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Sparsest Feedback Selection for Structurally Cyclic Systems with Dedicated Actuators and Sensors in Polynomial Time

Shana Moothedath*, Prasanna Chaporkar†, and Madhu N. Belur†

***Abstract*—This paper deals with optimal feedback selection problem in linear time-invariant (LTI) structured systems for arbitrary pole placement, an important open problem in structured systems. Specifically, we solve the *sparsest feedback selection* problem for LTI structured systems. In this paper, we consider *structurally cyclic* systems with *dedicated* inputs and outputs. For this class of systems, we prove that the sparsest feedback selection problem is equivalent to the strong connectivity augmentation problem in graph theory. We present an $O(n^2)$ algorithm to find a sparsest feedback matrix for structured systems when every state is actuated by a dedicated input and every state is sensed by a dedicated output, where n denotes the number of states in the system. If the inputs and the outputs are such that not every state is actuated by a dedicated input and/or not every state is sensed by a dedicated output, then we provide an $O(n^3)$ algorithm for the sparsest feedback selection problem. These results show that sparsest feedback selection with dedicated i/o is a specific case of the optimal feedback selection problem that is solvable in polynomial time. We later analyze the sparsest feedback selection problem for structurally cyclic systems when both the input and the output are dedicated and the feedback pattern is *constrained*. When some of the feedback links are forbidden, we prove that the problem is NP-hard. The results in this paper along with the previously known results conclude that the optimal feedback selection problem is polynomial-time solvable only for the dedicated input-output case without forbidden feedback links and also without weights for the feedback links.**

***Index Terms*—Linear structured systems, Arbitrary pole placement, Linear output feedback, Sparsest feedback selection.**

1. Introduction

Feedback selection for control systems that guarantees arbitrary placement of the closed-loop poles is a fundamental design problem in control theory. The challenging part of the design problem is to accomplish an optimal design, for example in the sense of number of connections or weight of connections. We consider feedback selection in large scale linear dynamical systems. The analysis done in this paper is based on the zero/nonzero pattern (*sparsity*) of the system. The rationale behind performing this analysis is, in most large scale systems and real time systems, the numerical values of the nonzero entries in the system description are either not known at all, like social networks, biological systems, or they are not known accurately, like electric networks, power grids and multi-agent systems. To this end, various system properties of these systems are studied using their sparsity pattern and the framework used is referred to as *structural analysis* [1].

Structural analysis of linear control systems, namely *structural controllability* was introduced by Lin in [2]. Over last few decades, this area has gained interest due to its applicability in various complex systems: see [3] and references therein for details. Structural analysis of behavioral systems is addressed in papers [4], [5]. This paper discusses sparsest feedback selection that guarantees arbitrary pole placement. A necessary and sufficient graph theoretic condition to guarantee arbitrary pole placement in a closed-loop system is given in [6] using the concept of fixed modes[1] [7].

Given a large scale dynamical system, *optimal feedback selection problem* consists of finding an optimal (in the sense of cardinality or weight) set of feedback edges, i.e., which output should be fed to which input, such that one has guaranteed arbitrary pole placement of the closed-loop system. In other words, given the digraph representing the state dynamics, the inputs, and the outputs of the system, the objective is to find an optimal set of feedback connections that ensures the desired design objective. There are different settings of the optimal feedback selection problem depending on whether the inputs and the outputs are *dedicated*[2] or non-dedicated and the feedback links are *weighted*[3] or not. Further, like most papers in this area, in this paper we consider a subclass of systems referred as structurally cyclic systems. A system is said to be *structurally cyclic* if all the vertices of the state digraph (see Section 2-C) are spanned by disjoint cycles. Before delving into the implication of the structurally cyclic assumption, the algorithmic complexity results across the literature and this paper of the optimal feedback selection problem for structurally cyclic systems for different problem settings are consolidated in the table below.

Table I: Algorithmic complexity results of the optimal feedback selection problem

Input and Output \ Feedback links	Non-weighted	Weighted
Dedicated	P (this paper)	NP-hard [8]
Non-dedicated	NP-hard [9]	NP-hard [9]

The class of structurally cyclic systems is wide: for example, *self-damped* systems (see [10]) including consensus dynamics in multi-agent systems and epidemic equations. Further, for systems whose state matrices are invertible, the state digraph is

*Department of Electrical and Computer Engineering, University of Washington, Seattle, USA. Email: sm15@uw.edu.
†Department of Electrical Engineering, Indian Institute of Technology Bombay, India. Email: {chaporkar, belur}@ee.iitb.ac.in.

[1] While absence of structurally fixed modes is *necessary* for arbitrary pole placement to be possible, output feedback in general has to be *dynamic* to achieve arbitrary pole placement.

[2] An input (output, resp.) is said to be *dedicated* if it can actuate (sense, resp.) a single state only.

[3] If a feedback pattern consists of feedback edges without weights, then it is understood that *all* outputs can possibly be fed back to all inputs, i.e., no feedback edge is 'forbidden'. On the other hand, if some of the outputs *cannot* be fed to some of the inputs and there is no weight associated with all the possible feedback links, then we refer to this case as 'sparsest feedback with forbidden set'.

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structurally cyclic. For the class of structurally cyclic systems, we prove that finding a sparsest feedback matrix has $O(n^2)$ complexity, where n denotes the number of states in the system, when each state is actuated by an input and each state is sensed by an output. For this class of systems, we also prove that when not every state is connected to an actuator and not every state is connected from a sensor, then finding a sparsest feedback matrix has $O(n^3)$ complexity. Note that here the feedback matrix is unconstrained, i.e., it is possible to connect any output to any input. Later we analyze the problem after imposing a restriction that some of the feedback links are forbidden. With this restriction, we show that the sparsest feedback selection problem on structurally cyclic systems is NP-hard when both the inputs and the outputs are dedicated. This paper along with the previously known NP-hardness results in [8], [9] concludes that the optimal feedback selection problem is polynomially solvable only for the case where both the inputs and the outputs are dedicated and the feedback pattern is unconstrained without link weights.

The organization of this paper is as follows. In Section 2, we formally define the sparsest feedback selection problem and describe related work in this area. We also provide some preliminaries and state some known results that we use subsequently. In Section 3, we provide algorithms for solving Problem 1 and prove their optimality. We also show that an additional constraint of the feedback pattern, i.e., few feedback links being forbidden, makes the problem NP-hard. Finally, we conclude in Section 4.

