International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: http://www.tandfonline.com/loi/tcon20

Closed-form solutions of singular KYP lemma: strongly passive systems, and fast lossless trajectories

Chayan Bhawal, Debasattam Pal & Madhu N. Belur

To cite this article: Chayan Bhawal, Debasattam Pal & Madhu N. Belur (2018): Closed-form solutions of singular KYP lemma: strongly passive systems, and fast lossless trajectories, International Journal of Control, DOI: 10.1080/00207179.2018.1500039

To link to this article: https://doi.org/10.1080/00207179.2018.1500039

Accepted author version posted online: 12 Jul 2018.
Published online: 26 Jul 2018.

INTERNATIONAL JOURNAL OF CONTROL
https://doi.org/10.1080/00207179.2018.1500039

Closed-form solutions of singular KYP lemma: strongly passive systems, and fast lossless trajectories

Chayan Bhawal, Debasattam Pal and Madhu N. Belur

Department of Electrical Engineering, Indian Institute of Technology Bombay, Mumbai, India

ABSTRACT

In this paper, we deal with a special class of passive systems, which possess the characteristic property of having no finite spectral zeros. We call these systems *strongly passive*. It is well known that, for these systems, storage functions, i.e. solutions to the linear matrix inequality (LMI) arising from the Kalman–Yakubovich–Popov (KYP) lemma, cannot be obtained by the conventional approach of algebraic Riccati equations (AREs) and Hamiltonian matrices. In this paper, we first show that a strongly passive system always admits a unique storage function. We then provide a closed-form expression for this unique storage function. Using the closed-form formula of the unique storage function we characterise the 'lossless' trajectories of strongly passive systems and show that such systems admit impulsive lossless trajectories on the half-line; we call them *fast lossless trajectories*. This adds to the existing notion that such systems do not admit any 'slow' lossless trajectories.

ARTICLE HISTORY
Received 1 February 2018
Accepted 8 July 2018

KEYWORDS
KYP lemma; Linear matrix inequality; storage functions; passivity

1. Introduction

One of the most important tools used in the analysis and design of control systems is the linear matrix inequality (LMI) arising from the Kalman–Yakubovich–Popov (KYP) lemma (see Kalman, 1963; Popov, 1964; Yakubovich, 1962); for the sake of brevity, henceforth in this paper, this LMI will be called the *KYP LMI*. The KYP LMI is well studied in the literature and its solutions find widespread applications in stability analysis, dissipativity, optimal control, stochastic control and filtering: see Brogliato, Lozano, Maschke, and Egeland (2007, Chapter 3) and references therein for various applications. The KYP lemma states that a system with a minimal input-state-output (i/s/o) representation $(\mathrm{d}/\mathrm{d}t)x = Ax + Bu, y = Cx + Du$, where $A \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}, B, C^T \in \mathbb{R}^{\mathrm{n}\times\mathrm{p}}$ and $D \in \mathbb{R}^{\mathrm{p}\times\mathrm{p}}$, is passive if and only if there exists a non-negative definite $K = K^T \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ such that

$$\begin{bmatrix} A^TK + KA & KB - C^T \\ B^TK - C & -(D + D^T) \end{bmatrix} \leqslant 0. \tag{1}$$

One of the most widely used methods to compute solutions of the KYP LMI involves the use of the corresponding algebraic Riccati equation (ARE): see Anderson and Vongpanitlerd (2006, Section 6.2) and Bini, Iannazzo, and Meini (2012) for methods to solve the ARE. However, the formulation of the ARE crucially depends on the nonsingularity of $D + D^T$. We call this condition of nonsingularity of $D + D^T$ the *feedthrough regularity condition*. In this paper, we deal with a special class of passive systems that *do not* satisfy this feedthrough regularity condition and hence do not admit an ARE.

Non-satisfaction of the feedthrough regularity condition causes a deficit in the number of finite spectral zeros[1] from being double the system's order in the regular case, i.e. when the feedthrough regularity condition is satisfied (Jugade, Pal, Kalaimani, & Belur, 2013, Lemma 3.2; Trentelman, Minh, & Rapisarda, 2009, Theorem 3.6). The class of passive systems that we deal with in this paper is characterised by an extreme deficit in the number of finite spectral zeros: *these systems have no finite spectral zeros at all*. We call this special class of passive systems, *strongly passive* systems (see Definition 2.1 below for the formal definition of strong passivity). At this point, an elaboration on the nomenclature is in order. The spectral zeros of a passive system are known to be precisely the modes of trajectories in the system that incur zero loss due to dissipation (Trentelman et al., 2009). As mentioned above, the characteristic property of strongly passive systems is that they have no finite spectral zeros. It then follows that such systems do not admit any lossless trajectory that has a finite mode. In other words, a strongly passive system admits no 'slow' lossless trajectories. Saying alternatively, *every* slow trajectory in a strongly passive system incurs dissipation loss. The adverb 'strongly' thus emphasises this inevitability of dissipation loss for slow trajectories in the system.

For every control problem based on the theory of passivity, non-satisfaction of the feedthrough regularity condition causes significant difficulty in obtaining a solution. This is mainly due to non-existence of the ARE and shortage of finite spectral zeros. This, perhaps, is the main reason why it is prevalent in the existing literature on passivity and passivity based control to assume that the feedthrough regularity condition is satisfied (Sorensen, 2005). However, it should be noted that non-satisfaction of the feedthrough regularity condition arises quite naturally in passive systems. In particular, there is a large class of passive systems that, in fact, exhibit strong passivity. Passivity

CONTACT Chayan Bhawal chayanbhawal@ee.iitb.ac.in

theory originated in the problem of synthesising linear systems by electrical circuits (Cauer, 1926). Going back to this origin, we can readily construct RLC circuits, which are strongly passive. One of the simplest such circuits is shown in Figure 1. Apart from such simple examples, one can also construct a family of strongly passive electrical circuits, as well. For example, the RLC network in Figure 2 is strongly passive for any arbitrary value of $b > 0$. Quite evidently, existence of strongly passive circuits is not confined to electrical circuits only: the characterising property of these systems, based on their transfer functions, has been presented in Definition 2.1. The abundance of strongly passive systems, and the fact that such systems do not satisfy the feedthrough regularity condition indicates that study of such systems are essential for any application that uses solutions of KYP LMI for design and analysis. Typical areas of analysis where strongly passive systems can show up is in passivity preserving model order reduction (Sorensen, 2005), in the design of controllers for various applications (Hoang, Tuan, & Nguyen, 2009), (Paszke, Rogers, & Gałkowski, 2013), in the analysis of uncertain systems (Megretski & Rantzer, 1997) and so on.

The motivation for our studying systems with strong passivity comes from two different perspectives: first, from a numerical point of view and second, from a system theoretic one. As mentioned earlier, solutions of the KYP LMI are essential in different fields of control and communication. It is natural, therefore, to seek for an algorithm to compute solutions of the KYP LMI efficiently. For passive systems that admit ARE, efficient ARE solvers already exist (Bini et al., 2012). However, such efficient ARE solvers are unusable for strongly passive systems, since they do not admit AREs. Besides ARE solvers, one of the most widely used methods to solve such LMIs is to use semi-definite programming (SDP) techniques. SDP solvers are based on decreasing the *gap*[2] between improved estimates of the solution to the LMI and hence, such algorithms have an inherent *tolerance* (algorithm stopping criterion) associated with them: see Boyd and Vandenberghe (2004, Section 5.5) for more on tolerance. This makes such solvers inefficient in terms of computational time and computational error (Vandenberghe, Balakrishnan, Wallin, Hansson & Roh, 2005). Therefore, there is always a scope for an algorithm to compute solutions of the KYP LMI that outperforms SDP-based LMI solvers. It turns out that, by exploiting the special properties of strongly passive systems, we can indeed obtain a much more efficient and accurate solver. This is achieved by the solver because it solves the KYP LMI with a closed-form expression: see Theorem 3.1 and Algorithm 1.

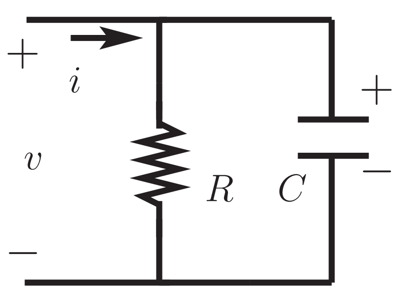

Figure 1. A parallel RC network with impedance function $Z(s) = \frac{R}{1+sRC}$.

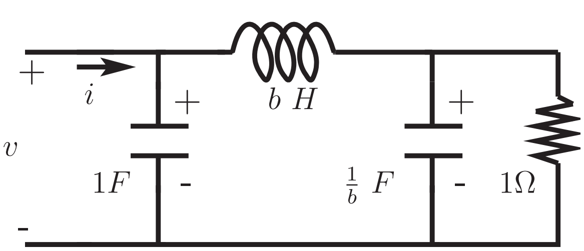

Figure 2. A RLC network with impedance function $Z(s) = \frac{(bs^2+b^2s+b)}{(bs^3+b^2s^2+(1+b)s+b)}$, $b > 0$.

As mentioned above, the system theoretic interpretation of the solutions of KYP LMI is the other motivating factor for studying strongly passive systems in this paper. It is known in the literature that there exists an interesting link between the solutions of the KYP LMI and the *optimal charging and discharging* policies of passive circuits: see Willems (1972) and Willems and Trentelman (1998, Remark 5.14) for more on this optimisation problem. Note that a characteristic property of passive systems is that the set of solutions of the KYP LMI are partially ordered with two extrema (Willems & Trentelman, 1998), i.e. there exist solutions of KYP LMI (1), $K_{\mathtt{min}}$ and $K_{\mathtt{max}}$, such that $K_{\mathtt{min}} \leqslant K \leqslant K_{\mathtt{max}}$ for all K that satisfy the KYP LMI (1). These extremal solutions are special since they reveal the optimal charging and discharging policies of an RLC network. Using these extremal solutions of the KYP LMI, controllers can be designed that restrict the system trajectories to its optimal charging and discharging trajectories. These trajectories are, in fact, the lossless trajectories of the system as discussed above. It is known in the literature that for passive systems that admit AREs, these lossless trajectories turn out to be infinitely differentiable (slow) (Trentelman et al., 2009). However, we show that for strongly passive systems such lossless trajectories are impulsive (Theorem 5.3). This is primarily due to the fact that such systems have no finite spectral zeros, and hence, no slow lossless voltage-current profile exists. Therefore, a natural question is: can these lossless trajectories be characterised? This is one of the primary motivations for studying such an extreme case of passive systems in this paper.

Our main contributions in this paper can therefore be broadly divided into two points: first, we bring out an explicit closed-form expression of the (unique) solution to the KYP LMI for strongly passive systems. Second, we show existence of impulsive lossless trajectories in these systems, although, as mentioned above, these systems do not admit any slow lossless trajectories as per the existing theory. We briefly summarise the contributions of this paper next.

