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Brief paper

Approximating constrained minimum cost input–output selection for generic arbitrary pole placement in structured systems ☆

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ABSTRACT

This paper deals with minimum cost constrained selection of inputs, outputs and feedback pattern in structured systems, referred to as *optimal input–output and feedback co-design problem*. The input–output set is constrained in the sense that the set of states that each input can influence and the set of states that each output can sense are pre-specified. Further, each input and each output are associated with a cost. The feedback pattern is unconstrained and the cost of a feedback edge is the sum of cost of the input and output associated with it. Our goal is to optimally select an input–output set and a feedback pattern such that the closed-loop system has no structurally fixed modes (SFMs). This problem is known to be NP-hard. In this paper, we show that the problem is inapproximable to factor $(1 - o(1)) \log n$, where n denotes the number of states in the system. Then we present an approximation algorithm of time complexity $O(n^3)$ to approximate the problem. We prove that the proposed algorithm gives a $(2 \log n)$-approximate solution to the problem. Thus the algorithm given in this paper is an order optimal approximation algorithm to approximate the optimal input–output and feedback co-design problem.

1. Introduction

Complex dynamical systems are indispensable constituents in social, biological, and technological fields such as robot swarms (Rafiee & Bayen, 2010), power grids (Terzija et al., 2011), biological systems (Gu et al., 2015), water distribution networks, and economic networks (Liu, Slotine, & Barabási, 2011). Complex systems are characterized by huge system dimension and a complex web like graph structure. The actuators and the sensors in the network are distributed which makes communication between them not easy. Establishing actuation, sensing and feedback connections in these networks are difficult, and thus a centralized decision making scheme is not desirable. To this end, we aim towards identifying the critical locations in the complex system, to be controlled and monitored for achieving the desired operational performance of the closed-loop system. Our focus is on optimal selection of inputs, outputs and feedback pattern to achieve satisfactory cost-effective decentralized control. Specifically, we consider systems with constrained input–output set and non-uniform costs on actuators and sensors and the aim is to select an optimal actuator–sensor set such that if feedback connections are made among these, then the poles of the closed-loop system can be placed at any desired location. We refer to this problem as *optimal input–output and feedback co-design problem* and it is known to be NP-hard (Pequito, Kar, & Pappas, 2015). The cost of inputs and outputs considered here comes from manufacturing, installation and monitoring costs associated with the network.

The main contributions of this paper are as follows:

- We prove that no polynomial time algorithm can approximate the optimal input–output and feedback co-design problem to factor $(1 - o(1)) \log n$, where n denotes the number of states in the system (Theorem 4.16(ii)).
- We provide a polynomial time approximation algorithm that gives approximation ratio $2 \log n$ for solving the optimal input–output and feedback co-design problem. Thus the proposed algorithm is order optimal (Theorem 4.16(i)).
- We show that the proposed approximation algorithm has computational complexity $O(n^3)$ (Theorem 4.17).
- We identify few special cases where the proposed algorithm attains the best possible bound, i,e., $\log n$ (Section 5).

In large scale systems, including biological systems, the web, power grids and social network to name a few, more often only

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the connections in the graph are known. The exact parameters are unavailable and only the sparsity pattern of the system matrices, i.e., the zero/non-zero pattern of the matrices, is known precisely. Many papers perform analysis of such networks using *structured matrices* instead of numerical matrices, referred to as *structural analysis* (Murota, 1987). Various optimization problems in linear structured systems, including minimum input selection (Commault & Dion, 2015; Liu et al., 2011; Olshevsky, 2015 and Pequito, Kar, & Aguiar, 2016), input addition for structural controllability (Commault & Dion, 2013), strong structural controllability (Chapman & Mesbahi, 2013), minimum cost control configuration selection (Pequito et al., 2015), are addressed in the literature. Paper by Pequito et al. (2016) discusses the problem of finding sparsest input, output and feedback pattern for a given system such that the closed-loop system has no SFMs. Note that in Pequito et al. (2016) input, output and feedback pattern are unconstrained and hence the problem is solvable in polynomial complexity. Given state, input and output matrices and cost associated with each feedback edge, finding a minimum cost feedback pattern such that the closed-loop system has no SFMs is NP-hard (Moothedath, Chaporkar, & Belur, 2018). In Moothedath et al. (2018) the NP-hardness is shown using a reduction from minimum set cover problem and then a dynamic programming-based algorithm is proposed for a special class of systems so-called line graphs. It is also shown in literature that when inputs and outputs are dedicated,[1] then finding sparsest feedback matrix has polynomial complexity (Moothedath, Chaporkar, & Belur, 2019). However, if inputs and outputs are non-dedicated, then sparsest feedback selection is NP-hard and even inapproximable to constant factor (Joshi, Moothedath, & Chaporkar, 2017). References Joshi et al. (2017), Moothedath et al. (2018), and Moothedath et al. (2019) deal with optimum feedback selection problem when the input and output matrices are prespecified. In contrast, this paper focuses on optimal input–output and feedback co-design problem in structured systems. Note that while Moothedath et al. (2018) present a polynomial time algorithm using dynamic programming to find an optimal solution to the minimum cost feedback selection problem for systems with line graph topology, this paper proposes an order approximate algorithm for input–output and feedback co-design problem for systems with general graph topology.

Optimal input–output and feedback co-design problem includes optimal selection of inputs and outputs from a constrained i/o set and a feedback design. This problem is addressed in Pequito et al. (2015) where Pequito et al. studied irreducible[2] systems. In contrast, this paper addresses the problem for general graph topology with non-uniform cost on inputs and outputs and the cost of a feedback edge is the sum of cost of the input and output associated with it. Unfortunately there do not exist polynomial algorithms for solving this problem unless P = NP (Pequito et al., 2015). We prove that the problem is inapproximable to a constant factor in polynomial time and then propose an approximation algorithm for order optimal approximation ratio.

The organization of this paper is as follows: Section 2 formulates the optimal input–output and feedback co-design problem. Section 3 discusses few preliminaries and some existing results. Section 4 explains our approach to solve the minimum cost input–output and feedback co-design problem and presents an approximation algorithm for solving the problem. Section 5 gives the approximation result in the context of few special cases. Section 6 presents an example illustrating the results of the paper. Section 7 gives the final concluding remarks.

[1] An input is said to be dedicated, if it directly actuates a single state only. An output is said to be dedicated, if it directly senses a single state only.

[2] A directed graph is said to be irreducible if there exists a directed path from each state to every other state.

2. Problem formulation

Consider structured matrices $\bar{A} \in \{\star, 0\}^{n\times n}$, $\bar{B} \in \{\star, 0\}^{n\times m}$ and $\bar{C} \in \{\star, 0\}^{p\times n}$, where $\star$ denotes free independent entries and 0 denotes fixed zero entries. The matrices $\bar{A}$, $\bar{B}$ and $\bar{C}$ *structurally represent* state, input and output matrices respectively of any control system $\dot{x} = Ax + Bu$, $y = Cx$, where

$$\begin{aligned} A_{ij} &= 0 \text{ whenever } \bar{A}_{ij} = 0, \text{ and} \\ B_{ij} &= 0 \text{ whenever } \bar{B}_{ij} = 0, \text{ and} \\ C_{ij} &= 0 \text{ whenever } \bar{C}_{ij} = 0. \end{aligned} \tag{1}$$

Any triple (A, B, C) that satisfies (1) is said to be a *numerical realization* of the *structured system* $(\bar{A}, \bar{B}, \bar{C})$. Further, the matrix $\bar{K} \in \{\star, 0\}^{m\times p}$, where $\bar{K}_{ij} = \star$ if the jth output is available for output feedback to the ith input is referred to as the *feedback matrix*. Let $[K]$ be the collection of all numerical realizations of $\bar{K}$, i.e., $[K] := \{K : K_{ij} = 0 \text{ if } \bar{K}_{ij} = 0\}$.

Definition 2.1. The structured system $(\bar{A}, \bar{B}, \bar{C})$ is said to have no SFMs with respect to an information pattern $\bar{K}$ if there exists one numerical realization (A, B, C) of $(\bar{A}, \bar{B}, \bar{C})$ such that $\cap_{K\in[K]}\sigma(A + BKC) = \emptyset$, where the function $\sigma(T)$ denotes the set of eigenvalues of a square matrix T.

Let $p_u \in \mathbb{R}^m$, where every entry $p_u(i)$, $i = 1, \ldots, m$, indicates the cost of using the ith input. Also, $p_y \in \mathbb{R}^p$, where every entry $p_y(j)$, $j = 1, \ldots, p$, indicates the cost of using the jth output. For $\mathcal{W} \subseteq \{1, \ldots, m\}$, $\mathcal{Z} \subseteq \{1, \ldots, p\}$, let $\bar{B}_{\mathcal{W}}$ be the restriction of $\bar{B}$ to columns only in $\mathcal{W}$, $\bar{C}_{\mathcal{Z}}$ be the restriction of $\bar{C}$ to rows only in $\mathcal{Z}$ and $\bar{K}_{(\mathcal{W}\times\mathcal{Z})}$ be the restriction of $\bar{K}$ to rows in $\mathcal{W}$ and columns in $\mathcal{Z}$. Furthermore, let $\mathcal{K} = \{(\mathcal{W}, \mathcal{Z}) : (\bar{A}, \bar{B}_{\mathcal{W}}, \bar{C}_{\mathcal{Z}}, \bar{K}_{(\mathcal{W}\times\mathcal{Z})}) \text{ have no SFMs}\}$. Our aim is to find $(\mathcal{I}, \mathcal{J}) \in \mathcal{K}$ such that the cost of inputs and outputs is minimized. For any $(\mathcal{I}, \mathcal{J})$, define $p(\mathcal{I}) = \sum_{i\in\mathcal{I}} p_u(i)$, $p(\mathcal{J}) = \sum_{j\in\mathcal{J}} p_y(j)$, and $p(\mathcal{I}, \mathcal{J}) = \sum_{i\in\mathcal{I}} p_u(i) + \sum_{j\in\mathcal{J}} p_y(j)$. We wish to solve the following optimization:

Problem 2.2. Given a structured system $(\bar{A}, \bar{B}, \bar{C})$, feedback matrix $\bar{K}$ and cost vectors p_u, p_y, find $(\mathcal{I}^\star, \mathcal{J}^\star) \in \arg\min_{(\mathcal{I},\mathcal{J})\in\mathcal{K}} p(\mathcal{I}, \mathcal{J})$, where $\mathcal{K} = \{(\mathcal{W}, \mathcal{Z}) : (\bar{A}, \bar{B}_{\mathcal{W}}, \bar{C}_{\mathcal{Z}}, \bar{K}_{(\mathcal{W}\times\mathcal{Z})}) \text{ has no SFMs}\}$.

