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Brief paper

Optimal selection of essential interconnections for structural controllability in heterogeneous subsystems☆

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ABSTRACT

This paper deals with structural controllability of a linear time-invariant *composite* system consisting of several *heterogeneous* subsystems. The interaction links through which the subsystems interact with other subsystems are referred to as *interconnections*. We assume the composite system to be structurally controllable if all possible interconnections are present. Our objective is to identify a minimum cardinality set of interconnections required to retain the structural controllability of the composite system. We refer to this problem as the *optimal essential interconnection selection* problem. We approach the problem in a structured framework, where the zero/nonzero structure of the subsystems is used in the analysis instead of the numerical matrices themselves. This analysis applies to an equivalence class of systems with the same sparsity pattern. Firstly, we propose a polynomial time algorithm to solve the optimal essential interconnection selection problem on a structured composite system when each subsystem is irreducible and no subsystem has a perfect matching in its state bipartite graph. Later, we consider the case where one or more subsystems have perfect matching in their state bipartite graphs. For this case, we first prove a lower bound on the number of minimum number of interconnections needed. Subsequently, we provide a polynomial time algorithm based on a minimum weight perfect matching algorithm and a so-called *stub-matching* algorithm that achieves this bound. We also discuss about how heterogeneity of the subsystems poses different challenges to the homogeneous counterpart and demonstrate the algorithms using illustrative examples.

1. Introduction

In recent years large-scale dynamical systems have become a subject of intensive research in systems and control theory because of its applications in diverse areas, including biological networks, transportation networks, water distribution networks, multi-agent systems, and internet (Liu, Slotine, & Barabási, 2011; Pequito, Preciado, Barabási, & Pappas, 2017). Most of the complex networks consist of spatially distributed units interconnected to constitute a large system. We refer to these entities as *subsystems*, the links connecting these subsystems as *interconnections* and the full system as the *composite* system.

The system-theoretic study of composite systems is of research interest for many decades. Controllability and observability of composite systems was introduced in Gilbert (1963), where it is related to the controllability and observability of its subsystems. Most of the earlier research in this area focuses on standard interconnections, namely series and parallel connections (Chen & Desoer, 1967; Davison & Wang, 1975; Wolovich & Hwang, 1974). However, in practice the interconnections in large complex systems may not be of these standard nature but is complicated. Interconnections other than the standard ones are also considered in the literature, for instance see Refs. Ikeda, Šiljak, and Yasuda (1983) and Zhou (2015). There is a diverse class of physical networks composed of subsystems interconnected in a complex manner. For example, in robotics a composite system consists of a swarm of robots (we may refer each as an *agent*) that collectively operate to perform a desired task. The agents in a swarm may not be homogeneous. They can be heterogeneous also, say manufactured by different companies. Although the agents must communicate with each other to achieve the desired goal, it is preferred to keep the interactions the least possible in regard of cost and privacy. Given a set of agents, it is a legitimate question to ask *which* agents should communicate and *what* information has to be exchanged so that the composite system achieves the intended performance with minimum interaction.

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The desired performance focused in this paper is the controllability of a composite system. Complex networks consist of many subsystems of large system dimension connected to each other. On account of the large size of these systems, devising efficient framework to design the interconnections of the subsystems in an optimal fashion is indispensable. A set of interconnection edges is referred as an *essential interconnection set* if breakage of any one connection in the set results in the system losing the desired property, i.e., controllability. We refer to the problem of finding a minimum cardinality essential interconnection set as the *optimal essential interconnection selection* problem.

For many complex networks the system parameters are not known precisely. Also, in most of the cases, either it is not possible to measure the link weights in a graph by knowing the graph structure (for example in social communication network and biological systems) or the link weights are time-dependent (like transportation networks, where the traffic in the lane changes with time (Liu & Barabási, 2016)). In such a case, the available system information is the sparsity pattern or the graph structure. Our approach in this paper uses the sparsity pattern, i.e., zero/nonzero pattern, of the system matrices instead of the exact numerical matrices. Given a set of subsystems (agents), our aim is to find a minimum cardinality set of interconnections between them so that the composite system is controllable. We provide an efficient framework to address the optimal essential interconnection selection problem using *structural systems theory* (Reinschke, 1988).

Recently, we considered the problem for the homogeneous case in Moothedath, Chaporkar, and Belur (2017). When subsystems are homogeneous, size of maximum matching in the state bipartite graph of all subsystems is the same. Hence some key issues and certain profound graph theoretic challenges of the heterogeneous case do not arise in the homogeneous case. For this reason, the algorithm given in Moothedath et al. (2017) does not extend to the heterogeneous case considered here. While optimizing the interconnections for a given set of subsystems we also find out which subsystems to be connected and what information to be communicated such that the composite system is structurally controllable.

We summarize this paper's contribution below.

- We propose a polynomial time algorithm to solve the essential interconnection selection problem (Problem 2.3) on structured subsystems when no subsystem has a perfect matching in its state bipartite graph (Algorithm 4.1). This algorithm is based on a minimum cost perfect matching algorithm with additional modifications.
- We provide a polynomial time algorithm to solve Problem 2.3 on structured subsystems when some subsystems have a perfect matching in their state bipartite graphs (Algorithm 5.1). This algorithm incorporates a minimum cost perfect matching algorithm along with a so-called stub-matching algorithm.
- We prove the optimality and the complexity results of both these algorithms (Theorems 4.6, 4.7, 5.2, and 5.3).

The organization of this paper is as follows: Section 2 gives the formulation of the problem and the related work. Section 3 contains few graph theoretic preliminaries and some existing results in the area of structural controllability. Section 4 presents a polynomial time algorithm for solving the essential interconnection selection problem when no subsystem has a perfect matching in its state bipartite graph. Section 5 gives an algorithm to solve the essential interconnection selection problem when one more subsystems have a perfect matching. Section 6 demonstrates the proposed algorithms using illustrative examples and also discusses possible extensions. Finally, Section 7 gives the concluding remarks.

2. Problem formulation and related work

In this section, we first present the formulation of the problem and then briefly describe the related work.

2.1. Problem formulation

Structural representation of an LTI system with dynamics $\dot{x} = Ax + Bu$, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, is represented by $\bar{A} \in \{0,\star\}^{n\times n}$ and $\bar{B} \in \{0,\star\}^{n\times m}$. Here $\mathbb{R}$ denotes the set of real numbers and $\{0,\star\}^{p\times q}$ denotes the set of $p \times q$ matrices whose entries are $\star$'s and zeros, where $\star$ is a free independent parameter. The pair $(\bar{A},\bar{B})$ structurally represents a system (A, B) if it satisfies:

$$\begin{aligned} \bar{A}_{pq} &= 0 \text{ whenever } A_{pq} = 0, \text{ and} \\ \bar{B}_{pq} &= 0 \text{ whenever } B_{pq} = 0. \end{aligned} \tag{1}$$

We refer to $(\bar{A},\bar{B})$ that satisfies Eq. (1) as the *structured system* corresponding to the *numerical system* (A, B). Note that $(\bar{A},\bar{B})$ does not have numerical values but only indicates locations of nonzero entries using $\star$'s. $(\bar{A},\bar{B})$ structurally represents a class of control systems corresponding to all possible numerical realizations (A, B) satisfying Eq. (1). The key idea in structural controllability is to determine controllability of the class of systems represented by $(\bar{A},\bar{B})$. Specifically, the following definition holds.

Definition 2.1. The structured system $(\bar{A},\bar{B})$ is said to be structurally controllable if there exists at least one controllable numerical realization (A, B).

Remark 2.2. Even though the definition of structural controllability requires only one controllable realization, it is known that if a system is structurally controllable, then 'almost all' numerical realizations of the same structure are controllable (Reinschke, 1988).

Consider k *structured subsystems* $(\bar{A}_i,\bar{B}_i)$, for $i = 1,\dots,k$, with $\bar{A}_i \in \{0,\star\}^{n_i\times n_i}$ and $\bar{B}_i \in \{0,\star\}^{n_i\times m_i}$. Let the ith subsystem $(\bar{A}_i,\bar{B}_i)$ be denoted by S_i. Then the dynamics of S_i is

$$\dot{x}_i(t) = \bar{A}_i x_i(t) + \bar{B}_i u_i(t), \text{ for } i = 1,\dots,k. \tag{2}$$

Each subsystem is individually not structurally controllable. To achieve structural controllability, one need to interconnect subsystems. Let $\bar{E}_{ij} \in \{0,\star\}^{n_i\times n_j}$ denote the *structured connection matrix* from S_j to S_i. After interconnecting, the composite structured system of k subsystems has the following dynamics.

$$\dot{x}(t) = \underbrace{\begin{bmatrix} \bar{A}_1 & \bar{E}_{12} & \cdots & \bar{E}_{1k} \\ \bar{E}_{21} & \bar{A}_2 & \cdots & \bar{E}_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ \bar{E}_{k1} & \bar{E}_{k2} & \cdots & \bar{A}_k \end{bmatrix}}_{\bar{A}_T} x(t) + \underbrace{\begin{bmatrix} \bar{B}_1 & 0 & \cdots & 0 \\ 0 & \bar{B}_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{B}_k \end{bmatrix}}_{\bar{B}_T} u(t), \tag{3}$$

where $\bar{A}_T \in \{0,\star\}^{n_T\times n_T}$ with $n_T = \sum_{i=1}^k n_i$ and $\bar{B}_T \in \{0,\star\}^{n_T\times m_T}$ with $m_T = \sum_{i=1}^k m_i$. Here, $x = [x_1^T,\dots,x_k^T]^T$ with $x_i = [x_1^i,\dots,x_{n_i}^i]^T$ and $u = [u_1^T,\dots,u_k^T]^T$ with $u_i = [u_1^i,\dots,u_{m_i}^i]^T$. The system $(\bar{A}_T,\bar{B}_T)$ is said to be a *structured composite system* formed by subsystems $(\bar{A}_1,\bar{B}_1),\dots,(\bar{A}_k,\bar{B}_k)$ interconnected through $\bar{E}_{ij}$'s, for $i,j \in \{1,\dots,k\}$.

Now we define the optimization problem considered in this paper. Given a set of structured subsystems $(\bar{A}_i,\bar{B}_i)$, $i = 1,\dots,k$, where $\bar{A}_i \in \{0,\star\}^{n_i\times n_i}$ and $\bar{B}_i \in \{0,\star\}^{n_i\times m_i}$, find a set of sparsest connection matrices, $\bar{E}_{ij} \in \{0,\star\}^{n_i\times n_j}$'s, such that the composite structured system, $(\bar{A}_T,\bar{B}_T)$, obtained corresponding to Eq. (3) is structurally controllable. Note that there are exponential number of ways that one can connect the subsystems. Our objective is to select the sparsest set of $\bar{E}_{ij}$'s, for $i, j = 1, \ldots, k$. Let $\mathcal{K}$ denote the set of all structurally controllable composite systems that can be formed by using subsystems $(\bar{A}_i, \bar{B}_i)$, for $i = 1, \ldots, k$. In other words, $\mathcal{K}$ consists of structured matrices $\bar{A}'$, where $\bar{A}' \in \{0, \star\}^{n_T \times n_T}$, satisfying $\bar{A}'_i = \bar{A}_i$ for $i = 1, \ldots, k$ and $\bar{E}'_{ij} \in \{0, \star\}^{n_i \times n_j}$ such that the composite system obtained $(\bar{A}', \bar{B}_T)$ is structurally controllable. Then, the optimization problem to be solved is as follows:

Problem 2.3. Given $(\bar{A}_i, \bar{B}_i)$, $i = 1, \ldots, k$, find

$$\bar{A}^\star \in \arg\min_{\bar{A}' \in \mathcal{K}} \left\| \bar{A}' \right\|_0 .$$

Here $\|\cdot\|_0$ denotes the zero matrix norm.[1] The set $\mathcal{K}$ is nonempty. This is because, if all entries in $\bar{E}_{ij}$'s are $\star$'s, for $i, j \in \{1, \ldots, k\}$, then the resulting structured system, $(\bar{A}_T, \bar{B}_T)$, is structurally controllable. In other words, $(\bar{A}_T, \bar{B}_T) \in \mathcal{K}$. Notice that the optimal essential interconnection selection problem is same as finding the sparsest $\bar{A}' \in \mathcal{K}$. Since the structured matrices in set $\mathcal{K}$ are constrained to have $\bar{A}_i$ as their block diagonal matrices, for $i \in \{1, \ldots, k\}$, optimization is possible only from entries in $\bar{E}_{ij}$'s. In other words, optimization is possible only over interconnections. Solving Problem 2.3 hence solves the optimal essential interconnection selection problem.