In this paper, we focus on strongly passive SISO systems. For such systems, the KYP LMI takes the form

$$\begin{bmatrix} A^TK + KA & KB - C^T \\ B^TK - C & 0 \end{bmatrix} \leqslant 0. \tag{2}$$

We call this the *singular KYP LMI*. The problem of solving singular KYP LMIs is not new. Besides SDP-based LMI solvers, there are at least two known theoretical methods in the literature for solving singular KYP LMIs. In one of these methods, spectral factorisation technique (see Anderson & Vongpanitlerd, 2006, Section 6.5; Willems & Trentelman, 1998) is used. A short-coming of this method is that it can characterise only slow lossless trajectories. Hence, for strongly passive systems, this method asserts that there are no lossless trajectories for any nonzero initial conditions (Trentelman et al., 2009). We overcome this short-coming in this paper and show that such systems do admit 'fast' lossless trajectories for every initial condition (Theorem 5.3).

The other method is a generalisation of the ARE/ Hamiltonian matrix method, that is commonly used for the regular case, to the singular case. In this method, the issue of non-invertibility of $D + D^T$ is circumvented by looking into a more fundamental object called the *Hamiltonian pencil*. The notion of invariant subspaces in the regular case, likewise, gets substituted by a more fundamental object called *deflating subspaces* (Reis, 2011; Reis, Rendel, & Voigt, 2015; Van Dooren, 1981). Our approach is similar to this approach in that we, too, use the Hamiltonian pencil extensively to bring out the main results. However, it is important to note here that we do not use properties of deflating subspaces for solving the singular KYP LMI. Instead, we show that a much simpler construction, based on Markov parameters of the strongly passive system, leads to a closed-form formula for the unique solution of the singular KYP LMI corresponding to the system (Theorem 3.1). The above-mentioned observation (Lemma 3.2) concerning Markov parameters of strongly passive systems plays a crucial role in this paper. This result also helps us in showing that the solution to the KYP LMI for the case of strongly passive systems is unique (Theorem 3.1). Interestingly, in this paper, we probe even further into the Hamiltonian pencil. We look into the dynamical system whose singular descriptor state-space equation is given by the Hamiltonian pencil. This singular descriptor dynamical system is known to be the result of an interconnection of the system and its adjoint (Jugade, Pal, Kalaimani, & Belur, 2017). Because of the singular descriptor nature, the interconnected system admits half-line impulsive solutions. We show in this paper (Theorem 5.3) that, these impulsive half-line solutions are 'lossless', which generalises the well-known fact for regular passive systems that solutions in the intersection of the system and its adjoint are lossless (Trentelman et al., 2009, Theorem 3.4).

The rest of the paper is organised as follows. The following section contains the notation and preliminaries required for this paper. Section 3 contains the first main result of the paper. In this main result, we propose a closed-form solution of the singular KYP LMI for a strongly passive SISO system and show that this solution is unique. Based on the first main result, an algorithm (Algorithm 1) to compute the storage function of a strongly passive system is proposed in Section 4. Further, we also compare the algorithm with standard LMI solvers in this section. In Section 5, we establish the link between this unique solution and lossless trajectories of a strongly passive system. Concluding remarks are presented in Section 6.

2. Notation and preliminaries

We use symbols $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ for the sets of natural, real and complex numbers, respectively. The symbol $\mathbb{R}_+$ denotes the set of non-negative real numbers. The symbol $\mathbb{R}^{\mathtt{n}\times\mathtt{p}}$ denotes the set of matrices with $\mathtt{n}$ rows and $\mathtt{p}$ columns, where the elements are from $\mathbb{R}$. We use $\mathbb{R}[s]$ and $\mathbb{R}(s)$, respectively, for denoting the sets of polynomials and rational functions in one-variable s with coefficients from $\mathbb{R}$. Likewise, we use $\mathbb{R}[s]^{\mathtt{n}\times\mathtt{p}}$ and $\mathbb{R}(s)^{\mathtt{n}\times\mathtt{p}}$ for the sets of $\mathtt{n}\times\mathtt{p}$ matrices with elements from $\mathbb{R}[s]$ and $\mathbb{R}(s)$, respectively. The symbol $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{n}})$ stands for the space of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{n}}$. For brevity, we use the symbol $\mathfrak{C}^\infty$ to denote the set $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$. We use the symbol $\mathfrak{C}^\infty|_{\mathbb{R}_+}$ to represent the set of all functions from $\mathbb{R}_+$ to $\mathbb{R}$ that are restrictions of smooth ($\mathfrak{C}^\infty$) functions to $\mathbb{R}_+$, i.e.

$$\mathfrak{C}^\infty|_{\mathbb{R}_+} := \{w : \mathbb{R}_+ \to \mathbb{R} \mid \exists v \in \mathfrak{C}^\infty \text{ such that } w = v|_{\mathbb{R}_+}\}.$$

Symbol $\mathtt{col}(B_1, B_2, \dots, B_{\mathtt{n}})$ represents a matrix of the form $\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{\mathtt{n}} \end{bmatrix}$ and $\mathtt{det}(A)$ represents the determinant of the matrix A. Symbol $\mathtt{img}(A)$ is used to denote the subspace spanned by the columns of matrix A. The degree of a polynomial $p(s) \in \mathbb{R}[s]$ is denoted by $\mathtt{deg}(p(s))$. A block diagonal matrix G is represented as $\mathtt{diag}(G_1, \dots, G_{\mathtt{m}})$, where each of $G_1, \dots, G_{\mathtt{m}}$ are square matrices of possibly different sizes. We use the symbol $A = [a_{mn}]_{m,n=0,1,\dots\mathtt{N}-1}$ to represent[3] a matrix $A \in \mathbb{R}^{\mathtt{N}\times\mathtt{N}}$ with element a_{mn} in the $(m+1)$-st row and $(n+1)$-st column. Symbol $\mathbf{0}_{\mathtt{n}} \in \mathbb{R}^{\mathtt{n}}$ is used for the vector having all elements equal to zero and symbol $\mathbf{0}_{\mathtt{n}\times\mathtt{n}} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is used to denote a zero matrix of size $\mathtt{n}\times\mathtt{n}$. The symbol δ represents the Dirac delta impulse function and $\delta^{(i)}$ represents the ith distributional derivative of δ with respect to t. We use the symbol $\sigma(A)$ to denote the set of eigenvalues of A (including multiplicity). Next we give a brief review of various preliminary concepts required for this paper.

As introduced in Section 1, a system with minimal i/s/o representation

$$\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu, \quad y = Cx + Du, \quad \text{where } A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B, C^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{p}} \text{ and } D \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}, \tag{3}$$

is said to be passive if there exists a non-negative definite $K = K^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that K satisfies the KYP LMI (1). Further, it is also known in the literature that for all $(x, u, y) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}+2\mathtt{p}})$ that satisfies Equation (3), any solution K of the KYP LMI (1) satisfies the following inequality (see Trentelman & Willems, 1997):

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^T Kx) \leqslant 2u^T y \quad \text{for all } t \in \mathbb{R}. \tag{4}$$

Inequality (4) is known as the *dissipation inequality* and the quadratic term $(x^T Kx)$ is called a *storage function* of the passive system[4] under consideration (Willems & Trentelman, 1998). Thus the solutions of the KYP LMI (1) induce the storage functions of a passive system. Henceforth, we use the term 'storage function' to also mean a solution matrix K of the KYP LMI. Note that a stable system with transfer function matrix $G(s)$ is passive if and only if both the following statements hold (Anderson & Vongpanitlerd, 2006, Section 2.7):

(1) $G(s)$ is analytic on the open right half of the $\mathbb{C}$-plane.
(2) For all $\omega \in \mathbb{R}$ such that $j\omega$ is not a pole of $G(s)$:

$$G(j\omega) + G(-j\omega)^T \geqslant 0. \tag{5}$$

As mentioned earlier, our main object of study in this paper is a special class of passive SISO systems, which we call *strongly passive* systems. We define such systems next.

Definition 2.1 (strong passivity): A passive SISO system with transfer function $G(s)$ (with no common poles and zeros) is called *strongly passive* if the numerator of $G(s) + G(-s)$ is a nonzero constant.

From Definition 2.1, it is clear that a necessary condition for a SISO system Σ to be strongly passive is $D = 0$. This also reconfirms the fact that a SISO system Σ with strong passivity admits a singular KYP LMI of the form (2). For the example shown in Figure 1, note that $G(s) + G(-s) = (1/C)/(s + 1/RC) + (1/C)/(-s + 1/RC) = (-2/RC^2)/(s^2 - 1/R^2C^2)$.

Lossless trajectories and adjoint systems: Given a system Σ with transfer function matrix $G(s)$ and a minimal i/s/o representation as in Equation (3), its *adjoint* is defined to be the system with a minimal i/s/o representation $\dot{z} = -A^T z + C^T e$ and $f = B^T z - D^T e$. (The adjoint of a system is also known as the *dual* and the *orthogonal complement* in the literature Van der Schaft & Rapisarda, 2011.) We use the symbol $\Sigma_{\mathtt{adj}}$ to represent the adjoint of the system Σ. We 'interconnect' the two systems Σ and $\Sigma_{\mathtt{adj}}$ such that $u = e$ and $y = f$. The new system obtained from this interconnection has a first-order representation of the form

$$\underbrace{\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E} \begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{u} \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & (D + D^T) \end{bmatrix}}_{H} \begin{bmatrix} x \\ z \\ u \end{bmatrix}. \tag{6}$$

We use the symbol Σ_{Ham} to represent this interconnected system; the subscript 'Ham' in the symbol Σ_{Ham} stands for 'Hamiltonian' and the pencil $(sE - H) \in \mathbb{R}[s]$ is called *Hamiltonian pencil*. It is a fact that for passive systems with $D + D^T > 0$ elimination of the variable u leads to a state-space representation of Σ_{Ham} given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A - B(D + D^T)^{-1}C & B(D + D^T)^{-1}B^T \\ -C^T(D + D^T)^{-1}C & -(A - B(D + D^T)^{-1}C) \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}. \tag{7}$$

The system matrix in Equation (7) is called a Hamiltonian matrix $\mathcal{H}$. This Hamiltonian matrix and its suitable invariant subspaces are used to compute storage functions of the system Σ (see Kučera, 1991). Note that, in this paper, all passive systems that satisfy the feedthrough regularity condition such that their corresponding Hamiltonian matrix $\mathcal{H}$ has no eigenvalues on the imaginary axis are called *regularly passive* systems. For regularly passive systems, it is known that degree of $\mathtt{det}(sE - H) = 2\mathtt{n}$ (see Trentelman et al., 2009, Theorem 3.6). The roots of $\mathtt{det}(sE - H)$ happen to be the eigenvalues of the Hamiltonian matrix $\mathcal{H}$ of Equation (7). In such a case, one of the standard methods to compute the solution of KYP LMI involves partitioning the set of the roots of $\mathtt{det}(sE - H)$ into two sets each of cardinality $\mathtt{n}$ subject to certain conditions (see Kučera, 1991). Such sets are called *Lambda sets*.[5] Bases of the $\mathtt{n}$-dimensional invariant subspaces of the Hamiltonian matrix corresponding to each of these Lambda sets are used to compute storage functions of the system: see Van Dooren (1981) for a detailed exposition. For easy reference, we present this method to compute storage functions of regularly passive systems as a proposition next. The significance of our main result (Theorem 3.1) is the close parallel with the statements below, though the concepts involved in the proposition are very different.