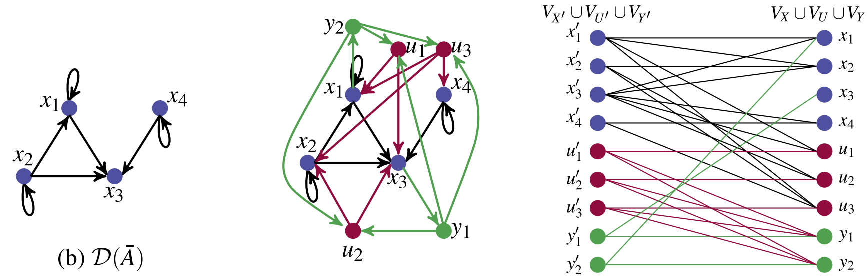
We refer to Problem 2.2 as the *optimal input–output and feedback co-design problem*. Without loss of generality, the set $\mathcal{K}$ is non-empty. Let $p^\star = p(\mathcal{I}^\star, \mathcal{J}^\star)$. Thus, $p^\star$ denotes the optimum cost of Problem 2.2. In this paper, $\bar{K}$ is unconstrained, i.e., $\bar{K}_{ij} = \star$ for all i, j. In the next section, we briefly give some preliminaries used in the sequel.

3. Preliminaries and existing results

In this section, we first discuss few graph theoretic preliminaries used in the sequel and some existing results. In structured systems there are two types of fixed modes, Type-1 and Type-2 (see Pichai, Sezer, & Šiljak, 1984; Wang & Davison, 1973 for more details). Presence of Type-1 SFMs can be checked using the concept of strong connectedness of the system digraph which is constructed as follows: firstly, we construct the state digraph $\mathcal{D}(\bar{A}) := (V_X, E_X)$, where $V_X = \{x_1, \ldots, x_n\}$ and $(x_j, x_i) \in E_X$ if $\bar{A}_{ij} = \star$. Thus a directed edge (x_j, x_i) exists if state x_j can influence state x_i. Now we construct the system digraph $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K}) := (V_X \cup V_U \cup V_Y, E_X \cup E_U \cup E_Y \cup E_K)$, where $V_U = \{u_1, \ldots, u_m\}$ and $V_Y = \{y_1, \ldots, y_p\}$. An edge $(u_j, x_i) \in E_U$ if $\bar{B}_{ij} = \star$, $(x_j, y_i) \in E_Y$ if $\bar{C}_{ij} = \star$ and $(y_j, u_i) \in E_K$ if $\bar{K}_{ij} = \star$. Thus a directed edge (u_j, x_i) exists if input u_j can actuate state x_i and a directed edge (x_j, y_i) exists if output y_i can sense state x_j. Construction of state digraph

$$\bar{A} = \begin{bmatrix} \star & \star & 0 & 0 \\ 0 & \star & 0 & 0 \\ \star & \star & 0 & \star \\ 0 & 0 & 0 & \star \end{bmatrix}, \bar{B} = \begin{bmatrix} \star & 0 & \star \\ 0 & \star & \star \\ \star & \star & 0 \\ 0 & 0 & \star \end{bmatrix},$$
$$\bar{C} = \begin{bmatrix} 0 & 0 & \star & 0 \\ \star & 0 & 0 & 0 \end{bmatrix}, \bar{K} = \begin{bmatrix} \star & \star \\ \star & \star \\ \star & \star \end{bmatrix}.$$

(a) Structured system $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$



(b) $\mathcal{D}(\bar{A})$ (c) $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ (d) $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$

Fig. 1. Structured system $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ and digraphs and bipartite graph associated with it.

$\mathcal{D}(\bar{A})$ and closed-loop system digraph $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ is illustrated in the example in Fig. 1. We next define the concepts of accessibility and sensability.

Definition 3.1. A state x_i is said to be *accessible* if there exists a directed path from some input u_j to x_i in the digraph $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$. Also, a state x_i is said to be *sensable* if there exists a directed path from x_i to some output y_j in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$.

A digraph is said to be strongly connected if for each ordered pair of vertices (v_i, v_j) there exists an elementary path from v_i to v_j. A strongly connected component (SCC) of a digraph is a maximal strongly connected subgraph of it. If $\mathcal{D}(\bar{A})$ is a single SCC, then the system is said to be *irreducible*. Using digraph $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ a necessary and sufficient graph theoretic condition for absence of SFMs is given in the following result.

Proposition 3.2 (*Pichai et al., 1984, Theorem 4*). *A structured system $(\bar{A}, \bar{B}, \bar{C})$ has no SFMs with respect to a feedback matrix $\bar{K}$ if and only if the following conditions hold:*

(a) in the digraph $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$, each state node x_i is contained in an SCC which includes an edge in E_K, and

(b) there exists a finite disjoint union of cycles $\mathcal{C}_g = (V_g, E_g)$ in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$, where g belongs to the set of natural numbers such that $V_X \subset \cup_g V_g$.

In Proposition 3.2, condition (a) corresponds to SFMs of Type 1 and condition (b) corresponds to SFMs of Type 2. In order to characterize condition (a) we first generate a *directed acyclic graph* (DAG) associated with $\mathcal{D}(\bar{A})$ by condensing each SCC to a supernode. In this DAG, vertex set comprises of all SCCs in $\mathcal{D}(\bar{A})$. A directed edge exists between two nodes of the DAG *if and only if* there exists a directed edge connecting two states in the respective SCCs in $\mathcal{D}(\bar{A})$. Using this DAG we have the following definition that characterizes SCCs in $\mathcal{D}(\bar{A})$.

Definition 3.3. An SCC is said to be non-top linked if it has no incoming edges to its vertices from the vertices of another SCC. Further, an SCC is said to be non-bottom linked if it has no outgoing edges from its vertices to the vertices of another SCC.

We have the following definition.

Definition 3.4. An SCC is said to be covered by input u_j if there exists a state x_i in that SCC such that $\bar{B}_{ij} = \star$. Similarly, an SCC is said to be covered by output y_j if there exists a state x_i in that SCC such that $\bar{C}_{ji} = \star$.

The following corollary holds from Definitions 3.1, 3.3 and 3.4.

Corollary 3.5. *All states are accessible in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ if all non-top linked SCCs of $\mathcal{D}(\bar{A})$ are covered by input. Also, all states are sensable in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ if all non-bottom linked SCCs of $\mathcal{D}(\bar{A})$ are covered by output.*

For a structured system $(\bar{A}, \bar{B}, \bar{C})$ with feedback matrix $\bar{K}$, verifying absence of SFMs has polynomial complexity. Specifically, condition (a) can be verified in $O(n^2)$ computations using the concept of SCCs in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ (Diestel, 2000). Condition (b) can be verified in $O(n^{2.5})$ computations using the concept of information paths given in Papadimitriou and Tsitsiklis (1984) or using the bipartite matching proposed in Pequito et al. (2015). In our work, we use bipartite matching condition and so we introduce a few notations and formally define the notion of matching in a bipartite graph.

Given an undirected bipartite graph $G_B = (V_B, \widetilde{V}_B, \mathcal{E}_B)$, where $V_B \cup \widetilde{V}_B$ denotes the vertex set such that $V_B \cap \widetilde{V}_B = \emptyset$ and $\mathcal{E}_B \subseteq V_B \times \widetilde{V}_B$ denotes the set of edges, a matching M is a collection of edges $M \subseteq \mathcal{E}_B$ such that for any two edges $(i, j), (u, v) \in M$, $i \neq u$ and $j \neq v$. A perfect matching is a matching M such that $|M| = \min(|V_B|, |\widetilde{V}_B|)$. For checking condition (b) in a structured system, we use the bipartite graph $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ constructed in Pequito et al. (2015) using Pichai et al. (1984). Define $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K}) := (V_{X'} \cup V_{U'} \cup V_{Y'}, V_X \cup V_U \cup V_Y, \mathcal{E}_X \cup \mathcal{E}_U \cup \mathcal{E}_Y \cup \mathcal{E}_K \cup \mathcal{E}_{\mathbb{U}} \cup \mathcal{E}_{\mathbb{Y}})$, where $V_{X'} = \{x'_1, \ldots, x'_n\}$, $V_{U'} = \{u'_1, \ldots, u'_m\}$, $V_{Y'} = \{y'_1, \ldots, y'_p\}$ and $V_X = \{x_1, \ldots, x_n\}$, $V_U = \{u_1, \ldots, u_m\}$ and $V_Y = \{y_1, \ldots, y_p\}$. Also, $(x'_i, x_j) \in \mathcal{E}_X \Leftrightarrow (x_j, x_i) \in E_X$, $(x'_i, u_j) \in \mathcal{E}_U \Leftrightarrow (u_j, x_i) \in E_U$, $(y'_j, x_i) \in \mathcal{E}_Y \Leftrightarrow (x_i, y_j) \in E_Y$ and $(u'_i, y_j) \in \mathcal{E}_K \Leftrightarrow (y_j, u_i) \in E_K$. Moreover, $\mathcal{E}_{\mathbb{U}}$ include edges (u'_i, u_i) for $i = 1, \ldots, m$ and $\mathcal{E}_{\mathbb{Y}}$ include edges (y'_j, y_j) for $j = 1, \ldots, p$. We show that there exists a perfect matching in the bipartite graph $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ if and only if the system $(\bar{A}, \bar{B}, \bar{C})$ along with feedback matrix $\bar{K}$ satisfies condition (b) (see Section 4.4). The system bipartite graph $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ for the structured system given in Fig. 1(a) is shown in Fig. 1(d).

A structured closed-loop system is said to have no SFMs if and only if all the state vertices lie in some SCC of $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ with an edge in E_K and the system bipartite graph $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ has a perfect matching. Thus, using the two graph theoretic conditions explained in this section, we conclude that presence of SFMs in a structured closed-loop system can be checked in $O(n^{2.5})$ computations. However, optimal selection of inputs, outputs and feedback pattern from a constrained set is NP-hard. In the next section we detail our approach to approximate this.

4. Approximating optimal input–output and feedback co-design problem

In this section, we present a polynomial time algorithm to approximate the optimal input–output and feedback co-design problem, which is known to be NP-hard. We prove that the proposed algorithm finds a 2 log n-approximate solution to the problem, where n denotes the dimension of the system. We also show that the problem is inapproximable to factor $(1 - o(1)) \log n$. Thus this section presents an order optimal approximation algorithm to solve Problem 2.2.

Our approach to find an approximate solution to Problem 2.2 is to split the problem into three subproblems listed below:

- Minimum cost accessibility problem
- Minimum cost sensability problem
- Minimum cost disjoint cycle problem

Broadly, minimum cost accessibility (sensability, resp.) problem aims at finding minimum cost — collection of inputs (outputs, resp.) that cover all SCCs of $\mathcal{D}(\bar{A})$. In minimum cost disjoint cycle problem, our aim is to find minimum cost collection of inputs and outputs such that condition (b) is satisfied.

