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Model reduction of AHWR space–time kinetics using balanced truncation

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ABSTRACT

Study of transient behavior and design of control for large reactors, such as the Advanced Heavy Water Reactor (AHWR) requires accurate mathematical models. An 80th order space–time kinetics model which describes the core neutronics behavior of AHWR is available but it is nonlinear and stiff. The modern control design and analysis studies with this model are accompanied by serious numerical ill-conditioning problems. Hence, there is a strong motivation for obtaining a reduced order model which preserves the input–output behavior accurately.

In this paper, we have explored the application of balanced truncation technique to obtain a reduced order model from the original high order model of the AHWR. The reduced order model thus obtained was examined from the view point of transient performance by comparing its response with that of the original model and it is found to yield a very good approximation. We also carried out the comparative study between different reduced order models of AHWR, namely, Davison's technique, Marshall's technique, singular perturbation analysis and balanced truncation by comparing their performances with respect to each other and with the original model. All of these methods are found to be effective, however the overall accuracy in the approximation using the balanced truncation approach is found to be far more superior.

1. Introduction

Description of large-scale systems by mathematical models involves a set of first order differential or difference equations. These models can be used to simulate the system response and predict the behavior. Sometimes, these mathematical models are also used to modify or control the system behavior to conform with certain desired performance. In practical control engineering applications with the increase in need for improved accuracy, mathematical models lead to high order and complexity. Although, the well established modern control concepts are valid for any system order, they may not give fruitful control algorithms in control design. Moreover, working with very high order model involves computational complexity and need for high storage capability. Sometimes, the presence of small time constants, masses, *etc.* may give rise to an interaction among slow and fast dynamic phenomena with attendant ill-conditioning or stiff numerical problems. When analyzing and controlling these large-scale dynamic systems, it is extremely important to look for and to rely upon efficient simplified reduced order models which capture the main features of the full order complex model.

In the past few decades, several analytical model reduction techniques have been proposed, such as retaining of the dominant modes (Davison, 1966; Marshall, 1966), model reduction by aggregation (Aoki, 1968) and decomposition of higher order model into slow and fast systems by two-time-scale methods and singular perturbation analysis (Kokotovic et al., 1976), *etc.* These methods dealt with the eigenvalues of the system and require the assessment of dominant modes present in the model. Various other methods such as balanced truncation, balancing free technique, *etc.*, are also available for model order reduction.

For the state-space models, model order reduction method based on the assessment of degree of controllability and observability has been suggested in Moore (1981) and Pernebo and Silverman (1982) which is popularly known as balanced truncation. In order to obtain the original system in balanced form, its basis should be transformed into another basis where the states which are difficult to reach are simultaneously difficult to observe.

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It can be achieved by simultaneously diagonalizing the reachability and the observability Gramians (Laub et al., 1987), which are the solutions to reachability and observability Lyapunov equations. The positive decreasing diagonal entries in the diagonal reachability and observability Gramians in the new basis are called the Hankel singular values of the system. The reduced order model is obtained simply by truncation of the states corresponding to the smallest singular values. The number of states that can be truncated depends on how accurate the approximate model should be. There are some other techniques to obtain the balanced truncation *viz.*, Schur method (Safonov and Chiang, 1989), balance square root method (Varga, 1991) similar to Moore (1981), however, they differ in the algorithms to obtain the balancing transformation. The aforesaid methods can be efficiently applied when the system is asymptotically stable and minimal, however, for the systems where the stabilization is the major concern their straightforward application is not possible. Balanced truncation for stable nonminimal systems has been attempted in Tombs and Postlethwaite (1987). Balanced truncation for unstable systems has also been attempted in Kenney and Hewer (1987), Safonov et al. (1987) and Benner et al. (2003). Usually unstable poles cannot be neglected, therefore model reduction in this situation can be treated by first separating the stable and unstable parts of the model and then reducing the order of the stable part using balanced truncation methods. Pertinent literature survey on balanced truncation methods and obtaining balanced transformation procedure can be found in Benner et al. (2003), Gugercin and Antoulas (2004) and Penzl (2006).

Advanced Heavy Water Reactor (AHWR) (Sinha and Kakodkar, 2006) is a 920 MW (thermal), vertical, pressure tube type, heavy water moderated, boiling light water cooled, natural circulation reactor. The physical dimensions of the AHWR core are large compared to the neutron migration length. Therefore from neutronic view-point, the behavior tends to be loosely coupled, due to which a serious situation called 'flux tilt' may arise in AHWR followed by an operational perturbation. Therefore it is necessary to provide online monitoring and control schemes during the reactor operation. A reasonably accurate space–time kinetics model for describing the behavior of AHWR is developed for control related studies using nodal methods (Shimjith et al., 2010; Sagar et al., 2013). An important characteristic of the model based on nodal methods is that the order of mathematical model depends on the number of nodes into which the reactor spatial domain is divided. A rigorous model with more number of nodes will give good accuracy in online monitoring and control, but its order is very high. At the same time, nuclear reactor models often exhibit simultaneous presence of dynamics of different speeds. Such behavior leads the mathematical model exhibiting multiple time-scales, which may be susceptible to numerical ill-conditioning (Rajasekhar et al., 2016; Kokotovic et al., 1976). Hence, it is very much essential to develop a suitable mathematical model of lower order which alleviates the dimensionality and numerical ill-conditioning problems in computations. The application of Davison's technique for model order reduction has been explored for PHWR in Talange et al. (2002) and for AHWR in Rajasekhar et al. (2015). The application of Marshall's dominant mode retention technique has been explored for AHWR in Rajasekhar et al. (2015). In Talange et al. (2006), the aggregation technique has been applied for obtaining simplified model of PHWR and model decomposition based on singular perturbation and time-scale methods for controller design have also been applied for PHWR in Tiwari et al. (1996); Tiwari et al., 2000 and for AHWR in Shimjith et al. (2011a,b), Munje et al. (2013), Rajasekhar et al. (2015), Munje et al. (2015a) and Munje et al. (2015b).

In this paper, we consider the 80th order linear model of AHWR (Rajasekhar et al., 2015) developed by considering the time dependent core neutronics equations with 17 node scheme and control rod dynamic equations as derived in Sagar et al. (2013). Methods based on retaining of dominant modes: Davison's, Marshall's and model decomposition based on singular perturbation and time-scale methods have already been attempted for the same model in Rajasekhar et al. (2015). However, application of balanced truncation method is not found in case of AHWR. Motivated by this, we try to obtain a reduced order model based on balanced reduction technique. However, the presence of multiple eigenvalues at origin of the complex *s*-plane in the AHWR model, restricts the straightforward application of the state-space balancing algorithm (Laub et al., 1987). Therefore, the model is first decomposed into stable and unstable subsystems (Safonov et al., 1987; Benner et al., 2003). Simplified model is obtained for the stable subsystem and finally the reduced order model is formulated by augmenting the unstable model with the simplified model of the stable subsystem. The balanced truncation method is found to be very effective in model order reduction. The transient performance of the simplified model is compared with that of the original model and also with transient performance of simplified models obtained using some other methods.

The rest of the paper is organized as follows: Section 2 describes the review of balanced truncation method and mathematical approach for the decomposition of stable and unstable systems. Section 3 presents the mathematical modeling of AHWR and its state-space representation. Application of balanced truncation technique to AHWR mathematical model is presented in Section 4. Comparison of the transient response of different reduced order models is presented in Section 5 and conclusions are drawn in Section 6.