Proposition 2.2: *Consider a regularly passive system Σ with an i/s/o representation as given in Equation (3) and let the corresponding Hamiltonian matrix $\mathcal{H}$ be as given in Equation (7). Let Λ be a Lambda-set of $\mathtt{det}(sI - \mathcal{H})$. Suppose the $\mathtt{n}$-dimensional $\mathcal{H}$-invariant subspace corresponding to Λ is given by*

$$\mathcal{V} := \mathtt{img}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \quad \textit{where } V_1, V_2 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}. \tag{8}$$

Then the following statements are true.

(1) *V_1 is invertible.*
(2) *$K := V_2 V_1^{-1}$ is symmetric.*
(3) *K is a solution to the ARE: $A^T K + KA + (KB - C^T)(D + D^T)^{-1}(B^T K - C) = 0$.*
(4) *$x^T Kx$ is a storage function of the system Σ, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}(x^T Kx) \leqslant 2u^T y$ for all $(x, u, y) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}+\mathtt{p}+\mathtt{p}})$ that satisfy Equation (3).*

On the other hand, for a strongly passive SISO system Σ, the system Σ_{Ham} has the following first-order representation

$$\begin{bmatrix} I_{\mathtt{n}} & 0 & 0 \\ 0 & I_{\mathtt{n}} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ 0 & -A^T & C^T \\ C & -B^T & 0 \end{bmatrix}\begin{bmatrix} x \\ z \\ u \end{bmatrix}. \tag{9}$$

Note that for strongly passive SISO systems, u cannot be expressed as a linear combination of x and z, and hence, Σ_{Ham} cannot be brought into the state-space form of Equation (7). Therefore, Σ_{Ham} in this scenario is a singular descriptor system. Further, a strongly passive system Σ does not admit an ARE of the form given in Statement (3) of Proposition 2.2 due to nonsingularity of $D + D^T$. Computation of $\mathtt{det}(sE - H)$ using Schur complement with respect to the top-left block $\mathtt{diag}(sI_{\mathtt{n}} - A, sI_{\mathtt{n}} + A^T)$ reveals that the numerator of $G(s) + G(-s)$ and $\mathtt{det}(sE - H)$ is the same (upto sign). Hence, for strongly passive SISO systems $(sE - H)$ is a unimodular matrix, i.e. $\mathtt{det}(sE - H)$ is a nonzero constant. Therefore, with strongly passive SISO systems, partitioning of the roots of $\mathtt{det}(sE - H)$ is not possible. This means that Proposition 2.2 is not applicable to strongly passive SISO systems. So a relevant question is: how to compute storage functions of such systems then? We answer this in the next section, and the answer eventually leads us to the claimed closed-form solution of the singular KYP LMI for strongly passive SISO systems. To summarise, given a strongly passive SISO system Σ, the following statements are true.

(1) $\mathtt{det}(sE - H) \in \mathbb{R}\backslash 0$, i.e. $(sE - H)$ is unimodular and the system Σ has no spectral zeros (see Endnote 1).

(2) The system Σ_{Ham} corresponding to Σ does not contain any exponential trajectories (slow trajectories).
(3) The system Σ does not satisfy the feedthrough regularity condition and hence does not admit an ARE.

On the contrary, for a regularly passive SISO system Σ, the following statements are true.

(1) $\det(sE-H) \in \mathbb{R}[s]\backslash 0$, i.e. $(sE-H)$ is a regular pencil[6] and degree of $\det(sE-H) = 2\mathrm{n}$, i.e. the system Σ has $2\mathrm{n}$ spectral zeros.
(2) The system Σ_{Ham} corresponding to Σ contains slow trajectories of dimension $2\mathrm{n}$.
(3) The system Σ satisfies the feedthrough regularity condition and therefore admits an ARE.

3. Storage functions of strongly passive SISO systems

We now state and prove our first main result, Theorem 3.1. The theorem provides closed-form solutions of the singular KYP LMI (2) for strongly passive SISO systems.

Theorem 3.1: *Consider a strongly passive SISO system Σ with a minimal i/s/o representation $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu, y = Cx$, where $A \in \mathbb{R}^{n\times n}, B, C^T \in \mathbb{R}^n$. Define*

$$\widehat{A} := \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}, \quad \widehat{B} := \begin{bmatrix} B \\ C^T \end{bmatrix} \quad \text{and} \quad \widehat{C} := \begin{bmatrix} C & -B^T \end{bmatrix}.$$

Define further $W := [\widehat{B} \ \ \widehat{A}\widehat{B} \ \ \cdots \ \ \widehat{A}^{n-1}\widehat{B}] \in \mathbb{R}^{2n\times n}$ and partition $W =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, where $X_1, X_2 \in \mathbb{R}^{n\times n}$.

Then the following statements are true.

(1) *X_1 is invertible.*
(2) *$K := X_2X_1^{-1}$ is symmetric.*
(3) *K is a solution to LMI (2), i.e. $KB - C^T = 0$ and $A^TK + KA \leqslant 0$.*
(4) *K is the unique solution to LMI (2).*
(5) *x^TKx is the storage function of the system Σ, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}(x^TKx) \leqslant 2uy$ for all $(x, u, y) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{n+1+1})$ that satisfy the i/s/o representation of Σ.*

Further, if eigenvalues of A are in the open left-half of the $\mathbb{C}$-plane, then $K \geqslant 0$.

The significance of the above result is that the properties of the storage function K have close parallel to the regularly passive case (see Proposition 2.2). However, the procedure to obtain K is very different. The proof needs more development and we prove the statements one by one.

Proof of Statement 1 of Theorem 3.1: Note that $X_1 = [B \ AB \ \cdots \ A^{\mathrm{n}-1}B] \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$. Invertibility of X_1 follows from controllability of (A, B). ■

To prove K is a symmetric matrix we use the concept of Markov parameters. Recall we use 0 as the starting index for this paper: see Endnote 3. Note that the Markov parameters of the system Σ are CA^kB for $k = 0, 1, \ldots, \mathrm{n}-1$: for a detailed exposition refer to Antoulas (2005, Section 4.2). It is known that the Markov parameters of a system are intrinsically linked to the relative degree[7] of the system. Interestingly, the Markov parameters of $G(s) + G(-s)$ are also related to the Markov parameters of Σ as shown in the next lemma. This is needed in the proof of Theorems 3.1 and 5.3.

Lemma 3.2: *Consider a strongly passive SISO system Σ with transfer function $G(s)$. Let a minimal i/s/o representation of Σ be $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$ and $y = Cx$, where $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^n$. Then,*

$$CA^kB - (-1)^k(CA^kB)^T = 0 \quad \textit{for } k = 0, 1, \ldots, 2\mathrm{n}-2.$$

Proof: Note that $G(s) + G(-s) = C(sI-A)^{-1}B - B^T(sI+A^T)^{-1}C^T$. Taking the Laplace inverse of $G(s) + G(-s)$ we get

$$h(t) := \mathcal{L}^{-1}(G(s) + G(-s)) = Ce^{At}B - B^Te^{-A^Tt}C^T. \qquad (10)$$

From Equation (10) it is clear that

$$\begin{aligned} h(t)|_{t=0^+} &= CB - B^TC^T \\ \dot{h}(t)|_{t=0^+} &= CAB + B^TA^TC^T \\ &\vdots \\ h^{(2\mathrm{n}-2)}(t)|_{t=0^+} &= CA^{2\mathrm{n}-2}B - (-1)^{2\mathrm{n}-2}B^T(A^T)^{2\mathrm{n}-2}C^T. \end{aligned}$$

Since the relative degree of $H(s) := G(s) + G(-s)$ is $2\mathrm{n}$, the initial value theorem implies that

$$\begin{aligned} h(t)|_{t=0^+} &= \lim_{s\to\infty} sH(s) = 0 \\ \dot{h}(t)|_{t=0^+} &= \lim_{s\to\infty} s^2H(s) = 0 \\ &\vdots \\ h^{(2\mathrm{n}-2)}(t)|_{t=0^+} &= \lim_{s\to\infty} s^{2\mathrm{n}-1}H(s) = 0. \end{aligned}$$

Therefore, $CA^kB - (-1)^k(CA^kB)^T = 0$ for $k = 0, 1, \ldots, 2\mathrm{n}-2$. This completes the proof of Lemma 3.2. ■

Proof of Statement 2 of Theorem 3.1: We need to prove that $X_2X_1^{-1}$ is symmetric, i.e. $X_2X_1^{-1} = (X_2X_1^{-1})^T$, i.e. $X_1^TX_2 = X_2^TX_1$. Hence, proving Statement 2 of Theorem 3.1 is equivalent to proving $X_1^TX_2 - X_2^TX_1 = 0$.

Let $X_1^TX_2 - X_2^TX_1 =: [\mathcal{J}_{\alpha\beta}]_{\alpha,\beta=0,1,2,\ldots,\mathrm{n}-1}$. Here

$$\begin{aligned} X_1 &= \begin{bmatrix} B & AB & \cdots & A^{\mathrm{n}-1}B \end{bmatrix} \quad \text{and} \\ X_2 &= \begin{bmatrix} C^T & -(CA)^T & \cdots & (-1)^{\mathrm{n}-1}(CA^{\mathrm{n}-1})^T \end{bmatrix}. \end{aligned}$$

Therefore, for $\alpha, \beta = 0, 1, 2, \ldots, \mathrm{n}-1$

$$\mathcal{J}_{\alpha\beta} = (-1)^\beta[(CA^{\alpha+\beta}B)^T - (-1)^{\alpha+\beta}(CA^{\alpha+\beta}B)]$$

Using Lemma 3.2, it is easy to see that $\mathcal{J}_{\alpha\beta} = 0$ for $\alpha, \beta = 0, 1, 2, \ldots, \mathrm{n}-1$. Therefore, $X_1^TX_2 - X_2^TX_1 = 0$. This proves that $X_2X_1^{-1}$ is symmetric. ■

To prove Statement 3 of Theorem 3.1, we need the following simple matrix result.

Lemma 3.3: *Consider $A \in \mathbb{R}^{n\times n}$ and $B, C^T \in \mathbb{R}^n$. Define $X_1, X_2 \in \mathbb{R}^{n\times n}$ such that*

$$X_1 := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \quad \text{and}$$
$$X_2 := \begin{bmatrix} C^T & -(CA)^T & \cdots & (-1)^{n-1}(CA^{n-1})^T \end{bmatrix}.$$

Then

$$X_2X_1^{-1}A^kB = (-1)^k(CA^k)^T \text{ for every } k = 0,1,\ldots,n\text{-}1.$$

Proof: Note that $X_2X_1^{-1}A^kB = X_2[B\ AB\ \cdots\ A^{n-1}B]^{-1}A^kB = X_2\mathrm{col}(\mathbf{0}_k, 1, \mathbf{0}_{n-k-1})$. Therefore, $X_2X_1^{-1}A^kB$ is the $(k+1)$-st element of X_2, i.e. $(-1)^k(CA^k)^T$. Therefore, $X_2X_1^{-1}A^kB = (-1)^k(CA^k)^T$. This completes the proof of Lemma 3.3. ∎

Proof of Statement 3 of Theorem 3.1: We first prove that $KB - C^T = 0$. Lemma 3.3 applied to the case $k = 0$ gives $X_2X_1^{-1}B = C^T$. Thus we have $KB - C^T = 0$.