2.2. Related work

Structural analysis was introduced by Lin in Lin (1974). Since then a wide range of problems associated with structural controllability is studied (see Liu & Barabási, 2016 and references therein). Papers (Liu et al., 2011; Pequito et al., 2017) are representatives which show application of these concepts on diverse class of real networks including electronic circuits, neural network, social communication network, power grid and food web. Structural analysis of composite systems is previously addressed in literature (see Anderson & Hong, 1982; Carvalho, Pequito, Aguiar, Kar, & Johansson, 2017; Davison, 1977; Li, Xi, & Zhang, 1996; Rech & Perret, 1991; Yang & Zhang, 1995 and references therein). These papers devised various conditions for checking structural controllability of composite systems in terms of the subsystems given. More recently, Carvalho et al. (2017) gave a distributed algorithm to verify structural controllability of a composite system. This paper deals with *optimizing* the set of interconnections for a system composed of several heterogeneous subsystems. In other words, our aim is to find a set of interconnections, whose cardinality is the least possible, such that the composite system is structurally controllable.

3. Review of essential graph theoretic results and graphical representations

This section briefly describes few graph theoretic concepts (Diestel, 2000), some existing results associated with structural controllability (Liu & Barabási, 2016) and few graph constructions used in the sequel.

3.1. Review of essential graph theoretic results

A system is said to be *controllable* if it is possible to drive the state of the system from arbitrary initial state to any desired state in finite time by applying appropriate input. For a system to be controllable, it is essential that every state is influenced by some input. In order to represent the influences of states and inputs on each state we analyze a structured system $(\bar{A}, \bar{B})$[2] using a digraph $\mathcal{D}(\bar{A}, \bar{B})$. Here $\bar{A} \in \{0, \star\}^{n \times n}$ and $\bar{B} \in \{0, \star\}^{n \times m}$. Now $\mathcal{D}(\bar{A}, \bar{B})$ is constructed as follows: firstly on account of the interactions between states the digraph $\mathcal{D}(\bar{A})$ is constructed with vertex set V_X and edge set E_X. Here, $V_X = \{x_1, \ldots, x_n\}$ and $(x_j, x_i) \in E_X$ if $\bar{A}_{ij} = \star$. If $(x_j, x_i) \in E_X$, then we say that state x_j directly influences state x_i. Similarly, on account of the interaction between inputs and states we construct the digraph $\mathcal{D}(\bar{A}, \bar{B})$, with vertex set $V_X \cup V_U$ and edge set $E_X \cup E_U$. Here $V_U = \{u_1, \ldots, u_m\}$ and edge $(u_j, x_i) \in E_U$ if $\bar{B}_{ij} = \star$. If $(u_j, x_i) \in E_U$, then we say that input u_j directly influences state x_i.

State x_j is said to be *accessible* if there exists a directed path in $\mathcal{D}(\bar{A}, \bar{B})$ from some input node u_i to x_j. An alternate method for checking if all states are accessible is by using a concept of *strong connectedness* of the graph. A digraph is said to be strongly connected if for each ordered pair of vertices (v_i, v_j) there exists a path from v_i to v_j. *Strongly connected component* (SCC) is a maximal strongly connected subgraph of a digraph that is strongly connected and is not properly contained in any other subgraph that is strongly connected. Thus all x_i's are accessible *if and only if* all SCCs of $\mathcal{D}(\bar{A})$ are accessible.

For structural controllability, in addition to accessibility the notion of *no-dilation* is also necessary. Digraph $\mathcal{D}(\bar{A}, \bar{B})$ is said to have dilation, if there exists a node set $S \subset V_X$ whose neighborhood node set $T(S)$ (where $v \in T(S)$, if there exists a directed edge from v to a node in S) satisfies $|T(S)| < |S|$. Here, $S \subset V_X$ and $T(S) \subset V_X \cup V_U$. Presence of dilations in $\mathcal{D}(\bar{A}, \bar{B})$ can be checked using a matching condition on the system bipartite graph denoted as $\mathcal{B}(\bar{A}, \bar{B})$. Define a bipartite graph $G_B := ((V, \widetilde{V}), E)$, where the vertex set satisfies $V \cap \widetilde{V} = \emptyset$ and the edge set satisfies $E \subseteq V \times \widetilde{V}$. Matching, perfect matching and minimum cost perfect matching in G_B are defined below.

Definition 3.1 (*Diestel, 2000*)**.** A matching M in a bipartite graph $G_B = ((V, \widetilde{V}), E)$ is a collection of edges $M \subseteq E$ such that no two edges in the collection share the same endpoint. In other words, if $\{(i, j), (\underline{w}, v)\} \in M$, then $i \neq w$ and $j \neq v$, where $i, w \in V$ and $j, v \in \widetilde{V}$. A matching M is said to be a perfect matching of G_B if $|M| = \min(|V|, |\widetilde{V}|)$. Further, given G_B and a cost function c from the set E to the set of non-negative real numbers $\mathbb{R}_+$, the cost of a matching M is defined as, $c(M) = \sum_{e \in M} c(e)$. Then a minimum cost perfect matching is a perfect matching M such that $c(M) \leqslant c(M')$, where M' is any perfect matching of G_B.

To establish the equivalent matching condition for the no-dilation condition, now we explain the construction of $\mathcal{B}(\bar{A}, \bar{B})$ which includes two stages. In the first stage, we construct the *state bipartite graph* $\mathcal{B}(\bar{A})$ with vertex set $(V_{X'}, V_X)$ and edge set $\mathcal{E}_X$. Here, $V_{X'} = \{x'_1, x'_2, \ldots, x'_n\}$, $V_X = \{x_1, x_2, \ldots, x_n\}$ and $(x'_j, x_i) \in \mathcal{E}_X \Leftrightarrow (x_i, x_j) \in E_X$. We extend this to the *system bipartite graph* $\mathcal{B}(\bar{A}, \bar{B})$ with vertex set given by $(V_{X'}, V_X \cup V_U)$ and edge set given by $\mathcal{E}_X \cup \mathcal{E}_U$. Here, $V_U = \{u_1, u_2, \ldots, u_m\}$ and $(x'_j, u_i) \in \mathcal{E}_U \Leftrightarrow (u_i, x_j) \in E_U$. Using $\mathcal{B}(\bar{A}, \bar{B})$ the following result holds.

Proposition 3.2 (*Olshevsky, 2015, Theorem 2*)**.** *The digraph $\mathcal{D}(\bar{A}, \bar{B})$ of the structured system $(\bar{A}, \bar{B})$ has no dilation if and only if the bipartite graph $\mathcal{B}(\bar{A}, \bar{B})$ has a perfect matching.*

Using accessibility condition and no-dilation conditions, Lin proved the following result for structural controllability.

Proposition 3.3 (*Lin, 1974, pp. 207*)**.** *The structured system $(\bar{A}, \bar{B})$ is structurally controllable if and only if the associated digraph $\mathcal{D}(\bar{A}, \bar{B})$ has no inaccessible states and has no dilations.*

Alternatively, a structured system is said to be structurally controllable if and only if all states are accessible by some input and there exists a perfect matching in $\mathcal{B}(\bar{A}, \bar{B})$.

[1] Although $\|\cdot\|_0$ does not satisfy some of the norm axioms, the number of nonzero entries in a matrix is conventionally referred to as the zero norm.

[2] Typical structured system is denoted by $(\bar{A}, \bar{B})$ and the related concepts can be extended to specific system under consideration.

3.2. Graphical representations in composite system

In this subsection, we describe few graphical constructions and notations used in the sequel. For every subsystem S_i with structured state matrix $\bar{A}_i$, we construct the state digraph $\mathcal{D}(\bar{A}_i)$ with vertex set V_{X_i} and edge set E_{X_i}. Here, $V_{X_i} = \{x^i_1, \ldots, x^i_{n_i}\}$ and $(x^i_q, x^i_p) \in E_{X_i}$ if $\bar{A}_{i_{pq}} = \star$. We assume that each subsystem is irreducible, i.e., state digraph $\mathcal{D}(\bar{A}_i)$ is an SCC, for $i = 1, \ldots, k$. Consequently, if one state in a subsystem is accessible, then the whole subsystem is accessible. However, for many subsystems $\bar{B}_i$'s will be a zero matrix, as in a multi-agent system where only few agents receive input. Now we define system digraph of S_i as $\mathcal{D}(\bar{A}_i, \bar{B}_i)$, with vertex set $V_{X_i} \cup V_{U_i}$ and edge set $E_{X_i} \cup E_{U_i}$. Here, $V_{U_i} = \{u^i_1, \ldots, u^i_{m_i}\}$ and $(u^i_q, x^i_p) \in E_{U_i}$ if $\bar{B}_{i_{pq}} = \star$. The state bipartite graph of subsystem $\bar{A}_i$ is denoted as $\mathcal{B}(\bar{A}_i)$ with vertex set $(V_{X'_i}, V_{X_i})$ and edge set $\mathcal{E}_{X_i}$. Here, $V_{X'_i} = \{x'^i_1, \ldots x'^i_{n_i}\}$ and $(x'^i_q, x^i_p) \in \mathcal{E}_{X_i} \Leftrightarrow (x^i_p, x^i_q) \in E_{X_i}$. Further, the bipartite graph $\mathcal{B}(\bar{A}_i, \bar{B}_i)$ is defined with vertex set $(V_{X'i}, V_{X_i} \cup V_{U_i})$ and edge set $\mathcal{E}_{X_i} \cup \mathcal{E}_{U_i}$, where $(x'^i_p, u^i_q) \in \mathcal{E}_{U_i} \Leftrightarrow (u^i_q, x^i_p) \in E_{U_i}$.