2. Review of balanced truncation method

Consider a large-scale dynamical system described by the linear time invariant model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= \Psi x(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors respectively; $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $\Psi \in \mathbb{R}^{p\times n}$ are system, input and output matrices respectively. For the rest of this paper, we use $G := \left[\begin{array}{c|c} A & B \\ \hline \Psi & 0 \end{array}\right]$ to denote state-space form as in (1). We use the same G to also denote the Transfer Function (TF) corresponding to (1). From the context there would be no ambiguity. The transfer function from u to y is $G(s) := \Psi(sI - A)^{-1}B$.

For state-space models represented by (1), a methodology for deriving reduced-order model is provided in terms of realization in balanced co-ordinates (Moore, 1981). Most of the balanced truncation methods available in literature can only be applied if the system is stable. But realistic models which are used for system analysis and design may not be stable. Hence, the straightforward application of balanced truncation methods is not possible. Therefore, model reduction of unstable system can be treated by first separating the stable and unstable parts of the system, and then reducing the stable part using balanced truncation methods.

2.1. Modal decomposition into stable and unstable subsystems

For the purpose of this paper, stable means the open left half of the complex *s*-plane *i.e.*, $\gamma \in \mathbb{C}_-$, $Re(\gamma) < 0$; unstable means right half of the complex *s*-plane including the imaginary axis *i.e.*, $\gamma \in \overline{\mathbb{C}}_+$, $Re(\gamma) \geqslant 0$. Different approaches for decomposition into

stable and unstable subsystems were reported in Safonov et al. (1987), Kenney and Hewer (1987), Zhou et al. (1999), Benner et al. (2003) and Nagar and Singh, 2004. Here, we consider the application of decomposition algorithm given in Safonov et al. (1987) and Benner et al. (2003). It is briefly introduced in the following.

We assume that the system (1) represented by G has eigenvalues both in left and right half of complex s-plane. It can be converted to block triangular form as

$$G_t := \left[\begin{array}{c|c} U^T A U & U^T B \\ \hline \Psi U & 0 \end{array}\right] = \left[\begin{array}{c|c} A_t & B_t \\ \hline \Psi_t & 0 \end{array}\right], \tag{2}$$

where

$$A_t = \begin{matrix} k \\ n-k \end{matrix}\left[\begin{array}{c|c} \overset{k}{A_{t_{11}}} & \overset{n-k}{A_{t_{12}}} \\ \hline 0 & A_{t_{22}} \end{array}\right]; \quad B_t = \begin{matrix} k \\ n-k \end{matrix}\left[\begin{array}{c} B_{t_1} \\ \hline B_{t_2} \end{array}\right];$$

$$\Psi_t = \left[\begin{array}{c|c} \overset{k}{\Psi_{t_{11}}} & \overset{n-k}{\Psi_{t_{12}}} \end{array}\right],$$

where $A_{t_{11}}$, $A_{t_{12}}$, $A_{t_{22}}$, B_{t_1}, B_{t_2}, $\Psi_{t_{11}}$ and $\Psi_{t_{12}}$ are respectively $k\times k$, $k\times(n-k)$, $(n-k)\times(n-k)$, $k\times m$, $(n-k)\times m$, $p\times k$ and $p\times(n-k)$ submatrices, obtained by partitioning A_t, B_t and Ψ_t as indicated.

A similarity transformation that is applied to the system given by (1) to obtain the system (2) is

$$x = U x_t, \tag{3}$$

where U is an orthogonal similarity transformation that brings A to the real Schur form such that the diagonal elements of A_t are the real parts of the eigenvalues of A arranged in order of increasing values of the real part. Further, k is the number of stable poles of G. By another similarity transformation, the system represented by (2) can be converted into block diagonal form as

$$G_d := \left[\begin{array}{c|c} W^{-1} A_t W & W^{-1} B_t \\ \hline \Psi_t W & 0 \end{array}\right] = \left[\begin{array}{c|c} A_d & B_d \\ \hline \Psi_d & 0 \end{array}\right], \tag{4}$$

where

$$A_d = \begin{matrix} k \\ n-k \end{matrix}\left[\begin{array}{c|c} \overset{k}{A_s} & \overset{n-k}{0} \\ \hline 0 & A_{us} \end{array}\right]; \quad B_d = \begin{matrix} k \\ n-k \end{matrix}\left[\begin{array}{c} B_s \\ \hline B_{us} \end{array}\right];$$

$$\Psi_d = \left[\begin{array}{c|c} \overset{k}{\Psi_s} & \overset{n-k}{\Psi_{us}} \end{array}\right],$$

with $\gamma(A_s)\subset\mathbb{C}_-, A_s\in\mathbb{R}^{k\times k}, \gamma(A_{us})\subset\overline{\mathbb{C}}_+$, and $A_{us}\in\mathbb{R}^{(n-k)\times(n-k)}$. Note that $A_s, A_{us}, B_s, B_{us}, \Psi_s$ and Ψ_{us} are respectively $k\times k, (n-k)\times(n-k), k\times m, (n-k)\times m, p\times k$ and $p\times(n-k)$ submatrices, obtained by partitioning A_d, B_d and Ψ_d as indicated.

The second stage transform that is applied to the system given by (2) to obtain the system (4) is

$$x_t = \begin{bmatrix} I_k & S \\ 0 & I_{n-k} \end{bmatrix}\begin{bmatrix} x_s \\ x_{us} \end{bmatrix} = W\begin{bmatrix} x_s \\ x_{us} \end{bmatrix}, \tag{5}$$

where S is the solution of the Sylvester equation

$$A_{t_{11}} S - S A_{t_{22}} + A_{t_{12}} = 0. \tag{6}$$

Therefore the system (1) can be written in additive decomposition form Safonov et al. (1987) of TF as

$$G(s) := G_-(s) + G_+(s), \tag{7}$$

where $G_-(s) := \Psi_s(sI - A_s)^{-1}B_s$ is stable TF, $G_+(s) := \Psi_{us}(sI - A_{us})^{-1}B_{us}$ is unstable TF and $G_- := \left[\begin{array}{c|c} A_s & B_s \\ \hline \Psi_s & 0 \end{array}\right]$, $G_+ := \left[\begin{array}{c|c} A_{us} & B_{us} \\ \hline \Psi_{us} & 0 \end{array}\right]$ are associated state-space realizations of G_-, G_+ respectively.

2.2. State-space balancing algorithm

Reachability and observability Gramians play a major role in obtaining system balancing transformation. Assuming that the pair (A_s, B_s) is reachable and (Ψ_s, A_s) is observable, the *reachability Gramian* W_R and *observability Gramian* W_O of G_- can be obtained by the solution of the following algebraic Lyapunov equations (Laub et al., 1987):

$$A_s W_R + W_R A_s^T + B_s B_s^T = 0, \tag{8}$$

$$A_s^T W_O + W_O A_s + \Psi_s^T \Psi_s = 0. \tag{9}$$

The goal of balancing is to find a co-ordinate transformation such that in the new co-ordinate system the reachability and the observability Gramians both are diagonal and equal. In a balanced co-ordinate system G_- can be represented as

$$G_-^{bal} := \left[\begin{array}{c|c} T^{-1}A_s T & T^{-1}B_s \\ \hline \Psi_s T & 0 \end{array}\right] = \left[\begin{array}{c|c} A_{bal} & B_{bal} \\ \hline \Psi_{bal} & 0 \end{array}\right], \tag{10}$$

where

$$A_{bal} = \begin{matrix} r \\ k-r \end{matrix}\left[\begin{array}{c|c} \overset{r}{A_{bal}^{(11)}} & \overset{k-r}{A_{bal}^{(12)}} \\ \hline A_{bal}^{(21)} & A_{bal}^{(22)} \end{array}\right]; \quad B_{bal} = \begin{matrix} r \\ k-r \end{matrix}\left[\begin{array}{c} B_{bal}^{(1)} \\ \hline B_{bal}^{(2)} \end{array}\right];$$

$$\Psi_{bal} = \left[\begin{array}{c|c} \overset{r}{\Psi_{bal}^{(1)}} & \overset{k-r}{\Psi_{bal}^{(2)}} \end{array}\right].$$

$A_{bal}^{(11)}$, $A_{bal}^{(12)}$, $A_{bal}^{(21)}$, $A_{bal}^{(22)}$, $B_{bal}^{(1)}$, $B_{bal}^{(2)}$, $\Psi_{bal}^{(1)}$ and $\Psi_{bal}^{(2)}$ are respectively $r\times r$, $r\times(k-r)$, $(k-r)\times r$, $(k-r)\times(k-r)$, $r\times m$, $(r-k)\times m$, $p\times r$ and $p\times(k-r)$ submatrices, obtained by partitioning A_{bal}, B_{bal} and Ψ_{bal} as indicated. Further, r is the number of states which are to be retained in G_-^{bal}. The procedure for obtaining G_-^{bal} and selection of r are discussed in the following.