Next we prove that $A^TK + KA \leqslant 0$. Instead of directly proving $A^TK + KA \leqslant 0$, we shall prove that $X_1^T(A^TK + KA)X_1 \leqslant 0$. Since X_1 is invertible, $A^TK + KA \leqslant 0$ if and only if $X_1^T(A^TK + KA)X_1 \leqslant 0$. This is true because congruence transformation on a symmetric matrix preserves its signature. Let us first define $X_1^T(A^TK + KA)X_1 =: [T_{\alpha\beta}]_{\alpha,\beta=0,1,\ldots,n-1}$. Therefore

$$T_{\alpha\beta} = B^T(A^\alpha)^T(A^TK + KA)A^\beta B$$

Note that

$$\begin{aligned} T_{\alpha\beta} &= B^T(A^\alpha)^T(A^TK + KA)A^\beta B \\ &= (B^T(A^{\alpha+1})^T)(KA^\beta B) + (B^T(A^\alpha)^TK)(A^{\beta+1}B) \end{aligned}$$

Using Lemma 3.3, we have

$$KA^\alpha B = (-1)^\alpha(CA^\alpha)^T \quad \text{and} \quad KA^\beta B = (-1)^\beta(CA^\beta)^T$$

for $\alpha, \beta = 0, 1, 2, \ldots, n-1$. Hence, $T_{\alpha\beta} =$

$$\begin{aligned} &(A^{\alpha+1}B)^T((-1)^\beta(CA^\beta)^T) + ((-1)^\alpha(CA^\alpha)^T)^T(A^{\beta+1}B) \\ &= (-1)^\beta(CA^{\alpha+\beta+1}B)^T + (-1)^\alpha(CA^{\alpha+\beta+1}B) \\ &= (-1)^\alpha(CA^{\alpha+\beta+1}B - (-1)^{\alpha+\beta+1}(CA^{\alpha+\beta+1}B)^T). \end{aligned}$$

For every $(\alpha, \beta) \in \{0, 1, \ldots, n-1\}^2 \setminus \{(n-1, n-1)\}$ we can infer the following from Lemma 3.2: $T_{\alpha\beta} = 0$. We now check for $\alpha = n-1$ and $\beta = n-1$. Recall the expression for $T_{(n-1)(n-1)} =$

$$\begin{aligned} &(-1)^{(n-1)}(CA^{2n-1}B - (-1)^{2n-1}(CA^{2n-1}B)^T) \\ &= -(-1)^n(CA^{2n-1}B + (CA^{2n-1}B)^T) \end{aligned} \tag{11}$$

Since Σ has been assumed to be strongly passive, it follows from Equation (5) that for all $\omega \in \mathbb{R}$,

$$G(j\omega) + G(-j\omega)^T \geqslant 0 \text{ i.e. } \omega^{2n}(G(j\omega) + G(-j\omega)^T) \geqslant 0.$$

Therefore,

$$\lim_{\omega\to\infty} \omega^{2n}(G(j\omega) + G(-j\omega)^T) \geqslant 0. \tag{12}$$

Note that $G(s) + G(-s)^T$ admits an infinite series expansion at $s = \infty$ of the following form:

$$\begin{aligned} G(s) + G(-s)^T &= C(sI - A)^{-1}B + B^T(-sI - A^T)^{-1}C^T \\ &= \sum_{k=0}^{\infty} \frac{1}{s^{k+1}}(CA^kB - (-1)^k(CA^kB)^T). \end{aligned}$$

Using Lemma 3.2, we have

$$G(s) + G(-s)^T = \sum_{k=2n-1}^{\infty} \frac{1}{s^{k+1}}(CA^kB - (-1)^k(CA^kB)^T).$$

From inequality (12) then we get

$$\begin{aligned} &\lim_{\omega\to\infty} \omega^{2n}(G(j\omega) + G(-j\omega)^T) \\ &= \lim_{\omega\to\infty} \left(\omega^{2n} \sum_{k=2n-1}^{\infty} \frac{1}{(j\omega)^{k+1}}(CA^kB - (-1)^k(CA^kB)^T) \right) \geqslant 0. \end{aligned} \tag{13}$$

Note that we have utilised the fact that the expansion of $G(j\omega) + G(-j\omega)^T$ at $\omega = \infty$ remains valid even after multiplication by ω^{2n} because $G(s) + G(-s)^T$ has relative degree $2n$. Rewriting inequality (13), we get

$$\begin{aligned} &\lim_{\omega\to\infty} \frac{\omega^{2n}}{(j\omega)^{2n-1+1}}(CA^{2n-1}B - (-1)^{2n-1}(CA^{2n-1}B)^T) \\ &+ \lim_{\omega\to\infty} \sum_{k=2n}^{\infty} \frac{\omega^{2n}}{(j\omega)^{k+1}}(CA^kB - (-1)^k(CA^kB)^T) \geqslant 0. \end{aligned}$$

Hence, we have

$$(-1)^n(CA^{2n-1}B + (CA^{2n-1}B)^T) \geqslant 0. \tag{14}$$

Using inequality (14) in Equation (11), we get

$$T_{(n-1)(n-1)} \leqslant 0.$$

Thus, we have

$$\begin{aligned} X_1^T(A^TK + KA)X_1 &= \begin{bmatrix} \mathbf{0}_{(n-1)\times(n-1)} & \\ & T_{(n-1)(n-1)} \end{bmatrix} \\ &= \mathtt{diag}(0, T_{(n-1)(n-1)}) \in \mathbb{R}^{n\times n}. \end{aligned} \tag{15}$$

Since $T_{(n-1)(n-1)} \leqslant 0$, it follows that $X_1^T(A^TK + KA)X_1 \leqslant 0$, and hence $(A^TK + KA) \leqslant 0$. This completes the proof of Statement 3 of Theorem 3.1. ∎

Proof of Statement 4 of Theorem 3.1: To prove the uniqueness of K, we first claim that any solution $K = K^T$ of the singular KYP LMI (2) satisfies $KA^\alpha B = (-1)^\alpha (CA^\alpha)^T$ for any $\alpha \in \{1, \dots, \mathsf{n}-1\}$. We prove it using principle of mathematical induction and Lemma 3.2.

Base case: $\alpha = 1$. Note that for any solution K of the LMI (2), we have

$$A^T K + KA \leqslant 0 \quad \text{and} \quad KB - C^T = 0. \tag{16}$$

Therefore, using Equation (16), we have

$$\begin{aligned} A^T K + KA \leqslant 0 &\implies B^T (A^T K + KA) B \leqslant 0 \\ &\implies (AB)^T (KB) + (KB)^T (AB) \leqslant 0 \\ &\implies (CAB)^T + CAB \leqslant 0 \end{aligned} \tag{17}$$

From Lemma 3.2, customised to $k=1$, we have $(CAB)^T + CAB = 0$. Therefore, $B^T(A^T K + KA)B = 0$. Since $A^T K + KA \leqslant 0$, we have

$$\begin{aligned} B^T(A^T K + KA)B = 0 &\implies (A^T K + KA)B = 0 \\ &\implies A^T KB + KAB = 0 \\ &\implies A^T C^T + KAB = 0 \\ &\implies KAB = -(CA)^T. \end{aligned} \tag{18}$$

Induction case. We assume $KA^{(\alpha-1)}B = (-1)^{(\alpha-1)}(CA^{(\alpha-1)})^T$ and we want to show that $KA^\alpha B = (-1)^\alpha (CA^\alpha)^T$. From Equation (16), we have

$$\begin{aligned} A^T K + KA \leqslant 0 &\implies (A^{(\alpha-1)}B)^T (A^T K + KA)(A^{(\alpha-1)}B) \leqslant 0 \\ &\implies (A^\alpha B)^T KA^{(\alpha-1)}B + (KA^{(\alpha-1)}B)^T A^\alpha B \leqslant 0 \\ &\implies (A^\alpha B)^T (-1)^{(\alpha-1)} (CA^{(\alpha-1)})^T \\ &\quad + (-1)^{(\alpha-1)} (CA^{(\alpha-1)}) A^\alpha B \leqslant 0 \\ &\implies (-1)^{(\alpha-1)} ((CA^{2\alpha-1}B)^T + (CA^{2\alpha-1}B)) \leqslant 0. \end{aligned} \tag{19}$$

Using Lemma 3.2 in Equation (19), we have for $\alpha \in \{1, 2, \dots, \mathsf{n}-1\}$,

$$\begin{aligned} &((CA^{2\alpha-1}B)^T + (CA^{2\alpha-1}B)) = 0 \\ &\implies (A^{(\alpha-1)}B)^T (A^T K + KA)(A^{(\alpha-1)}B) = 0. \end{aligned} \tag{20}$$

Once again, since $A^T K + KA \leqslant 0$, we infer from Equation (20)

$$\begin{aligned} &(A^T K + KA) A^{(\alpha-1)} B = 0 \\ &\implies (-1)^{(\alpha-1)} A^T (CA^{(\alpha-1)})^T + KA^\alpha B = 0 \\ &\implies KA^\alpha B = (-1)^\alpha (CA^\alpha)^T. \end{aligned} \tag{21}$$

Thus, by the principle of mathematical induction and Lemma 3.2, we infer that for any solution K of the singular KYP LMI (2), $KA^\alpha B = (-1)^\alpha (CA^\alpha)^T$ for $\alpha \in \{1, 2, \dots, \mathsf{n}-1\}$. Writing the matrix equation $KB - C^T = 0$ and Equations (21) together in matrix form, we have

$$KX_1 = X_2, \tag{22}$$

where $X_1 = [B \; AB \; \cdots A^{(\mathsf{n}-1)}B]$ and $X_2 = [C^T \; -(CA)^T \; \cdots \; (-1)^{(\mathsf{n}-1)}(CA^{(\mathsf{n}-1)})^T]$. We now prove the uniqueness of K. Let K and $\widehat{K}$ be two solutions of the singular KYP LMI corresponding to the passive system Σ. From Equation (22), we know that

$$KX_1 = X_2 \quad \text{and} \quad \widehat{K}X_1 = X_2$$

Therefore, $(K - \widehat{K})X_1 = 0$. From Statement 1 of Theorem 3.1, X_1 is invertible and thus we infer that $K - \widehat{K} = 0 \implies K = \widehat{K}$. Thus, K is the unique solution of the singular KYP LMI (2). This completes the proof of Statement 4 of Theorem 3.1 ■

Proof of Statements 5 of Theorem 3.1: From Statement 3 of Theorem 3.1 it is clear that K is a solution of the singular KYP LMI (2). Therefore, $x^T Kx$ is a storage function of Σ. This proves Statement 5 of Theorem 3.1.

Lastly, if A has eigenvalues in the open left-half of the $\mathbb{C}$-plane, then K is the solution of the Lyapunov inequality $A^T K + KA \leqslant 0$, and hence, we must have $K \geqslant 0$.

This completes the proof of Theorem 3.1. ■

Thus we have constructed a closed-form formula for the unique solution of the singular KYP LMI corresponding to a strongly passive SISO system.