We assume that all states of the jth subsystem can connect to all states of the ith subsystem, for $i \neq j$. In other words, for any subsystem S_i, there is no restriction on the set of interconnections through which S_i can connect to other subsystems. The possible set of interconnection edges, $E_{\mathcal{I}}$, consists of $(x^i_p, x^j_q) \in E_{\mathcal{I}}$ for all $i, j \in \{1, \ldots, k\}$, $i \neq j$, $x^i_p \in V_{X_i}$ and $x^j_q \in V_{X_j}$. Now we define the digraph $\mathcal{D}(\bar{A}_T)$ with vertex set $\cup_{i=1}^k V_{X_i}$ and edge set $\cup_{i=1}^k E_{X_i} \cup E_{\mathcal{I}}$. $\mathcal{D}(\bar{A}_T)$ includes all k subsystems along with all possible interconnections. The bipartite graph of the composite system $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ is defined with vertex set $(\cup_{i=1}^k V_{X'_i}, \cup_{i=1}^k V_{X_i} \cup \cup_{i=1}^k V_{U_i})$ and edge set $\cup_{i=1}^k \mathcal{E}_{X_i} \cup \cup_{i=1}^k \mathcal{E}_{U_i} \cup \mathcal{E}_{\mathcal{I}}$, where $(x'^i_p, x^j_q) \in \mathcal{E}_{\mathcal{I}} \Leftrightarrow (x^j_q, x^i_p) \in E_{\mathcal{I}}$.

We describe briefly the graph theoretic reasons that cause Problem 2.3 to not be straightforward. Notice that if there exists a perfect matching in $\mathcal{B}(\bar{A}_i, \bar{B}_i)$, for $i = 1, \ldots, k$, then there exists a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ without using any interconnection edge. In other words, the composite system has no dilation even without using any interconnection edge. In that case, a minimum cardinality set of interconnections that guarantees accessibility can be obtained by running a minimum weight spanning tree algorithm (Davison, 1977). On the other hand, if $\bar{B}_i \neq 0$, for all $i = 1, \ldots, k$, then the composite system is accessible without using any interconnection edge (since all subsystems are irreducible). In such a case a minimum cardinality set of interconnections that guarantees the no-dilation condition can be obtained using a minimum cost perfect matching algorithm on $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ with nonzero cost only on the interconnection edges. However, when some of the subsystems are such that their bipartite graphs denoted by $\mathcal{B}(\bar{A}_i, \bar{B}_i)$ do not have a perfect matching and some subsystems are such that their input matrix $\bar{B}_j = 0$, then solving Problem 2.3 is not straightforward. We analyze the complexity of Problem 2.3 and describe our approach to solve Problem 2.3 in the next section.

4. Algorithm and results when no subsystem has perfect matching in its state bipartite graph

In this section, we propose a polynomial time algorithm to solve Problem 2.3 when every subsystem is irreducible and no subsystem has a perfect matching in its state bipartite graph. In other words, we give a polynomial time algorithm to solve Problem 2.3 on a set of structured subsystems $S_1, \ldots, S_k$, where for all $i \in \{1, \ldots, k\}$, $\mathcal{D}(\bar{A}_i)$ is an SCC and $\mathcal{B}(\bar{A}_i)$ has no perfect matching. If subsystems are irreducible and if $\mathcal{B}(\bar{A}_i)$ has a perfect matching, for all $i \in \{1, \ldots, k\}$, and $\bar{B}_i$'s are nonzero matrices, for all $i \in \{1, \ldots, k\}$, then the composite system is structurally controllable without using any interconnections. However, in practice only few subsystems receive input due to which $\bar{B}_i$ is a zero matrix for many values of $i \in \{1, \ldots, k\}$. Interconnections are essential for achieving accessibility. Further, $\mathcal{B}(\bar{A}_i)$'s for all $i \in \{1, \ldots, k\}$ may not have a perfect matching. However, some subsystems may have a perfect matching and this case is considered in Section 5. The algorithm proposed in this section is based on the bipartite graph denoted by $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ constructed using the subsystems $(\bar{A}_1, \bar{B}_1), \ldots, (\bar{A}_k, \bar{B}_k)$ and the interconnection edge set $E_{\mathcal{I}}$ defined above. Define cost vector

$$c_W(e) := \begin{cases} 1, \text{ for } e \in \cup_{i=1}^k \mathcal{E}_{X_i}, \\ 0, \text{ for } e \in \cup_{i=1}^k \mathcal{E}_{U_i}, \\ 2, \text{ for } e \in \mathcal{E}_{\mathcal{I}}. \end{cases} \tag{4}$$

We explain the algorithm and results for the single input case and later explain how to extend these to the multi-input case (see Section 6.2). Without loss of generality, we assume that this single input corresponds to the first subsystem. Thus $\cup_{i=1}^k U_i = u^1_1$. For notational brevity, we denote u^1_1 as u_1. For $\mathcal{B}(\bar{A}_T, \bar{B}_T)$, using cost vector c_W the following results hold.

Lemma 4.1. *Consider the composite structured system $(\bar{A}_T, \bar{B}_T)$ obtained by composing k irreducible subsystems with all possible interconnections and let $\bar{B}_T$ be a single input with input node u_1. Let $M^\star_W$ be a minimum cost perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ under cost vector c_W given in Eq. (4). Then, $(x'^j_i, u_1) \in M^\star_W$ for some $i \in \{1, \ldots, n_j\}$ and some $j \in \{1, \ldots, k\}$.*

Proof. Given $M^\star_W$ is an optimum perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ and $\bar{B}_T$ is a single input. We prove the result using a contradiction argument. Suppose $(x'^j_i, u_1) \notin M^\star_W$ for all $i \in \{1, \ldots, n_j\}$ and $j \in \{1, \ldots, k\}$. Since $M^\star_W$ is a perfect matching, there exists an edge $(x'^j_i, x^t_r) \in M^\star_W$ for some node x^t_r. Construct a new matching M'_W by breaking the edge (x'^j_i, x^t_r) and making the edge (x'^j_i, u_1), i.e., $M'_W = \{M^\star_W \setminus (x'^j_i, x^t_r)\} \cup \{x'^j_i, u_1\}$. Notice that $c_W(M'_W) < c_W(M^\star_W)$. This contradicts the assumption that $M^\star_W$ is an optimum perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ and this implies $(x'^j_i, u_1) \in M^\star_W$. □

Note that the above result holds for the multi-input case also. In a multi-input case, any optimum matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ has all the input nodes matched. For any matching M_W in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ define $\mathcal{D}(M_W) := (V_{M_W}, E_{M_W})$, where $V_{M_W} = \{\cup_{i=1}^k U_i, S_1, \ldots, S_k\}$, $(u^i_p, S_i) \in E_{M_W}$ for all $i \in \{1, \ldots, k\}$ and for some p, and $(S_j, S_i) \in E_{M_W}$ if $(x'^i_p, x^j_q) \in M_W$. The vertices in $\mathcal{D}(M_W)$ are partitioned into two sets: $\widetilde{S}(M_W)$ and $S \setminus \widetilde{S}(M_W)$, where $\widetilde{S}(M_W) = \{S_i : S_i$ is input accessible in $\mathcal{D}(M_W)\} \cup \{\cup_{i=1}^k U_i\}$. The following three results crucially help in showing a certain cut is empty, which helps our main result on optimality.

Lemma 4.2. *Consider the composite structured system $(\bar{A}_T, \bar{B}_T)$ obtained by composing k irreducible subsystems with all possible interconnections. Let M_W be a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. Then, for any $S_i \in \widetilde{S}(M_W)$ and for any $S_j \in S \setminus \widetilde{S}(M_W)$ the subsystem digraph $\mathcal{D}(M_W)$ satisfies: $(S_i, S_j) \notin E_{M_W}$.*

Proof. We prove the above claim using a contradiction argument. Suppose there exists an edge $(S_i, S_j) \in E_{M_W}$, where $S_i \in \widetilde{S}(M_W)$ and $S_j \in S \setminus \widetilde{S}(M_W)$. Then, since $S_i \in \widetilde{S}(M_W)$ and all subsystems are irreducible, $S_j \in \widetilde{S}(M_W)$. This contradicts the assumption that $S_j \in S \setminus \widetilde{S}(M_W)$ and hence proves the result. □

Lemma 4.3. *Consider the composite structured system $(\bar{A}_T, \bar{B}_T)$ obtained by composing k irreducible subsystems with all possible interconnections and let $\bar{B}_T$ be a single input. Let $M^\star_W$ be a minimum cost perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ using cost vector c_W. Then, (a) there exists exactly one right unmatched node, say x^t_r of subsystem S_t, and (b) further, $S_t \in \widetilde{S}(M^\star_W)$.*

Proof. (a) part: Given $M_W^\star$ is an optimum perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ is a bipartite graph with n_T nodes on the left side and $n_T + 1$ nodes on the right side, one extra node being the input node u_1. Therefore, $|M_W^\star| = n_T$. By definition $u_1 \in \widetilde{S}(M_W^\star)$ and by Lemma 4.1 input node u_1 is matched in $M_W^\star$. Since every subsystem has equal number of state nodes on both the sides of $\mathcal{B}(\bar{A}_T, \bar{B}_T)$, corresponding to some subsystem, say S_t, there exists a right unmatched state node, say x_r^t, in $M_W^\star$.

(b) part: Now we need to show that $S_t \in \widetilde{S}(M_W^\star)$. In other words, the subsystem to which the unmatched node belongs to is accessible in the digraph $\mathcal{D}(M_W^\star)$. By Lemma 4.1, all optimum perfect matchings in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ consist of an edge $(x_i'^j, u_1)$ for some node $x_i'^j$. Let $(x_i'^j, u_1) \in M_W^\star$ and the node x_i^j is accessible in the specified digraph. Now in the matching $M_W^\star$, the node x_i^j satisfies one of the following: (a) x_i^j is unmatched, or (b) x_i^j is matched. In case (a), $x_i^j = x_r^t$, $S_t = S_j$ and proof follows. In case (b), let $(x_q'^p, x_i^j) \in M_W^\star$. Then the node x_q^p is accessible. Note that all subsystems are irreducible and hence $S_p \in \widetilde{S}(M_W^\star)$. Now recursively using the same argument as before, we can say that the subsystem to which the unmatched node in $M_W^\star$ belongs to is input accessible in $\mathcal{D}(M_W^\star)$. □

For the multi-input case, along similar lines, one can show that there are m_T right unmatched nodes in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ with respect to any optimum matching and all those m_T state nodes are input accessible, where m_T is the total number of inputs.

Lemma 4.4. *Consider a set of irreducible structured subsystems $(\bar{A}_1, \bar{B}_1), \ldots, (\bar{A}_k, \bar{B}_k)$. Let M_W be any perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ and let $\mathcal{D}(M_W)$ be the corresponding subsystem digraph with partitions $\widetilde{S}(M_W)$ and $S \setminus \widetilde{S}(M_W)$ respectively. Then, $S_j \in S \setminus \widetilde{S}(M_W)$, implies $\bar{B}_j = 0$.*

Proof. We assume $S_j \in S \setminus \widetilde{S}(M_W)$. This implies subsystem S_j is inaccessible. Assume to the contrary, $\bar{B}_j \neq 0$. Then, since $\mathcal{D}(\bar{A}_j)$ is irreducible, S_j is accessible and hence $S_j \in \widetilde{S}(M_W)$. Thus $S_j \in S \setminus \widetilde{S}(M_W)$ implies $\bar{B}_j = 0$. This is a contradiction and proof follows. □

Lemmas 4.2, 4.4, and 4.5 together prove that corresponding to a perfect matching M_W in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$, the cut of sets $\widetilde{S}(M_W)$ and $S \setminus \widetilde{S}(M_W)$ of the subsystem digraph $\mathcal{D}(M_W)$ is empty.