A similarity transformation to obtain G_-^{bal} from G_- is

$$x_s = T x_{s_b}, \tag{11}$$

where $T\in\mathbb{R}^{k\times k}$ is nonsingular. Following (Laub et al., 1987), an algorithm for computation of a balancing transformation is as follows:

(1) Compute the Gramians W_R and W_O for G_-.
(2) Compute the Cholesky factors of W_R and W_O *i.e.*, $W_R = L_R L_R^T$, $W_O = L_O L_O^T$, where L_R and L_O denote lower triangular Cholesky factors.
(3) Compute singular value decomposition (SVD) of product of Cholesky factors, *i.e.*, $L_O^T L_R = U\Sigma V^T$

(4) Finally, a co-ordinate transformation that results in balanced realization can be obtained as

$$T = L_R V \Sigma^{-1/2}. \tag{12}$$

G_-^{bal} is asymptotically stable and is in balanced realization form with

$$\overline{W}_R = \overline{W}_O = \text{diag. } [\Sigma_1 \quad \Sigma_2], \tag{13}$$

where

$$\Sigma_1 = \text{diag. } [\sigma_1\ \sigma_2\ \ldots\ \sigma_r], \quad \Sigma_2 = \text{diag. } [\sigma_{r+1}\ \sigma_{r+2}\ \ldots\ \sigma_k],$$

$\sigma_r > \sigma_{r+1}$ and $\sigma_i > 0$, $i = 1, 2, \ldots, k$, σ_i are the Hankel singular values of G_-^{bal}. One usually tries to choose r so that we have $\sigma_r \gg \sigma_{r+1}$, in addition to other criteria like desired accuracy and sought order of the reduced order model. Therefore, the system (1) can be represented in additive TF form as

$$G(s) := G_+(s) + G_-^{bal}(s), \tag{14}$$

where $G_{bal}^- := \left[\begin{array}{c|c} A_{bal} & B_{bal} \\ \hline \Psi_{bal} & 0 \end{array}\right]$ is balanced and stable. Let

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{bmatrix} \begin{bmatrix} x_{us} \\ x_{s_{b1}} \\ x_{s_{b2}} \end{bmatrix}, \tag{15}$$

where $x_{us} \in \mathbb{R}^{n-k}$, $x_{s_{b1}} \in \mathbb{R}^r$, and $x_{s_{b2}} \in \mathbb{R}^{k-r}$ be a similarity transformation to obtain (14) from (1). Hankel singular values for the system are defined as the square roots of the eigenvalues of the product $W_R W_O$. The balanced basis has the property that the states which are difficult to reach are simultaneously difficult to observe. The states in G_-^{bal} corresponding to the largest singular values are most important in the input–output behavior. Truncation of the states corresponding to the smaller Hankel singular values *i.e.,* Σ_2 will result in a reduced order model $\widehat{G}_r$ whose input–output behavior closely approximates the behavior of the original model. More precisely, the H_∞ norm of the difference between full-order system G and the reduced order system $\widehat{G}_r$ is upper bounded by twice the sum of the neglected Hankel singular values (Antoulas, 2005) and given as

$$\|G - \widehat{G}_r\|_{H_\infty} \leqslant 2(\sigma_{r+1} + \cdots + \sigma_k). \tag{16}$$

Therefore a reduced order model for the system (1) can be obtained as

$$\widehat{G}_r := \left[\begin{array}{c:c} A_r & B_r \\ \hdashline C_r & 0 \end{array}\right], \tag{17}$$

where

$$A_r = \begin{matrix} \\ r \end{matrix}\overset{\begin{matrix} n-k & r \end{matrix}}{\begin{bmatrix} A_{us} & 0 \\ 0 & A_{bal}^{(11)} \end{bmatrix}}; \quad B_r = \begin{matrix} n-k \\ r \end{matrix}\begin{bmatrix} B_{us} \\ B_{bal}^{(1)} \end{bmatrix};$$

$$\Psi_r = \begin{bmatrix} \overset{n-k}{\Psi_{us}} & \overset{r}{\Psi_{bal}^{(1)}} \end{bmatrix}.$$

Reduced order model of (1) in terms of original co-ordinate system can be obtained by setting $x_{s_{b2}} = 0$ in (15) as

$$\dot{\tilde{x}} = \Lambda A_r \Lambda^{-1} \tilde{x} + \Lambda B_r u, \tag{18}$$

$$y = \Psi_r \Lambda^{-1} \tilde{x}, \tag{19}$$

where $\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\Lambda = \begin{bmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{bmatrix}$. Moreover, from (15) we have

$$x_3 = \xi \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \text{ where } \xi = [\eta_{31} \quad \eta_{32}]\Lambda^{-1}. \tag{20}$$

Thus, the original nth order model represented by (1) is reduced to $(n-k+r)$th order model. The state variables of the reduced order model are defined as the first $(n-k+r)$ state variables of the original model. Though we do not need the remaining $(k-r)$ state variables of original model in this paper, if they are required in an application, they can be expressed in terms of the first $(n-k+r)$ state variables by using (20).

Remark: Model reduction by balanced truncation requires balancing the whole system G_- followed by truncation. The Lyapunov Eqs. (8) and (9) play a prominent role in obtaining system balancing transformation T and are required to be solved to obtain W_R and W_O. The Bartels-Stewart and Hammarling methods are direct standard methods for the solution of Lyapunov equations of small to moderate size. These methods rely on initial Schur decomposition of A_S followed by additional factorization schemes. In general and especially for large-scale systems, it is unwise to solve for W_R and W_O directly since these require arithmetic operations of order N^3 representing computational complexity and storage of order N^2, where N is the original system order. This approach may turn out to be numerically inefficient and ill-conditioned as the Gramians W_R and W_O often have numerically low rank *i.e.*, the eigenvalues of W_R and W_O decay rapidly. However, results on low rank approximations to the solutions of Lyapunov equations based on iterative methods (SVD-Krylov methods) make the balanced truncation model reduction approach feasible for large-scale systems (Penzl, 2006; Antoulas, 2005; Benner et al., 2005).

3. Mathematical model of AHWR

An extensive derivation of AHWR mathematical model is given in Shimjith et al. (2010) and Sagar et al. (2013) and the same has been used for the study carried out in this paper. However, for brevity the model is discussed briefly in the following.