Remark 3.4: Note that for controllable, lossless systems, the singular KYP LMI (2) reduces to $A^T K + KA = 0$ and $KB - C^T = 0$ (see Anderson & Vongpanitlerd, 2006, Section 6.5). For the K defined in Statement (2) of Theorem 3.1, it is evident that $KB - C^T = 0$ by Lemma 3.3. Further, it is known that for a lossless SISO system, $G(j\omega) + G(-j\omega) = 0$ for all $\omega \in \mathbb{R}_+$ (see Anderson & Vongpanitlerd, 2006, Theorem 2.7.4). Therefore, $T_{(\mathsf{n}-1)(\mathsf{n}-1)}$ in Equation (11) is zero and this proves that $A^T K + KA = 0$. Thus, K defined in Theorem 3.1 is the solution of the singular KYP LMI for lossless systems too. This shows that Theorem 3.1 is applicable to lossless systems as well. Thus as a special case of our algorithm obtained from Theorem 3.1, we retrieve a known algorithm to compute storage functions of lossless systems: see Anderson and Vongpanitlerd (2006, Section 6.5).

4. Algorithmic aspects and illustrative example

At the very outset of this section, we present the algorithm to compute the storage function of a strongly passive system based on Theorem 3.1.

Algorithm 1 Algorithm to compute the storage function of a strongly passive system.

Input: $A \in \mathbb{R}^{\mathsf{n}\times\mathsf{n}}, B, C^T \in \mathbb{R}^{\mathsf{n}}$.
Output: $K = K^T \in \mathbb{R}^{\mathsf{n}\times\mathsf{n}}$.

1: Construct $\widehat{A} := \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}$ and $\widehat{B} := \begin{bmatrix} B \\ C^T \end{bmatrix}$.

2: Construct $W := [\widehat{B} \;\; \widehat{A}\widehat{B} \;\; \cdots \;\; \widehat{A}^{\mathsf{n}-1}\widehat{B}] \in \mathbb{R}^{2\mathsf{n}\times\mathsf{n}}$

3: Partition W as $W =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, where $X_1, X_2 \in \mathbb{R}^{\mathsf{n}\times\mathsf{n}}$

4: Compute the storage function: $K = X_2 X_1^{-1} \in \mathbb{R}^{\mathsf{n}\times\mathsf{n}}$

Now that we have the algorithm to compute the storage function of a strongly passive system, we discuss the algorithmic aspects of Algorithm 1 next. Note that each step in Algorithm 1 can be efficiently implemented using certain standard algorithms (Watkins, 2002). We use these standard algorithms to compute the total flop count of Algorithm 1 in Table 1.

From Table 1 it is evident that the total flop count for Algorithm 1 is $\mathcal{O}(\mathrm{n}^3)$. Recall that as motivated in Section 1, a standard method to compute solutions of LMI is to use semi-definite programming (SDP) techniques. It is known in the literature that solving an LMI using SDP techniques requires generically $\mathcal{O}(\mathrm{n}^6)$ flops, while exploitation of certain structures in the problem may lead to an improvement up to $\mathcal{O}(\mathrm{n}^{4.5})$ flops (Vandenberghe et al., 2005). Hence, Algorithm 1 is expected to perform faster when compared with SDP-based optimisation packages. To demonstrate this we compare the time required by Algorithm 1 to compute a solution of KYP LMI (2) to that required by two standard Matlab based optimisation packages, namely *CVX: Matlab Software for Disciplined Convex Programming* (CVX) (Grant & Boyd, 2013) and *Yet Another LMI Parser* (YALMIP) (L'ofberg, 2004). Apart from these two packages, we also compare Algorithm 1 with the spectral factorisation technique (SFT) described in Willems and Trentelman (1998). We use a one-variable Euclidean division algorithm to implement this technique. We do not compare Algorithm 1 with the deflating subspace based method in Reis (2011); Reis et al. (2015) due to the absence, to the best of our knowledge, of standard packages to implement it. The experimental setup for the comparison of Algorithm 1 with standard methods is as follows.

Experimental setup and procedure: The experiment has been carried out on an Intel(R) Xeon(R) computer operating at 3.50 GHz with 64 GB RAM using Ubuntu 16.04 LTS operating system. Numerical computational package Matlab has been used to implement Algorithm 1 and the standard `tic-toc` command of Matlab is used to record the computational time. Execution time for the Euclidean division based spectral factorisation algorithm is also computed using the `tic-toc` command. The SDP solver used for both CVX and YALMIP is `sedumi`. The predefined numerical precision for the solver has been set to 10^{-12}. The total computational time for CVX is obtained by the command `cvx_cputime`, which includes both CVX modelling time and solver time. Similarly, the field `yalmiptime` is used to obtain the total computational time, which includes modelling and solver time, for YALMIP.

Randomly generated transfer functions corresponding to strongly passive systems of five different orders are used to compare the computational time of Algorithm 1 with that of CVX, YALMIP and SFT. The computation time for each order has been averaged over 15 randomly generated transfer functions.

Table 1. Flop count of each step in Algorithm 1.

Step	Operations	Algorithm	Flops
1	Matrix concatenation	Merely bookkeeping	0
2	(n − 1) Matrix-vector multiplication	Normal matrix-vector multiplication	$\mathcal{O}(\mathrm{n}^3)$
3	Matrix partitioning	Merely bookkeeping	0
4	Matrix inversion	Cholesky, LU, Gaussian elimination	$\mathcal{O}(\mathrm{n}^3)$
	Matrix-matrix multiplication	Normal matrix-matrix multiplication	$\mathcal{O}(\mathrm{n}^3)$

Further, in order to nullify the effect of CPU delays the computational time to calculate solutions of the KYP LMI (2) for each transfer function is further averaged over hundred iterations.

4.1 Experimental results

Computational time: Figure 3 demonstrates the time taken to compute the storage functions of strongly passive systems using CVX, YALMIP, SFT and Algorithm 1. From Figure 3, it is evident that Algorithm 1 is approximately 10^3 times faster compared to CVX and YALMIP. Further, it is also clear that although the execution time for Algorithm 1 is better than that of SFT, it is comparable.

Computational error: Since SDP solvers have an inherent numerical tolerance associated with them, the solutions of LMI (2) found using CVX and YALMIP are within a predefined numerical precision. However, all the operations performed in Algorithm 1 are implementable using algorithms that are not only numerically stable (Watkins, 2002) but also do not admit any numerical tolerance. A few of such numerically stable algorithms are suggested in Table 1. Evidently, Algorithm 1 is better than CVX and YALMIP from a numerical precision viewpoint as well. On the other hand, since SFT-based algorithms use matrix-matrix multiplication for its implementation, the error associated with SFT is comparable to that of Algorithm 1. However, one of the major drawbacks of SFT is that the solution obtained from the SFT algorithm corresponds to the KYP

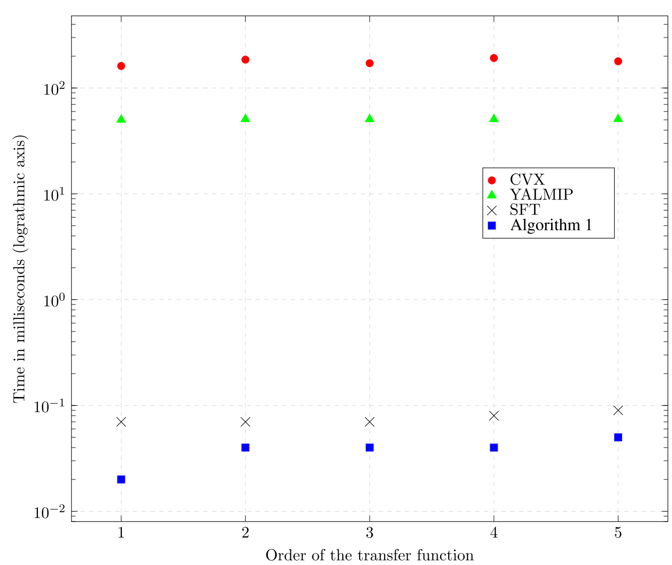

Figure 3. Plot of computational time to solve KYP LMI (2) for strongly passive SISO systems using CVX, YALMIP, SFT and Algorithm 1.

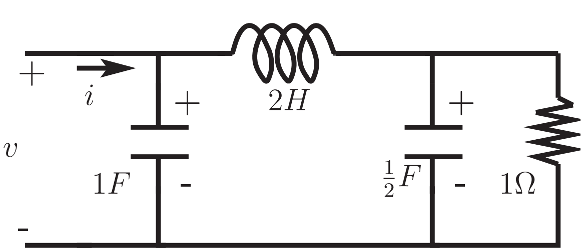

Figure 4. A RLC network with impedance transfer function as given in Example 4.1.

LMI (2) when the system matrices (A, B, C) are in the controller canonical form. Further, as mentioned in Section 1, SFT does not reveal the impulsive lossless trajectories of a strongly passive system. Unlike SFT, Theorem 3.1 directly leads to a complete characterisation of the impulsive lossless trajectories of a strongly passive system: see Section 5.

Next we construct a family of strongly passive systems and for one of them compute the storage function using Algorithm 1.

Example 4.1: Consider a strongly passive system with transfer function $G(s) = \frac{s^2+bs+1}{s^3+bs^2+ds+1}$ such that $b, d \in \mathbb{R}_+ \setminus 0$ and $d = 1 + \frac{1}{b}$. For example, let $b=2$ and $d = \frac{3}{2}$(see Figure 4. Then, an i/s/o representation of the system is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1.5 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} x.$$

Using Theorem 3.1, we construct W and obtain K as below

$$W = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 2.5 \\ \hline 1 & 1 & 0 \\ 2 & 0.5 & -1 \\ 1 & 0 & -0.5 \end{array}\right] = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

$$K = X_2 X_1^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0.5 & -1 \\ 1 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 2.5 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 3.5 & 3 & 1 \\ 3 & 4.5 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Note that $A^T K + KA = \mathtt{diag}(-2, 0, 0) \leqslant 0$ and $KB - C^T = 0$.

5. Impulses in lossless trajectories of strongly passive SISO systems

In this section, we establish the relation between lossless trajectories of a strongly passive SISO system and its storage function. To do so, we need the notion of trajectories of a system, which we define now. We first define the following function space of *impulsive-smooth* distributions, and denote it by $\mathfrak{C}_{\mathrm{imp}}$: see Hautus and Silverman (1983) for details.

Definition 5.1: An *impulsive-smooth distribution* is a distribution f of the form $f = f_1 + f_2$ where $f_1 \in \mathfrak{C}^\infty|_{\mathbb{R}_+}$ and $f_2 = \sum_{i=0}^{k} a_i \delta^{(i)}$ where $a_i \in \mathbb{R}, k \in \mathbb{N}$.

The symbol $\mathfrak{C}^{\mathtt{n}}_{\mathrm{imp}}$ denotes the set of $\mathtt{n}$-tuples of functions $(f_1, f_2, \ldots, f_{\mathtt{n}})$, where $f_i \in \mathfrak{C}_{\mathrm{imp}}$ for all $1 \leqslant i \leqslant \mathtt{n}$. We call $(x, u, y) \in \mathfrak{C}^{\mathtt{n}+1+1}_{\mathrm{imp}}$ to be a *trajectory* in a SISO system Σ if it is a distributional solution of the differential equation of Σ: $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$ and $y = Cx$. We define lossless trajectories next.