Lemma 4.5. *Consider the composite structured system $(\bar{A}_T, \bar{B}_T)$ obtained by composing k irreducible subsystems with all possible interconnections. Let M_W be a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. Then, for any $S_i \in \widetilde{S}(M_W)$ and for any $S_j \in S \setminus \widetilde{S}(M_W)$, the subsystem digraph $\mathcal{D}(M_W)$ satisfies: $(S_j, S_i) \notin E_{M_W}$.*

Proof. We prove the above claim using a contradiction argument. Suppose there exists an edge $(S_j, S_i) \in E_{M_W}$, where $S_i \in \widetilde{S}(M_W)$ and $S_j \in S \setminus \widetilde{S}(M_W)$. Note that by Lemma 4.4 the number of nodes in the bipartite graph $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ corresponding to subsystems in $S \setminus \widetilde{S}(M_W)$ is equal on both the sides. By Lemma 4.2, there exists no edge from some $S_p \in \widetilde{S}(M_W)$ to some $S_q \in S \setminus \widetilde{S}(M_W)$. As a result, if $(S_j, S_i) \in E_{M_W}$, where $S_i \in \widetilde{S}(M_W)$ and $S_j \in S \setminus \widetilde{S}(M_W)$, at least one left node of $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ corresponding to some subsystem $S_v \in S \setminus \widetilde{S}(M_W)$ is unmatched in M_W. This contradicts the assumption that M_W is a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ and hence $(S_j, S_i) \notin E_{M_W}$. □

From Lemmas 4.2 and 4.5 we conclude that corresponding to a perfect matching M_W in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$, there are no edges between the vertex sets $\widetilde{S}(M_W)$ and $S \setminus \widetilde{S}(M_W)$ in $\mathcal{D}(M_W)$. Algorithm 4.1 solves Problem 2.3 for composite system composed of irreducible subsystems with no perfect matching in $\mathcal{B}(\bar{A}_i)$, for all $i \in \{1, \ldots, k\}$.

Algorithm 4.1 Pseudo-code for solving Problem 2.3 on structured subsystems when all subsystems are irreducible and no subsystem has perfect matching in its state bipartite graph

Input: Structured subsystems $(\bar{A}_1, \bar{B}_1), \ldots, (\bar{A}_k, \bar{B}_k)$
Output: Interconnections $E_{\mathcal{I}_A}$

1: Define the bipartite graph $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ as $((\cup_{i=1}^k V_{X_i'}, \cup_{i=1}^k V_{X_i} \cup \cup_{i=1}^k V_{U_i}), \cup_{i=1}^k \mathcal{E}_{X_i} \cup \cup_{i=1}^k \mathcal{E}_{U_i} \cup \mathcal{E}_{\mathcal{I}})$
2: Find minimum cost maximum matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ using c_W given in Eq. (4), say $M_W^\star$
3: For any matching M_W in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$, $\mathcal{D}(M_W) := (V_{M_W}, E_{M_W})$, where $V_{M_W} = \{\cup_{i=1}^k U_i, S_1, \ldots, S_k\}$, $(u_p^i, S_i) \in E_{M_W}$ for all $i \in \{1, \ldots, k\}$ and some p and $(S_j, S_i) \in E_{M_W}$ if $(x'^i_p, x^j_q) \in M_W$
4: $\widetilde{S}(M_W) \leftarrow \{S_i : S_i \text{ is input accessible in } \mathcal{D}(M_W)\} \cup \{\cup_{i=1}^k U_i\}$
5: **for** $S \setminus \widetilde{S}(M_W^\star) \neq \emptyset$ **do**
6: Find a right unmatched node in $M_W^\star$, say x_r^t
7: For $S_i, S_j \in S \setminus \widetilde{S}(M_W^\star)$ and $(S_j, S_i) \in E_{M_W^\star}$, find edge $(x'^i_p, x^j_q) \in M_W^\star$
8: $M_W' \leftarrow \{M_W^\star \setminus (x'^i_p, x^j_q)\} \cup \{(x'^i_p, x^t_r)\}$
9: $S_{\rm new} \leftarrow S_i \cup \{S_g : S_g \text{ is reachable from } S_i \text{ in } \mathcal{D}(M_W^\star)\}$
10: $M_W^\star \leftarrow M_W'$
11: $\widetilde{S}(M_W^\star) \leftarrow \widetilde{S}(M_W^\star) \cup S_{\rm new}$
12: **end for**
13: $E_{\mathcal{I}_A} \leftarrow \{(x^i_p, x^j_q) : (x'^j_q, x^i_p) \in M_W^\star \text{ and } i \neq j\}$

We provide description of why the steps in Algorithm 4.1 indeed solve Problem 2.3. The optimality aspect is proved in Theorem 4.6.

Steps 1–2: Given a set of structured subsystems we first construct the bipartite graph $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ (Step 1) and subsequently solve the minimum cost perfect matching on it using the cost vector c_W defined in Eq. (4). Let $M_W^\star$ be an optimum matching obtained (Step 2). Notice that $M_W^\star \cap \mathcal{E}_{\mathcal{I}}$ consists of the minimum number of interconnection edges to satisfy the 'no-dilation' condition. However, these interconnection edges may not make all subsystems accessible. Hence the idea is to reconstruct the matching $M_W^\star$, keeping the cardinality of $M_W^\star \cap \mathcal{E}_{\mathcal{I}}$ same, to obtain a final matching such that it is a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ and all subsystems become input accessible.

Steps 3–6: Let $S = \{S_1, \ldots, S_k\}$. Corresponding to a matching M_W we define the subsystem digraph $\mathcal{D}(M_W) := (V_{M_W}, E_{M_W})$. Note that here $\cup_{i=1}^k U_i = u_1$. This implies $V_{M_W} = \{u_1, S_1, \ldots, S_k\}$, $(u_1, S_1) \in E_{M_W}$ and $(S_j, S_i) \in E_{M_W}$ if $(x'^i_p, x^j_q) \in M_W$ for some $p \in \{1, \ldots, n_i\}$ and $q \in \{1, \ldots, n_j\}$ (Step 3). The vertex set V_{M_W} is now partitioned into two sets $\widetilde{S}(M_W)$ and $S \setminus \widetilde{S}(M_W)$. Here $\widetilde{S}(M_W)$ consists of input node u_1 and all subsystems that are input accessible in $\mathcal{D}(M_W)$ (Step 4). Notice that $S_1 \in \widetilde{S}(M_W)$. Since $M_W^\star$ is an optimum matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$, by Lemma 4.3 there exists a unique right unmatched vertex x_r^t in $M_W^\star$ (Step 6). Also, by Lemma 4.3 the node x_r^t is input accessible. Thus, $S_t \in \widetilde{S}(M_W^\star)$.

Consider the digraph $\mathcal{D}(M_W^\star) = (V_{M_W^\star}, E_{M_W^\star})$, corresponding to the matching $M_W^\star$. By Lemmas 4.2 and 4.5 there exists no edge in $E_{M_W^\star}$ from a vertex in $\widetilde{S}(M_W^\star)$ to a vertex in $S \setminus \widetilde{S}(M_W^\star)$ and vice-versa. The cut corresponding to the partitioning $\widetilde{S}(M_W^\star)$ and $S \setminus \widetilde{S}(M_W^\star)$ of the digraph $\mathcal{D}(M_W^\star)$ is empty. Further, by Lemma 4.4, for all $S_i \in \widetilde{S}(M_W^\star)$, the input matrix $\bar{B}_i = 0$. Hence $S_i \in S \setminus \widetilde{S}(M_W^\star)$ satisfies one of the following: (a) S_i is an isolated vertex in $\mathcal{D}(M_W^\star)$, or (b) S_i connects to some $S_j \in S \setminus \widetilde{S}(M_W^\star)$ in $\mathcal{D}(M_W^\star)$. Case (a) does not exist since by assumption $\mathcal{B}(\bar{A}_i)$ does not have a perfect matching, for $i \in \{1, \ldots, k\}$. Hence we consider only case (b). By Lemma 4.4, all subsystems in $S \setminus \widetilde{S}(M_W^\star)$ have equal number of nodes on both the sides of the bipartite graph $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ (since their input matrices are zero matrices) and by assumption no $\mathcal{B}(\bar{A}_i)$ has a perfect matching. Therefore, each $S_i \in S \setminus \widetilde{S}(M_W^\star)$ has both incoming and outgoing edges in $\mathcal{D}(M_W^\star)$. In other words, for every $S_i \in S \setminus \widetilde{S}(M_W^\star)$ there exist $(S_j, S_i) \in E_{M_W^\star}$ and $(S_i, S_r) \in E_{M_W^\star}$, where $S_j, S_r \in S \setminus \widetilde{S}(M_W^\star)$. However, for $(S_j, S_i) \in E_{M_W^\star}$, the corresponding interconnection edge caters only no-dilation condition and not accessibility.

Steps 7–13: Our focus is to reconstruct these interconnections by breaking them and making new ones such that the new interconnections cater both the no-dilation condition and the accessibility condition. For achieving this, we first identify sets $S_j, S_i \in S \setminus \widetilde{S}(M_W^\star)$ that have interconnection edge $(x'^i_p, x^j_q) \in M_W^\star$ (Step 7). Now we construct a matching $M'_W = \{M_W^\star \setminus (x'^i_p, x^j_q)\} \cup \{(x'^i_p, x^t_r)\}$, where $S_t \in \widetilde{S}(M_W^\star)$ (Step 8). Notice that M'_W is also an optimum matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ with $c_W(M'_W) = c_W(M_W^\star)$ and the number of interconnections remains the same as in $M_W^\star$. By this reconstruction process subsystem S_i becomes accessible. Moreover, all subsystems that are reachable from S_i in $D(M_W^\star)$ are also accessible. We update the matching $M_W^\star$ and the set $\widetilde{S}(M_W^\star)$ (Steps 10 and 11). We recursively do this until $S \setminus \widetilde{S}(M_W^\star) = \emptyset$. The interconnection edges of the final optimum matching $M_W^\star$ are then given by $E_{\mathcal{I}_A}$ (Step 13 of Algorithm 4.1).

Let $E_{\mathcal{I}^\star}$ denote the optimum set of interconnections corresponding to an optimum solution $\bar{A}^\star$ of Problem 2.3. With slight abuse of notation, we refer to $E_{\mathcal{I}^\star}$ as an optimum solution to Problem 2.3. Now we prove the optimality of Algorithm 4.1.

Theorem 4.6. *Output of Algorithm 4.1, which takes as input a set of irreducible structured subsystems $(\bar{A}_i, \bar{B}_i)$, $i = 1, \ldots, k$, is an optimal solution to Problem 2.3, i.e., $|E_{\mathcal{I}_A}| = |E_{\mathcal{I}^\star}|$, where $|E_{\mathcal{I}^\star}|$ denotes the optimum number of interconnections that solves Problem 2.3.*

Proof. The optimum matching $M_W^\star$ obtained by solving the minimum cost perfect matching on $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ using cost vector c_W in Step 2 of Algorithm 4.1 consists of the least number of feedback edges to satisfy the no-dilation condition in the composite structured system. Let $E_{\mathcal{I}}(M_W^\star)$ be the interconnection edges corresponding to matching $M_W^\star$, i.e., $(x^i_p, x^j_q) \in E_{\mathcal{I}}(M_W^\star)$ if $(x'^j_q, x^i_p) \in M_W^\star$ and $i \neq j$. $|E_{\mathcal{I}}(M_W^\star)|$ is the minimum number of interconnections required to achieve the no-dilation condition and $|E_{\mathcal{I}^\star}| \geqslant |E_{\mathcal{I}}(M_W^\star)|$. However, $E_{\mathcal{I}}(M_W^\star)$ may not satisfy the accessibility of all the subsystems. Then, $S \setminus \widetilde{S}(M_W^\star) \neq \emptyset$. Hence in Step 10 of Algorithm 4.1 we obtain a new matching M'_W by breaking an interconnection edge and making another interconnection edge. The new interconnection made connects the unique unmatched node x^t_r in $M_W^\star$ to a state in some subsystem $S_i \in S \setminus \widetilde{S}(M_W^\star)$. By Lemma 4.3, x^t_r lies in a subsystem that is accessible and hence subsystem S_i also becomes accessible. Moreover, $c_W(M'_W) = c_W(M_W^\star)$ and the number of interconnections in M'_W and $M_W^\star$ are both equal. However, more number of subsystems are accessible in $\mathcal{D}(M'_W)$ when compared to $\mathcal{D}(M_W^\star)$. In every iteration of the algorithm, cardinality of the set of accessible subsystems increases.