3.1. Core neutronics model

The AHWR core is considered to be divided into 17 nodes, as shown in Fig. 1 by the segments labelled from 1 to 17. The top and bottom reflector regions are divided into 17 nodes in identical pattern as the core, labelled from 18 to 51, whereas side reflector is divided into 8 nodes labelled from 52 to 59, giving 59 nodes in total. Ignoring the dynamical effects of xenon and iodine, the following set of nonlinear time dependent equations represents the nodal core model of the AHWR:

$$\frac{d\phi_h}{dt} = -\omega_{hh}\upsilon_h\phi_h + \sum_{k=1}^{N_h}\omega_{hk}\upsilon_h\phi_k + (\rho_h - \beta)\frac{\phi_h}{\ell_h} + \sum_{i=1}^{m_d}\upsilon_h\lambda_i C_{ih}, \quad h = 1, 2, \cdots, Z_p, \tag{21}$$

$$\frac{dC_{ih}}{dt} = \frac{\beta_i\phi_h}{\upsilon_h\ell_h} - \lambda_i C_{ih}, \quad i = 1, 2, \cdots, m_d, \tag{22}$$

$$\frac{d\phi_h}{dt} = -\omega_{hh}\upsilon_h\phi_h + \sum_{k=1}^{N_h}\omega_{hk}\upsilon_h\phi_k, \quad h = Z_p + 1, \cdots, Z_p + Z_r, \tag{23}$$

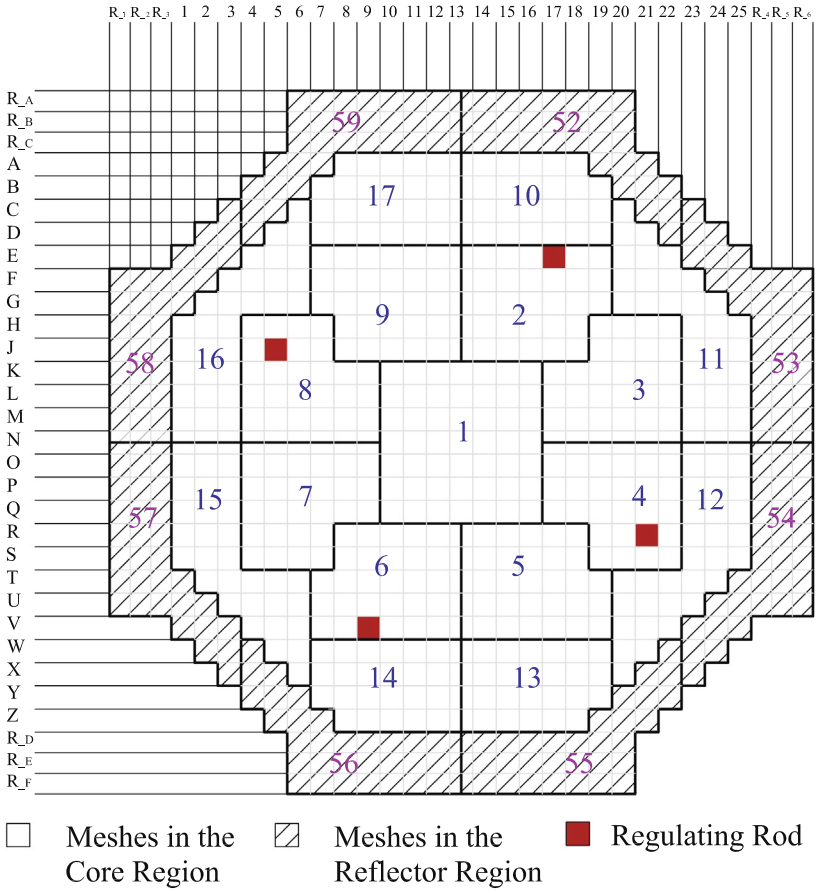

Fig. 1. 17 Node AHWR nodalization scheme with 17 nodes in the core, 17 nodes each in top and bottom reflectors and 8 nodes in surrounding reflector.

$$\frac{dH_k}{dt} = K_{RR}\vartheta_k, \quad k = 2,\ 4,\ 6,\ 8. \tag{24}$$

where ϕ_h is the neutron flux in node h; C_{ih} the delayed neutron precursor concentration for group i; H_k the %-in position of kth control rod; β_i and λ_i denote neutron fraction yield and decay constant for group i; m_d denotes total number of delayed neutron precursor groups; υ_h the mean velocity of neutrons in node h; and ℓ the prompt neutron life time; ϑ_k the control signal applied to kth regulating rod (RR) drive; and K_{RR}=0.56, a constant. N_h denotes total number of neighboring nodes to node h. Z_p and Z_r denote the total number of nodes in reactor core and surrounding reflector regions respectively. The coupling coefficients ω_{ij}, depend on the geometry and material composition and characteristic distance between the nodes. ρ_h, the reactivity contributed by the movement of the RRs around their equilibrium positions, is expressed as

$$\rho_h = \begin{cases} (-10.234H_k + 676.203) \times 10^{-6} & \text{if } k = 2,\ 4,\ 6,\ 8 \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

3.2. Linearization and state-space representation

For simplicity, only one group of delayed neutron precursors is considered instead of six groups. Now, the set of equations given by (21)–(24) can be linearized around the steady state operating conditions $(\phi_{h0},\ C_{h0},\ H_{j0})$, and the linear equations so obtained can be represented in standard state-space form. For this, define the state vector as

$$x := \begin{bmatrix} x_{\phi_C}^T & x_C^T & x_{\phi_R}^T & x_H^T \end{bmatrix}^T, \tag{26}$$

where

$$x_{\phi_C} := \begin{bmatrix} \delta\phi_1/\phi_{1_0} & \cdots & \delta\phi_{17}/\phi_{17_0} \end{bmatrix}^T$$
$$x_C := \begin{bmatrix} \delta C_1/C_{1_0} & \cdots & \delta C_{17}/\phi_{17_0} \end{bmatrix}^T$$
$$x_{\phi_R} := \begin{bmatrix} \delta\phi_{18}/\phi_{18_0} & \cdots & \delta\phi_{59}/\phi_{59_0} \end{bmatrix}^T$$
$$x_H := \begin{bmatrix} \delta H_2 & \delta H_4 & \delta H_6 & \delta H_8 \end{bmatrix}^T$$

in which δ denotes the deviation from respective steady state value of the variable. Likewise define the input vector as $u = [\delta\vartheta_2 \quad \delta\vartheta_4 \quad \delta\vartheta_6 \quad \delta\vartheta_8]^T$ and the output vector as $y = [y_1 \quad \cdots \quad y_{17}]^T$, where $y_i = \delta\phi_i/\phi_{i_0}$ denotes the corresponding deviation in nodal flux. Then the system of Eqs. (21)–(24) can be expressed in linear standard state-space form (1), with

$$A = \begin{bmatrix} A_{\phi_C\phi_C} & A_{\phi_C C} & A_{\phi_C\phi_R} & A_{\phi_C H} \\ A_{C\phi_C} & A_{CC} & 0 & 0 \\ A_{\phi_R\phi_C} & 0 & A_{\phi_R\phi_R} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & B_H^T \end{bmatrix}^T, \tag{28}$$

$$\text{and } \Psi = \begin{bmatrix} \Psi_{\phi_C} & 0 & 0 & 0 \end{bmatrix} \tag{29}$$

where

$$A_{\phi_C\phi_C}(i,j) := \begin{cases} -\omega_{ij}\upsilon_i + \frac{\rho_{i0}}{\ell_i} - \frac{\beta}{\ell_i} & \text{if } i = j \\ \omega_{ij}\upsilon_i \frac{\phi_{j0}}{\phi_{i0}} & \text{if } i \neq j \end{cases}$$

$$A_{\phi_C C} := -\beta \times \text{diag.} \left[\frac{1}{\ell_1} \quad \frac{1}{\ell_2} \quad \cdots \quad \frac{1}{\ell_{Z_p}} \right]$$