Firstly, we show that the minimum cost accessibility (sensability, resp.) problem is "equivalent to" the weighted set cover problem. On account of the equivalence, one can use any algorithm for the weighted set cover problem to solve the minimum cost accessibility and sensability problems with the same performance guarantees and vice versa. Weighted set cover problem is a well studied NP-hard problem. There exist approximation algorithms that give solution to the weighted set cover problem to log factor in problem size (Chvatal, 1979). However, there also exists inapproximability result showing that it cannot be approximated to a constant factor (Feige, 1998). Using the equivalence of the problems, we provide an order optimal approximation algorithm to solve the minimum cost accessibility and sensability problems.

We also give a necessary and sufficient condition to the solution of the minimum cost disjoint cycle problem using a minimum cost perfect matching defined on $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ with a suitable cost function. Bipartite matching is also a well studied area and there exists polynomial time algorithm of complexity $O(\ell^3)$ that finds minimum cost perfect matching in a bipartite graph with ℓ nodes on one side (Diestel, 2000). Using the minimum cost perfect matching algorithm we provide a polynomial time algorithm to solve the minimum cost disjoint cycle problem. Later we show that combining the solutions to these subproblems gives an approximate solution to Problem 2.2. Now we formally define and tackle each of these subproblems separately.

4.1. Approximating minimum cost accessibility problem

In this subsection, we show that when the inputs are constrained and each input is associated with a cost, then satisfying minimum cost accessibility condition is *equivalent* to solving a weighted set cover problem defined on the structured system.

Consider a structured system $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ and cost vectors p_u, p_y. The closed-loop system $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ is said to satisfy the minimum cost accessibility condition if all the non-top linked SCCs in $\mathcal{D}(\bar{A})$ are covered using the least cost input set; i.e. we need to find a set of inputs $\mathcal{I}^\star_{\mathcal{A}} \subseteq \{1, \ldots, m\}$ such that all state nodes are accessible in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}^\star_{\mathcal{A}}}, \bar{C}, \bar{K})$ and $p(\mathcal{I}^\star_{\mathcal{A}}) \leqslant p(\mathcal{I}_{\mathcal{A}})$ for any $\mathcal{I}_{\mathcal{A}} \subseteq \{1, \ldots, m\}$ that satisfy accessibility of all state nodes in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}_{\mathcal{A}}}, \bar{C}, \bar{K})$. Specifically, we need to solve the following optimization: for any $\mathcal{I}_{\mathcal{A}} \subseteq \{1, \ldots, m\}$, define $p(\mathcal{I}_{\mathcal{A}}) = \sum_{i \in \mathcal{I}_{\mathcal{A}}} p_u(i)$.

Problem 4.1. Given $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ and cost vectors p_u, p_y, find $\mathcal{I}^\star_{\mathcal{A}} \in \arg\min_{\mathcal{I}_{\mathcal{A}} \subseteq \{1,\ldots,m\}} p(\mathcal{I}_{\mathcal{A}})$, such that all state nodes are accessible in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}_{\mathcal{A}}}, \bar{C}, \bar{K})$.

We refer to Problem 4.1 as *the minimum cost accessibility problem*. Before showing the equivalence between Problem 4.1 and the weighted set cover problem, we first describe the weighted set cover problem for the sake of completeness. Given a universe of N elements $\mathcal{U} = \{1, 2, \ldots, N\}$, a set of r sets $\mathcal{P} = \{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_r\}$ with $\mathcal{S}_i \subset \mathcal{U}$ and $\bigcup_{i=1}^r \mathcal{S}_i = \mathcal{U}$ and a weight function w from $\mathcal{P}$ to the set of non-negative real numbers, weighted set cover problem consists of finding a set $\mathcal{S}^\star \subseteq \mathcal{P}$ such that $\cup_{\mathcal{S}_i \in \mathcal{S}^\star} \mathcal{S}_i = \mathcal{U}$ and $\sum_{\mathcal{S}_i \in \mathcal{S}^\star} w(i) \leqslant \sum_{\mathcal{S}_i \in \widetilde{\mathcal{S}}} w(i)$ for any $\widetilde{\mathcal{S}}$ that satisfies $\cup_{\mathcal{S}_i \in \widetilde{\mathcal{S}}} = \mathcal{U}$.

Algorithm 4.1 Pseudocode for reducing minimum cost accessibility problem to a weighted set cover problem

Input: Structured system $(\bar{A}, \bar{B})$ and input cost vector p_u
Output: Input set $\mathcal{I}(\mathcal{S})$ and cost $p(\mathcal{I}(\mathcal{S}))$

1: Find all non-top linked SCCs in $\mathcal{D}(\bar{A})$, $\mathcal{N} := \mathcal{N}_1, \ldots, \mathcal{N}_q$
2: Define weighted set cover problem as follows:
3: Universe $\mathcal{U} \leftarrow \{\mathcal{N}_1, \ldots, \mathcal{N}_q\}$
4: Sets $\mathcal{S}_i \leftarrow \{\mathcal{N}_j : \bar{B}_{ri} = \star$ and $x_r \in \mathcal{N}_j\}$
5: Weights $w(i) \leftarrow p_u(i)$ for $i \in \{1, \ldots, m\}$
6: Given a cover $\mathcal{S}$ such that $\cup_{\mathcal{S}_i \in \mathcal{S}} \mathcal{S}_i \subseteq \mathcal{U}$, define:
7: Weight of the cover $w(\mathcal{S}) \leftarrow \sum_{\mathcal{S}_i \in \mathcal{S}} w_u(i)$
8: Define $\mathcal{I}(\mathcal{S}) \leftarrow \{i : \mathcal{S}_i \in \mathcal{S}\}$
9: Cost of $\mathcal{I}(\mathcal{S})$, $p(\mathcal{I}(\mathcal{S})) \leftarrow \sum_{i \in \mathcal{I}(\mathcal{S})} p_u(i)$

Now we reduce Problem 4.1 to an instance of the weighted set cover problem in polynomial time and prove inapproximability of Problem 4.1. This inapproximability result is later used to prove the inapproximability of Problem 2.2.

The pseudocode showing a reduction of Problem 4.1 to an instance of the weighted set cover problem is presented in Algorithm 4.1. Given $(\bar{A}, \bar{B})$ and cost vector p_u, we define a weighted set cover problem as follows: the universe $\mathcal{U}$ consists of all non-top linked SCCs $\{\mathcal{N}_1, \ldots, \mathcal{N}_q\}$ in $\mathcal{D}(\bar{A})$ (Step 3). The sets $\mathcal{S}_1, \ldots, \mathcal{S}_m$ are defined in such a way that set $\mathcal{S}_i$ consists of all non-top linked SCCs that are covered by the ith input (Step 4). Further, for each set $\mathcal{S}_i$ we define weight $w(i)$ as shown in Step 5. Given a solution $\mathcal{S}$ to the weighted set cover problem, we define the associated weight $w(\mathcal{S})$ as the sum of the weights of all sets selected under $\mathcal{S}$ (Step 7). Also, the indices of the sets selected in $\mathcal{S}$ are denoted as $\mathcal{I}(\mathcal{S})$ and its cost is denoted as $p(\mathcal{I}(\mathcal{S}))$ as shown in Steps 8 and 9 respectively. We denote an optimal solution to Problem 4.1 as $\mathcal{I}^\star_{\mathcal{A}}$ and its cost as $p^\star_{\mathcal{A}}$. Also an optimal solution to the weighted set cover problem given in Algorithm 4.1 is denoted by $\mathcal{S}^\star_{\mathcal{A}}$ and its weight is denoted by $w^\star_{\mathcal{A}}$.

Lemma 4.2. *Consider any structured system $(\bar{A}, \bar{B}, \bar{C})$, feedback matrix $\bar{K}$ and cost vectors p_u, p_y and the corresponding weighted set cover problem obtained using Algorithm 4.1. Let $\mathcal{S}$ be a feasible solution to the weighted set cover problem and $\mathcal{I}(\mathcal{S})$ be the index set selected in Step 8. Then, all states are accessible in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}(\mathcal{S})}, \bar{C}, \bar{K})$ and $p(\mathcal{I}(\mathcal{S})) = w(\mathcal{S})$.*

See the Appendix for the proof. We next prove that the weighted set cover problem's ϵ-approximation algorithm as obtained in Algorithm 4.1 gives such an algorithm to Problem 4.1 too.

Theorem 4.3. *Consider any structured system $(\bar{A}, \bar{B}, \bar{C})$, feedback matrix $\bar{K}$ and cost vectors p_u, p_y and the corresponding weighted set cover problem obtained using Algorithm 4.1. Then, for any $\epsilon \geqslant 1$, if $\mathcal{S}$ is an ϵ-optimal solution to the weighted set cover problem, then $\mathcal{I}(\mathcal{S})$ is an ϵ-optimal solution to the minimum cost accessibility problem.*

Proof. The proof of this lemma is twofold: (i) we show that an optimal solution $\mathcal{S}^\star_{\mathcal{A}}$ to the weighted set cover problem gives an optimal solution $\mathcal{I}^\star_{\mathcal{A}}$ to Problem 4.1, and (ii) we show that if $w(\mathcal{S}) \leqslant \epsilon\, w^\star_{\mathcal{A}}$, then $p(\mathcal{I}(\mathcal{S})) \leqslant \epsilon\, p^\star_{\mathcal{A}}$.

Given $\mathcal{S}^\star_{\mathcal{A}}$ is an optimal solution to the weighted set cover problem with cost $w^\star_{\mathcal{A}}$. For (i) we show that the input set $\mathcal{I}(\mathcal{S}^\star_{\mathcal{A}})$ selected under $\mathcal{S}^\star_{\mathcal{A}}$ is a minimum cost input set that satisfies the accessibility of all states, i.e., all the states of $\bar{A}$ are accessible in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}(\mathcal{S}^\star_{\mathcal{A}})}, \bar{C}, \bar{K})$ and $p(\mathcal{I}(\mathcal{S}^\star_{\mathcal{A}})) = p^\star_{\mathcal{A}}$. Since $\mathcal{S}^\star_{\mathcal{A}}$ is a solution to the weighted set cover problem, using Lemma 4.2 all states are accessible in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}(\mathcal{S}^\star_\mathcal{A})}, \bar{C}, \bar{K})$. Thus $\mathcal{I}(\mathcal{S}^\star_\mathcal{A})$ is a feasible solution to Problem 4.1. To prove minimality, we use a contradiction argument. Let us assume that $\mathcal{S}^\star_\mathcal{A}$ is an optimal solution to the weighted set cover problem but $\mathcal{I}(\mathcal{S}^\star_\mathcal{A}) = \{i : \mathcal{S}_i \in \mathcal{S}^\star_\mathcal{A}\}$ is not a minimum cost input set that satisfies the accessibility of all states. Then there exists $\mathcal{I}'_\mathcal{A} \subseteq \{1, \ldots, m\}$ such that all state nodes are accessible in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}'_\mathcal{A}}, \bar{C}, \bar{K})$ and $p(\mathcal{I}'_\mathcal{A}) < p(\mathcal{I}(\mathcal{S}^\star_\mathcal{A}))$. Note that for $\mathcal{S}' = \{\mathcal{S}_i : i \in \mathcal{I}'_\mathcal{A}\}$, $\cup_{\mathcal{S}_i \in \mathcal{S}'} \mathcal{S}_i = \mathcal{U}$. Using Lemma 4.2, $w(\mathcal{S}') < w^\star_\mathcal{A}$. This gives a contradiction to the assumption that $\mathcal{S}^\star_\mathcal{A}$ is a minimal solution to the weighted set cover problem. This completes the proof of (i).