Now we show that Algorithm 4.1 terminates in a finite number of iterations. Notice that at the start of the iterations the maximum cardinality of the set $S \setminus \widetilde{S}(M_W^\star)$ is at most $k-1$ (since $S_1 \in \widetilde{S}(M_W^\star)$). Also, indexed by the iteration count of Algorithm 4.1, the sequence of cardinality of the set $S \setminus \widetilde{S}(M_W^\star)$ is a monotonically strictly decreasing sequence. More precisely, we prove that the size decreases by two or more. By Lemmas 4.2 and 4.5 there are no edges between subsystems in $\widetilde{S}(M_W^\star)$ and $S \setminus \widetilde{S}(M_W^\star)$ in $\mathcal{D}(M_W^\star)$. Also by Lemma 4.3, the unique right unmatched node in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ lies in $\widetilde{S}(M_W^\star)$. Hence if a subsystem in $S \setminus \widetilde{S}(M_W^\star)$ has an incoming edge it must have an outgoing edge as well. Moreover, since no $\mathcal{B}(\bar{A}_i)$ has a perfect matching and $\bar{B}_j = 0$ for all $S_j \in \widetilde{S}(M_W^\star)$, every subsystem in $S \setminus \widetilde{S}(M_W^\star)$ has an incoming and an outgoing edge in $\mathcal{D}(M_W^\star)$. In every iteration the cardinality of the set $S \setminus \widetilde{S}(M_W^\star)$ decreases at least by 2. Algorithm 4.1 terminates in finite number of iterations with the same number of interconnections. Also, these interconnections correspond to an optimum perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ and all subsystems are accessible and $|E_{\mathcal{I}^\star}| \leqslant |E_{\mathcal{I}_A}|$. Since $|E_{\mathcal{I}^\star}| \geqslant |E_{\mathcal{I}_A}|$ and $|E_{\mathcal{I}^\star}| \leqslant |E_{\mathcal{I}_A}|$, combining both, we get $|E_{\mathcal{I}^\star}| = |E_{\mathcal{I}_A}|$ and this completes the proof. □

Now we give the complexity result of Algorithm 4.1.

Theorem 4.7. *Algorithm 4.1 which takes as input irreducible structured subsystems $(\bar{A}_1, \bar{B}_1), \ldots, (\bar{A}_k, \bar{B}_k)$ of state dimensions $n_1, \ldots, n_k$ respectively and gives as output an optimum set of interconnection edges, $E_{\mathcal{I}_A}$, has complexity $O(n_T^3)$, where $n_T = \sum_{i=1}^k n_i$.*

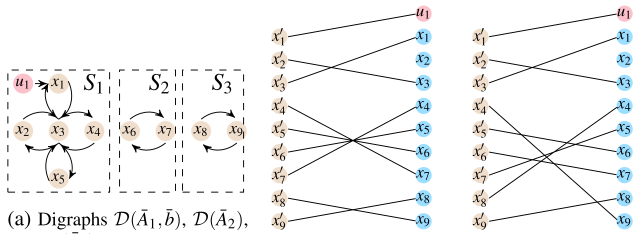
Proof. Constructing the bipartite graph $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ and solving the minimum cost perfect matching problem has complexity $O(n_T^3)$, where $n_T = \sum_{i=1}^k n_i$. Construction of $\mathcal{D}(M_W^\star)$ and finding partitions $\widetilde{S}(M_W^\star)$ and $S \setminus \widetilde{S}(M_W^\star)$ has $O(n_T)$ complexity. Also each iteration is of linear complexity and there are at most $\lfloor \frac{k-1}{2} \rfloor$ iterations. Since $k = O(n_T)$, the total complexity of the iterations is $O(n_T^2)$. Finally finding the interconnections $E_{\mathcal{I}_A}$ is linear in n_T. Complexity of Algorithm 4.1 is $O(n_T^3)$. □

This concludes the discussion on solving Problem 2.3 on irreducible subsystems when no subsystem has a perfect matching in its state bipartite graph. However, if some subsystems have a perfect matching in their state bipartite graph, then the analysis is more involved. We discuss this case in the next section.

5. Algorithm and results when some subsystems have perfect matching in their state bipartite graphs

In this section, we discuss the case where every subsystem is irreducible and some subsystems have a perfect matching in their state bipartite graphs. Algorithm 4.1 may not terminate in this case and then may not give an optimal solution to Problem 2.3. The key idea in Algorithm 4.1 is the reconstruction of the interconnections in an optimal matching $M_W^\star$ of $\mathcal{B}(\bar{A}_T, \bar{B}_T)$, by keeping the number of interconnections unaltered, such that all subsystems become accessible. Note that in $\mathcal{D}(M_W^\star)$ all subsystems $S_i \in S \setminus \widetilde{S}(M_W^\star)$ have corresponding input matrix $\bar{B}_i = 0$. Otherwise, since all subsystems are irreducible $S_i \in \widetilde{S}(M_W^\star)$. Also note that in Algorithm 4.1 we assume that no subsystem has a perfect matching in its state bipartite graph and based on this assumption the reconstruction process crucially uses the fact that in $\mathcal{D}(M_W^\star)$ all subsystems in $S \setminus \widetilde{S}(M_W^\star)$ have at least one incoming edge (Step 7). However, when some subsystems have a perfect matching in their $\mathcal{B}(\bar{A}_i)$'s, it is not necessary that the optimum matching $M_W^\star$ obtained has at least one interconnection edge for every $S_i \in S \setminus \widetilde{S}(M_W^\star)$. In other words, $\mathcal{D}(M_W^\star)$ can contain isolated nodes. In this case, Algorithm 4.1 cannot be used.

We illustrate this through an example in Fig. 1a. For the structured subsystems considered, S_2 and S_3 have perfect matchings in $\mathcal{B}(\bar{A}_2)$ and $\mathcal{B}(\bar{A}_3)$ respectively. Fig. 1 illustrates two optimum matchings $M_W^\star$ and $\hat{M}^\star$ in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ for the example given in Fig. 1a. Note that, $c_W(M_W^\star) = c_W(\hat{M}^\star)$. However, notice that in their respective subsystem digraphs given in Fig. 2, $\mathcal{D}(M_W^\star)$ has S_3 as an isolated node, but $\mathcal{D}(\hat{M}^\star)$ has no isolated nodes. Hence, when subsystems have perfect matching in their $\mathcal{B}(\bar{A}_i)$, it is not always possible to arrive at an optimal solution to Problem 2.3 using Algorithm 4.1. To this end, we propose another polynomial time algorithm to solve Problem 2.3 on irreducible subsystems when one or more subsystems have a perfect matching in their respective state bipartite graphs.

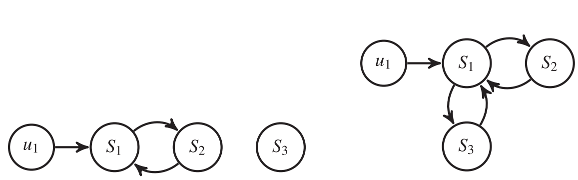


(a) Digraphs $\mathcal{D}(\bar{A}_1, \bar{b})$, $\mathcal{D}(\bar{A}_2)$, and $\mathcal{D}(\bar{A}_3)$ of structured subsystems S_1, S_2, and S_3, respectively. Subsystems S_2 and S_3 have perfect matching in their state bipartite graphs.

(b) Bipartite matching $M_W^\star$ in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ for the structured system given in Figure 1a

(c) Bipartite matching $\hat{M}^\star$ in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ for the structured system given in Figure 1a

Fig. 1. Figs. 1b and 1c demonstrate two different optimum matchings corresponding to the same subsystems given in Fig. 1a.



(a) Subsystem digraph $\mathcal{D}(M_W^\star)$ for the matching $M_W^\star$ given in Figure 1b

(b) Subsystem digraph $\mathcal{D}(\hat{M}^\star)$ for the matching $M_W^\star$ given in Figure 1c

Fig. 2. Illustrative figure demonstrating limitation of Algorithm 4.1 when some of the subsystems have perfect matching in their state bipartite graphs.

The proposed algorithm, Algorithm 5.1, consists of three stages. In Stage 1, we run Algorithm 4.1. Note that, for subsystems with no perfect matching Algorithm 4.1 terminates when $S \setminus \widetilde{S}(M_W^\star)$ is empty. In this case (when few subsystems have perfect matching in their state bipartite graph) we terminate Stage 1 when the optimum matching obtained in Algorithm 4.1, i.e., $M_W^\star$, in two consecutive iterations are the same. At the end of Stage 1 we get a partitioning of subsystems $\widetilde{S}(M_W^\star)$ and $S \setminus \widetilde{S}(M_W^\star)$, where every subsystem $S_i \in S \setminus \widetilde{S}(M_W^\star)$ is an isolated node in $\mathcal{D}(M_W^\star)$ and $\mathcal{B}(\bar{A}_i)$ has a perfect matching. Notice that at the end of this stage it is not possible to make an $S_i \in S \setminus \widetilde{S}(M_W^\star)$ accessible by reconstructing the interconnections as done before in Algorithm 4.1. In Stage 2 we run a 'stub-matching' algorithm. At the end of Stage 2 we update $M_W^\star$. Stage 2 is also executed without increasing the number of interconnections and it results in more subsystems being accessible. However, even after completing Stage 2 there can be some subsystems that are inaccessible. In Stage 3, we add extra interconnection edges, one each to every $S_i \in S \setminus \widetilde{S}(M_W^\star)$ from some $S_j \in \widetilde{S}(M_W^\star)$.

The pseudo-code of the proposed algorithm is presented in Algorithm 5.1. It involves three stages as elaborated below.

Stage 1 (Steps 1–2): In Stage 1, we run Algorithm 4.1 on the given system (Step 1). We terminate Stage 1 when Algorithm 4.1 gives the same $M_W^\star$ for two consecutive iterations. Note that at the instant of termination of Stage 1 the partition $\widetilde{S}(M_W^\star)$ and $S \setminus \widetilde{S}(M_W^\star)$ with respect to the matching $M_W^\star$ is such that $S \setminus \widetilde{S}(M_W^\star)$ consists of subsystems that has no interconnection edge connected to it (Step 2). In other words, subsystems in $S \setminus \widetilde{S}(M_W^\star)$ are isolated nodes in $\mathcal{D}(M_W^\star)$. We denote the number of isolated nodes in $\mathcal{D}(M_W^\star)$ by $t_{M_W^\star}$. In other words, the number of non-accessible subsystems is $t_{M_W^\star}$. Our aim in Stage 2 is to reconstruct the interconnections in $M_W^\star$, in a different way than in Algorithm 4.1, to get a new matching M_{new} such that the number of interconnections are the same in $M_W^\star$ and M_{new} and more subsystems are accessible. More precisely, we achieve $t_{M_{\text{new}}} < t_{M_W^\star}$ using the same number of interconnections as before. Finally we update $M_W^\star$ as M_{new}.