2. Problem Formulation, Related Work and Preliminaries

In this section, we formulate the sparsest feedback selection problem and then describe related work in this area and few preliminaries used in the sequel.

A. Problem Formulation

Consider a linear time-invariant system $\dot{x} = Ax + Bu$, $y = Cx$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$. Here $\mathbb{R}$ denotes the set of real numbers. The structural representation of this system referred to as *structured system* is denoted by $(\bar{A}, \bar{B}, \bar{C})$, where $\bar{A} \in \{0,\star\}^{n\times n}, \bar{B} \in \{0,\star\}^{n\times m}$, and $\bar{C} \in \{0,\star\}^{p\times n}$ has the same structure as that of A, B and C, respectively. Here 0 entries are fixed zeros while $\star$ entries denote free independent parameters. More precisely,

$$\begin{aligned} A_{ij} &= 0 \text{ whenever } \bar{A}_{ij} = 0, \text{ and} \\ B_{ij} &= 0 \text{ whenever } \bar{B}_{ij} = 0, \text{ and} \\ C_{ij} &= 0 \text{ whenever } \bar{C}_{ij} = 0. \end{aligned} \tag{1}$$

Any (A, B, C) that satisfies (1) is referred to as a *numerical realization* of the structured system $(\bar{A}, \bar{B}, \bar{C})$. Let $\bar{K} \in \{0,\star\}^{m\times p}$ denotes a feedback matrix, where $\bar{K}_{ij} = \star$ if the j^{th} output is fed to the i^{th} input. We define, $[K] := \{K : K_{ij} = 0, \text{ if } \bar{K}_{ij} = 0\}$.

Definition 1. *The structured system $(\bar{A}, \bar{B}, \bar{C})$ and the feedback matrix $\bar{K}$ is said to have no structurally fixed modes (SFMs) if there exists a numerical realization (A, B, C) of $(\bar{A}, \bar{B}, \bar{C})$ such that $\cap_{K\in[K]} \sigma(A + BKC) = \emptyset$, where $\sigma(T)$ denotes the set of eigenvalues of a square matrix T.*

Consider a given structured matrix $\bar{K}^G \in \{0,\star\}^{m\times p}$. We say $\bar{K} \preceq \bar{K}^G$ if $\bar{K}$ satisfies the following conditions: (i) the dimension of $\bar{K}$ is same as that of $\bar{K}^G$, and (ii) $\bar{K}_{ij} = \star$ only if $\bar{K}^G_{ij} = \star$. Given a structured system $(\bar{A}, \bar{B}, \bar{C})$ and a feedback configuration $\bar{K}^G$, our aim in this paper is to find a feedback pattern $\bar{K}$ with minimum number of nonzero entries (i.e., sparsest) such that $\bar{K} \preceq \bar{K}^G$ and the closed-loop structured system $(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ has no SFMs. Let $\mathcal{K}_s := \{\bar{K} \in \{0,\star\}^{m\times p} : \bar{K} \preceq \bar{K}^G \text{ and } (\bar{A}, \bar{B}, \bar{C}, \bar{K}) \text{ has no SFMs}\}$. For a structured system $(\bar{A}, \bar{B}, \bar{C})$, without loss of generality, we assume that $\mathcal{K}_s$ is non-empty. Specifically, $\bar{K}^G \in \mathcal{K}_s$. Next, we describe the problem addressed in this paper.

Problem 1. *Given a structured system $(\bar{A}, \bar{B}, \bar{C})$ and feedback configuration $\bar{K}^G$, find*

$$\bar{K}^\star \in \operatorname*{arg\,min}_{\bar{K}\in\mathcal{K}_s} \|\bar{K}\|_0 .$$

Here $\|\cdot\|_0$ denotes the zero matrix norm[4]. We refer to Problem 1 as the *sparsest feedback selection problem*.

B. Related Work

Finding sparsest feedback matrix for a given structured system is considered in [11]. The approach given in [11] requires a minimum cardinality input-output to be found, which in itself is an NP-hard problem [12], [13]. The authors in [14] discuss minimum weight feedback selection, which is a more general problem. The approach given in [14] requires solving a multi-commodity network flow problem: an NP-hard problem. Thus neither [11] nor [14] yield a polynomial time algorithm to the feedback selection problem, and hardness of the sparsest feedback selection problem remained unsolved. For a structured state matrix, the problem of finding *jointly* sparsest input, output and feedback matrices is addressed in [15]. In [15] there is no restriction on the structure of the input and output matrices. The approach in [15] first finds a minimum i/o set for accessibility and sensability and then performs so-called mixed pairing between them to get a sparsest feedback matrix. This approach is possible as the i/o structure is unconstrained and non-dedicated. Finding a minimum i/o set for accessibility and sensability itself is NP-hard [12] when the i/o structure is constrained. Further, the mixed pairing concept critically uses the non-dedicated nature of the i/o set. Due to these two factors, the approach given in [15] can not solve Problem 1. On the contrary, [16] considered the problem when there is *no flexibility* in choosing the input and output matrices. Given structured state, input and output matrices and weights associated with each of them (i.e, each input, output and feedback edge is associated with a weight), finding a minimum weight input set, a minimum weight output set, and a minimum weight feedback matrix is addressed in [16]. This problem is known to be NP-hard [12] and hence [16] considered a special class of systems where the state matrix is *irreducible*[5]. For the case when the state, input and output matrices are fixed, the problem of finding a sparsest feedback matrix has been studied in [17].

[4]Although $\|\cdot\|_0$ does not satisfy some of the norm axioms, the number of nonzero entries in a matrix is conventionally referred to as the *zero norm*.

[5]A directed graph is said to be irreducible if there exists a directed path between any two vertices, equivalently, the graph is strongly connected.

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In the context of the complexity of the optimal feedback selection problem for structurally cyclic systems, we proved the NP-hardness and the constant factor inapproximability for the non-dedicated input-output case (when feedback links are weighted and non-weighted) recently in [9]. Paper [9] then considered a special graph topology called line graph systems where an optimal solution is obtained using a dynamic programming-based algorithm in polynomial time. We also proved the NP-hardness and the constant factor inapproximability for the dedicated input-output case with weighted feedback links in [8]. Special graph topologies called back-edge feedback structure and hierarchical networks are studied in [8], where polynomial time solutions are obtained. This paper addresses the case where both the inputs and the outputs are dedicated and feedback links are non-weighted. The paper [17] claim NP-hardness of this case for structurally cyclic systems. We present a polynomial time algorithm to obtain an optimal solution and prove that this case is in P. We describe below briefly the error in the proof and the result given in [17].

