is 4: using the definition, we conclude that there are no zeros at infinity.

A conclusion inferred from the above example is that, while the states could have impulsive behavior, the manifest variables need not have, as suggested by the kernel representation. More crucially, in spite of the singular descriptor system satisfying observability at ∞ with respect to two different definitions, the impulsive behavior in the states is not revealed in the manifest variable. Perhaps a different notion of impulse observability needs to be formulated to explain this.

7. Concluding remarks

We studied the interconnection of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$ and studied half-line solutions in the interconnected system. While the full-line solutions are the same for three different representations of this set (namely, the latent variable representation, the kernel representation and the state space representation), the fast-modes in the half-line solutions set need not be the same. We formulated necessary and sufficient conditions for the interconnected system to be well-posed and for it to be autonomous.

When the interconnection of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$ is not well-posed, under suitable regularizing assumptions, we formulated necessary and sufficient conditions for existence of inadmissible initial conditions for the interconnected system $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ in terms of the first moment about $s = \infty$ of the transfer matrix: our second main result. We also related these conditions to the one-step convergence of the well-known subspace iteration algorithms for obtaining the strongly reachable and weakly unobservable subspaces. We noted that the condition on the skew-symmetric part of the first moment was opposite to that for the MIMO transfer matrix to be an all-pass filter.

We finally saw two examples, and one of them had the feature that while the state space representation of $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ had impulsive solutions, the kernel representation (in just the manifest variable) did not reveal any impulsive behavior. Further, the states were observable at infinity in the sense of both Verghese et al. [12] and Rosenbrock [30], thus raising questions about why impulses in the states were not revealed in the manifest variables.

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