$$A_{\phi_C\phi_R}(i,j) := \begin{cases} -\omega_{ij}\upsilon_i & \text{if } i = j \\ \omega_{ij}\upsilon_i \frac{\phi_{j0}}{\phi_{i0}} & \text{if } i \neq j \end{cases}$$

$$A_{\phi_C H}(i,j) := \begin{cases} -1.0234 \times 10^{-5} \times \frac{\phi_{i0}}{\ell} & \text{for } i = 2,\ 4,\ 6,\ 8,\ \left(\text{j} = \frac{\text{i}}{2},\right) \\ 0 & \text{otherwise.} \end{cases}$$

$$A_{C\phi_C} := \text{diag.} \left[\lambda_1 \quad \lambda_2 \quad \ldots \quad \lambda_{Z_p} \right]$$

$$A_{CC} := -\text{diag.} \left[\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_{Z_p} \right]$$

$$A_{\phi_R\phi_R}(i,j) := \begin{cases} -\omega_{ij}\upsilon_i & \text{if } i = j \\ \omega_{ij}\upsilon_i \frac{\phi_{j0}}{\phi_{i0}} & \text{if } i \neq j \end{cases}$$

$$A_{\phi_R\phi_C} := A^T_{\phi_C\phi_R}$$

$$B_H := \text{diag.} \left[K_{RR} \quad K_{RR} \quad K_{RR} \quad K_{RR} \right]$$

$\Psi_{\phi_C} := I_{Z_p}$, and I_{Z_p} denotes an Identity matrix.

The neutronic parameters, nodal volumes and necessary data under full power operation are given in Shimjith et al. (2010) and Sagar et al. (2013). The eigenvalues of the system matrix A of AHWR are reported in Rajasekhar et al. (2015). An observation of these eigenvalues reveals that they fall into two distinct clusters. First cluster has 21eigenvalues consisting of 5 eigenvalues at the origin and the other 16 eigenvalues ranging from -6.2977×10^{-2} to -5.1852×10^{-2} and the second one is of 59 eigenvalues ranging from -4.751×10^{2} to -8.4578. This indicates the presence of two-time-scales. The distance between these two eigenvalue clusters, computed by dividing the largest absolute value of the slow (first) group by the smallest absolute value of the fast (second) group, is $\varepsilon = 0.0074$. This value is small enough to motivate the use of two-time-scale based techniques.

4. Application to AHWR space–time kinetics model reduction

The balanced truncation technique discussed in Section 2 can be applied to the AHWR model developed in Section 3.2 for obtaining the reduced order model with lesser dimension than the original. The model has 80 states, 4 inputs and 17 outputs. The presence of the five eigenvalues at the origin restricts the straightforward application of the state-space balancing method described in Section 2.2. Therefore, we carry out stable and unstable decomposition of the AHWR model described by (1) to obtain an unstable subsystem of order 5 and a stable subsystem of order 75. Thereafter, we calculate the similarity transformation T in (10) such that the reachability and observability Gramians in the transformed coordinate system are diagonal and equal. Corresponding Hankel singular values of G_-^{bal} are shown in Table 1. Fig. 2 shows the Hankel singular values which represent the "energy" of each state in the balanced system of AHWR. Before applying the truncation of the system G_-^{bal}, it is necessary to determine the partitioning of the Hankel singular values. We consider the first five Hankel singular values as listed in Table 1 because the ratio σ_1/σ_6 is considerably larger than σ_1/σ_5, σ_1/σ_4, ..., *etc.* Hence, the order of $\widehat{G}_r$ is 5 and considering unstable dynamics also an approximate model of order 10 is obtained. The eigenvalues of the reduced order system matrix A_r of AHWR are shown in Table 2, from which it is evident that the eigenvalues of A_r also fall into two distinct clusters. First cluster has 9 eigenvalues consisting of 5 eigenvalues at the origin and other four, ranging from -0.0601 to -0.0529 and the second cluster has only one eigenvalue, *i.e.,* -8.5034. Thus, the reduced order model also possesses two-time-scale property. The distance between these clusters, ε_r, of the reduced order model is 0.0071, which is almost equal to ε of the original model.

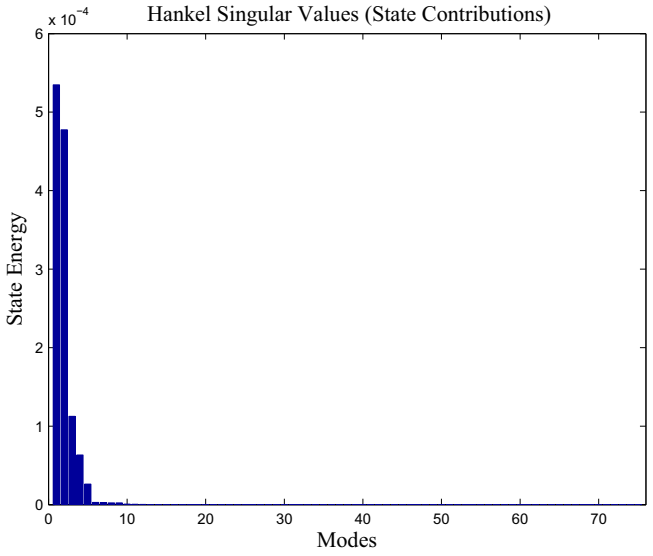

Fig. 2. Hankel singular values.

To illustrate the dynamic behavior of the 10th order reduced model obtained as described above, the open loop response for a short–time control relevant transient is presented here. Response obtained by the simulation of original 80th order is considered as reference. In the simulation, the reactor was assumed to be initially operating at full power and each RR is at 66.7% in position. At time $t = 0$ s, a control signal of 1 V is applied to RR drive in node 2 and maintained for 5 s, under which the RR moved linearly into the reactor core. After a short interval of 5 s, the control signal is made -1 V and is maintained at this level for 10 s. Then the control voltage is made 1 V for 5 s to bring back the RR to its nominal position. Fig. 3 shows the position of RR and, the reactivity introduced by it during the transient. From the deviations in the nodal fluxes, the deviation in the core average flux is calculated as

Table 1
Hankel singular values.

S. No.	HSV ($\times 10^{-3}$)	S. No.	HSV ($\times 10^{-3}$)
1	0.5348	10	0.0006
2	0.4775	11	0.0006
3	0.1124	12	0.0002
4	0.0634	13	0.0002
5	0.0263	14	0.0001
6	0.0028	15	0.0001
7	0.0025	16	0.0001
8	0.0023	17–75	$< 10^{-8}$
9	0.0022		

Table 2
Eigenvalues of system matrix A_r.

S. No.	Eigenvalue	S. No.	Eigenvalue
1	0	6	−0.0529
2	0	7	−0.0530
3	0	8	−0.0596
4	0	9	−0.0601
5	0	10	−8.5034

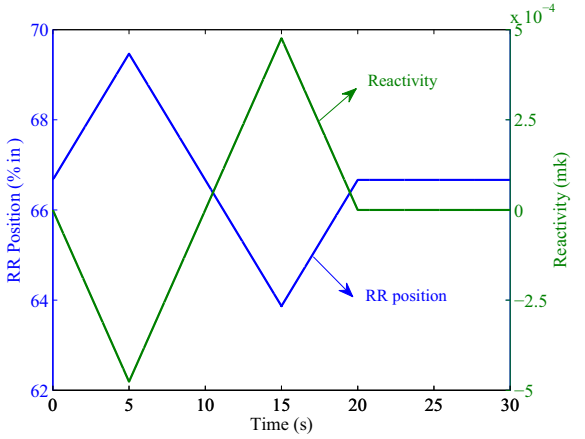

Fig. 3. Position of RR and reactivity introduced during the movement of RR.