Remark 1. *Reference [17] shows that when $\bar{K}^G_{ij} = \star$ for all i, j (i.e., all feedback links allowed), an optimal solution $\bar{K}^\star$ to Problem 1 which has two SCCs in $\mathcal{D}(\bar{A},\bar{B},\bar{C},\bar{K}^\star)$ gives a solution to a known NP-hard problem, the graph decomposition problem. This does not answer the case where an optimal solution $\bar{K}^\star$ results in a different number of SCCs in $\mathcal{D}(\bar{A},\bar{B},\bar{C},\bar{K}^\star)$, specifically one SCC. For the special case considered in the NP-hardness proof in [17], there always exists an optimal solution, $\bar{K}^\star$, that results in a single SCC in $\mathcal{D}(\bar{A},\bar{B},\bar{C},\bar{K}^\star)$ (Corollary 1). The (modified) problem that is shown to be indeed NP-hard in [17] can be stated as: given a structured system $(\bar{A},\bar{B},\bar{C})$, find $\bar{K}$ such that $\mathcal{D}(\bar{A},\bar{B},\bar{C},\bar{K})$ has two SCCs.*

Paper [17] also claimed the NP-hardness of the non-dedicated i/o case by the straightforward application of their NP-hardness claim and proof for the dedicated i/o case. This claim does not hold, since dedicated i/o case is not NP-hard. The complexity result for the non-dedicated i/o case is given in [9].

C. Preliminaries

Graph theory is a key tool in the analysis of structured systems since a structured system can be represented as a digraph and there exist necessary and sufficient graph theoretic conditions for various structural properties of the system [1]. Given a structured system $(\bar{A},\bar{B},\bar{C})$ we first construct the system digraph denoted as $\mathcal{D}(\bar{A},\bar{B},\bar{C})$ which is constructed as follows: we define the *state digraph* $\mathcal{D}(\bar{A}) := (V_X, E_X)$ where $V_X = \{x_1,\ldots,x_n\}$ and an edge $(x_j,x_i) \in E_X$ if $\bar{A}_{ij} = \star$. A directed edge (x_j,x_i) exists if state x_j can directly influence state x_i. Define the system digraph $\mathcal{D}(\bar{A},\bar{B},\bar{C}) := (V_X \cup V_U \cup V_Y, E_X \cup E_U \cup E_Y)$, where $V_U = \{u_1,\ldots,u_m\}$ and $V_Y = \{y_1,\ldots,y_p\}$. An edge $(u_j,x_i) \in E_U$ if $\bar{B}_{ij} = \star$ and an edge $(x_j,y_i) \in E_Y$ if $\bar{C}_{ij} = \star$. A directed edge (u_j,x_i) exists if input u_j can *actuate* state x_i and a directed edge (x_j,y_i) exists if output y_i can *sense* state x_j and this completes the construction of the system digraph. Given a structured system $(\bar{A},\bar{B},\bar{C})$ and a feedback matrix $\bar{K}$, we define the *closed-loop system digraph* $\mathcal{D}(\bar{A},\bar{B},\bar{C},\bar{K}) := (V_X \cup V_U \cup V_Y, E_X \cup E_U \cup E_Y \cup E_K)$, where $(y_j,u_i) \in E_K$ if $\bar{K}_{ij} = \star$. Here a directed edge (y_j,u_i) exists if output y_j can be *fed to* input u_i.

A set of vertices $\hat{V}$ of a digraph is said to be strongly connected if for each ordered pair of vertices (v_i,v_k), $v_i,v_k \in \hat{V}$, there exists an elementary directed path from v_i to v_k. A strongly connected component (SCC) is a subgraph that consists of a maximal set of strongly connected vertices. Now, using the closed-loop system digraph $\mathcal{D}(\bar{A},\bar{B},\bar{C},\bar{K})$ the following result has been shown in [6].

Proposition 1 ([6], Theorem 4)**.** *A structured system $(\bar{A},\bar{B},\bar{C})$ has no structurally fixed modes with respect to an information pattern $\bar{K}$ if and only if the following conditions hold:*
a) in the digraph $\mathcal{D}(\bar{A},\bar{B},\bar{C},\bar{K})$, each state node x_i is contained in an SCC which includes an edge from E_K, and
b) there exists a finite node disjoint union of cycles[6] $\mathcal{C}_g = (V_g,E_g)$ in $\mathcal{D}(\bar{A},\bar{B},\bar{C},\bar{K})$ for some positive integer g such that $V_X \subseteq \cup_g V_g$.

Given a closed-loop structured system $(\bar{A},\bar{B},\bar{C},\bar{K})$, one can check condition a) in Proposition 1 in $O(n^2)$ computations [18] (by checking if all the SCCs in $\mathcal{D}(\bar{A},\bar{B},\bar{C},\bar{K})$ has at least one feedback edge in it), and condition b) in Proposition 1 in $O(n^{2.5})$ computations (using the information path concept given in [19]). Verification of existence of SFMs in a structured system has complexity $O(n^{2.5})$ [18]. Our objective here is to find a sparsest feedback matrix such that the resulting closed-loop system has no SFMs. Note that, for designing a feedback matrix that ensures no-SFMs criteria, all state nodes in $\mathcal{D}(\bar{A})$ must be actuated and sensed by some input and output, respectively. While for a structured system that is structurally controllable and structurally observable connecting all outputs to all inputs would guarantee the no-SFMs criteria, this approach may not necessarily give a sparsest feedback matrix. In this paper, we consider structurally cyclic systems with dedicated inputs and outputs. A structurally cyclic system is defined as follows.

Definition 2. *A structured system $\bar{A}$ is said to be structurally cyclic if all the vertices of the state digraph $\mathcal{D}(\bar{A})$ are spanned by disjoint union of cycles.*

In a structurally cyclic system, condition b) in Proposition 1 is satisfied. The feedback selection problem for a structurally cyclic system only need to obtain a feedback matrix that satisfies condition a) in Proposition 1. Henceforth, we consider structurally cyclic systems with dedicated inputs and outputs. The following assumption holds.

Assumption 1. *The state digraph $\mathcal{D}(\bar{A})$ is structurally cyclic.*

In the next section, we give our algorithms of polynomial complexity for solving Problem 1 when Assumption 1 holds and both the input and the output sets are dedicated.