Given, $w(\mathcal{S}) \leqslant \epsilon w^\star_\mathcal{A}$. By Lemma 4.2, $p(\mathcal{I}(\mathcal{S})) \leqslant \epsilon w^\star_\mathcal{A} = \epsilon p(\mathcal{I}^\star_\mathcal{A}) = \epsilon p^\star$. This completes the proof of (ii). □

Using the above result, we have the following theorem.

Theorem 4.4. *Consider any structured system $(\bar{A}, \bar{B}, \bar{C})$, feedback matrix $\bar{K}$ and cost vectors p_u, p_y. Then, there exists a polynomial time algorithm that approximates Problem 4.1 to factor $\log n$, where n denotes the number of states.*

Proof. Algorithm 4.1 reduces Problem 4.1 to weighted set cover problem $(\mathcal{U}, \mathcal{P}, w)$ in polynomial time. From Theorem 4.3, a polynomial time ϵ-optimal algorithm for the weighted set cover problem gives a polynomial time ϵ-optimal algorithm for Problem 4.1. The maximum cardinality of a set $\mathcal{S}_i \in \mathcal{P}$ constructed in Algorithm 4.1 is q, where q is the number of non-top linked SCCs in $\mathcal{D}(\bar{A})$. Since $q \leqslant n$, the greedy approximation algorithm for solving the weighted set cover problem given in Chvatal (1979), gives a $\log n$-optimal solution to Problem 4.1. □

We next prove a complexity result regarding a reduction.

Lemma 4.5. *Consider a structured system $(\bar{A}, \bar{B}, \bar{C})$, feedback matrix $\bar{K}$ and cost vectors p_u, p_y. Then, Algorithm 4.1 reduces Problem 4.1 to a weighted set cover problem in $O(n^2)$ time, where n denotes the number of states in the system.*

See Appendix for the proof. Using Algorithm 4.1 one can reduce any instance of Problem 4.1 to an instance of the weighted set cover problem in polynomial time. Further, any approximation algorithm for the weighted set cover problem gives an approximation algorithm to Problem 4.1, with the same approximation ratio. Now, we prove the polynomial time inapproximability of Problem 4.1. To achieve this we give a polynomial time reduction of the weighted set cover problem to an instance of Problem 4.1 in Algorithm 4.2. Then we show that any polynomial time ϵ-optimal approximation algorithm for solving Problem 4.1 gives an ϵ-optimal approximate algorithm for the weighted set cover problem. Since weighted set cover problem cannot be approximated to constant factor (Feige, 1998), Problem 4.1 also cannot be approximated to constant factor.

The pseudocode showing a reduction of the weighted set cover problem to an instance of Problem 4.1 is presented in Algorithm 4.2. Given $\mathcal{U}, \mathcal{P}$ and w, we reduce the weighted set cover problem to an instance of the minimum cost accessibility problem. Here, $\bar{A}$ is a diagonal $N \times N$ matrix with all diagonal entries $\star$s (Step 2). $\bar{B}$ is defined in such a way that its jth column corresponds to the set $\mathcal{S}_j$ (Step 3) and cost of jth input is same as the weight $w(j)$ of $\mathcal{S}_j$ (Step 4). Notice that $\bar{C}$ and $\bar{K}$ have all entries as $\star$s (Step 1). Given a solution $\mathcal{I}$ to the accessibility problem, we define the associated cost $p(\mathcal{I})$, the sets selected $\mathcal{S}(\mathcal{I})$ and its weight $w(\mathcal{S}(\mathcal{I}))$ as shown in Steps 7–9. We denote an optimal solution to the set cover problem in Algorithm 4.2 as $\mathcal{S}^\star$ and its weight as $w^\star$. Now we prove the following results.

Algorithm 4.2 Pseudocode for reducing the weighted set cover problem to a minimum cost accessibility problem

Input: Weighted set cover problem with universe $\mathcal{U} = \{1, \ldots, N\}$, sets $\mathcal{P} = \{\mathcal{S}_1, \ldots, \mathcal{S}_r\}$ and weight function w
Output: Structured system $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ and cost vectors p_u, p_y

1: Define a minimum cost accessibility problem instance with $\bar{A} \in \{0, \star\}^{N\times N}$, $\bar{B} \in \{0, \star\}^{N\times r}$, $\bar{C} \in \{\star\}^{r\times N}$, $\bar{K} \in \{\star\}^{r\times r}$ and cost vectors p_u, p_y as follows:
2: $\bar{A}_{ij} \leftarrow \begin{cases} \star, \text{ for } i = j, \\ 0, \text{ otherwise}. \end{cases}$
3: $\bar{B}_{ij} \leftarrow \begin{cases} \star, \text{ for } i \in \mathcal{S}_j, \\ 0, \text{ otherwise}. \end{cases}$
4: $p_u(i) \leftarrow w(i)$, for $i \in \{1, \ldots, r\}$
5: $p_y(j) \leftarrow 0$, for $j \in \{1, \ldots, r\}$
6: Given a set $\mathcal{I}$ such that all states are accessible in $\mathcal{D}(\bar{A}, \bar{B}_\mathcal{I}, \bar{C}, \bar{K})$, define:
7: Cost of the set $p(\mathcal{I}) \leftarrow \sum_{i\in\mathcal{I}} p_u(i)$,
8: Define $\mathcal{S}(\mathcal{I}) \leftarrow \{\mathcal{S}_i : i \in \mathcal{I}\}$,
9: Weight of $\mathcal{S}(\mathcal{I})$, $w(\mathcal{S}(\mathcal{I})) \leftarrow \sum_{\mathcal{S}_i \in \mathcal{S}(\mathcal{I})} w(i)$.

Lemma 4.6. *Consider an weighted set cover problem given by $(\mathcal{U}, \mathcal{P}, w)$ and the corresponding structured system obtained using Algorithm 4.2. Let $\mathcal{I}$ be a feasible solution to Problem 4.1 and $\mathcal{S}(\mathcal{I})$ consists of the sets selected under $\mathcal{I}$. Then, $\mathcal{S}(\mathcal{I})$ covers $\mathcal{U}$ and $w(\mathcal{S}(\mathcal{I})) = p(\mathcal{I})$.*

Proof of Lemma 4.6 is given in the Appendix. In the following theorem we show that an ϵ-approximation algorithm for Problem 4.1 gives an ϵ-approximate solution to the weighted set cover problem also.

Theorem 4.7. *Consider any weighted set cover problem and the corresponding structured system $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ and cost vectors p_u, p_y obtained using Algorithm 4.2. For $\epsilon \geqslant 1$, if $\mathcal{I}$ is an ϵ-optimal solution to the minimum cost accessibility problem, then $\mathcal{S}(\mathcal{I})$ is an ϵ-optimal solution to the weighted set cover problem.*

Proof. Proof of this lemma consists of two steps: (i) we show that an optimal solution $\mathcal{I}^\star_\mathcal{A}$ to Problem 4.1 gives an optimal solution $\mathcal{S}^\star_\mathcal{A}$ to the weighted set cover problem, and (ii) we show that, if $p(\mathcal{I}) \leqslant \epsilon\, p^\star_\mathcal{A}$, then $w(\mathcal{S}(\mathcal{I})) \leqslant \epsilon\, w^\star_\mathcal{A}$.

For proving (i) we assume that $\mathcal{I}^\star_\mathcal{A}$ is an optimal solution to Problem 4.1 and then prove that $\mathcal{S}(\mathcal{I}^\star_\mathcal{A})$ is an optimal solution to the weighted set cover problem, i.e., $\cup_{\mathcal{S}_i \in \mathcal{S}(\mathcal{I}^\star_\mathcal{A})} = \mathcal{U}$ and $w(\mathcal{S}(\mathcal{I}^\star_\mathcal{A})) = w^\star_\mathcal{A}$. Given $\mathcal{I}^\star_\mathcal{A}$ is an optimal solution to Problem 4.1. Thus all states are accessible in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}^\star_\mathcal{A}}, \bar{C}, \bar{K})$. Hence, by Lemma 4.6, $\mathcal{S}(\mathcal{I}^\star_\mathcal{A})$ is a feasible solution to the weighted set cover problem. Now we prove optimality using a contradiction argument. Let $\mathcal{I}^\star_\mathcal{A}$ be an optimal solution to Problem 4.1, but $\mathcal{S}(\mathcal{I}^\star_\mathcal{A})$ is not an optimal solution to the weighted set cover problem. Then there exists $\widetilde{\mathcal{S}} \subset \{\mathcal{S}_1, \ldots, \mathcal{S}_r\}$ such that $\cup_{\mathcal{S}_i \in \widetilde{\mathcal{S}}} \mathcal{S}_i = \mathcal{U}$ and $w(\widetilde{\mathcal{S}}) < w(\mathcal{S}(\mathcal{I}^\star_\mathcal{A}))$. Then $\widetilde{\mathcal{I}} = \{i : \mathcal{S}_i \in \widetilde{\mathcal{S}}\}$ covers all the non-top linked SCCs in $\mathcal{D}(\bar{A})$. Also, from Lemma 4.2, $p(\widetilde{\mathcal{I}}) < p^\star_\mathcal{A}$. This gives a contradiction to the assumption that $\mathcal{I}^\star_\mathcal{A}$ is a minimum cost input set that satisfies accessibility condition. This completes the proof of (i). Now (ii) follows directly from Step 4 of Algorithm 4.2 and Lemma 4.6. □

Theorems 4.3 and 4.7 together prove the equivalence of Problem 4.1 and the weighted set cover problem. There exist various approximation algorithms to find approximate solution to the weighted set cover problem. Specifically, the greedy approximation algorithm given in Chvatal (1979) gives a $\log d$ approximation, where d is the cardinality of the largest set $\mathcal{S}_i$ in $\mathcal{P}$. In addition to this, we also know strong negative approximability result for the set cover problem using which we have the following result.