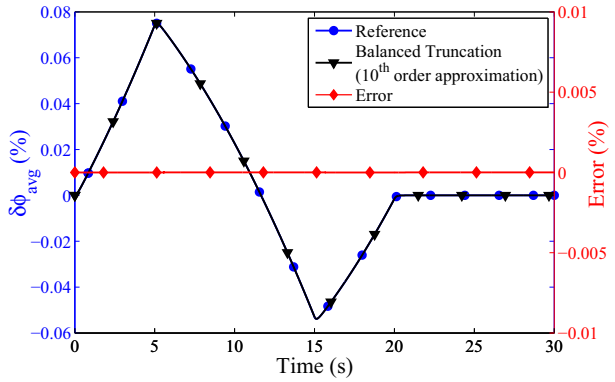

Fig. 4. Change in core average flux from its steady state value during the movement of RR.

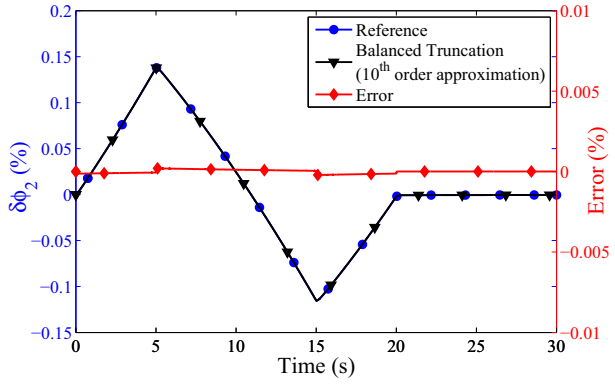

Fig. 5. Change in flux in node 2 from its steady state value during the movement of RR.

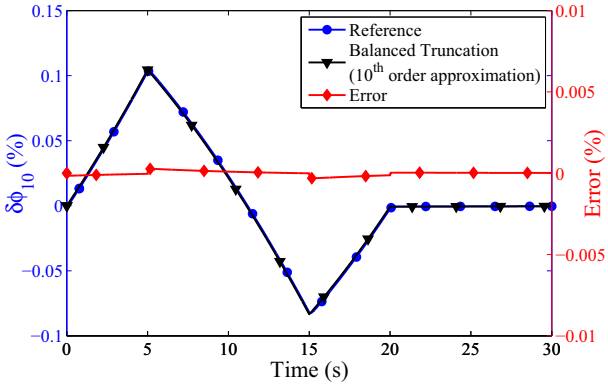

Fig. 6. Change in flux in node 10 from its steady state value during the movement of RR.

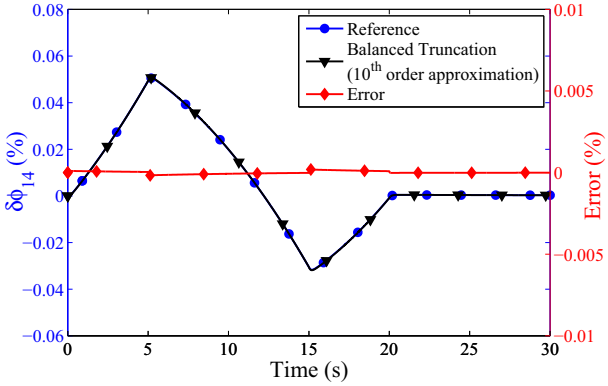

Fig. 7. Change in flux in node 14 from its steady state value during the movement of RR.

$$\delta\phi_{avg} = \frac{\sum_{i=1}^{17} \delta\phi_i V_i}{\sum_{i=1}^{17} V_i}, \tag{30}$$

where V_i denotes the volume of ith node. To characterize the accuracy of approximation of the reduced order model, we compute the error using

$$e_{y_i} = (y_{i_{ref}} - y_i) \times 100, \quad i = 1, 2, \cdots, 17, \tag{31}$$

where $y_{i_{ref}}$ denotes the reference value of deviation in flux in node i and y_i denotes the approximate value of deviation in flux in node i. Fig. 4 shows the core average flux alongwith error. Fig. 5 shows the variation of neutron flux in node 2 from the respective equilibrium value. Fig. 6 shows the variation of neutron flux in node 10, which is neighboring to node 2. Fig. 7 shows the variation of neutron flux in node 14 which is far away from node 2.

Table 3
Error (%) in deviation in fluxes.

Method	Order of the Simplified model	L_2-norm of error (%) in			
		Core average flux	Node-2 flux	Node-10 flux	Node-14 flux
Davison's technique	22	0.0954	3.7349	1.9481	1.0641
Marshall's technique	21	0.1518	0.1646	0.1582	0.1324
Singular perturbation analysis	21	0.1559	0.1682	0.1616	0.1347
Balanced truncation technique	10	0.1680×10^{-3}	0.0132	0.0157	0.0099

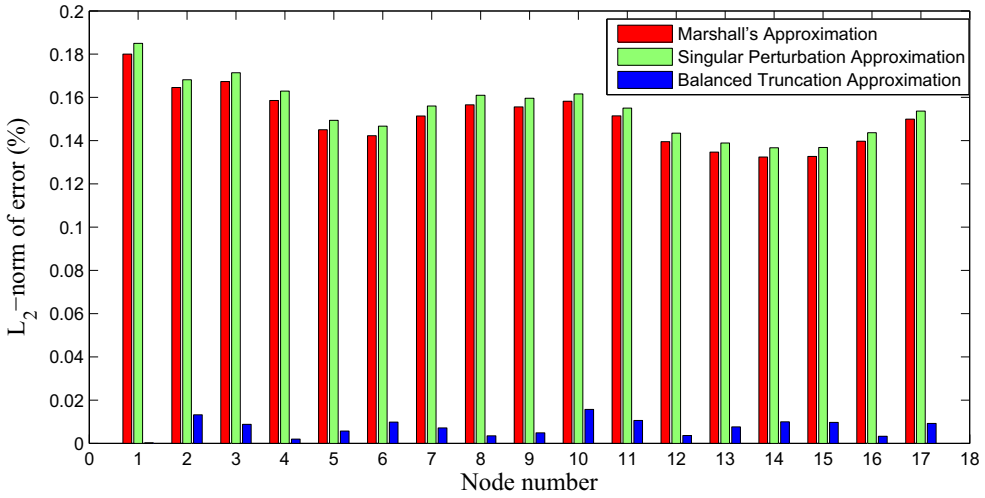

Fig. 8. Comparison of L_2-norm of the error (%) in deviation in neutron fluxes in all the 17 nodes of the AHWR.