3. Solving Sparsest Feedback Selection Problem with Dedicated Input-Output

In this section, we solve the sparsest feedback selection problem when both the input and the output sets are dedicated.

[6]In a digraph, a cycle is a directed path whose starting and ending vertices are the same such that there are no node repetitions.

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In the context of dedicated inputs and/or outputs, $\bar{B}$ or $\bar{C}$ are structured diagonal matrices with a specific number of nonzero entries. Let $\mathbb{I}_n$ be an $n \times n$ diagonal matrix with all diagonal entries nonzero and Z_r be an $n \times n$ diagonal matrix with exactly r diagonal entries nonzero. We address two cases here: (i) complete-dedicated input-output, i.e., $\bar{B} = \bar{C} = \mathbb{I}_n$, and (ii) dedicated input-output, i.e., $\bar{B} = Z_m$ and $\bar{C} = Z_p$ for some $m, p \leqslant n$.

The feedback configuration $\bar{K}^G$ satisfies the following assumption.

Assumption 2. *Given a structured system* $(\bar{A}, \bar{B}, \bar{C})$, *the feedback configuration* $\bar{K}^G$ *satisfies* $\{\bar{K}^G_{ij} = \star$ *if* $\bar{B}_{ii} = \star$ *and* $\bar{C}_{jj} = \star\}$. *Then,* $\bar{K}^G$ *is said to be complete.*

A feedback configuration satisfying Assumption 2 has no constraint on the feedback connections and it permits feedback of any output to any input. In this section, we show that Problem 1 has polynomial time solution if Assumption 2 holds. Later in Theorem 4, we relax Assumption 2 and show that the sparsest feedback selection problem is NP-hard when some of the feedback links are forbidden. Note that, with respect to $\bar{B} = \bar{C} = \mathbb{I}_n$, a complete $\bar{K}^G$ satisfies $\bar{K}^G_{ij} = \star$ for all i, j. On the other hand, with respect to $\bar{B} = Z_m$ and $\bar{C} = Z_p$, a complete $\bar{K}^G$ consists of zero entries, where $\bar{K}^G_{ij} = 0$ when $\bar{B}_{ii} = 0$ or $\bar{C}_{jj} = 0$ or both.

In the context of structurally cyclic systems satisfying Assumption 2, for case (i) ($\bar{B} = \bar{C} = \mathbb{I}_n$) we propose an $O(n^2)$ algorithm and for case (ii) ($\bar{B} = Z_m$ and $\bar{C} = Z_p$) we propose an $O(n^3)$ algorithm. The proposed solutions are the consequence of an important observation that there exists an optimal solution $\bar{K}^\star$ to Problem 1 such that all state nodes lie in a single SCC in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K}^\star)$. This is illustrated in Figure 1 and proved using Lemma 1. Note that in Figure 1a, the optimal solution $\bar{K}'$ results in three SCCs. However, there is another optimal solution, $\bar{K}''$, given in Figure 1b which results in two SCCs. On the other hand, the optimal solution $\bar{K}^\star$ given in Figure 1c results in a single SCC. Note that in this example, many optimal feedback configurations are possible and there exists one which results in a single SCC in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K}^\star)$. Formally, we prove the following result.

Lemma 1. *Consider a structured system* $(\bar{A}, \bar{B}, \bar{C})$ *satisfying Assumption 1 and a feedback configuration* $\bar{K}^G$ *satisfying Assumption 2. Let* $\bar{K}^\star$ *be an optimal solution to Problem 1 such that all state nodes lie in* β *number of SCCs in* $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K}^\star)$, *where* $\beta > 1$. *Then, there exists another optimal solution* $\bar{K}^\star_{\rm new}$ *such that* $\|\bar{K}^\star\|_0 = \|\bar{K}^\star_{\rm new}\|_0$ *and all state nodes lie in* $\beta - 1$ *number of SCCs in* $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K}^\star_{\rm new})$.

Proof. Given $\bar{K}^\star$ is an optimal solution to Problem 1 and all state nodes lie in the β number of SCCs in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K}^\star)$, say $\mathscr{C}_1, \ldots, \mathscr{C}_\beta$. Pick two SCCs, say $\mathscr{C}_i, \mathscr{C}_j$. Since $\bar{K}^\star$ satisfies condition a) in Proposition 1, both $\mathscr{C}_i$ and $\mathscr{C}_j$ have at least one feedback edge in them. Let $(y_a, u_b) \in \mathscr{C}_i$ and $(y_c, u_d) \in \mathscr{C}_j$. Now, break the edges $(y_a, u_b), (y_c, u_d)$ and make the edges $(y_a, u_d), (y_c, u_b)$. The inclusion of edges $(y_a, u_d), (y_c, u_b)$ is allowed as $\bar{K}^G$ satisfies Assumption 2. We claim that now all the nodes in $\mathscr{C}_i$ and $\mathscr{C}_j$ lie in a single SCC with feedback edges $(y_a, u_d), (y_c, u_b)$. To prove this we need to show that there exists a directed path between two arbitrary vertices in them. Consider any four arbitrary vertices $v_r, v_q \in \mathscr{C}_i$ and $v_\zeta, v_\delta \in \mathscr{C}_j$. Since $\mathscr{C}_i$ is an SCC, notice that there exists a directed path from u_b to v_r. Similarly, there exists a directed path from v_r to y_a also. Thus there exists a directed path from u_b to y_a passing through v_r. Similarly, we can show a directed path from u_b to y_a passing through v_q. Further, using the same argument on $\mathscr{C}_j$ we can show that there exists a directed path from u_d to y_c passing through v_ζ and from u_d to y_c passing through v_δ. These paths are shown using dotted lines in Figure 2. Now, on adding edges $(y_a, u_d), (y_c, u_b)$, there exists directed paths between all these vertices v_r, v_q, v_ζ and v_δ as shown in Figure 2. Therefore, there exists a path between any two arbitrary vertices in $\mathscr{C}_i$, any two arbitrary vertices in $\mathscr{C}_j$, and any two arbitrary vertices in $\mathscr{C}_i, \mathscr{C}_j$. As a result, by breaking edges $(y_a, u_b), (y_c, u_d)$ and making edges $(y_a, u_d), (y_c, u_b)$ all the vertices in $\mathscr{C}_i$ and $\mathscr{C}_j$ lie in a single SCC. Thus given an optimal feedback matrix, there exists another feedback matrix with the same number of edges, hence optimal, such that all the state nodes are spanned by one less number of SCCs in the closed-loop system digraph. This completes the proof. $\square$

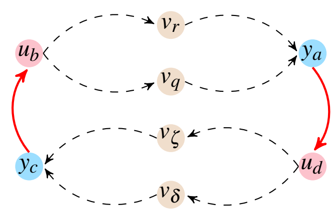

Figure 2: Schematic diagram depicting the construction used in the proof of Lemma 1. Dotted lines between two vertices denote existence of a directed path between them.