Definition 5.2: Consider a passive SISO system Σ with a minimal i/s/o representation $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$ and $y = Cx$, where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B, C^T \in \mathbb{R}^{\mathtt{n}}$. Let $(x, u, y) \in \mathfrak{C}_{\mathrm{imp}}{}^{\mathtt{n}+1+1}$ be a trajectory in Σ. Then, (x, u, y) is called a *lossless trajectory* if there exists a storage function $x^T Kx$ of Σ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^T Kx) = 2u^T y \quad \text{for all } t \in \mathbb{R}_+. \tag{23}$$

Definition 5.2 means that lossless trajectories of Σ are those for which the rate of change of stored energy is *equal* to the power supplied. It is crucial to note here that Definition 5.2 does not preclude the possibility of multiplication of Dirac delta impulse δ and its derivatives with themselves. We treat Equation (23) only formally here. By this we mean that Equation (23) is said to hold if and only if the expression $\frac{\mathrm{d}}{\mathrm{d}t}(x^T Kx) - 2u^T y$ is zero as a function for $t \in (0, \infty)$, and each of the coefficients of the monomials in the quadratic expression involving symbols $\delta, \dot{\delta}, \ldots, \delta^{(k)}, \ldots$ in the expression $\frac{\mathrm{d}}{\mathrm{d}t}(x^T Kx) - 2u^T y$ is zero.

Interestingly, for the regularly passive case, i.e. passive systems having minimal i/s/o representation $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$, $y = Cx + Du$ with $D + D^T$ nonsingular, it is known that Σ_{Ham} contains all the lossless trajectories of Σ (see Rapisarda, Trentelman, & Minh, 2013). Further, these lossless trajectories are intrinsically linked to the storage functions of the system. Taking a clue from this, we address the following questions on strongly passive SISO systems in this section.

(1) Are there any nontrivial lossless trajectories of a strongly passive SISO system? Does Σ_{Ham} contain such trajectories?
(2) If there are lossless trajectories in Σ, in what way are they linked with the storage functions of Σ?

To get to the main results of this section, we first rewrite the first order representation of Σ_{Ham} given in Equation (9) in an output-nulling representation in the following manner:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ C^T \end{bmatrix} u, \quad \text{and}$$
$$0 = \begin{bmatrix} C & -B^T \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}. \tag{24}$$

Like before, we use $(\widehat{A}, \widehat{B}, \widehat{C})$, as defined in Theorem 3.1, for the system matrices of Σ_{Ham}. Recall that, for a strongly passive SISO system, $(sE - H)$ is unimodular. This means Σ_{Ham}, represented by Equation (24), is autonomous (see Hautus & Silverman, 1983; Willems, Kitapci, & Silverman, 1986) and hence, given an initial condition (x_0, z_0), there is a unique trajectory in Σ_{Ham} starting from (x_0, z_0). Note that these trajectories in Σ_{Ham} must be in $\mathfrak{C}_{\mathrm{imp}}$ because $(sE - H)$ being unimodular prevents Σ_{Ham} from containing full-line nonzero smooth solutions (Dai, 1989). Now, let K be a solution of the singular KYP LMI (2) for a strongly passive system. We claim that, for a strongly passive SISO system Σ, the trajectories in Σ_{Ham} subject to initial conditions in $\mathtt{img\,col}(I, K)$, provide the lossless trajectories in Σ; this is the second main result of this paper.

Theorem 5.3: *Consider a strongly passive SISO system Σ with a minimal i/s/o representation $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu, y = Cx$, where $A \in \mathbb{R}^{n\times n}$ and $B, C^T \in \mathbb{R}^n$. The first-order representation of Σ_{Ham} is given by Equation (24). Let $K = K^T \in \mathbb{R}^{n\times n}$ be the solution of the corresponding singular KYP LMI (2). Let $(x, z, u) \in \mathfrak{C}_{\text{imp}}{}^{2n+1}$ be a trajectory in Σ_{Ham} corresponding to initial condition $(x_0, z_0) \in \texttt{img col}(I, K)$. Then, the following are true*

(1) *The corresponding $(x, u, Cx) \in \mathfrak{C}_{\text{imp}}{}^{n+1+1}$ is a trajectory in Σ.*
(2) *Further, this trajectory (x, u, Cx) is lossless in the sense of Definition 5.2.*

In order to prove this theorem we shall first characterise the trajectories of Σ_{Ham} when their initial conditions are restricted to the subspace $\texttt{img col}(I, K)$. Note that it follows from the construction of K in Theorem 3.1 that $(x_0, z_0) \in \texttt{img col}(I, K)$ is equivalent to $(x_0, z_0) = \sum_{k=0}^{n-1} \alpha_k \widehat{A}^k\widehat{B}$, where $\alpha_k \in \mathbb{R}$ for $k = 0, 1, \ldots, n-1$.

Theorem 5.4: *Consider a strongly passive SISO system Σ with a minimal i/s/o representation $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu, y = Cx$, where $A \in \mathbb{R}^{n\times n}$ and $B, C^T \in \mathbb{R}^n$. Let the output-nulling representation of Σ_{Ham} be as given in Equation (24). Then the following are true:*

(1) *For $k \in \{1, 2 \ldots, n-1\}$ and $\alpha_k \in \mathbb{R}$, consider an initial condition $(x_0, z_0) = \alpha_k \widehat{A}^k\widehat{B}$. Then,*

$$\begin{bmatrix} x(t) \\ z(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} -\alpha_k \sum_{i=0}^{k-1} (\widehat{A}^{(k-1-i)}\widehat{B}\delta^{(i)}) \\ -\alpha_k \delta^{(k)} \end{bmatrix} \in \mathfrak{C}_{\text{imp}}{}^{2n+1} \tag{25}$$

is a trajectory in Σ_{Ham}.
(2) *For $k = 0$ and $\alpha_0 \in \mathbb{R}$, consider an initial condition $(x_0, z_0) = \alpha_0 \widehat{B}$. Then, $(x, z, u) = (0, 0, -\alpha_0\delta)$ is a trajectory in Σ_{Ham}.*

Proof: **(1)** For initial condition $(x_0, z_0) = \alpha_k \widehat{A}^k\widehat{B}$:

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}_{\text{Unforced}} = \alpha_k e^{\widehat{A}t}\widehat{A}^k\widehat{B}. \tag{26}$$

Define $s(t)$ to be the unit step function.[8] For input $u(t) = -\alpha_k\delta^{(k)}$, the forced response is

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}_{\text{Forced}} = -\alpha_k \frac{d^k}{dt^k}(e^{\widehat{A}t}\widehat{B}s(t)). \tag{27}$$

We prove that

$$\frac{d^k}{dt^k}(e^{\widehat{A}t}\widehat{B}s(t)) = e^{\widehat{A}t}\widehat{A}^k\widehat{B}s(t) + \sum_{i=0}^{k-1} \widehat{A}^{(k-1-i)}\widehat{B}\delta^{(i)}. \tag{28}$$

We use the principle of mathematical induction to prove it.

Base case: $k = 1$.

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{\widehat{A}t}\widehat{B}s(t)) = e^{\widehat{A}t}\widehat{A}\widehat{B}s(t) + \widehat{B}\delta.$$

Induction step: We assume that

$$\frac{d^k}{dt^k}(e^{\widehat{A}t}\widehat{B}s(t)) = e^{\widehat{A}t}\widehat{A}^k\widehat{B}s(t) + \sum_{i=0}^{k-1} \widehat{A}^{(k-1-i)}\widehat{B}\delta^{(i)}.$$

We want to show that

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}}(e^{\widehat{A}t}\widehat{B}s(t)) = e^{\widehat{A}t}\widehat{A}^{(k+1)}\widehat{B}s(t) + \sum_{i=0}^{(k+1)-1} \widehat{A}^{((k+1)-1-i)}\widehat{B}\delta^{(i)}.$$

Now,

$$\begin{aligned} \frac{d^{k+1}}{dt^{k+1}}(e^{\widehat{A}t}\widehat{B}s(t)) &= \frac{d^k}{dt^k}(e^{\widehat{A}t}\widehat{A}\widehat{B}s(t) + \widehat{B}\delta) \\ &= \frac{\mathrm{d}^k}{\mathrm{d}t^k}(e^{\widehat{A}t}\widehat{A}\widehat{B}s(t)) + \widehat{B}\delta^{(k)} \\ &= e^{\widehat{A}t}\widehat{A}^{(k+1)}\widehat{B}s(t) + \sum_{i=0}^{k-1} \widehat{A}^{(k-i)}\widehat{B}\delta^{(i)} + \widehat{B}\delta^{(k)} \\ &= e^{\widehat{A}t}\widehat{A}^{(k+1)}\widehat{B}s(t) + \sum_{i=0}^{(k+1)-1} \widehat{A}^{((k+1)-1-i)}\widehat{B}\delta^{(i)}. \end{aligned}$$

We used the fact that $\widehat{A}$ commutes with $e^{\widehat{A}t}$ in the analysis above. This proves Equation (28) by the principle of mathematical induction. Therefore, adding the forced and unforced responses (Equations (26) and (27)), we get that the overall response due to the initial condition $(x_0, z_0) = \alpha_k \widehat{A}^k\widehat{B}$ and input $u(t) = -\alpha_k\delta^{(k)}$ is given by

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = -\alpha_k \sum_{i=0}^{k-1} \widehat{A}^{(k-1-i)}\widehat{B}\delta^{(i)}. \tag{29}$$

Thus the trajectory (25) satisfies equation $\left[\begin{smallmatrix} \dot{x} \\ \dot{z} \end{smallmatrix}\right] = \widehat{A}\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] + \widehat{B}u$ (Equation (24)). Now we check whether this trajectory satisfies the output-nulling equation within (24).

$$\begin{aligned} \widehat{C}\begin{bmatrix} x \\ z \end{bmatrix} &= -\widehat{C}\left(\alpha_k \sum_{i=0}^{k-1} (\widehat{A}^{(k-1-i)}\widehat{B}\delta^{(i)})\right) \\ &= -\alpha_k \sum_{i=0}^{k-1} (\widehat{C}\widehat{A}^{(k-1-i)}\widehat{B}\delta^{(i)}). \end{aligned} \tag{30}$$

From Lemma 3.2, $\widehat{C}\widehat{A}^{\ell}\widehat{B} = 0$ for $\ell \in \{0, 1, 2, \ldots, n-1\}$. Therefore, the right-hand side of Equation (30) is equal to 0. Hence, the trajectory in Equation (25) satisfies Equation (24) for $k \in \{1, 2, \ldots, n-1\}$.

(2) When $k = 0$ i.e. $(x_0, z_0) = \alpha_0\widehat{B}$ and $u(t) = -\alpha_0\delta$, from Equations (26) and (27) we have

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \alpha_0 e^{\widehat{A}t}\widehat{B} - \alpha_0 e^{\widehat{A}t}\widehat{B} = 0.$$

Clearly, $\widehat{C}\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] = 0$. Therefore, $(0, 0, -\alpha_0\delta)$ is a trajectory in Σ_{Ham}.