Algorithm 5.1 Pseudo-code for solving Problem 2.3 on structured subsystems when all subsystems are irreducible and some subsystems have perfect matching in their state bipartite graphs

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Input: Structured subsystems $(\bar{A}_1, \bar{B}_1), \ldots, (\bar{A}_k, \bar{B}_k)$
Output: Interconnections $E_{\mathcal{I}_B}$

Stage 1
1: Run Algorithm 4.1 on the structured system and terminate when $M_W^\star$ in two iterations are the same
2: Find the partition $\widetilde{S}(M_W^\star)$ and $S \setminus \widetilde{S}(M_W^\star)$

Stage 2
3: For $i \in \{1, \ldots, k\}$, find max matching in $\mathcal{B}(\bar{A}_i, \bar{B}_i)$, say M_i
4: Let $d_i \leftarrow |M_i|$
5: Define $\mathcal{J} \leftarrow \{j : \bar{B}_j = 0\}$
6: $\beta_i \leftarrow |\{(x'^i_p, x^j_q) \in M_W^\star : i \neq j\}|$
7: **for** $S \setminus \widetilde{S}(M_W^\star) \neq \emptyset$ and $\beta_i > \begin{cases} \max\{n_i - d_i, 1\}, \text{for i} \in \mathcal{J}, \\ (n_i - d_i), \text{otherwise.} \end{cases}$ **do**
8: Find $S_i \in \widetilde{S}(M_W^\star)$ s.t $\beta_i > \begin{cases} \max\{n_i - d_i, 1\}, \ i \in \mathcal{J}, \\ (n_i - d_i), \text{otherwise.} \end{cases}$
9: $\check{M}_i \leftarrow \{(x'^i_t, x^i_r), (x'^i_h, u^i_g) : (x'^i_t, x^i_r) \in M_W^\star, (x'^i_h, u^i_g) \in M_W^\star\}$
10: $\gamma_i \leftarrow |\check{M}_i|$
11: $\hat{M}_i \leftarrow \{(x'^i_t, x^j_r), (x'^j_v, x^i_w), (x'^i_h, u^i_g) \in M_W^\star : j \in \{1, \ldots, k\}\}$
12: $\mathcal{S}_\ell \leftarrow \{(x'^j_q, \cdot) (x'^j_q, x^i_p) \in M_W^\star$ and $i \neq j\}$
13: $\mathcal{S}_r \leftarrow \{(\cdot, x^j_q) : (x'^i_p, x^j_q) \in M_W^\star$ and $i \neq j\}$
14: Select $\widetilde{M}_i \subset M_i : |\widetilde{M}_i| = \gamma_i + 1$, where M_i is a perfect matching in $\mathcal{B}(\bar{A}_i, \bar{B}_i)$
15: $\widetilde{M} = \{M_W^\star \setminus \hat{M}_i\} \cup \{\widetilde{M}_i\}$
16: $L_i \leftarrow \{x'^i_p : x'^i_p$ is unmatched in $\widetilde{M}\}$
17: $R_i \leftarrow \{x^i_p : x^i_p$ is unmatched in $\widetilde{M}\}$
18: **while** $L_i \neq \emptyset$ and $\mathcal{S}_r \neq \emptyset$ **do**
19: $L_{\text{edges}} \leftarrow \{(x'^i_p, x^j_q) : x'^i_p \in L_i$ and $x^j_q \in \mathcal{S}_r\}$
20: $L_i \leftarrow L_i \setminus x'^i_p$
21: $\mathcal{S}_r \leftarrow \mathcal{S}_r \setminus x^j_q$
22: **end while**
23: **while** $R_i \neq \emptyset$ and $\mathcal{S}_\ell \neq \emptyset$ **do**
24: $R_{\text{edges}} \leftarrow \{(x'^j_q, x^i_p) : x^i_p \in R_i$ and $x'^j_q \in \mathcal{S}_\ell\}$
25: $R_i \leftarrow R_i \setminus x^i_p$
26: $\mathcal{S}_\ell \leftarrow \mathcal{S}_\ell \setminus x'^j_q$
27: **end while**
28: Choose an edge $(x'^h_a, x^h_b) \in \widetilde{M}$, where $S_h \in S \setminus \widetilde{S}(M_W^\star)$
29: Let $x^r_c \in \mathcal{S}_r$
30: **if** $\mathcal{S}_\ell \neq \emptyset$ **then**
31: Let $x'^t_d \in \mathcal{S}_\ell$
32: $M_W \leftarrow \{\widetilde{M} \setminus (x'^h_a, x^h_b)\} \cup \{(x'^h_a, x^r_c), (x'^t_d, x^h_b)\}$
33: **else**
34: $M_W \leftarrow \{\widetilde{M} \setminus (x'^h_a, x^h_b)\} \cup \{(x'^h_a, x^r_c)\}$
35: **end if**
36: $M_{\text{new}} \leftarrow \{M_W\} \cup \{L_{\text{edges}}\} \cup \{R_{\text{edges}}\}$
37: $M_W^\star \leftarrow M_{\text{new}}$
38: $\beta_i \leftarrow |\{(x'^i_p, x^j_q) \in M_W^\star : i \neq j\}|$
39: **end for**
40: **if** $S \setminus \widetilde{S}(M_W^\star) = \emptyset$ **then**

Stage 3
41: $E_{\mathcal{I}_B} \leftarrow \{(x^i_p, x^j_q) : (x'^j_q, x^i_p) \in M_W^\star$ and $i \neq j\}$
42: **else**
43: $E_{\mathcal{I}_B} \leftarrow \{(x^i_p, x^j_q) : (x'^j_q, x^i_p) \in M_W^\star$ and $i \neq j\} \cup \{(x^i_p, x^r_t) : S_r \in S \setminus \widetilde{S}(M_W^\star)$ and $S_i \in \widetilde{S}(M_W^\star)\}$
44: **end if**

Stage 2 (Steps 3–8): Now we begin Stage 2, i.e., the stub-matching algorithm. For all S_i's in $\widetilde{S}(M_W^\star)$, we find a maximum matching in

$\mathcal{B}(\bar{A}_i, \bar{B}_i)$ for $i = 1, \ldots, k$, say M_i. Let $|M_i| = d_i$ (Steps 3 and 4). The number of unmatched nodes in M_i for $S_i \in \widetilde{S}(M^\star_W)$ equals $n_i - d_i$. The bipartite graph $\mathcal{B}(\bar{A}_i, \bar{B}_i)$ has a perfect matching if and only if $d_i = n_i$. Since $M^\star_W$ is a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$, in $M^\star_W$ the number of left side nodes of S_i that are matched to right side nodes of some S_j's, $i \neq j$, is at least $n_i - d_i$. This is because in $M^\star_W$ few subsystems may end up with more than $n_i - d_i$ left side nodes matched to other subsystems, so as to form a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$.

Let β_i be the number of incoming interconnection edges corresponding to S_i in $M^\star_W$ (Step 6). The procedure is to appropriately choose those subsystems that have more number of incoming interconnections than $n_i - d_i$ and reconstruct the matching such that the number of interconnections in S_i decreases but new interconnections are added to the isolated subsystems. However, note that if $i \in \mathcal{J}$, where $\mathcal{J} := \{j \in \mathcal{J} : \bar{B}_j = 0\}$, then at least one interconnection is needed for the accessibility of subsystem S_i. For $i \in \mathcal{J}$, the condition is $\beta_i > \max\{n_i - d_i, 1\}$. Stage 2 is relevant if there exists one or more isolated subsystem in $\mathcal{D}(M^\star_W)$, i.e., a not accessible subsystem, and there exists one or more accessible subsystem that has $\beta_i > \max\{n_i - d_i, 1\}$, when $i \in \mathcal{J}$ and $\beta_i > (n_i - d_i)$, otherwise (Step 7).

The reconstruction in Stage 2 is done without altering the number of interconnections of Stage 1. We first identify a subsystem $S_i \in \widetilde{S}(M^\star_W)$ that satisfies the condition given in Step 8. There are two possible cases. In case (a), $i \in \mathcal{J}$. Note that for $i \in \mathcal{J}$, the condition is $\beta_i > \max\{n_i - d_i, 1\}$. Here, $\max\{n_i - d_i, 1\} \geqslant 1$ and hence $\beta_i > 1$. In this case, if $\beta_i = 1$, there is only one incoming interconnection edge and so breaking this edge will make S_i inaccessible (since $i \in \mathcal{J}$). Such subsystems do not participate in the reconstruction process. On the other hand, if $\beta_i > \max\{n_i - d_i, 1\}$ we construct a new matching $\widetilde{M}$ from $M^\star_W$. In case (b): $i \notin \mathcal{J}$. Then, the condition is $\beta_i > (n_i - d_i)$. Since accessibility is already achieved through input, if the number of interconnections is greater than $(n_i - d_i)$, we perform edge reconstruction.

Steps 9–13: The reconstruction of matching in Stage 2 is done as follows: let $\check{M}_i$ be the set of edges between nodes of S_i themselves in $M^\star_W$ and γ_i be its cardinality (Steps 9 and 10). If $\beta_i > n_i - d_i$, then $\gamma_i < d_i$. Hence, even though the size of maximum matching in S_i is d_i, some nodes in S_i are matched to nodes in some S_j for $i \neq j$ so that $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ has a perfect matching. Note that the identity of the nodes in the interconnections does not matter and hence one can reconstruct $M^\star_W$ into another perfect matching $M_{\rm new}$ in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ without changing the number of interconnections.

For this we first construct a new matching $\widetilde{M}$, where $\widetilde{M}$ is not a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$, such that $\widetilde{M}$ has $\gamma_i + 1$ edges whose both end points are nodes of S_i. For constructing $\widetilde{M}$, we first break all the edges in $M^\star_W$ associated with S_i (both interconnections and edges within), defined by $\hat{M}_i$ (Step 11). For $i \neq j$, let $(x'^i_p, x^j_q) \in M^\star_W$. After breaking the edge (x'^i_p, x^j_q), we retain a 'stub' (this is used in similar context in literature), i.e., a half-edge stemming from node x^j_q, denoted as $(\cdot, x^j_q)$. The set of right side stubs is defined by set $\mathcal{S}_r$ (Step 13). Similarly, if $(x'^j_q, x^i_p) \in M^\star_W$, we break it and retain a stub from node x'^j_q, $(x'^j_q, \cdot)$. The set of left side stubs is defined by set $\mathcal{S}_\ell$ (Step 12). On the other hand, for edges within S_i in $\hat{M}_i$, we break them resulting in unmatched nodes and there are no stubs resulting from this breakage.

Steps 14–17: Our approach is to increase the number of edges between nodes in S_i by one. For this we select $\gamma_i + 1$ number of edges from a perfect matching M_i, in $\mathcal{B}(\bar{A}_i, \bar{B}_i)$, to form $\widetilde{M}_i$ (Step 14). Using $\widetilde{M}_i$ we construct matching $\widetilde{M}$ as shown in Step 15. Notice that previously there were only γ_i edges between nodes of S_i in $M^\star_W$, but $\widetilde{M}$ has $\gamma_i + 1$. We denote the set of left (right, resp.) unmatched nodes of S_i in $\widetilde{M}$ as L_i (R_i, resp.) (Steps 16 and 17). Here $|L_i| < |\mathcal{S}_r|$ and $|R_i| \leqslant |\mathcal{S}_\ell|$. More precisely, $|\mathcal{S}_r| = |L_i| + 1$, and $|R_i| = |\mathcal{S}_\ell|$ if the unique right unmatched vertex of the matching $M^\star_W$ obtained in Step 1 of Algorithm 5.1 belongs to subsystem S_i and $|\mathcal{S}_\ell| = |L_i| + 1$, otherwise.