As a consequence of Lemma 1, we have the following corollary.

Corollary 1. *Consider a structured system* $(\bar{A}, \bar{B}, \bar{C})$ *satisfying Assumption 1 and a feedback configuration* $\bar{K}^G$ *satisfying Assumption 2. Then, there exists an optimal solution* $\bar{K}^\star$ *to Problem 1 such that all state nodes lie in a single SCC in* $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K}^\star)$.

The above corollary is true, since given any optimal solution $\bar{K}'$ to Problem 1 (recall that the set $\mathcal{K}_s$ is nonempty), one can apply Lemma 1 recursively and obtain another optimal solution $\bar{K}^\star$ such that $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K}^\star)$ is a single SCC. Consequently, solving Problem 1 on a structured system $(\bar{A}, \bar{B}, \bar{C})$ is same as finding a set of minimum number of feedback edges to add in the digraph $\mathcal{D}(\bar{A}, \bar{B}, \bar{C})$ such that in the resulting digraph all state nodes lie in an SCC. We relate Problem 1 to a known problem in graph theory, namely the *strong connectivity augmentation problem* [20]. The problem of finding the minimum number of edges to add in a digraph such that the resulting digraph is strongly connected is referred to as the strong connectivity augmentation problem. Next, we formally state the strong connectivity augmentation problem for the sake of completeness.

Problem 2 (Strong connectivity augmentation problem [20])**.** *Given a directed graph* $\mathcal{D} = (V_D, E_D)$, *the strong connectivity augmentation problem aims at finding the minimum cardinality*

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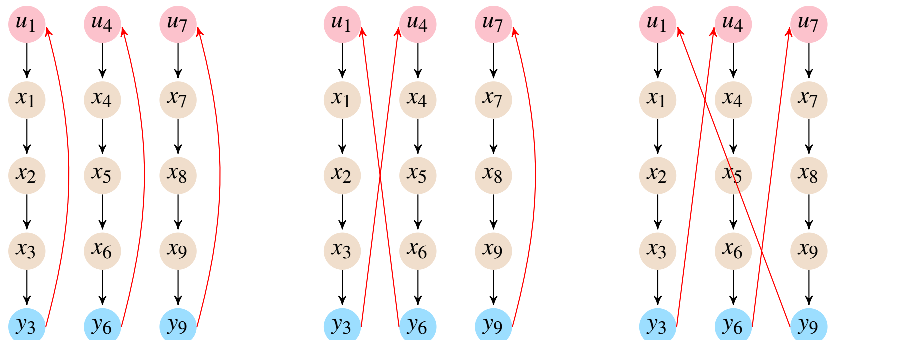

(a) Digraph $\mathcal{D}(\bar{A},\mathbb{I}_n,\mathbb{I}_n,\bar{K}')$, where $\bar{K}'_{13}=\bar{K}'_{46}=\bar{K}'_{79}=\star$ and zero otherwise, has three SCCs.

(b) Digraph $\mathcal{D}(\bar{A},\mathbb{I}_n,\mathbb{I}_n,\bar{K}'')$, where $\bar{K}''_{16}=\bar{K}''_{43}=\bar{K}''_{79}=\star$ and zero otherwise, has two SCCs.

(c) Digraph $\mathcal{D}(\bar{A},\mathbb{I}_n,\mathbb{I}_n,\bar{K}^\star)$, where $\bar{K}^\star_{19}=\bar{K}^\star_{43}=\bar{K}^\star_{76}=\star$ and zero otherwise, is a single SCC.

Figure 1: Illustrative figure demonstrating the existence of an optimal solution to Problem 1 with a single SCC in $\mathcal{D}(\bar{A},\bar{B},\bar{C},\bar{K}^\star)$. For simplicity, self-loop of each state vertex x_k is not shown in all the figure. Also, each state vertex x_k has input u_k and output y_k connected which are omitted for many x_k's for the sake of clarity.

set of edges E'_D such that $\mathcal{D}'=(V_D,E_D\cup E'_D)$ is strongly connected.

First note that if $\mathcal{D}(\bar{A})$ is irreducible, then $E'_D=\emptyset$ and any $\bar{K}$ with a single nonzero entry is optimal. Hence, from now on we only focus on the non-trivial cases such that $\mathcal{D}(\bar{A})$ has at least two SCCs. There exists an algorithm for solving Problem 2 optimally with complexity linear in $(|V_D|+|E_D|)$ [21]. Given a directed graph $\mathcal{D}$, the algorithm proposed in [21] outputs a minimum set of edges E'_D such that $\mathcal{D}'$ is a single SCC. In the subsections below, we present algorithms for solving Problem 1 in cases (i) and (ii) using the strong connectivity augmentation algorithm given in [21].

A. Algorithm for $\bar{B}=\bar{C}=\mathbb{I}_n$

In this subsection, we show that if the structured system satisfies Assumption 1 and $\bar{B}=\bar{C}=\mathbb{I}_n$, then Problem 1 can be solved in $O(n^2)$ time. The feedback pattern $\bar{K}^G$ here satisfies Assumption 2, i.e., $\bar{K}^G_{ij}=\star$ for all i,j. Using Corollary 1 now we give the polynomial time algorithm to solve Problem 1 on structured systems that satisfy Assumption 1:
Step 1: Given a structured system $(\bar{A},\bar{B}=\mathbb{I}_n,\bar{C}=\mathbb{I}_n)$, solve the strong connectivity augmentation problem on the digraph $\mathcal{D}(\bar{A})$. Let $E_{\mathcal{X}}$ denotes the optimal solution obtained.
Step 2: Define $\bar{K}^S=\{\bar{K}^S_{ij}=\star:(x_j,x_i)\in E_{\mathcal{X}}\}$.

Note that defining $\bar{K}^S$ as given in Step 2 is possible since $\bar{B}=\bar{C}=\mathbb{I}_n$. Now we prove that $\bar{K}^S$ obtained in Step 2 is an optimal solution to Problem 1.