Theorem 4.8. *Consider a structured system* $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ *and cost vectors* p_u, p_y*. Then, there does not exist a polynomial time algorithm that approximates Problem 4.1 to factor* $(1 - o(1)) \log n$*, where* n *denotes the number of states.*

Proof. From Theorem 4.7, a polynomial time ϵ-optimal algorithm for solving Problem 4.1 gives a polynomial time ϵ-optimal algorithm for solving the weighted set cover problem. The set cover problem cannot be approximated to factor $(1 - o(1)) \log N$, where N is the cardinality of the universe (Feige, 1998). The set cover problem is a special case of the weighted set cover problem, where all weights are non-zero and uniform. Thus the inapproximability result of the set cover problem applies to the weighted set cover problem also. The weighted set cover reduction of Problem 4.1 given in Algorithm 4.1 has $|\mathcal{U}| = q$ and $q \leqslant n$. Thus Problem 4.1 cannot be approximated to factor $(1-o(1)) \log n$. □

Lemma 4.9. *Consider a weighted set cover problem with universe* $\mathcal{U}$*, set* $\mathcal{P}$ *and weight* w *with* $|\mathcal{U}| = N$*. This problem is reduced to Problem 4.1 by Algorithm 4.2 in* $O(N^2)$ *computations.*

The proof of Lemma 4.9 is given in the Appendix. This subsection thus concludes that, one can obtain a $(\log n)$-approximate solution to Problem 4.1 in polynomial time and Problem 4.1 is inapproximable to factor $(1 - o(1)) \log n$ in polynomial time. In the following subsection we discuss briefly about the minimum cost sensability problem.

4.2. Approximating minimum cost sensability problem

In this section, we establish a relation between the minimum cost sensability problem and the weighted set cover problem. Specifically, we show that when the outputs are constrained and each output is associated with a cost, then satisfying minimum cost sensability condition is equivalent to solving a weighted set cover problem defined on the structured system.

Consider a structured system $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ and cost vectors p_u and p_y. This system is said to satisfy the minimum cost sensability condition if all the non-bottom linked SCCs in $\mathcal{D}(\bar{A})$ are covered by the least cost output set. That is, we need to find a set of outputs $\mathcal{J}^\star_{\mathbb{A}} \subseteq \{1, \ldots, p\}$ such that all state nodes are sensable in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}_{\mathcal{J}^\star_{\mathbb{A}}}, \bar{K})$ and $p(\mathcal{J}^\star_{\mathbb{A}}) \leqslant p(\mathcal{J}_{\mathbb{A}})$ for any $\mathcal{J}_{\mathbb{A}} \subseteq \{1, \ldots, p\}$ that satisfy sensability of all state in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}_{\mathcal{J}_{\mathbb{A}}}, \bar{K})$. We refer to the above problem as *the minimum cost sensability problem*.

However, because of duality between controllability and observability solving minimum cost sensability problem is equivalent to solving minimum cost accessability problem of the structured system $(\bar{A}^T, \bar{C}^T, \bar{B}^T, \bar{K}^T)$ with cost vectors p_y, p_u. Thus the weighted set cover reformulation of Problem 4.1 for the system $(\bar{A}^T, \bar{C}^T, \bar{B}^T, \bar{K}^T)$ with cost vectors p_y, p_u solves the minimum cost sensability problem of $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ with cost vectors p_u, p_y. Hence the following result immediately follows from the analysis done in the previous subsection.

Corollary 4.10. *Consider a structured system* $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ *and cost vectors* p_u, p_y*. Then, there exists a greedy algorithm that gives a* $(\log n)$*-optimal solution to the minimum cost sensability problem, where* n *is the number of states in the system. Also, there does not exist a polynomial time algorithm that approximates minimum cost sensability problem to factor* $(1 - o(1)) \log n$*.*

We next formulate a relation between minimum cost disjoint cycle condition and a bipartite matching problem.

Algorithm 4.3 Pseudocode for reducing minimum cost disjoint cycle problem to a minimum cost perfect matching problem

Input: Structured system $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ and cost vectors p_u, p_y
Output: Input-output set $(\mathcal{I}(M_c), \mathcal{J}(M_c))$ and cost $p(\mathcal{I}(M_c), \mathcal{J}(M_c))$

1: Construct the bipartite graph $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$
2: For $e \in \mathcal{E}_X \cup \mathcal{E}_U \cup \mathcal{E}_Y \cup \mathcal{E}_K \cup \mathcal{E}_{\mathbb{U}} \cup \mathcal{E}_{\mathbb{Y}}$ define:
3: Cost, $c(e) \leftarrow \begin{cases} p_u(i) + p_y(j), & \text{for } e = (u'_i, y_j) \in \mathcal{E}_K, \\ 0, & \text{otherwise.} \end{cases}$
4: Find minimum cost perfect matching of $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ under cost c, say M_c
5: Cost of M_c, $c(M_c) \leftarrow \sum_{e \in M_c} c(e)$
6: Input index set selected under $M_c : \mathcal{I}(M_c) \leftarrow \{i : (x'_j, u_i) \in M_c\}$
7: Input cost $p(\mathcal{I}(M_c)) \leftarrow \sum_{i \in \mathcal{I}(M_c)} p_u(i)$
8: Output index set selected under $M_c : \mathcal{J}(M_c) \leftarrow \{j : (y'_j, x_i) \in M_c\}$
9: Output cost $p(\mathcal{J}(M_c)) \leftarrow \sum_{j \in \mathcal{J}(M_c)} p_y(j)$.

4.3. Solving minimum cost disjoint cycle problem

In this subsection, we establish a relation between disjoint cycle condition and perfect matching problem. Specifically, we show that when inputs and outputs are constrained and each input and output are associated with cost, then satisfying disjoint cycle condition using a minimum cost input–output set is equivalent to solving a minimum cost perfect matching problem on a bipartite graph defined on the structured system.

A structured system $(\bar{A}, \bar{B}, \bar{C})$ with feedback matrix $\bar{K}$ and cost vectors p_u, p_y is said to satisfy the minimum cost disjoint cycle condition if all the state vertices are spanned by disjoint union of cycles in the system digraph by using an input–output set of the least cost. That is, we need to find an input set $\mathcal{I}^\star_c \subseteq \{1, \ldots, m\}$ and an output set $\mathcal{J}^\star_c \subseteq \{1, \ldots, p\}$ such that all x_i's are spanned by disjoint cycles in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}^\star_c}, \bar{C}_{\mathcal{J}^\star_c}, \bar{K}_{(\mathcal{I}^\star_c \times \mathcal{J}^\star_c)})$ and $p(\mathcal{I}^\star_c, \mathcal{J}^\star_c) \leqslant p(\mathcal{I}, \mathcal{J})$ for any input set $\mathcal{I} \subseteq \{1, \ldots, m\}$ and output set $\mathcal{J} \subseteq \{1, \ldots, p\}$ that satisfy disjoint cycle condition in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}}, \bar{C}_{\mathcal{J}}, \bar{K}_{(\mathcal{I} \times \mathcal{J})})$. Specifically, we consider the optimization problem.

Problem 4.11. Given $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ and cost vectors p_u and p_y, find $(\mathcal{I}^\star_c, \mathcal{J}^\star_c) \in \arg \min_{\substack{\mathcal{I}_C \subseteq \{1, \ldots, m\} \\ \mathcal{J}_C \subseteq \{1, \ldots, p\}}} p(\mathcal{I}_c, \mathcal{J}_c)$, such that all x_i's lie in finite disjoint union of cycles in the digraph
$\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}_C}, \bar{C}_{\mathcal{J}_C}, \bar{K}_{(\mathcal{I}_C \times \mathcal{J}_C)})$.

We refer to Problem 4.11 as *the minimum cost disjoint cycle problem*. Now we reduce the minimum cost disjoint cycle problem to a minimum cost perfect matching problem.

Pseudocode for reducing the minimum cost disjoint cycle problem to a minimum cost perfect matching problem is presented in Algorithm 4.3. The bipartite graph $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ constructed in Pequito et al. (2015) for a special case is used here to guarantee condition (b) of Proposition 3.2 for a general case. Given the bipartite graph $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ and the cost function c defined as in Step 3, we find a perfect matching M_c. On obtaining a perfect matching M_c, we define the associated cost $c(M_c)$ as the sum of the costs of edges that are present in M_c (Step 5). The input index set selected under M_c defined as $\mathcal{I}(M_c)$ is the set of indices of u_i's that are connected to some state vertices in M_c (Step 6) and its cost is defined as $p(\mathcal{I}(M_c))$ (Step 7). Now, the output index set selected under M_c defined as $\mathcal{J}(M_c)$ consists of indices of all outputs y_j's that are connected to some state vertices in M_c (Step 8) and its cost is defined as $p(\mathcal{J}(M_c))$ (Step 9).

We denote an optimal solution to the minimum cost perfect matching problem as $M^\star$ and the optimal cost as $c^\star$. Also, an optimal solution to Problem 4.3 is denoted as $(\mathcal{I}_c^\star, \mathcal{J}_c^\star)$ and the optimal input–output cost is denoted as $(p^\star_{cu}+p^\star_{cy})$. We prove the following theorem to give a necessary and sufficient condition for condition (b) in Proposition 3.2 for the sake of completeness.

Theorem 4.12. *Consider a structured system $(\bar{A}, \bar{B}, \bar{C})$ with feedback matrix $\bar{K}$. Then, the bipartite graph $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ has a perfect matching if and only if all states are spanned by disjoint union of cycles in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$.*

Proof. Only-if part: We assume that the system bipartite graph $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ has a perfect matching and prove that all state nodes are spanned by disjoint union of cycles in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$. Let M be a perfect matching in $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$. Define an edge set $\mathcal{E}' := \{(u_i', u_i), (y_j', y_j)\} \in M$ for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, p\}$. Thus edges in $M \setminus \mathcal{E}'$ correspond to edges in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ such that there exist one incoming edge and one outgoing edge corresponding to every vertex in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ except nodes u_i's and y_j's that has edges in $\mathcal{E}'$. Since corresponding to edges in $M \setminus \mathcal{E}'$ every vertex has both in-degree and out-degree one, these edges correspond to disjoint cycles in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$. Note that all state vertices lie in $M \setminus \mathcal{E}'$. Hence, all x_i's are spanned by disjoint union of cycles.

If part: We assume that there exists disjoint union of cycles that span all state nodes in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ and prove that there exists a perfect matching in $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$. Since the cycles are disjoint, each of its node has one incoming edge and one outgoing edge. Each edge in the cycle corresponds to an edge in the bipartite graph. Vertices in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ that are uncovered by these cycles belong to the set of input and output nodes only. For such nodes there exist edges (u_i', u_i) for all $i \in \{1, \dots, m\}$ and (y_j', y_j) for all $j \in \{1, \dots, p\}$ in $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$. These edges along with the cycle edges yield a perfect matching. □

Lemma 4.13. *Consider a structured system $(\bar{A}, \bar{B}, \bar{C})$ with feedback matrix $\bar{K}$ and cost vectors p_u and p_y. Let M_c be a perfect matching in $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ and $\mathcal{I}(M_c)$, $\mathcal{J}(M_c)$ denote the index set of inputs and index set of outputs selected under M_c respectively. Then, all x_i's lie in disjoint cycles in the digraph $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}(M_c)}, \bar{C}_{\mathcal{J}(M_c)}, \bar{K}_{(\mathcal{I}(M_c) \times \mathcal{J}(M_c))})$ and $p(\mathcal{I}(M_c), \mathcal{J}(M_c)) = c(M_c)$.*

See the Appendix for the proof. Now we prove that minimum cost perfect matching problem on $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ with cost c can be used to solve the minimum cost disjoint cycle problem.