This completes the proof of Theorem 5.4. ■

Since Σ_{Ham} is a linear system, from Theorem 5.4, it is clear that the trajectory in Σ_{Ham}, corresponding to initial condition $(x_0, z_0) = \sum_{i=0}^{\mathrm{n}-1} \alpha_i \widehat{A}^i \widehat{B}$, can be characterised as

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = -\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & \delta & \delta^{(1)} & \delta^{(2)} & \cdots & \delta^{(\mathrm{n}-2)} \\ 0 & 0 & \delta & \delta^{(1)} & \cdots & \delta^{(\mathrm{n}-3)} \\ 0 & 0 & 0 & \delta & \cdots & \delta^{(\mathrm{n}-4)} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \delta \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\Delta} \times \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{\mathrm{n}-1} \end{bmatrix}}_{\alpha}, \tag{31}$$

$$u(t) = -\begin{bmatrix} \delta & \delta^{(1)} & \delta^{(2)} & \cdots & \delta^{(\mathrm{n}-1)} \end{bmatrix} \alpha, \tag{32}$$

where $X_1 = \begin{bmatrix} B & AB & \cdots & A^{\mathrm{n}-1}B \end{bmatrix}$, $X_2 = [C^T \ \ -(CA)^T \ \cdots \ (-1)^{\mathrm{n}-1}(CA^{\mathrm{n}-1})^T]$.

Note that, since $(sE - H)$ is a unimodular matrix, it is known that the trajectories of Σ_{Ham} are a linear combinations of δ and its derivatives (see Dai, 1989). This conforms with our observation manifested in Equations (31) and (32). Recall that Σ_{Ham} was obtained by interconnecting Σ with its adjoint $\Sigma_{\mathtt{adj}}$ (see Section 2). Therefore, it is expected that (x, u) from Equations (31) and (32) would satisfy the differential equation of Σ in a distributional sense. We explicitly show this in Table 2. It is important to note here that since we are dealing with solutions defined over the half-line $\mathbb{R}_+$, the function that is zero for all $t \in \mathbb{R}_+$ and the function defined as

$$x(t) = \begin{cases} x_0 \text{ at } t = 0^-, \\ 0 \text{ at } t > 0, \end{cases}$$

are the same in the distributional sense. However, the distributional derivative of $x(t)$ turns out to be $-x_0\delta$, and hence depends on the initial condition x_0. We utilise this fact crucially while preparing Table 2. Now we prove Theorem 5.3.

Proof of Theorem 5.3:

(1) Note that $(x_0, z_0) \in \mathtt{img\,col}(I, K)$ means that $(x_0, z_0) = \sum_{k=0}^{\mathrm{n}-1} \alpha_k \widehat{A}^k \widehat{B}$ for $\alpha_k \in \mathbb{R}$. Since Σ is a linear system, from Table 2 it is clear that (x, u) satisfies the input-state equation $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$ of Σ. Further, the output equation of Σ is $y = Cx$. Therefore, (x, u, Cx) is a trajectory in Σ.

(2) From Equation (31), we have

$$x(t) = -X_1 \Delta \alpha. \text{ (with } \Delta \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}} \text{ defined in Equation (31))} \tag{33}$$

Recall that $K = K^T \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ is a solution to the KYP LMI. Evaluating $\frac{\mathrm{d}}{\mathrm{d}t}(x^T K x)$ and utilising the fact that $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^T K x) = \dot{x}^T K x + x^T K \dot{x}$$
$$= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + 2u^T C x.$$

Since $K = K^T \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ is a solution of the KYP LMI, $KB - C^T = 0$. Further, $y = Cx$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^T K x) = x^T (A^T K + KA) x + 2u^T y. \tag{34}$$

Using Equation (33) in Equation (34), we get $\frac{\mathrm{d}}{\mathrm{d}t}(x^T K x) = \alpha^T \Delta^T X_1^T (A^T K + KA) X_1 \Delta \alpha + 2u^T y$.

Recall from Equation (15), we have $X_1^T (A^T K + KA) X_1 = \mathtt{diag}(0, T_{(\mathrm{n}-1)(\mathrm{n}-1)})$, where $T_{(\mathrm{n}-1)(\mathrm{n}-1)} \in \mathbb{R}$. Since the first column and the last row of Δ contain only zeros, it is easy to see that $\alpha^T \Delta^T \mathtt{diag}(0, T_{(\mathrm{n}-1)(\mathrm{n}-1)}) \Delta \alpha = 0$. Thus $x^T (A^T K + KA) x = 0$ in Equation (34) and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^T K x) = 2u^T y.$$

Therefore, by Definition 5.2, the trajectories (x, u, Cx) are lossless. ■

Using Theorem 5.3, one can now formally assert that there indeed are *nontrivial* (impulsive) lossless trajectories in Σ, albeit there are no nontrivial slow lossless trajectories. These impulsive lossless trajectories can be found when we restrict the initial conditions of Σ_{Ham} to the subspace $\mathtt{img\,col}(I, K)$. For the regularly passive case, the lossless trajectories are exponential in nature (see Rapisarda et al., 2013). However, for strongly passive SISO case, all lossless trajectories are impulsive in nature. Hence, we call these trajectories *fast lossless trajectories*.

From Table 2, it is clear that there are many scenarios when we encounter products of δ and its derivatives while evaluating the power supply $2u^T y$ or the stored energy $x^T K x$. Multiplication of δ and its derivatives has been defined in

Table 2. Table to show the validity of $(\frac{\mathrm{d}}{\mathrm{d}t})x = Ax + Bu$ for different initial conditions.

x_0	$x(t)$	$u(t)$	$(\frac{\mathrm{d}}{\mathrm{d}t})x$	$Ax + Bu$
$\alpha_0 B$	0	$-\alpha_0\delta$	$-\alpha_0 B\delta$	$-\alpha_0 B\delta$
$\alpha_1 AB$	$-\alpha_1 B\delta$	$-\alpha_1\delta^{(1)}$	$-\alpha_1 B\delta^{(1)} - \alpha_1 AB\delta$	$-\alpha_1 AB\delta - \alpha_1 B\delta^{(1)}$
$\alpha_2 A^2 B$	$-\alpha_2(B\delta^{(1)} + AB\delta)$	$-\alpha_2\delta^{(2)}$	$-\alpha_2(B\delta^{(2)} + AB\delta^{(1)} + A^2 B\delta)$	$-\alpha_2(AB\delta^{(1)} + A^2 B\delta + B\delta^{(2)})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\alpha_{\mathrm{n}-1} A^{\mathrm{n}-1} B$	$-\alpha_{\mathrm{n}-1} \sum_{i=0}^{\mathrm{n}-2} A^{\mathrm{n}-2-i} B\delta^{(i)}$	$-\alpha_{\mathrm{n}-1}\delta^{(i)}$	$-\alpha_{\mathrm{n}-1} \sum_{i=0}^{\mathrm{n}-1} A^{\mathrm{n}-1-i} B\delta^{(i)}$	$-\alpha_{\mathrm{n}-1} \sum_{i=0}^{\mathrm{n}-1} A^{\mathrm{n}-1-i} B\delta^{(i)}$

(Fuchssteiner, 1984; Trenn, 2009). Such multiplications are defined in the literature using *Fuchssteiner multiplication*. However, physical interpretation of the products of δ and its derivatives is an open question to the best of our knowledge. We do not dwell into the physical interpretation of such products here. However, for a special case of Theorem 5.3, it is possible to rule out such products and therefore get a physical interpretation of lossless trajectories. We report it below as a corollary to Theorem 5.3.

Corollary 5.5: *Consider a strongly passive SISO system Σ with a minimal i/s/o representation $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$, $y = Cx$, where $A \in \mathbb{R}^{n\times n}$ and $B, C^T \in \mathbb{R}^n$. Let the initial condition x_0 of Σ be from $\mathtt{img}(B)$, i.e. $x_0 = \alpha_0 B$ for $\alpha_0 \in \mathbb{R}$. Then, there exists a lossless trajectory (x, u, y) in Σ with initial condition x_0. This lossless trajectory is given by $(0, -\alpha_0\delta, 0)$.*

We skip the proof of this corollary as it directly follows from Statement 2 of Theorem 5.3. We compute explicitly the lossless trajectories of a parallel RC circuit in the example presented next.

Example 5.6: Consider the strongly passive SISO system Σ with transfer function $G(s) = \frac{1}{s+1}$ as shown in Figure 5 (with $R = 1\Omega, C = 1F$). An i/s/o representation of the system is $\frac{\mathrm{d}}{\mathrm{d}t}x = -x + u$ and $y = x$, where $x = v_c$. Using Theorem 3.1, we have

$$X_1 = B = 1 \quad \text{and} \quad X_2 = C^T = 1.$$

Therefore, it is evident that the storage function for this circuit is $K = X_2X_1^{-1} = 1$. Note that subject to the initial condition $z_0 = Kx_0 = \alpha_0$, the trajectories of the system Σ are $u(t) = -\delta\alpha_0$ for $t \geqslant 0$ and $x(t) = \alpha_0$ for $t = 0$, and $x(t) = 0$ for $t > 0$.

From Corollary 5.5, we claim these trajectories are lossless. We show the validity of Equation (23) for these trajectories next.

- At $t = 0^-$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^TKx) = 2\dot{x}^Tx = -2\delta\alpha_0^2, \quad \text{and} \quad 2u^Ty = -2\delta\alpha_0^2.$$

- At $t > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^TKx) = 0, \quad \text{and} \quad 2u^Ty = 0.$$

Thus we have $\frac{\mathrm{d}}{\mathrm{d}t}(x^TKx) = 2u^Ty$ for all $t \geqslant 0$.

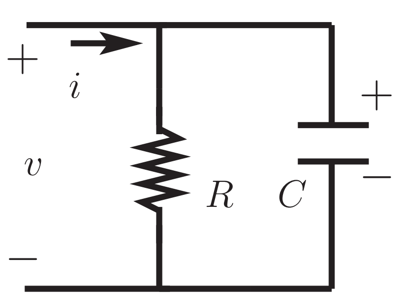

Figure 5. A parallel RC network with $Z(s) = G(s) = R/(1 + sRC)$.

Physically, it means that for a parallel RC circuit of the form given in the example with the capacitor initially charged to α_0, if one discharges the capacitor very fast, i.e. in the limit of a sequence of exponential decays: instantaneously,[9] then it is possible to extract the capacitor's entire stored energy ($\alpha_0K\alpha_0 = \alpha_0^2$) through the port. This is the optimal discharging policy for the parallel RC circuit in Figure 5. On the other hand, nonzero dissipation at the resistor R is inevitable if one does not discharge instantaneously. Similarly for the case of charging also, unlike regularly passive systems, with the same amount of energy, i.e. α_0^2 the capacitor can be charged from rest to α_0, as well, provided charging is done instantaneously.

Remark 5.7: Consider a strongly passive SISO system Σ with a minimal i/s/o representation $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$, $y = Cx$. Let K be the unique storage function of Σ. Assume Σ_{Ham} to be the system formed by the interconnection of Σ and $\Sigma_{\mathtt{adj}}$. Note that, corresponding to any initial condition x_0 of Σ, the vector $\left[\begin{smallmatrix} I \\ K \end{smallmatrix}\right]x_0$ is an initial condition of Σ_{Ham}. As given in Theorem 5.3, it follows that corresponding to such an initial condition in Σ_{Ham}, there exists a lossless trajectory in Σ. Although it seems restrictive to choose the initial conditions of Σ_{Ham} from the subspace $\mathtt{img\,col}(I, K)$ to obtain lossless trajectories in Σ, as far as the system Σ is concerned, its initial conditions x_0 are still free. Hence, for every initial condition x_0 of Σ there exists a lossless trajectory in Σ. Theorem 5.3 and Theorem 5.4 above shows that all these lossless trajectories are impulsive.