Theorem 1. *Consider a structured system $(\bar{A},\bar{B},\bar{C})$ satisfying Assumption 1 and a feedback configuration $\bar{K}^G$ satisfying Assumption 2. Then, solving Problem 1 has $O(n^2)$ complexity, where n denotes the number of states in the structured system.*

Proof. Here we prove that solving strong connectivity augmentation problem on $\mathcal{D}(\bar{A})$ gives an optimal solution to Problem 1. As Assumption 1 holds, condition b) in Proposition 1 is satisfied without using any feedback edge and our aim is to only satisfy condition a) in Proposition 1. The structured system given satisfies one of the following cases: i) $\mathcal{D}(\bar{A})$ is irreducible; ii) $\mathcal{D}(\bar{A})$ is reducible. In case i), solution to the strong connectivity augmentation problem, $E_{\mathcal{X}}=\emptyset$. Then, an optimal solution to Problem 1 is given by $\{\bar{K}^S:\bar{K}^S_{11}=\star$ and 0 otherwise$\}$. In case ii), we now prove that $\bar{K}^S$ obtained in Step 2 corresponding to $E_{\mathcal{X}}$ is an optimal solution to Problem 1. We first show that $\bar{K}^S$ is a feasible solution, i.e., $\bar{K}^S\in\mathcal{K}_s$. By the construction of $\bar{K}^S$ given in Step 2 notice that all state nodes lie in a single SCC in $\mathcal{D}(\bar{A},\bar{B},\bar{C},\bar{K}^S)$. Also, $\mathcal{D}(\bar{A})$ is not irreducible. Thus condition a) in Proposition 1 is satisfied for all states and hence $\bar{K}^S\in\mathcal{K}_s$. Now we prove that $\bar{K}^S$ is an optimal solution to Problem 1, i.e., $\|\bar{K}^S\|_0=\|\bar{K}^\star\|_0$. Suppose not. Then there exists $\bar{K}''\in\mathcal{K}_s$ such that $\|\bar{K}''\|_0<\|\bar{K}^S\|_0$ and all state nodes lie in a single SCC in $\mathcal{D}(\bar{A},\bar{B},\bar{C},\bar{K}'')$ (by Corollary 1). Consider edges $E'_{\mathcal{X}}$, where $(x_j,x_i)\in E'_{\mathcal{X}}$ if $\bar{K}''_{ij}=\star$. Notice that $|E'_{\mathcal{X}}|<|E_{\mathcal{X}}|$ and $\mathcal{D}'=(V_X,E_X\cup E'_{\mathcal{X}})$ is an SCC. This is a contradiction to the assumption that $E_{\mathcal{X}}$ is an optimal solution to the strong connectivity augmentation problem. This proves that the feedback matrix obtained by solving the strong connectivity augmentation problem on $\mathcal{D}(\bar{A})$, $\bar{K}^S=\{\bar{K}^S_{ij}=\star:(x_j,x_i)\in E_{\mathcal{X}}\}$, is an optimal solution to Problem 1.

Now the complexity of the strong connectivity augmentation algorithm is linear in the number of nodes and edges in the digraph. Since $|V_X|=n$ and $|E_X|=O(n^2)$, the result follows. $\square$

In the next subsection, we consider the case where both the inputs and the outputs are dedicated but not $\mathbb{I}_n$.

B. Algorithm for $\bar{B}=Z_m$ and $\bar{C}=Z_p$

In this subsection, we solve Problem 1 on structured systems that have *dedicated* input and output matrices with $\bar{B}=Z_m$ and $\bar{C}=Z_p$. Z_m and Z_p are $n\times n$ diagonal matrices with m and p nonzero entries, respectively. Here not every state is influenced (sensed, resp.) directly by an input (output, resp.) and $\bar{B}$ ($\bar{C}$, resp.) consists of $n-m$ zero columns ($n-p$ zero rows, resp.) and our aim is to solve Problem 1. We assume that

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Algorithm 1 Pseudo-code for solving Problem 1 on structurally cyclic systems when both the input and the output sets are dedicated

Input: Structured system $(\bar{A}, \bar{B} = \mathbb{Z}_m, \bar{C} = \mathbb{Z}_p)$
Output: Feedback matrix $\bar{K}^{s'}$

1: Solve strong connectivity augmentation problem on $\mathcal{D}(\bar{A})$, say $E_{\mathcal{X}}$ is the solution obtained
2: Find $E_{\mathcal{F}} \subset E_{\mathcal{X}}$, where $(x_j, x_i) \in E_{\mathcal{F}}$ if either $\bar{B}_{ii} = 0$ or $\bar{C}_{jj} = 0$ or both
3: **while** $E_{\mathcal{F}} \neq \emptyset$ **do**
4: Let $(x_j, x_i) \in E_{\mathcal{F}}$
5: Find x_b, x_t such that there exist directed paths from x_t to x_i and x_j to x_b and $\bar{B}_{bb} = \bar{C}_{tt} = \star$
6: $E'_{\mathcal{X}} \leftarrow \{E_{\mathcal{X}} \setminus (x_j, x_i)\} \cup (x_b, x_t)\}$
7: Let $E'_{\mathcal{F}} \subset E'_{\mathcal{X}}$ be the infeasible edges in $E'_{\mathcal{X}}$
8: $E_{\mathcal{X}} \leftarrow E'_{\mathcal{X}}$
9: $E_{\mathcal{F}} \leftarrow E'_{\mathcal{F}}$
10: **end while**
11: $\bar{K}^{s'} \leftarrow \bar{K}^{s'}_{ij} = \star$ if $(x_j, x_i) \in E'_{\mathcal{X}}$

the feedback pattern $\bar{K}^G$ satisfies Assumption 2, i.e., $\bar{K}^G_{ij} = \star$ whenever $\bar{B}_{ii} = \bar{C}_{jj} = \star$.

We propose an $O(n^3)$ complexity algorithm for solving Problem 1 on structured systems with both the input and the output sets are dedicated. The proposed method solves Problem 2, the strong connectivity augmentation, on $\mathcal{D}(\bar{A})$. However, the optimal solution obtained for Problem 2 does not immediately translate to an optimal solution to Problem 1. This is because the resulting $\bar{K}^s$ may consist of infeasible feedback links. So the obtained $\bar{K}^s$ is updated to get another feedback matrix $\bar{K}^{s'}$. The pseudo-code for the proposed solution procedure is presented in Algorithm 1.