5. Comparison of different reduced order models with the application of space–time kinetics model of AHWR

As already stated, due to the interaction of slow and fast dynamics present in AHWR, the mathematical model describing its behavior exhibits the multiple time-scales bringing in susceptibility to numerical ill-conditioning in control design and analysis. In Rajasekhar et al. (2015), a systematic method has been suggested to handle the numerical ill-conditioning occurring in the computations due to the presence of the slow control rod dynamics by decoupling the higher order model into very slow and fast mod-

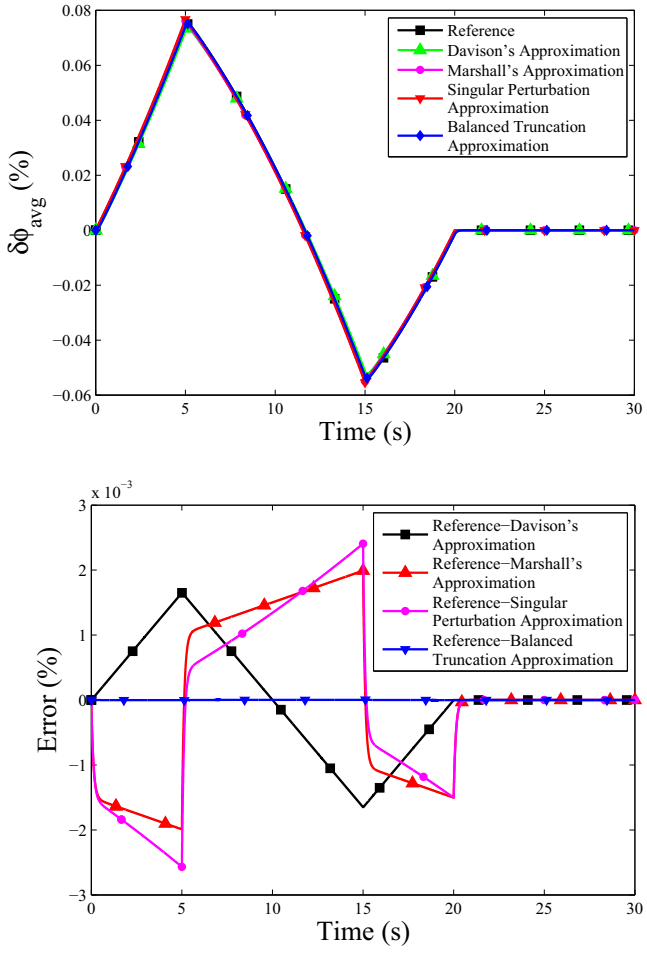

Fig. 9. Comparison of core average flux during the movement of RR.

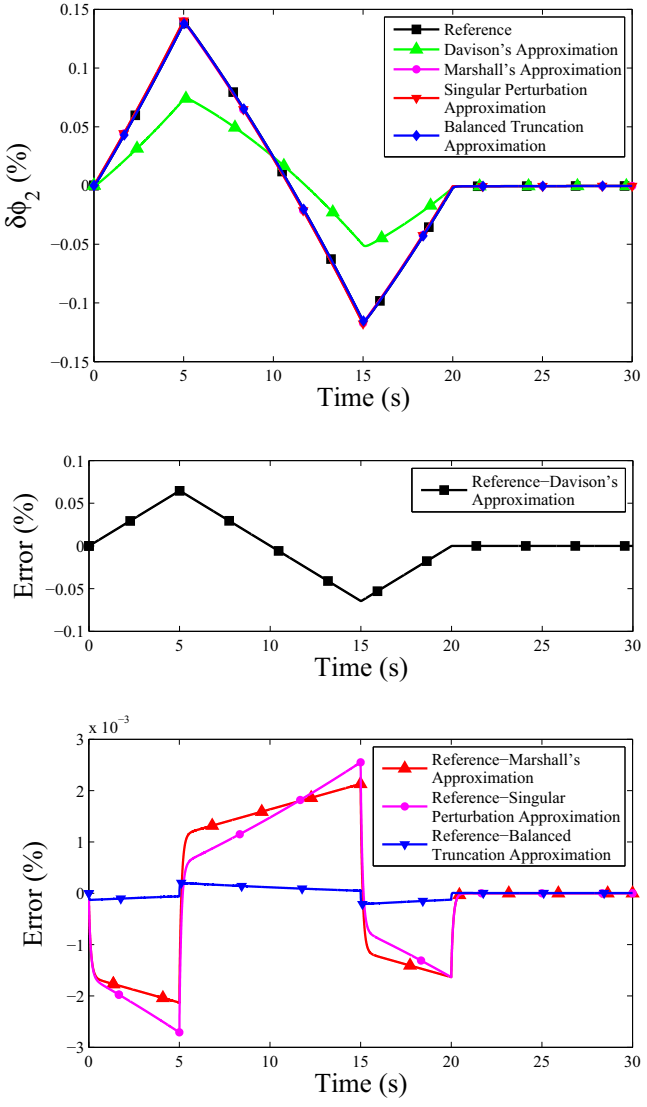

Fig. 10. Comparison of neutron flux in node-2 during the movement of RR.

els. Model order reduction techniques based on Davison's and Marshall's dominant mode retention are then applied to retain the slow dynamics. Finally, the reduced order model was formulated by augmenting the control rod dynamics. Model decomposition into slow and fast subsystems based on singular perturbation analysis has also been explored by suggesting regrouping of state variables.

In this section, we compare the performance characteristics of different reduced order models of AHWR obtained from Davison's, Marshall's, singular perturbation analysis and balanced truncation methods, with respect to each other and with the performance characteristics of the original higher order model. To illustrate the dynamic behavior of the aforesaid reduced order models, the open loop response is simulated for a short–time transient as described in Section 4 and is compared with the open loop response of the original 80th order model. Accuracy analysis as described earlier has also been carried out to determine the error in the approximation. We also quantified the error in approximation for different simplified models by computing the L_2-norm of the error defined by (31) and this is shown in Table 3. Fig. 8 compares the L_2-norm of error vector for all the 17-nodes of the AHWR. Plot for Davison's approach has not been shown, as it yields large error compared to other methods.

The response of different reduced order models shown in Fig. 9 reveals that the core average flux obtained from the approximate models is nearly same as that of the original model. Fig. 10 shows the variation of flux in node 2, from the respective equilibrium value. Fig. 11 shows the variation of neutron flux in node 10, which is neighboring to node 2. Fig. 12 shows the variation of neutron flux in node 14 which is far away from node 2. The comparison of responses makes it clear that Davison's technique fails to reproduce the accurate response characteristics as that of original model in node 2, 10 and 14 with the reduced order model obtained by retaining first 22 eigenvalues of A. Marshall's, singular perturbation and balanced truncation methods yield better approximation for deviation in core average flux as well as nodal fluxes with the order of 21, 21 and 10 respectively, compared to the Davison's technique. However, the application of Davison's and Marshall's method increases computational burden in obtaining approximate model in reactor applications due to the presence of multiple eigenvalues at origin of the complex s-plane, whereby the diagonalization of AHWR space–time kinetics model is difficult. Balanced truncation method requires the decomposition of stable and unstable dynamics while singular perturbation technique requires reordering of state variables and block diagonalization.

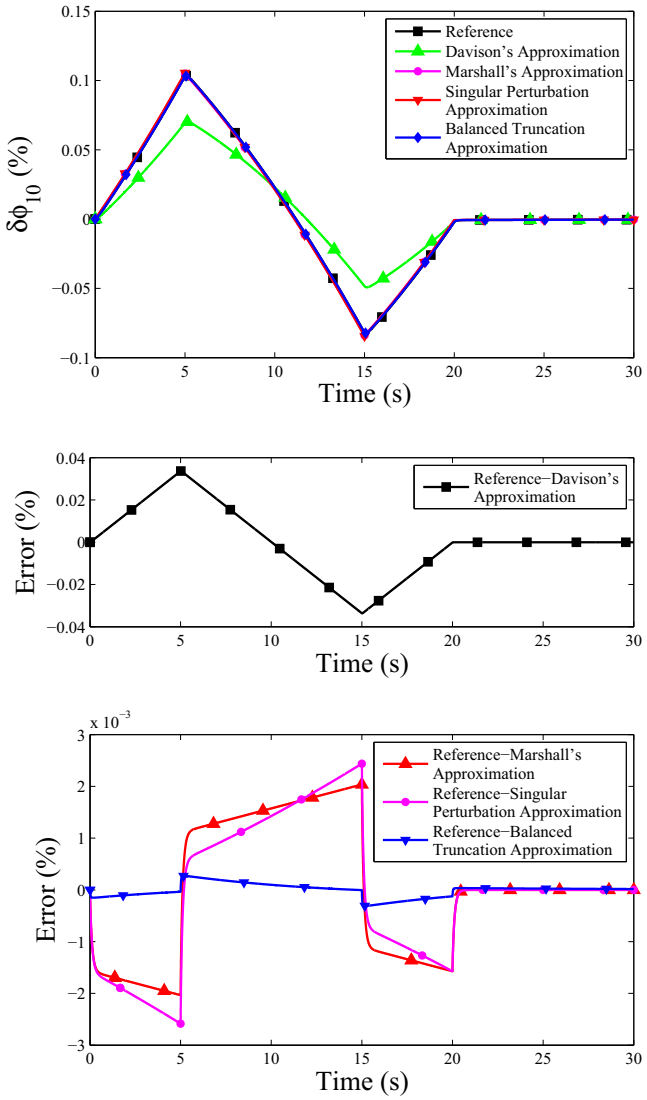

Fig. 11. Comparison of neutron flux in node-10 during the movement of RR.