Theorem 4.14. *Consider a structured system $(\bar{A}, \bar{B}, \bar{C})$ with feedback matrix $\bar{K}$ and cost vectors p_u, p_y. Let $(\mathcal{I}_c^\star, \mathcal{J}_c^\star)$ be an optimal solution to Problem 4.11 and let $c^\star$ be the optimal cost of the minimum cost perfect matching problem on $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$. Then, $c^\star = p(\mathcal{I}_c^\star, \mathcal{J}_c^\star)$. Moreover, the input index set and output index set selected under Algorithm 4.3 provide an optimal solution to Problem 4.11.*

Proof. Given $(\mathcal{I}_c^\star, \mathcal{J}_c^\star)$ is an optimal solution to Problem 4.11. Then, from Theorem 4.12 there exists a perfect matching in $\mathcal{B}(\bar{A}, \bar{B}_{\mathcal{I}_c^\star}, \bar{C}_{\mathcal{J}_c^\star}, \bar{K}_{(\mathcal{I}_c^\star \times \mathcal{J}_c^\star)})$. Let M be an optimum matching in $\mathcal{B}(\bar{A}, \bar{B}_{\mathcal{I}_c^\star}, \bar{C}_{\mathcal{J}_c^\star}, \bar{K}_{(\mathcal{I}_c^\star \times \mathcal{J}_c^\star)})$. Then, $c(M) \leqslant p(\mathcal{I}_c^\star, \mathcal{J}_c^\star)$. Note that $\widetilde{M} = M \cup \{(u_i', u_i) : i \notin \mathcal{I}_c^\star\} \cup \{(y_j', y_j) : j \notin \mathcal{J}_c^\star\}$ is a matching in $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$. Also $c(\widetilde{M}) = c(M)$. Thus $c(\widetilde{M}) = c^\star \leqslant p(\mathcal{I}_c^\star, \mathcal{J}_c^\star)$.

Now let $M^\star$ be an optimal solution to the minimum cost perfect matching problem in $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$. Then $c(M^\star) = c^\star$. By Theorem 4.12 there exist disjoint cycles whose union span all x_i's in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}(M^\star)}, \bar{C}_{\mathcal{J}(M^\star)}, \bar{K}_{(\mathcal{I}(M^\star) \times \mathcal{J}(M^\star))})$. Let the input–output set used in these cycles be $(\mathcal{I}, \mathcal{J})$. Now $p(\mathcal{I}, \mathcal{J}) \leqslant c^\star$. Also, $p(\mathcal{I}_c^\star, \mathcal{J}_c^\star) \leqslant p(\mathcal{I}, \mathcal{J})$. Thus $p(\mathcal{I}_c^\star, \mathcal{J}_c^\star) \leqslant c^\star$. Combining both, we get $p(\mathcal{I}_c^\star, \mathcal{J}_c^\star) = c^\star$.

Algorithm 4.4 Pseudocode for solving minimum cost accessibility, sensability and disjoint cycle problems

Input: Structured system $(\bar{A}, \bar{B}, \bar{C})$, unconstrained feedback matrix $\bar{K}$, input cost vector p_u, output cost vector p_y
Output: Input set $\mathcal{I}_a$ and output set $\mathcal{J}_a$

1: Find approximate solution to the minimum cost accessibility problem on $(\bar{A}, \bar{B}, p_u)$, say $\hat{\mathcal{I}}_{\mathcal{A}}^\star$
2: Find approximate solution to the minimum cost sensability problem on $(\bar{A}, \bar{C}, p_y)$, say $\hat{\mathcal{J}}_{\mathcal{A}}^\star$
3: Find optimal solution to the minimum cost disjoint cycle problem on $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ under cost function c, say $(\mathcal{I}_c^\star, \mathcal{J}_c^\star)$
4: $\mathcal{I}_a \leftarrow \hat{\mathcal{I}}_{\mathcal{A}}^\star \cup \mathcal{I}_c^\star$ and $\mathcal{J}_a \leftarrow \hat{\mathcal{J}}_{\mathcal{A}}^\star \cup \mathcal{J}_c^\star$

Now assume that $M^\star$ is an optimal solution to the minimum cost perfect matching problem with cost $c^\star$ and then show that input–output set $(\mathcal{I}(M^\star), \mathcal{J}(M^\star))$ selected under $M^\star$ is an optimal solution to Problem 4.11, i.e., all x_i's lie in disjoint union of cycles in the digraph $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}(M^\star)}, \bar{C}_{\mathcal{J}(M^\star)}, \bar{K}_{(\mathcal{I}(M^\star) \times \mathcal{J}(M^\star))})$ and $p(\mathcal{I}(M^\star), \mathcal{J}(M^\star)) = p(\mathcal{I}_c^\star, \mathcal{J}_c^\star)$.

Since $M^\star$ is a solution to the minimum cost perfect matching problem, by Lemma 4.13, there are disjoint cycles in the digraph $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}(M^\star)}, \bar{C}_{\mathcal{J}(M^\star)}, \bar{K}_{(\mathcal{I}(M^\star), \mathcal{J}(M^\star))})$ such that all state nodes lie in their union. Thus $(\mathcal{I}(M^\star), \mathcal{J}(M^\star))$ is a feasible solution to Problem 4.11. To prove minimality we use a contradiction argument. Let us assume that $M^\star$ is an optimal matching but $(\mathcal{I}(M^\star), \mathcal{J}(M^\star))$ is not an optimal solution to Problem 4.11. Then there exist $\mathcal{I}_c' \subset \{1, \dots, m\}$ and $\mathcal{J}_c' \subset \{1, \dots, p\}$ that satisfy the disjoint cycle condition in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}_c'}, \bar{C}_{\mathcal{J}_c'}, \bar{K}_{(\mathcal{I}_c' \times \mathcal{J}_c')})$ and $p(\mathcal{I}_c', \mathcal{J}_c') < p(\mathcal{I}(M^\star), \mathcal{J}(M^\star))$. Then by Theorem 4.12 there exists a perfect matching M' in $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ such that $\mathcal{I}(M') = \mathcal{I}_c'$ and $\mathcal{J}(M') = \mathcal{J}_c'$. Using Lemma 4.13, $c(M') < c^\star$. This gives a contradiction to the assumption that $M^\star$ is an optimal matching. This completes the proof. □

Hence, an optimal solution $M^\star$ to the minimum cost perfect matching problem gives a minimum cost input–output set $(\mathcal{I}_c^\star, \mathcal{J}_c^\star)$ that satisfies the disjoint cycle condition. There exist efficient polynomial time minimum cost perfect matching algorithms to solve Problem 4.11 (Diestel, 2000). In the next subsection we give a polynomial time algorithm to approximate Problem 2.2.

4.4. Approximation algorithm for the optimal input–output and feedback co-design problem

In this section, we give a polynomial time three stage algorithm to approximate Problem 2.2. The pseudocode for the proposed algorithm is given in Algorithm 4.4. In the first stage of Algorithm 4.4 we solve a weighted set cover problem using the greedy approximation algorithm given in Chvatal (1979) to obtain an approximate solution to the minimum cost accessibility problem. The input index set selected under its solution is denoted by $\hat{\mathcal{I}}_{\mathcal{A}}^\star$ (Step 1). Subsequently, in stage two we solve a weighted set cover problem to approximate the minimum cost sensability problem. The output index set selected under its solution is denoted by $\hat{\mathcal{J}}_{\mathcal{A}}^\star$ (Step 2). In the third stage of the algorithm a minimum cost perfect matching problem is solved on $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ using cost function c. The input–output index set selected under solution to this problem is denoted by $(\mathcal{I}_c^\star, \mathcal{J}_c^\star)$ (Step 3). In one of our main results we prove that $(\hat{\mathcal{I}}_{\mathcal{A}}^\star \cup \mathcal{I}_c^\star, \hat{\mathcal{J}}_{\mathcal{A}}^\star \cup \mathcal{J}_c^\star)$ is an approximate solution to Problem 2.2. Firstly, we prove the following preliminary result.

Lemma 4.15. *Let $\mathcal{D}(\bar{A})$ denote the state digraph of a structured system. Then, exactly one of the following happens:*
- *an SCC in $\mathcal{D}(\bar{A})$ is both non-top linked and non-bottom linked,*
- *an SCC in $\mathcal{D}(\bar{A})$ lies in a path starting at some non-top linked SCC and ending at some non-bottom linked SCC.*

Proof of Lemma 4.15 is presented in the Appendix. The main result of our paper is the following:

Theorem 4.16. *Consider a structured system $(\bar{A}, \bar{B}, \bar{C})$, an unconstrained feedback matrix $\bar{K}$ and cost vectors p_u, p_y. Let n be the number of states in the system and $(\mathcal{I}_a, \mathcal{J}_a)$ be an output of Algorithm 4.4. Then the following hold:*

(i) $(\mathcal{I}_a, \mathcal{J}_a) \in \mathcal{K}$, i.e., $(\bar{A}, \bar{B}_{\mathcal{I}_a}, \bar{C}_{\mathcal{J}_a}, \bar{K}_{(\mathcal{I}_a\times\mathcal{J}_a)})$ has no SFMs,

(ii) $p(\mathcal{I}_a, \mathcal{J}_a) \leqslant (2\log n)p^\star$.

Moreover, there does not exist any polynomial time algorithm that approximates Problem 2.2 to factor $(1 - o(1))\log n$. Thus Algorithm 4.4 is an order optimal approximation algorithm.

Proof. Given $(\mathcal{I}_a, \mathcal{J}_a)$ is an output of Algorithm 4.4. Hence, all states are accessible in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}_a}, \bar{C}, \bar{K})$ and all states are sensable in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}_{\mathcal{J}_a}, \bar{K})$. Thus, all states are both accessible and sensable in the digraph $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}_a}, \bar{C}_{\mathcal{J}_a}, \bar{K}_{(\mathcal{I}_a\times\mathcal{J}_a)})$. Consider an arbitrary state x which belongs to some SCC $\mathscr{N}$. By Lemma 4.15, $\mathscr{N}$ lies on some path from a non-top linked SCC, say $\mathscr{N}$, to a non-bottom linked SCC, say $\mathscr{B}$, in the SCC DAG. Since $U = \{u_i : i \in \mathcal{I}_a\}$ are enough for accessibility, there exists $u \in U$ such that u covers $\mathscr{N}$. Similarly, since $Y = \{y_j : j \in \mathcal{J}_a\}$ are enough for sensability there exists $y \in Y$ such that y covers $\mathscr{B}$. Since $\bar{K}$ is unconstrained (y, u) belongs to the edge set of $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}_a}, \bar{C}_{\mathcal{J}_a}, \bar{K}_{(\mathcal{I}_a\times\mathcal{J}_a)})$. Thus in this digraph all states in all the SCCs of $\mathcal{D}(\bar{A})$ that lie in the path from $\mathscr{N}$ to $\mathscr{B}$ now belong to a single SCC in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}_a}, \bar{C}_{\mathcal{J}_a}, \bar{K}_{(\mathcal{I}_a\times\mathcal{J}_a)})$ which has edge (y, u). Thus x belongs to an SCC in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}_a}, \bar{C}_{\mathcal{J}_a}, \bar{K}_{(\mathcal{I}_a\times\mathcal{J}_a)})$ with (y, u) edge. Since x is arbitrary condition (a) in Proposition 3.2 follows. Since $(\mathcal{I}_a, \mathcal{J}_a)$ is an output of Algorithm 4.4, by Theorem 4.12, there exist disjoint cycles that cover all state nodes using inputs whose indices are in $\mathcal{I}_a$ and outputs whose indices are in $\mathcal{J}_a$. Thus $(\mathcal{I}_a, \mathcal{J}_a)$ satisfies condition (b) in Proposition 3.2. Thus $(\mathcal{I}_a, \mathcal{J}_a) \in \mathcal{K}$. This completes the proof of (i).