Theorem 2. *Consider a structurally cyclic system $\bar{A}$ with dedicated input and output sets $\bar{B} = \mathbb{Z}_m$ and $\bar{C} = \mathbb{Z}_p$, respectively. Let $E_{\mathcal{X}}$ be an optimal solution to the strong connectivity augmentation problem on the digraph $\mathcal{D}(\bar{A})$ and $E_{\mathcal{F}} \subset E_{\mathcal{X}}$ be the infeasible set of edges. Then, (i) there exists an edge set $E'_{\mathcal{X}}$ and infeasible edge set $E'_{\mathcal{F}} \subset E'_{\mathcal{X}}$ with $|E'_{\mathcal{F}}| < |E_{\mathcal{F}}|$ such that (ii) $\bar{K}^{s'} = \{\bar{K}^{s'}_{ij} = \star : (x_j, x_i) \in E'_{\mathcal{X}}\}$ is an optimal solution to Problem 1.*

Proof. Given $E_{\mathcal{X}}$ is an optimal solution to the strong connectivity augmentation problem on the digraph $\mathcal{D}(\bar{A})$. For $\bar{K}^s := \{\bar{K}^s_{ij} = \star : (x_j, x_i) \in E_{\mathcal{X}}\}$ all state nodes lie in a single SCC in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K}^s)$. However, it may happen that there exists an infeasible edge set $E_{\mathcal{F}} \subset E_{\mathcal{X}}$, $E_{\mathcal{F}} \neq \emptyset$ (see Step 2). Note that if $(x_j, x_i) \in E_{\mathcal{F}}$ then at least one of the following has to happen: 1) $\bar{B}_{ii} = 0$, 2) $\bar{C}_{jj} = 0$. Then, the feedback edge $\bar{K}^s_{ij}$ is not feasible. Without loss of generality, let us assume that both 1) and 2) hold. Then state x_i has no input connecting to it and state x_j has no output sensing it. However, the set $\mathcal{K}_s$ is nonempty as $\bar{K}^G \in \mathcal{K}_s$. This implies that all states are actuated by some input and sensed by some output. Therefore, there exists some state x_t which has input u_t connected to it and there is a directed path from x_t to x_i. Similarly, there exists some state x_b which has output y_b connected and there is a directed path from x_j to x_b. Now consider $E'_{\mathcal{X}} = \{E_{\mathcal{X}} \setminus (x_j, x_i)\} \cup \{(x_b, x_t)\}$. The feedback edge is (x_b, x_t) is feasible as Assumption 2 holds. Let the infeasible edge set in $E'_{\mathcal{X}}$ be $E'_{\mathcal{F}}$. Notice that edge (x_b, x_t) is feasible and hence $E'_{\mathcal{F}} \subset E_{\mathcal{F}}$. Therefore, $|E'_{\mathcal{F}}| < |E_{\mathcal{F}}|$. This proves (i).

Now we prove (ii), i.e., the feedback matrix $\bar{K}^{s'}$, where $\{\bar{K}^{s'}_{ij} = \star : (x_j, x_i) \in E'_{\mathcal{X}}\}$, is an optimal solution to Problem 1. First, we prove that $\bar{K}^{s'}$ is a feasible solution and then we prove that it is minimal. Since the system is structurally cyclic condition b) in Proposition 1 is satisfied without any feedback edges. To prove $\bar{K}^{s'} \in \mathcal{K}_s$ one need to show that condition a) in Proposition 1 is satisfied in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K}^{s'})$. With feedback matrix $\bar{K}^s$ all state nodes in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K}^s)$ satisfy condition a) in Proposition 1. However, $\bar{K}^s$ may consist of infeasible feedback links. We need to show that breaking edge (y_j, u_i) and making edge (y_b, u_t) do not result in any state vertex not satisfying condition a) in Proposition 1. Let x_k be an arbitrary state that satisfies condition a) in Proposition 1 using the feedback edge (y_j, u_i). Since all state nodes lie in a single SCC in $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K}^s)$ there exist directed paths from x_i to x_k and from x_k to x_j that do not use feedback edge (y_j, u_i). This implies that there exist directed paths from x_t to x_k and x_k to x_b. Hence x_k lies in an SCC with x_t, x_b, u_t and y_b in it. Further, x_k continues to satisfy condition a) in Proposition 1 even after breaking (y_j, u_i) and making (y_b, u_t). Since x_k is arbitrary $\bar{K}^{s'} \in \mathcal{K}_s$.

Now for proving minimality, we argue that no lesser set of edges satisfies condition a) in Proposition 1. This is true because $E_{\mathcal{X}}$ is an optimal solution to the strong connectivity augmentation problem. By Theorem 1 the resulting $\bar{K}^s$ is an optimal solution to Problem 1 if all feedback links are feasible. When $\bar{B} = \mathbb{Z}_m$ and $\bar{C} = \mathbb{Z}_p$, no lesser set of edges can keep all state nodes in SCCs with a feedback edge in it. Consequently, an optimal solution $\bar{K}^{s'}$ satisfies $\|\bar{K}^{s'}\|_0 \geqslant \|\bar{K}^s\|_0$. Since $\|\bar{K}^{s'}\|_0 = \|\bar{K}^s\|_0$, minimality follows and this completes the proof. □

Theorem 3. *Consider a structurally cyclic system $\bar{A}$ with dedicated input and output sets $\bar{B} = \mathbb{Z}_m, \bar{C} = \mathbb{Z}_p$, respectively. Then, Algorithm 1 gives an optimal solution to Problem 1 in $O(n^3)$ computations, where n denotes the number of states.*

Proof. Let $E_{\mathcal{X}}$ be a solution to the strong connectivity augmentation problem on the digraph $\mathcal{D}(\bar{A})$. Let $E_{\mathcal{F}} \subset E_{\mathcal{X}}$ be the infeasible set of edges in $E_{\mathcal{X}}$. Recursively applying Theorem 2 on $E_{\mathcal{X}}$, as shown in Algorithm 1, we get edge set $E'_{\mathcal{X}}$ where infeasible edge set $E'_{\mathcal{F}} \subset E'_{\mathcal{X}}$ is empty and $\bar{K}^{s'} = \{\bar{K}^{s'}_{ij} = \star : (x_j, x_i) \in E'_{\mathcal{X}}\}$ is an optimal solution to Problem 1. The algorithm will terminate in finite steps since the infeasible edge set is finite and on termination, it gives an optimal solution to Problem 1.