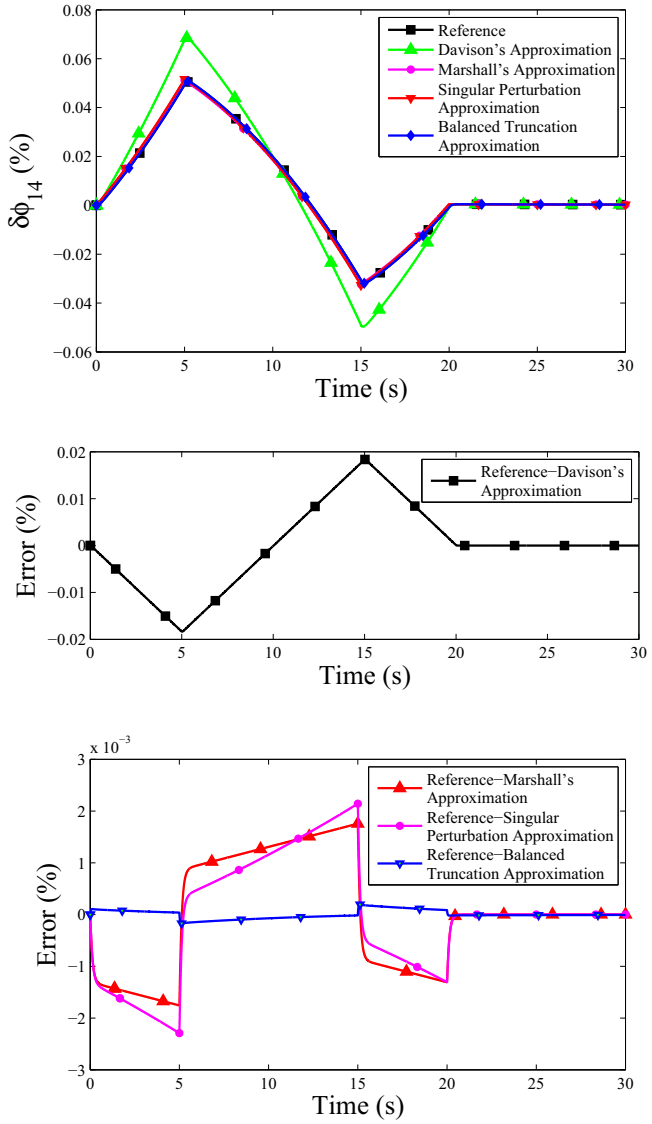

Fig. 12. Comparison of neutron flux in node-14 during the movement of RR.

Table 4(a)
Computation time for simplified models.

Method	Order of the Simplified model	Computation time in (s)		
		Algorithm	Simulation	Total
Davison's technique	22	0.0203	0.1569	0.1772
Marshall's technique	21	0.0174	0.1493	0.1667
Singular perturbation analysis	21	0.0025	0.1849	0.1874
Balanced truncation technique	10	0.3169	0.0823	0.3992
Original model	80	NA	0.3328	0.3328

Singular perturbation and balanced truncation methods preserve the two-time-scale property of the model, however transient response of balanced truncation method is marginally superior to the singular perturbation method. It may be noted from the Table 3 that the error in approximation using the balanced truncation approach is far lower in comparison with other methods.

We compare the computation time and memory for the different model order reduction techniques. The computation was performed on Matlab R2015b and Windows-7, 64 bit computer with intel(R) Core(TM) i3-4130 CPU @ 3.40 GHz processor and 4 GB RAM memory. The computation time for processing the model reduction algorithm is just a one-time requirement, while simulation is to be carried for different situations and multiple cases may be studied. Hence, comparison of simulation time would be appropriate. Nevertheless, the Table 4(a) gives a comparison of algorithm processing time as well as simulation time, for balanced truncation and other methods. Although the total computation time for balanced truncation approach is slightly higher compared to the original model, simulation time is very less compared to the original method as well as the other methods.

Also, the number of floating point operations (FLOPs) and memory requirement are given in Table 4(b) for the different methods. Singular perturbation analysis is the most efficient requiring the minimum memory as well as FLOPs while the balanced truncation method is the least efficient with requirement of about 17.2 Mega FLOPs and 1192.7 MB memory. However, the most important thing is its feasibility of working in reduced order modeling of AHWR space–time kinetics.

Table 4(b)
Computational cost and memory requirement for simplified models.

Method	Number of FLOPs (Mega FLOPs)	Memory requirement (MB)
Davison's technique	3.8	441.90
Marshall's technique	5.5	489.64
Singular perturbation analysis	2.2	140.58
Balanced truncation technique	17.2	1192.77

6. Conclusions

For the AHWR system, various model order reduction techniques, *viz.*, Davison's and Marshall's dominant mode retention techniques; balanced truncation technique and model decomposition into slow and fast subsystems based on singular perturbation analysis, have been applied. Among these, Davison's and Marshall's techniques require diagonalization and balanced truncation technique requires a modal decomposition into unstable and stable subsystems. Also, it is essential for model order reduction based on Davison's and Marshall's techniques, to identify the modes to retain and those to truncate/reduce. In contrast, singular perturbation techniques require a decomposition of the state-space systems into fast/slow subsystems using block diagonalization methods. Davison's and Marshall's techniques result into a simplified model that retains the slowly varying dynamics while the application of singular perturbation analysis and two-time-scale methods decompose the model into two subsystems *viz.*, slow and fast, thus providing better approximation of dynamics of the system. Quite similar to this, application of balanced truncation yields a reduced model in which both the slow and fast dynamic characteristics are simultaneously retained yielding good accuracy in approximation of high order model by reduced order model.

The order of the reduced order model obtained by the application of Davison's, Marshall's, and singular perturbation approaches is 22, 21 and 21 respectively. However, balanced truncation approach is most effective in model order reduction yielding the simplified model with an order 10. The transient response of simplified models based on Marshall's, singular perturbation and balanced truncation techniques were in good agreement with the transient response of the original high order model.

In summary, we have proposed a novel approach for the application of balanced truncation technique to nuclear reactor systems which have a nontrivial unstable part based on stable and unstable decomposition. This is accomplished easily while the direct application of balanced truncation for model order reduction is not feasible. The computation involved in obtaining a balanced basis is typically considered too high and this method has not been adequately used in model order reduction applications. However, our work shows that the computation is required to be done offline and only once. The computation time is only marginally higher, but the advantages outweigh the marginally higher computation time. It is evidenced that the final reduced order model based on the balanced truncation method is of much lower order in spite of negligible error between the response of the original system and that obtained from the reduced order model. This provides significant advantage from a practical perspective because a lower-order controller or estimator needs to be designed for the purpose of controlling the AHWR. In general, the singular perturbation and balanced truncation methods are expected to perform better in cases where time-scales are widely separated.

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