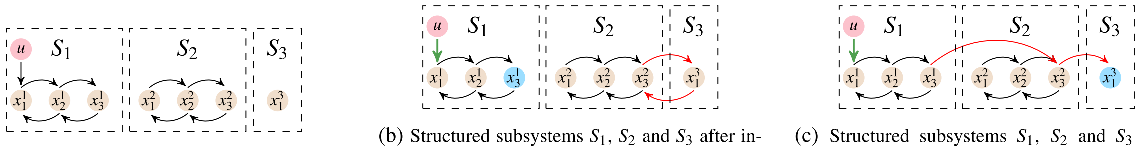
Steps 18–39: The left and right unmatched nodes of S_i in $\widetilde{M}$ are matched to the right and left stubs to form edges $L_{\rm edges}$ and $R_{\rm edges}$ respectively (Steps 19 and 24). Now the sets $\mathcal{S}_r$ and $\mathcal{S}_\ell$ are updated as shown in Steps 21 and 26. Notice that this will certainly result in one stub node in $\mathcal{S}_r$ unmatched, i.e., $|\mathcal{S}_r| = 1$ (since $\widetilde{M}_i \subset \widetilde{M}$, where $|\widetilde{M}_i| = \gamma_i + 1$). However, $|\mathcal{S}_\ell|$ can be either 1 or 0. Now we select a subsystem, say $S_h \in S \setminus \widetilde{S}(M^\star_W)$. Notice that corresponding to S_h there exists a perfect matching M_h and $M_h \subset \widetilde{M}$. Select an edge $(x'^h_a, x^h_b) \in \widetilde{M} \cap M_h$ (Step 28). Now we break this edge and connect the unmatched stubs appropriately as shown in Steps 32 and 34 to get $M_{\rm new}$ (Step 36). Note that $M_{\rm new}$ is a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. Now $M^\star_W$ is updated with $M_{\rm new}$ and this completes Stage 2.

Stage 3 (Steps 40–43): At the end of Stage 2, if $S \setminus \widetilde{S}(M^\star_W) = \emptyset$, then the interconnection edges are given by $E_{\mathcal{I}_B}$ as given in Step 41. At this stage, the number of interconnections is the same as the number of interconnections in the matching $M^\star_W$ obtained in Step 1. If $S \setminus \widetilde{S}(M^\star_W)$ is not empty in Step 42, then in addition to the already existing interconnections we add one extra interconnection to each subsystem in $S \setminus \widetilde{S}(M^\star_W)$ from some subsystem in $\widetilde{S}(M^\star_W)$ (Step 43). This completes Stage 3. The output of Algorithm 5.1 is denoted by $E_{\mathcal{I}_B}$.

Lemma 5.1. *Consider a set of irreducible structured subsystems $(\bar{A}_1, \bar{B}_1), \ldots, (\bar{A}_k, \bar{B}_k)$. For $i = 1, \ldots, k$, define the set $\mathcal{J} \subseteq \{1, \ldots, k\}$, as $\mathcal{J} := \{j \in \mathcal{J} : \bar{B}_j = 0\}$. Let ρ be the minimum number of interconnections required by the composite system to satisfy the no-dilation condition and let d_i be the size of the maximum matching in $\mathcal{B}(\bar{A}_i, \bar{B}_i)$. Let $\bar{A}^\star$ be an optimal solution to Problem 2.3 with the set of interconnections $E^\star_{\mathcal{I}}$. Then, $|E^\star_{\mathcal{I}}| \geqslant \max\Big(\rho, \sum_{i=1}^k (n_i - d_i) + |i \in \mathcal{J} : n_i = d_i|\Big)$.*

Proof. A solution $\bar{A}'$ to Problem 2.3 must satisfy two conditions: (a) $\mathcal{B}(\bar{A}', \bar{B}_T)$ must have a perfect matching, and (b) all subsystems should be accessible. Given ρ is the minimum number of interconnections required to satisfy the no-dilation condition in the composite system. Note that Stage 1 of Algorithm 5.1 is Algorithm 4.1. Hence the number of interconnections at the end of Stage 1 is ρ. Also, by Theorem 4.6, ρ number of interconnections are essential to guarantee condition (a). Thus $|E^\star_{\mathcal{I}}| \geqslant \rho$.

Let M_i, for $i = 1, \ldots, k$, be a maximum matching in $\mathcal{B}(\bar{A}_i, \bar{B}_i)$ such that $|M_i| = d_i$. $\cup_{i=1}^k M_i$ is a matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. Further, with respect to this matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ there are $\sum_{i=1}^k (n_i - d_i)$ number of left side unmatched nodes and at least $\sum_{i=1}^k (n_i - d_i)$ number of right side unmatched nodes. For satisfying condition (a) at least $\sum_{i=1}^k (n_i - d_i)$ number of interconnections are needed. Also, for satisfying condition (b) at least $|i \in \mathcal{J} : n_i = d_i|$ interconnections are needed. This is because for $i \notin \mathcal{J}$ the subsystems are accessible without using any interconnection edge since the subsystems are irreducible. We prove $|E^\star_{\mathcal{I}}| \geqslant \sum_{i=1}^k (n_i - d_i) + |i \in \mathcal{J} : n_i = d_i|$ using a contradiction argument. To the contrary, assume that there exists a solution to Problem 2.3, say $\bar{A}''$, such that the number of interconnections in $\bar{A}''$ is less than $\sum_{i=1}^k (n_i - d_i) + |i \in \mathcal{J} : n_i = d_i|$. Without loss of generality, assume $\bar{A}''$ consists of $\sum_{i=1}^k (n_i - d_i) + |i \in \mathcal{J} : n_i = d_i| - 1$ interconnections. This implies that one interconnection edge connects a subsystem S_j with $n_j \neq d_j$ to a subsystem S_i with $i \in \mathcal{J}$ and $n_i = d_i$. Note that this edge is from S_j to S_i and hence this edge does not match any of the $n_j - d_j$ left unmatched nodes of S_j and thereby does not contribute to matching any left node in S_j. This contradicts the assumption that



(a) Structured subsystems S_1, S_2 and S_3 considered for demonstrating Algorithm 4.1

(b) Structured subsystems S_1, S_2 and S_3 after including interconnections obtained in $M_W^\star$ shown in Figure 4a

(c) Structured subsystems S_1, S_2 and S_3 after reconstruction of interconnections to obtain $M_W^\star$ shown in Figure 4b

Fig. 3. Illustrative figure demonstrating Algorithm 4.1. Blue colored node denotes the unmatched accessible node . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

this edge serves both the purposes. Hence $|E^\star_{\mathcal{I}}| \geqslant \sum_{i=1}^{k}(n_i - d_i) + |i \in \mathcal{J} : n_i = d_i|$.

Since $|E^\star_{\mathcal{I}}| \geqslant \rho$ and $|E^\star_{\mathcal{I}}| \geqslant (\sum_{i=1}^{k}(n_i - d_i) + |i \in \mathcal{J} : n_i = d_i|)$, we get $|E^\star_{\mathcal{I}}| \geqslant \max\Big(\rho, \sum_{i=1}^{k}(n_i - d_i) + |i \in \mathcal{J} : n_i = d_i|\Big)$. $\square$

We next show that Algorithm 5.1 achieves the lower bound given in Lemma 5.1.

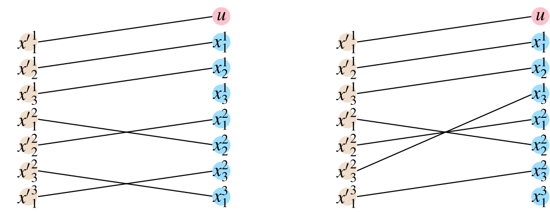
Theorem 5.2. *Output of Algorithm 5.1, which takes as input a set of irreducible structured subsystems $A_1, \ldots, A_k$ and structured input matrices $\bar{B}_1, \ldots, \bar{B}_k$, is an optimal solution to Problem 2.3, i.e., $|E_{\mathcal{I}_B}| = |E^\star_{\mathcal{I}}|$.*

Proof. Let $\bar{A}^\star$ be an optimal solution to Problem 2.3 and let $E^\star_{\mathcal{I}}$ be the set of interconnection edges corresponding to $\bar{A}^\star$. From Lemma 5.1 we know that $|E^\star_{\mathcal{I}}| \geqslant \max\Big(\rho, \sum_{i=1}^{k}(n_i - d_i) + |i \in \mathcal{J} : n_i = d_i|\Big)$. Using the interconnection edges obtained as output in Algorithm 5.1, all states are accessible and there exists a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. Now if we show that Algorithm 5.1 attains this bound, then optimality follows. Recall that Algorithm 5.1 consists of three stages and at the end of each stage the succeeding stage is executed if the partition $S \setminus \widetilde{S}(M^\star_W)$ is non-empty. Note that at the end of Stage 1 and Stage 2 the number of interconnections used, which is equal to ρ, is the minimum number of interconnections to achieve the no-dilation condition. At the end of Stage 2 there are two possible cases: (i) $S \setminus \widetilde{S}(M^\star_W) = \emptyset$, and (ii) $S \setminus \widetilde{S}(M^\star_W) \neq \emptyset$. In case (i) algorithm terminates at Stage 2 with $|E_{\mathcal{I}_B}| = \rho$.

In case (ii), Algorithm 5.1 proceeds to Stage 3. Notice that at the end of Stage 2, all subsystems with $n_i \neq d_i$ have exactly $n_i - d_i$ interconnections. However, there are few subsystems with $n_i = d_i$ that have one interconnection edge each at the end of Stage 2. Note that these subsystems have $\bar{B}_i = 0$, where $i \in \mathcal{J}$, $\mathcal{J} \subseteq \{1, \ldots, k\}$ defined as $\mathcal{J} := \{j \in \mathcal{J} : \bar{B}_j = 0\}$ (see definition of β_i and condition on β_i in Steps 6, 8 respectively of Algorithm 5.1). In Stage 3, extra interconnections are added to only those subsystems with $n_i = d_i$ and $i \in \mathcal{J}$ and that subsystem which has no interconnection edge connected at the end of Stage 2. The number of interconnections used in Algorithm 5.1 is exactly $\sum_{i=1}^{k}(n_i - d_i) + |i \in \mathcal{J} : n_i = d_i|$. Hence $|E_{\mathcal{I}_B}| = \sum_{i=1}^{k}(n_i - d_i) + |i \in \mathcal{J} : n_i = d_i|$ and Algorithm 5.1 attains the lower bound given in Lemma 5.1 and optimality holds. $\square$

Theorem 5.3. *Algorithm 5.1 which takes as input irreducible structured subsystems $(\bar{A}_1, \bar{B}_1), \ldots, (\bar{A}_k, \bar{B}_k)$ of state dimensions $n_1, \ldots, n_k$ respectively and gives as output an optimum set of interconnection edges $E_{\mathcal{I}_B}$ has complexity $O(n_T^3)$, where $n_T = \sum_{i=1}^{k} n_i$.*

Proof. Construction of $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ and solving the minimum cost perfect matching problem has complexity $O(n_T^3)$, where $n_T = \sum_{i=1}^{k} n_i$. Construction of $\mathcal{D}(M^\star_W)$ and finding partitions $\widetilde{S}(M^\star_W)$ and $S \setminus \widetilde{S}(M^\star_W)$ has $O(n_T)$ complexity. Complexity of Stage 1 is $O(n_T^3)$. In Stage 2, finding maximum matchings M_i's have $O({n_i}^3)$ complexity for each subsystem; there are k subsystems and $n_T = \sum_{i=1}^{k} n_i$.