6. Concluding remarks

In this paper, we dealt with an extreme case of passivity, namely, strongly passive SISO systems, where the system has no finite spectral zeros. We first proposed a closed-form formula of a solution to the singular KYP LMI corresponding to a strongly passive SISO system (Theorem 3.1). We then showed that this solution is unique (Theorem 3.1). In both these results a property of the Markov parameters of strongly passive systems played a crucial role (Lemma 3.2). Note that, when the feedthrough regularity condition is satisfied, i.e. $D + D^T > 0$, the Hamiltonian matrix provides the suitable $\mathtt{n}$-dimensional invariant subspaces needed for computation of K. Interestingly, we have shown in this paper that, for strongly passive SISO systems, where the feedthrough regularity condition is violated, a construction very much akin to this can be provided by a suitable arrangement of the controllability and observability matrices. In order to illustrate the main result Theorem 3.1, we also constructed a family of third order strongly passive systems and found the storage function for one member of this family following the construction presented in Theorem 3.1.

We further showed that this unique solution of the singular KYP LMI (storage function) of a strongly passive SISO system is intrinsically linked with the lossless trajectories of the system. We showed that, if the initial condition of the system formed by interconnection of the primal system Σ and its adjoint $\Sigma_{\mathtt{adj}}$ (this interconnected system is denoted by Σ_{Ham}), are restricted to the subspace $\mathtt{img\,col}(I, K)$, we get lossless trajectories of Σ. Note that, since these special initial conditions of the interconnected system Σ_{Ham} are of the form $(x_0, z_0) \in \mathtt{img\,col}(I, K)$, the projection of the subspace $\texttt{img col}(I, K)$ on the primal system's state-space is full. Therefore, for every initial condition x_0 in the primal system there exists a lossless trajectory. We characterised these lossless trajectories and showed that strongly passive SISO systems admit only fast lossless trajectories.

It is noteworthy that, all the main results in this paper are not specific to only passivity; analogous results hold for other notions of power supply as well, e.g. bounded-realness, LQR problem and so on. In this paper, we do not delve deeper into this analogy; this will be the topic of our immediate future work. Also, the results presented in this paper are restricted to only SISO systems. While the essential ideas presented in this paper are suspected to hold for MIMO systems also, it remains to show formally the corresponding results for MIMO systems. We plan to present these extensions elsewhere in a future paper. Interestingly, between the two extremities, i.e. strongly passive systems and regularly passive systems, there exists a class of passive systems that admit finite spectral zeros, but of cardinality less than double the system's order. These passive systems, too, do not admit the ARE/Hamiltonian matrix because they do not satisfy the feedthrough regularity condition. Likewise, finding solutions to singular KYP LMIs corresponding to these passive systems is equally challenging. An extension of the construction given in this paper is suspected to exist even for these systems as well. This will be taken up as a topic of our future research.

Notes

1. For a system, having equal number of inputs and outputs, with transfer function $G(s)$, the *spectral zeros* are defined to be the zeros of the determinant of the rational function matrix $G(s) + G(-s)^T$.
2. Optimisation algorithms begin with an initial guess of the unknown variable and generate a sequence of improved estimates. Such algorithms terminate when the 'distance' (gap) between these estimates are within a specified tolerance.
3. Conventionally, the elements of any matrix A, i.e. a_{mn} are indexed as $m, n = 1, 2, \ldots, \mathtt{n}$. However, to match the indexing to the exponent of A in the controllability matrix and Markov parameters (used in the proof of the main results of this paper), we use 0 as the starting index.
4. In this paper, since we focus only on fast solutions, we do not dwell on stability and therefore, we relax the non-negativity requirement of $x^T Kx$, i.e. positive semi-definiteness of K: the link with stability can be seen in (Willems & Trentelman, 1998, Theorem 6.3).
5. Let the characteristic polynomial of $\mathcal{H}$ be denoted as $\mathcal{X}(s)$. A Lambda-set $\Lambda \subset \sigma(\mathcal{H})$, if it exists, is the set of roots of a polynomial $p(s) \in \mathbb{R}[s]$ such that $\mathcal{X}(s) = p(s)p(-s)$ with $p(s)$ and $p(-s)$ coprime (sets are counted with multiplicity).
6. A matrix pencil $(sP - Q)$ is called regular if $\texttt{det}(sP - Q) \neq 0$.
7. Relative degree of a rational function $n(s)/d(s)$ is defined as $(\texttt{deg}\,[d(s)] - \texttt{deg}\,[n(s)])$.
8. The unit step function $s(t)$ is defined as $s(t) = 1$ for $t \geqslant 0$ and $s(t) = 0$ for $t < 0$.
9. Instantaneous discharge of capacitor C (by a controller at the port, which is, in this case 'short') can be viewed as a limit of a sequence of exponentially decaying extractions, with increasing magnitudes of decay rates. This paper focusses on the limiting case of a sequence of exponential trajectories: in the limit we have fast trajectories. These fast trajectories being lossless is the key inference from our analysis.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work is supported by Indian Institute of Technology (IIT) Bombay, India under [grant number 15IRCCSG 012] and DST-INSPIRE Faculty Grant, Department of Science and Technology (DST), Govt. of India under [grant number IFA14-ENG-99].

ORCID

Chayan Bhawal http://orcid.org/0000-0003-3477-3725
Debasattam Pal http://orcid.org/0000-0002-4886-3003
Madhu N. Belur http://orcid.org/0000-0002-1942-6293

References

Anderson, B., & Vongpanitlerd, S. (2006). *Network analysis and synthesis: A modern systems theory approach*. New York, NY: Dover.

Antoulas, A. (2005). *Approximation of large-scale dynamical systems, advances in design and control*. Philadelphia, PA: SIAM.(Vol. 6).

Bini, D., Iannazzo, B., & Meini, B. (2012). *Numerical solution of algebraic Riccati equations*. Philadelphia, PA: SIAM.

Boyd, S., & Vandenberghe, L. (2004). *Convex optimization*. Cambridge: Cambridge University Press.

Brogliato, B., Lozano, R., Maschke, B., & Egeland, O. (2007). *Dissipative systems analysis and control: Theory and applications*. London: Springer-Verlag.

Cauer, W. (1926). Die Verwirklichung der Wechselstromwiderstände vorgeschriebener Frequenzabhängigkeit. *Archiv für Elektrotechnik*, *17*, 355–388.

Dai, L. (1989). *Singular control systems*. New York, NY: Springer-Verlag.

Fuchssteiner, B. (1984). Algebraic foundation of some distribution algebras. *Studia Mathematica*, *77*(5), 439–453.

Grant, M., & Boyd, S. (2013). *CVX: Matlab software for disciplined convex programming*, version 2.0 beta. http://cvxr.com/cvx, September 2013.

Hautus, M., & Silverman, L. (1983). System structure and singular control. *Linear Algebra and Its Application*, *50*, 369–402.

Hoang, H., Tuan, H., & Nguyen, T. (2009). Frequency-selective KYP lemma, IIR filter, and filter bank design. *IEEE Transactions on Signal Processing*, *57*(3), 956–965.

Jugade, S., Pal, D., Kalaimani, R., & Belur, M. (2013). Stationary trajectories, singular Hamiltonian systems and ill-posed interconnection. In *Proceedings of the European Control Conference (ECC)* (pp. 1740–1745). Zurich: IEEE.

Jugade, S., Pal, D., Kalaimani, R., & Belur, M. (2017). Fast modes in the set of minimal dissipation trajectories. *Systems & Control Letters*, *101*, 28–36.

Kalman, R. (1963). Lyapunov functions for the problem of Luré in automatic control. *Proceedings of the National Academy of Sciences*, *49*(2), 201–205.

Kučera, V. (1991). Algebraic Riccati equation: Hermitian and definite solutions. In S. Bittanti, A. Laub, & J. Willems (Eds.), *The Riccati equation* (pp. 53–88). Berlin Heidelberg: Springer.

Löfberg, J. (2004). YALMIP: A toolbox for modeling and optimization in MATLAB. In *Proceedings of the computer aided control systems design conference* (pp. 284–289). New Orleans, LA: IEEE.

Megretski, A., & Rantzer, A. (1997). System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control*, *42*(6), 819–830.

Paszke, W., Rogers, E., & Gałkowski, K. (2013). KYP lemma based stability and control law design for differential linear repetitive processes with applications. *Systems & Control Letters*, *62*(7), 560–566.

Popov, V. (1964). Hyperstability and optimality of automatic systems with several control functions. *Revue Roumaine des Sciences Techniques – Serie Électrotechnique et Énergétique*, *9*(4), 629–690.

Rapisarda, P., Trentelman, H., & Minh, H. (2013). Algorithms for polynomial spectral factorisation and bounded-real balanced state space representations. *Mathematics of Control Signals and Systems*, *25*(2), 231–255.

Reis, T. (2011). Luré equations and even matrix pencils. *Linear Algebra and Its Applications*, *434*(1), 152–173.

Reis, T., Rendel, O., & Voigt, M. (2015). The Kalman-Yakubovich-Popov inequality for differential-algebraic systems. *Linear Algebra and Its Applications*, *485*, 153–193.

Sorensen, D. (2005). Passivity preserving model reduction via interpolation of spectral zeros. *Systems* & *Control Letters*, *54*(4), 347–360.

Trenn, S. (2009). Regularity of distributional differential algebraic equations. *Mathematics of Control, Signals, and Systems*, *21*(3), 229–264.

Trentelman, H., Minh, H., & Rapisarda, P. (2009). Dissipativity preserving model reduction by retention of trajectories of minimal dissipation. *Mathematics of Control, Signals, and Systems*, *21*(3), 171–201.

Trentelman, H., & Willems, J. (1997). Every storage function is a state function. *Systems* & *Control Letters*, *32*(5), 249–259.

Vandenberghe, L., Balakrishnan, V., Wallin, R., Hansson, A., & Roh, T. (2005). Interior-point algorithms for semidefinite programming problems derived from the KYP lemma. In D. Henrion & A. Garulli (Eds.), *Positive polynomials in control, lecture notes in control and information science* (Vol. 312, pp. 195–238). Berlin Heidelberg: Springer.

Van der Schaft, A., & Rapisarda, P. (2011). State maps from integration by parts. *SIAM Journal on Control and Optimization*, *49*(6), 2415–2439.

Van Dooren, P. (1981). A generalized eigenvalue approach for solving Riccati equation. *SIAM Journal on Scientific and Statistical Computing*, *2*(2), 121–135.

Watkins, D. (2002). *Fundamentals of matrix computation*. New York, NY: John Wiley & Sons.

Willems, J. (1972). Dissipative dynamical systems part I: General theory. *Archive for Rational Mechanics and Analysis*, *45*(5), 321–351.

Willems, J., Kitapci, A., & Silverman, L. (1986). Singular optimal control: A geometric approach. *SIAM Journal of Control and Optimization*, *24*(2), 323–337.

Willems, J., & Trentelman, H. (1998). On quadratic differential forms. *SIAM Journal on Control and Optimization*, *36*(5), 1703–1749.

Yakubovich, V. (1962). The solution of certain matrix inequalities in automatic control theory. *Soviet Mathematics Doklady*, *3*, 620–623.