Let $\mathcal{I}^\star_{\mathcal{A}}$ and $\mathcal{J}^\star_{\mathcal{A}}$ be optimal solutions to the minimum cost accessibility problem and minimum cost sensability problem respectively. Given $(\mathcal{I}_a, \mathcal{J}_a)$ is an output of Algorithm 4.4. Let $\mathcal{I}_a = \hat{\mathcal{I}}^\star_{\mathcal{A}} \cup \mathcal{I}^\star_{\mathcal{C}}$, where $\hat{\mathcal{I}}^\star_{\mathcal{A}}$ is an ϵ_1-optimal solution to the minimum cost accessibility problem and $\mathcal{I}^\star_{\mathcal{C}}$ is a minimum cost set that satisfies the disjoint cycle condition. Similarly, $\mathcal{J}_a = \hat{\mathcal{J}}^\star_{\mathcal{A}} \cup \mathcal{J}^\star_{\mathcal{C}}$, where $\hat{\mathcal{J}}^\star_{\mathcal{A}}$ is an ϵ_2-optimal solution to the minimum cost sensability problem and $\mathcal{J}^\star_{\mathcal{C}}$ is a minimum cost set that satisfies the disjoint cycle condition. Now by Theorem 4.4, $\epsilon_1 \leqslant \log n$ and by Corollary 4.10, $\epsilon_2 \leqslant \log n$. Since $(\mathcal{I}^\star, \mathcal{J}^\star)$ is an optimal solution to Problem 2.2 its cost is at least the cost of satisfying the two conditions in Proposition 3.2 separately. This gives Eqs. (2) and (3).

$$p(\mathcal{I}^\star, \mathcal{J}^\star) \geqslant p(\mathcal{I}^\star_{\mathcal{A}}) + p(\mathcal{J}^\star_{\mathcal{A}}), \tag{2}$$

$$p(\mathcal{I}^\star, \mathcal{J}^\star) \geqslant p(\mathcal{I}^\star_{\mathcal{C}}, \mathcal{J}^\star_{\mathcal{C}}), \tag{3}$$

$$2p(\mathcal{I}^\star, \mathcal{J}^\star) \geqslant p(\mathcal{I}^\star_{\mathcal{A}}) + p(\mathcal{J}^\star_{\mathcal{A}}) + p(\mathcal{I}^\star_{\mathcal{C}}) + p(\mathcal{J}^\star_{\mathcal{C}}),$$

$$p(\hat{\mathcal{I}}^\star_{\mathcal{A}}) + p(\hat{\mathcal{J}}^\star_{\mathcal{A}}) \leqslant \log n\,(p(\mathcal{I}^\star_{\mathcal{A}}) + p(\mathcal{J}^\star_{\mathcal{A}})), \tag{4}$$

$$\begin{aligned} p(\mathcal{I}^\star, \mathcal{J}^\star) &\geqslant \frac{p(\hat{\mathcal{I}}^\star_{\mathcal{A}}) + p(\hat{\mathcal{J}}^\star_{\mathcal{A}})}{2\,(\log n\,)} + \frac{p(\mathcal{I}^\star_{\mathcal{C}}, \mathcal{J}^\star_{\mathcal{C}})}{2}, \\ &\geqslant \frac{p(\hat{\mathcal{I}}^\star_{\mathcal{A}}, \hat{\mathcal{J}}^\star_{\mathcal{A}}) + p(\mathcal{I}^\star_{\mathcal{C}}, \mathcal{J}^\star_{\mathcal{C}})}{2\,(\log n\,)} = \frac{p(\mathcal{I}_A, \mathcal{J}_A)}{2\,(\log n\,)}. \end{aligned} \tag{5}$$

Eq. (4) holds as $\hat{\mathcal{I}}^\star_{\mathcal{A}}$ and $\hat{\mathcal{J}}^\star_{\mathcal{A}}$ are approximate solutions to the minimum cost accessibility problem and the minimum cost sensability problem respectively, obtained using greedy approximation of their weighted set cover formulations. Eq. (5) holds as $(2\log n\,) \geqslant 1$. This proves (ii).

The weighted set cover problem cannot be approximated to factor $(1 - o(1))\log N$, where N is the cardinality of the universe. Hence, there does not exist any polynomial algorithm that has approximation ratio $(1 - o(1))\log(\max(q, k))$ for Problem 2.2. Note that $\max(q, k) \leqslant n$. Thus there does not exist any polynomial algorithm that has approximation ratio $(1 - o(1))\log n$ for solving Problem 2.2. Thus the proposed algorithm is order optimal approximation algorithm for Problem 2.2. □

In the following theorem we give the complexity of the proposed approximation algorithm.

Theorem 4.17. *Algorithm 4.4 which takes as input a structured system $(\bar{A}, \bar{B}, \bar{C})$ with unconstrained feedback matrix $\bar{K}$ and cost vectors p_u, p_y and gives as output an approximate solution $(\mathcal{I}_a, \mathcal{J}_a)$ to Problem 2.2 has complexity $O(n^3)$, where n denotes the number of states in the system.*

Proof. Given state digraph $\mathcal{D}(\bar{A}) = (V_X, E_X)$ all the non-top linked SCCs can be found in $O(\max(|V_X|, |E_X|))$ computations. Here $|V_X| = n$ and $|E_X|$ is at most $|V_X|^2$. Thus set cover problems can be formulated in $O(n^2)$ computations. The greedy selection scheme for finding the approximate solution to the set cover problem has $O(n)$ complexity (Chvatal, 1979). The minimum cost bipartite matching can be solved in $O(n^3)$ computations. Thus Algorithm 4.4 has $O(n^3)$ complexity. □

In the next section we discuss few special class of systems in the context of Problem 2.2.

5. Special cases

In this section we consider few special cases. Using the approximation algorithm given in Section 4.4 we obtain the approximation results for these cases. In the following subsections we explain each of these cases briefly.

5.1. Irreducible systems

In this subsection we consider systems whose digraph $\mathcal{D}(\bar{A})$ is irreducible, i.e., $\mathcal{D}(\bar{A})$ is a single SCC. Note that for this class of systems Problem 2.2 is not NP-hard (Pequito et al., 2015). Pequito et al. addressed Problem 2.2 along with costs for feedback edges in Pequito et al. (2015) and obtained a polynomial time optimal algorithm. In the following result we prove that the polynomial time algorithm given in this paper also gives an optimal solution to Problem 2.2.

Theorem 5.1. *Consider a structured system $(\bar{A}, \bar{B}, \bar{C})$, feedback matrix $\bar{K}$ and cost vectors p_u, p_y. Let $\mathcal{D}(\bar{A})$ be irreducible. Then, Algorithm 4.4 returns an optimal solution to Problem 2.2.*

Proof. Given $\mathcal{D}(\bar{A})$ is irreducible. Thus condition (a) is satisfied by any (y_j, u_i) edge. Hence Algorithm 4.4 solves only the minimum cost perfect matching problem for satisfying condition (b) optimally. Without loss of generality, let u_i be an input and y_j be an output obtained in the solution, i.e., $i \in \mathcal{I}_a$ and $j \in \mathcal{J}_a$. Then edge (y_j, u_i) satisfies both conditions in Proposition 3.2. In case if $\mathcal{B}(\bar{A})$ has a perfect matching, then connecting the minimum cost input to the minimum cost output satisfies both the conditions in Proposition 3.2. Thus $p(\mathcal{I}_a, \mathcal{J}_a) = p^\star$ and Algorithm 4.4 gives an optimal solution to Problem 2.2. □

5.2. Structurally cyclic systems

In this subsection we consider structurally cyclic systems. These are systems whose state bipartite graph has a perfect matching. There exists a wide class of systems, including multi-agent systems and epidemic dynamical system, which have a so-called *self-damped* property: see Chapman and Mesbahi (2013). In this case condition (b) in Proposition 3.2 is satisfied without using any feedback edge. Thus condition (a) alone has to be considered, i.e. only minimum cost accessibility and minimum cost sensability problems need to be solved. We have the following result for these class of systems.

Theorem 5.2. *Consider a structured system $(\bar{A}, \bar{B}, \bar{C})$, unconstrained feedback matrix $\bar{K}$ and cost vectors p_u, p_y. Let $\mathcal{B}(\bar{A})$ have a perfect matching. Then, Algorithm 4.4 gives a $(\log n)$-optimal solution to Problem 2.2, where n denotes the number of states in the system.*

Proof. Given $\mathcal{B}(\bar{A})$ has a perfect matching. Thus condition (b) is satisfied. Thus we need to solve only the minimum cost accessibility problem and the minimum cost sensability problem. Now following the similar lines given in the proof of Theorem 4.16, we get $p(\mathcal{I}_a, \mathcal{J}_a) \leqslant (\log n)p^\star$. Hence, Algorithm 4.4 gives a $\log n$-optimal solution to Problem 2.2. □

For structurally cyclic systems the proposed algorithm gives a $\log n$ approximate solution. Problem 2.2 is inapproximable to factor $(1 - o(1)) \log n$ even for structurally cyclic systems. Thus, for practically important wide class of systems, the proposed scheme gives the best possible approximation.

5.3. Discrete linear systems

In this subsection we consider Linear Time-Invariant (LTI) discrete control system given by, $x(t+1) = Ax(t)+Bu(t), y(t) = Cx(t)$. For discrete systems we have the following result.

Theorem 5.3. *Consider a discrete structured system $(\bar{A}, \bar{B}, \bar{C})$, unconstrained feedback matrix $\bar{K}$ and cost vectors p_u, p_y. Then, Algorithm 4.4 gives a $(\log n)$-optimal solution to Problem 2.2.*

Proof. In discrete linear time invariant systems, only condition (a) in Proposition 3.2 has to be satisfied, since uncontrollable and unobservable modes of the system at origin are not of concern. Thus Algorithm 4.4 needs to solve only the minimum cost accessibility problem and the minimum cost sensability problem. Hence, we get a $(\log n)$-optimal solution. □

Theorems 5.2 and 5.3 thus conclude that Algorithm 4.4 outputs a solution with the best possible approximation guarantee to Problem 2.2 for structurally cyclic systems and discrete linear systems. This completes the discussion of the approximation results for various special classes of systems considered.