(a) Bipartite matching $M^\star_W$ obtained in Step 2 of Algorithm 4.1 for the structured system given in Figure 3a

(b) Bipartite matching $M^\star_W$ obtained at the end of Algorithm 4.1 for the structured system given in Figure 3a

Fig. 4. Bipartite matchings obtained at different stages of Algorithm 4.1 for the example given in Fig. 3a.

Further, the stub-matching algorithm has linear complexity there by giving total complexity of Stage 2 $O(n_T^3)$. Stage 3 has linear complexity. Combining, the overall complexity of Algorithm 5.1 is $O(n_T^3)$. $\square$

This concludes the case of solving Problem 2.3 on irreducible subsystems when some of the subsystems have perfect matching in their state bipartite graph. Now we demonstrate Algorithms 4.1 and 5.1 using illustrative examples.

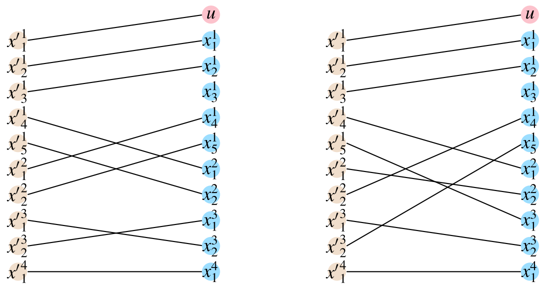
6. Illustrative examples and multi-input case

In this section, we first illustrate the two algorithms given in the paper, Algorithms 4.1 and 5.1, through examples and then give a brief outline on the extension to the multi-input case.

6.1. Illustrative example

In this subsection, we consider an example and apply results of this paper. Red colored edges in Figs. 3 and 6 are the interconnection edges. Initially, we demonstrate Algorithm 4.1 through an example given in Fig. 3a. The minimum cost perfect matching $M^\star_W$ obtained in Step 2 of Algorithm 4.1, shown in Fig. 4a, consists of two interconnection edges. Fig. 3b shows the resulting composite system corresponding to the interconnections established in $M^\star_W$ shown in Fig. 4a. The resulting partitioning is $\widetilde{S}(M^\star_W) = \{u, S_1\}$ and $S \setminus \widetilde{S}(M^\star_W) = \{S_2, S_3\}$. Here $x^t_r = x^1_3$. This gives $M'_W = \{M^\star_W \setminus (x'^2_3, x^3_1)\} \cup \{(x'^2_3, x^1_3)\}$. At the end of the first iteration $M^\star_W = M'_W$ as shown in Fig. 4b. Fig. 3c shows the resulting composite system corresponding to the interconnections established in $M^\star_W$ shown in Fig. 4b. $S \setminus \widetilde{S}(M^\star_W) = \emptyset$ and this completes Algorithm 4.1. The solution obtained is $E_{\mathcal{I}_A} = \{(x^1_3, x^2_3), (x^2_3, x^3_1)\}$.

Now we demonstrate Algorithm 5.1 through an example given in Fig. 6. Fig. 5a is the perfect matching $M^\star_W$ obtained in Step 1 of Algorithm 5.1. The resulting interconnections in the subsystems are shown in red edges in Fig. 6a. Note that the number of interconnections is four. Here $\widetilde{S}(M^\star_W) = \{u, S_1, S_2\}$ and $S \setminus \widetilde{S}(M^\star_W) = \{S_3, S_4\}$.



(a) Bipartite matching $M_W^\star$ obtained at the end of Stage 1 of Algorithm 5.1 for the structured system given in Figure 6

(b) Bipartite matching $M_W^\star$ obtained at the end of Stage 2 of Algorithm 5.1 for the structured system given in Figure 6

Fig. 5. Bipartite matchings obtained at different stages of Algorithm 5.1 for the example given in Fig. 6.

Also, $\mathcal{J} = \{2, 3, 4\}$. Note that $S_2 \in \widetilde{S}(M_W^\star)$ has a perfect matching in $\mathcal{B}(\bar{A}_2, \bar{B}_2)$. Here $d_2 = n_2 = 2$ and hence $n_2 - d_2 = 0$. Further, $2 \in \mathcal{J}$ and $\beta_2 = 2$. Here $\check{M}_2 = \emptyset$ and hence $\gamma_2 = 0$. $\hat{M}_2 = \{(x'^1_4, x^2_1), (x'^1_5, x^2_2), (x'^2_1, x^1_4), (x'^2_2, x^1_5)\}$ and $\mathcal{S}_\ell = \{x'^1_4, x'^1_5\}$ and $\mathcal{S}_r = \{x^1_4, x^1_5\}$. Now $\widetilde{M}_i = (x'^2_1, x^2_2)$. The matching $\widetilde{M} = \{(x'^1_1, u), (x'^1_2, x^1_1), (x'^1_3, x^1_2), (x'^2_1, x^2_2), (x'^3_1, x^3_2), (x'^3_2, x^3_1), (x'^4_1, x^4_1)\}$. $L_2 = \{x'^2_2\}$ and $R_2 = \{x^2_1\}$. This gives $L_{\rm edges} = \{(x'^2_2, x^1_4)\}$ and $R_{\rm edges} = \{(x'^1_4, x^2_1)\}$. At the end of Steps 22 and 27, $\mathcal{S}_r = x^1_5$ and $\mathcal{S}_\ell = x'^1_5$ respectively.

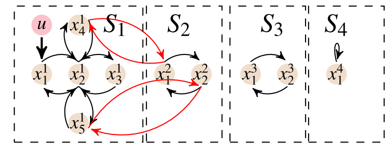
Now we select (x'^3_2, x^3_1) in Step 28. The matching $M_W = \{(x'^1_1, u), (x'^1_2, x^1_1), (x'^1_3, x^1_2), (x'^2_1, x^2_2), (x'^3_1, x^3_2), (x'^3_2, x^1_5), (x'^1_5, x^3_1), (x'^4_1, x^4_1)\}$. Now $M_{\rm new}$ is obtained as in Step 36 and $M_W^\star$ is updated as $M_{\rm new}$. The $M_W^\star$ obtained is shown in Fig. 5b. Here $\beta_2 = 1$. At the end of iteration 1 of Stage 2 $S \setminus \widetilde{S}(M_W^\star) = S_4$. Also $\beta_1 = 2$, $\beta_2 = \beta_3 = 1$ and $\beta_4 = 0$. Condition $\beta_i > \{\max_{i \in \mathcal{J}}\{n_i - d_i, 1\}, (n_i - d_i)\}$ is not satisfied. This completes Stage 2 and the number of interconnections remains as four. The corresponding subsystem graph is shown in Fig. 6b.

Since $S \setminus \widetilde{S}(M_W^\star) \neq \emptyset$, we enter Stage 3, where an extra edge (x'^4_1, x^3_2) is added as shown in Fig. 6c. The solution to Problem 2.3, $E_{\mathcal{I}_B} = \{(x^2_1, x^1_4), (x^3_2, x^4_1), (x^3_1, x^1_5), (x^1_4, x^2_2), (x^1_5, x^3_2)\}$.

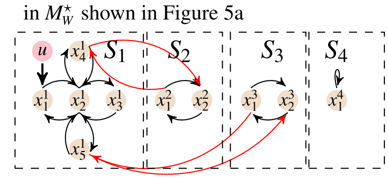
6.2. Multi-input case

In this subsection, we briefly give an outline for extending the algorithms and results given in this paper for the multi-input case. Firstly, we analyze Algorithm 4.1. Note that in Algorithm 4.1, the reconstruction process relies on two ideas: (i) in any optimum matching $M_W^\star$ in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$, there is an unmatched accessible node, and (ii) in $\mathcal{D}(M_W^\star)$ all subsystems in $S \setminus \widetilde{S}(M_W^\star)$ have an incoming and an outgoing interconnection. In the multi-input case, there exists at least one unmatched accessible node, and (i) continues to hold. Further, by Lemma 4.4 all subsystems in $S \setminus \widetilde{S}(M_W^\star)$ have equal number of nodes on the left and right sides of $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. Therefore, (ii) also holds and the results in Section 4 and Algorithm 4.1 extend to the multi-input case.

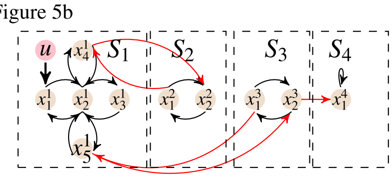
Now we analyze Algorithm 5.1. The key idea in Algorithm 5.1 is the stub-matching algorithm. It is based on the observation that, using a properly defined stub-matching scheme, from any optimum matching of the bipartite graph $\mathcal{B}(\bar{A}_T, \bar{B}_T)$, one can carefully reconstruct the interconnections in such a way that no subsystem has more number of interconnections than that are needed to achieve the no-dilation condition and the accessibility condition. The stub-matching scheme removes the redundancy in the interconnections, while making more subsystems accessible. This is how our scheme, results and algorithms extend to the multi-input case.



(a) Structured subsystems S_1, S_2 and S_3 after including interconnections obtained in $M_W^\star$ shown in Figure 5a



(b) Structured subsystems S_1, S_2 and S_3 after including interconnections obtained in $M_W^\star$ shown in Figure 5b



(c) Structured subsystems S_1, S_2 and S_3 at the end of Stage 3 of Algorithm 5.1

Fig. 6. Structured subsystems S_1, S_2 and S_3 considered for demonstrating Algorithm 5.1. Red colored edges correspond to the interconnections and black colored edges are connections within the subsystems . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

7. Conclusion

This paper dealt with structural controllability of linear time-invariant (LTI) composite system. Though we focused on structural controllability of systems, because of duality between controllability and observability in LTI systems, all results and algorithms of this paper directly follow in the structural observability problem as well. Considering structural controllability, the primary focus of this paper is to identify a minimum cardinality set of interconnection edges the subsystems should establish amongst each other such that the composite system is structurally controllable, i.e., the *optimal essential interconnection selection* problem. We approach the problem from a structured framework, where the zero/nonzero entries of the matrices are used instead of the numerical matrices themselves. Under the assumption that all subsystems are irreducible, we tackle the problem in two different settings: (i) when no subsystem has a perfect matching in its state bipartite graph, and (ii) when one or more subsystems have a perfect matching in their state bipartite graphs. For case (i), we proposed a polynomial time algorithm for solving the optimal essential interconnection selection problem. The proposed algorithm based on a minimum weight perfect matching algorithm followed by an edge reconstruction process gives as output an optimal solution in $O({n_T}^3)$ computations. In case (ii), we first identified a lower bound on the cardinality of the minimum number of interconnections needed for a given set of subsystems. Subsequently, we proposed an algorithm based on minimum weight perfect matching and a stub-matching algorithm that achieves this lower bound while satisfying the accessibility and no-dilation conditions. We proved the optimality of the algorithm and also showed that the computational complexity is $O({n_T}^3)$.

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