6. Illustrative example

In this section, we present an example that illustrates Algorithm 4.4. Consider a structured system $(\bar{A}, \bar{B}, \bar{C})$:

$$\bar{A} = \begin{bmatrix} 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \star & \star & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \star & \star & \star & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & \star & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 & \star \end{bmatrix}, \bar{B} = \begin{bmatrix} \star & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \star & 0 & 0 \\ 0 & \star & 0 & 0 \\ 0 & 0 & \star & 0 \\ 0 & 0 & \star & 0 \\ 0 & 0 & \star & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \bar{C}^\top = \begin{bmatrix} 0 & 0 & 0 \\ \star & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \star & \star \\ 0 & 0 & \star \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let the input and output cost vectors be $p_u = [50, 10, 100, 25]^\top$ and $p_y = [10, 100, 45]^\top$, respectively. Fig. 2 shows the digraph of the structured system described above. The structured system

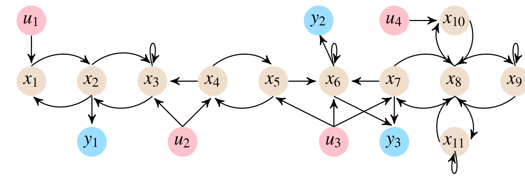


Fig. 2. The digraph $\mathcal{D}(\bar{A}, \bar{B}, \bar{C})$ of the structured system $(\bar{A}, \bar{B}, \bar{C})$ given in Section 6.

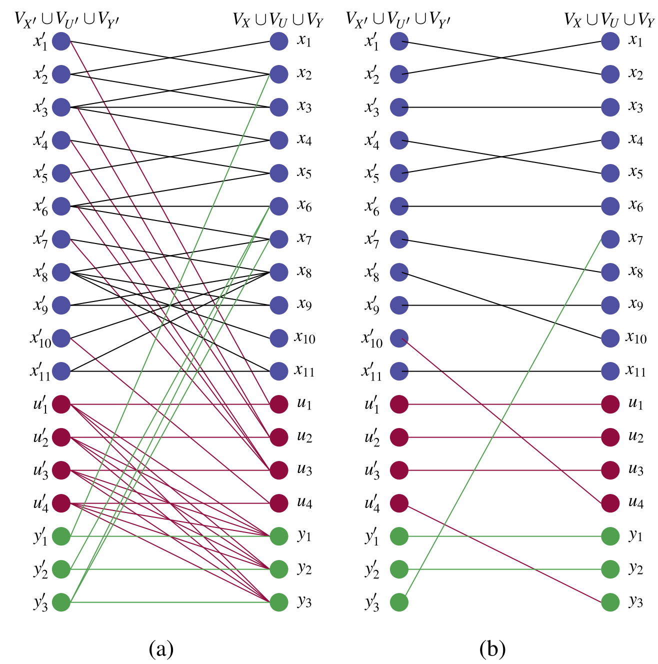


Fig. 3. (a) The bipartite graph $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ and (b) A minimum cost perfect matching in the bipartite graph $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ corresponding to the structured system $(\bar{A}, \bar{B}, \bar{C})$ given in Fig. 2.

has two non-top linked SCCs, $\{x_4, x_5\}$, $\{x_7, x_8, x_9, x_{10}, x_{11}\}$, and two non-bottom linked SCCs, $\{x_1, x_2, x_3\}$, and $\{x_6\}$. The bipartite graph $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ corresponding to the structured system in Fig. 2 is given in Fig. 3(a).

All the edges in the bipartite graph have zero cost except the following: $c(x'_1, u_1) = 50$, $c(x'_3, u_2) = 10$, $c(x'_4, u_2) = 10$, $c(x'_5, u_3) = 100$, $c(x'_6, u_3) = 100$, $c(x'_7, u_3) = 100$, $c(x'_{10}, u_4) = 25$, $c(y'_1, x_2) = 10$, $c(y'_2, x_6) = 100$, $c(y'_3, x_6) = 45$, $c(y'_3, x_7) = 45$, $c(u'_1, y_1) = 60$, $c(u'_1, y_2) = 150$, $c(u'_1, y_3) = 95$, $c(u'_2, y_1) = 20$, $c(u'_2, y_2) = 110$, $c(u'_2, y_3) = 55$, $c(u'_3, y_1) = 110$, $c(u'_3, y_2) = 200$, $c(u'_3, y_3) = 145$, $c(u'_4, y_1) = 35$, $c(u'_4, y_2) = 125$, and $c(u'_4, y_3) = 70$. Fig. 3(b) shows a minimum cost perfect matching. The edges in the matching with nonzero cost are: (x'_{10}, u_4), (u'_4, y_3), (y'_3, x_7). The cost of the matching is $p_u(4) + p_y(3) = 70$. The set of inputs selected under the matching is $\{u_4\}$ and the set of outputs selected under the matching is $\{y_3\}$. Thus $\mathcal{I}^\star_c = \{4\}$ and $\mathcal{J}^\star_c = \{3\}$. Combining the results, we get $\mathcal{I}_a = \hat{\mathcal{I}}^\star_A \cup \mathcal{I}^\star_c = \{2, 4\}$ and $\mathcal{J}_a = \hat{\mathcal{J}}^\star_A \cup \mathcal{J}^\star_c = \{1, 3\}$. This completes the minimum cost input–output and feedback co-design for the given structured system using Algorithm 4.4.

7. Conclusion

This paper dealt with optimal input–output and feedback co-design problem (Problem 2.2) for structured LTI systems when the input and output matrices are constrained and each input and output is associated with a cost. The feedback pattern is unconstrained and the cost of a feedback edge is the sum of cost of the input and output associated with it. We focused on finding a minimum cost input–output set and feedback pattern such that the closed-loop system has no SFMs. This problem is known to be NP-hard (Pequito et al., 2015). In this paper, we proposed a polynomial time algorithm to find an approximate solution to the problem by splitting the problem into three subproblems: minimum cost accessibility, minimum cost sensability, and minimum cost disjoint cycle. We proved that the minimum cost accessibility problem and the minimum cost sensability problem are *equivalent* to the weighted set cover problem. Further, we proved that the minimum cost disjoint cycle problem is equivalent to a minimum cost perfect matching problem on a bipartite graph constructed from the structured system and with a suitably defined cost function. Using these equivalence results, we proposed a polynomial time algorithm that returns a $(2\log n)$-optimal solution to Problem 2.2. We also proved that there exists no polynomial time algorithm that approximates Problem 2.2 to a factor $(1-o(1))\log n$; thus the proposed algorithm gives an order optimal approximate solution. We applied our general result on special class of systems of practical interest, like structurally cyclic systems and discrete systems. In these cases, we showed that output of the algorithm is a $(\log n)$-optimal solution, which is the optimal possible approximation factor.

Appendix

Proof of Lemma 4.2. Given that $\mathcal{S}$ is a feasible solution to the weighted set cover problem. Thus $\cup_{\mathcal{S}_i\in\mathcal{S}}\mathcal{S}_i = \mathcal{U}$. Hence, $\mathcal{I}(\mathcal{S}) = \{i : \mathcal{S}_i \in \mathcal{S}\}$ covers all the non-top linked SCCs in $\mathcal{D}(\bar{A})$. By Corollary 3.5 this implies that all states are accessible in the digraph $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}(\mathcal{S})}, \bar{C}, \bar{K})$. Now steps 5, 7, 8 and 9 of Algorithm 4.1 prove $p(\mathcal{I}(\mathcal{S})) = w(\mathcal{S})$. □

Proof of Lemma 4.5. Given state digraph $\mathcal{D}(\bar{A}) = (V_X, E_X)$, all the non-top linked SCCs can be found in $O(\max(|V_X|, |E_X|))$ computations. Here $|V_X| = n$ and $|E_X|$ is at most $|V_X|^2$. Thus the reduction in Algorithm 4.1 has $O(n^2)$ computations. Also, given a cover $\mathcal{S}$ one can obtain $\mathcal{I}(\mathcal{S})$ and $p(\mathcal{I}(\mathcal{S}))$ in linear time. □

Proof of Lemma 4.6. Given $\mathcal{I}$ is a feasible solution to Problem 4.1. Thus all states are accessible in $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}}, \bar{C}, \bar{K})$. This implies for $\mathcal{S}(\mathcal{I}) = \{\mathcal{S}_i : i \in \mathcal{I}\}$, $\cup_{\mathcal{S}_i\in\mathcal{S}(\mathcal{I})}\mathcal{S}_i = \mathcal{U}$. Thus by Corollary 3.5 $\mathcal{S}(\mathcal{I})$ covers $\mathcal{U}$. Now Steps 4, 7, 8 and 9 of Algorithm 4.2 give $w(\mathcal{S}(\mathcal{I})) = p(\mathcal{I})$. □

Proof of Lemma 4.9. Given any weighted set cover problem $\mathcal{U}, \mathcal{P}, w$, matrices $\bar{A}, \bar{B}$ can be found in $O(N)$, $O(N^2)$ computations respectively. Also, cost vector p_u can be found in linear time. Thus the reduction of the set cover problem to an instance of Problem 4.1 given in Algorithm 4.2 has $O(N^2)$ computations. Also, given a set $\mathcal{I}$ we can obtain $\mathcal{S}(\mathcal{I})$ and $w(\mathcal{S}(\mathcal{I}))$ in linear time and this completes the proof. □

Proof of Lemma 4.13. Given M_c is a perfect matching in the bipartite graph $\mathcal{B}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ with cost function c. By Theorem 4.12, there exist disjoint cycles that cover all state nodes in the digraph $\mathcal{D}(\bar{A}, \bar{B}_{\mathcal{I}(M_c)}, \bar{C}_{\mathcal{J}(M_c)}, \bar{K}_{(\mathcal{I}(M_c)\times\mathcal{J}(M_c))})$. Now we get $p(\mathcal{I}(M_c)) + p(\mathcal{J}(M_c)) = p(\mathcal{I}(M_c), \mathcal{J}(M_c)) = c(M_c)$ by Steps 3, 6, 9 of Algorithm 4.3. This completes the proof. □

Proof of Lemma 4.15. Consider the Directed Acyclic Graph (DAG) whose vertices are the SCCs in $\mathcal{D}(\bar{A})$ and an edge exists between two nodes if and only if there exists an edge connecting two states in those respective SCCs in $\mathcal{D}(\bar{A})$. The nodes in the DAG are of two types: (i) isolated, and (ii) has an incoming and/or outgoing edge. In case (i) the corresponding SCC is both non-top linked and non-bottom linked. In case (ii) it has either an incoming edge or an outgoing edge or both. Thus those SCCs lie in some path from some non-top linked SCC to some non-bottom linked SCC since the DAG is acyclic. □

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