Strong connectivity augmentation problem has complexity $O(n^2)$. For each infeasible edge present in the solution obtained, $E_{\mathcal{X}}$, finding x_t, x_b has $O(n)$ complexity each. At most, there can be n^2 infeasible edges. Finding set $E'_{\mathcal{X}}$ from $E_{\mathcal{X}}$ involves $O(n^3)$ computations. The remaining steps in the algorithm are of linear complexity. Combining together, solving Problem 1 has complexity $O(n^3)$. □

Theorem 3 concludes that if inputs and outputs are dedicated and feedback links are non-weighted, then optimal feedback selection problem is in P for structurally cyclic systems.

Remark 2. *If the structured system is not structurally cyclic, then a 2-optimal solution can be obtained for Problem 1 by combining the algorithms proposed in this paper with a minimum weight perfect matching algorithm (to satisfy condition b) in Proposition 1) using the approach given in [9].*

In the next subsection, we analyze the sparsest feedback selection problem under an additional constraint that some of the feedback links are forbidden.

C. Sparsest Feedback Selection: with Forbidden Feedback Links

In this subsection, we consider Problem 1 with dedicated input and output matrices and a constrained feedback matrix. Unlike the cases considered previously in subsections 3-A and 3-B, here Assumption 2 does not hold and feedback configuration $\bar{K}^G$ consists of forbidden feedback links. We show that with this restriction the sparsest feedback selection problem on structurally cyclic systems with dedicated input-output is NP-hard. We obtain the hardness result by reducing a well known NP-hard problem, the minimum hitting set problem [22], to an instance of Problem 1 with forbidden feedback connections. The minimum hitting set problem is described as follows: given a universe $\mathcal{U}$ consisting of N elements and a collection of sets $\mathcal{H} = \{\mathcal{S}_1, \ldots, \mathcal{S}_r\}$, where each $\mathcal{S}_i \subseteq \mathcal{U}$, a minimum hitting set for $\mathcal{H}$ consists of finding a set $\mathcal{U}' \subseteq \mathcal{U}$ such that $\mathcal{U}'$ contains at least one element from each set in $\mathcal{H}$ and cardinality of $\mathcal{U}'$ is minimum.

The pseudo-code showing a reduction of the minimum hitting set problem to an instance of Problem 1 along with a set of forbidden feedback connections is given in Algorithm 2. An illustrative example showing the construction of a structured system $(\bar{A}, \bar{B}, \bar{C})$ and forbidden set F from a hitting set problem is given in Figure 3. The result below proves that Problem 1 is NP-hard for structurally cyclic systems with dedicated input-output when Assumption 2 does not hold.

Theorem 4. *Consider a structurally cyclic structured system $(\bar{A}, \bar{B}, \bar{C})$ with dedicated input and output matrices Z_m and Z_p, respectively, constructed using Algorithm 2 corresponding to the minimum hitting set problem. Also, consider the feedback configuration $\bar{K}^G$ obtained in Algorithm 2. Then, finding an optimal solution to Problem 1 is NP-hard.*

Proof. We prove the NP-hardness of the problem using reduction from the minimum hitting set problem. Consider a general instance of the hitting set problem, with universe $\mathcal{U} = \{1, \ldots, N\}$ and sets $\mathcal{H} = \{\mathcal{S}_1, \ldots, \mathcal{S}_r\}$. We construct a structured system corresponding to the hitting set problem $(\mathcal{U}, \mathcal{H})$ as follows: define states $\{x_1, \ldots, x_{3N+r+2}\}$, inputs $\{u_1, \ldots, u_{2N}\}$, outputs $\{y_1, \ldots, y_{N+1}\}$ interconnected by matrices $\bar{A}, \bar{B}$ and $\bar{C}$ as shown in Steps 3, 4 and 5, respectively, of Algorithm 2. Let $\bar{K}$ be an optimal solution to Problem 1 for the structured system $(\bar{A}, \bar{B}, \bar{C})$ constructed and feedback configuration $\bar{K}^G$ (defined in Step 7 of Algorithm 2). Since the system is structurally cyclic, condition b) in Proposition 1 is satisfied without using any feedback edge. However, $\bar{K}$ satisfies condition a) in Proposition 1. We first show that the hitting set selected corresponding to $\bar{K}$, i.e., $\mathcal{U}'(\bar{K})$, is a feasible solution to the minimum hitting set problem. Note that an optimal solution to Problem 1 consists of feedback connections $\{(i,j) : i \in \{N+r+3, \ldots, 2N+r+2\}$ and $j = i+N\}$, since there is no other way that the state nodes $\{x_{N+r+3}, \ldots, x_{3N+r+2}\}$ can satisfy condition a) in Proposition 1. These feedback links also result in state nodes $x_1, \ldots, x_N$ satisfying condition a) in Proposition 1. Thus, optimal selection of feedback edges has to be done such that nodes $x_{N+1}, \ldots, x_{N+r}$ satisfy condition a) in Proposition 1.

Note that $\mathcal{U}'(\bar{K}) := \{i : \bar{K}_{ij} = \star$ and $j = N+r+2\}$. Since $\bar{K}$ is a feasible solution to the feedback selection problem, all states satisfy condition a) in Proposition 1. Consider an arbitrary node $x_{N+j} \in \{x_{N+1}, \ldots, x_{N+r}\}$. Then x_{N+j} satisfies condition a) in Proposition 1 using some feedback link from output node y_{N+r+2} to some input node, say $u_i \in \{u_1, \ldots, u_N\}$, that has a directed path to it. Based on the construction of $\bar{A}$, this is possible only if $i \in \mathcal{S}_j$. Thus $\mathcal{U}'(\bar{K}) \cap \mathcal{S}_j \neq \emptyset$. Since x_{N+j} is an arbitrary node in $\{x_{N+1}, \ldots, x_{N+r}\}$, for all $\mathcal{S}_g \in \mathcal{H}$, $\mathcal{S}_g \cap \mathcal{U}'(\bar{K}) \neq \emptyset$. Hence $\mathcal{U}'(\bar{K})$ is a feasible solution to the minimum hitting set problem.

Now we show that $\mathcal{U}'(\bar{K})$ is an optimal solution to the minimum hitting set problem. To the contrary assume that

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Algorithm 2 Pseudo-code for reducing the minimum hitting set problem to an instance of Problem 1

Input: Minimum hitting set problem with universe $\mathcal{U} = \{1, \ldots, N\}$ and sets $\mathcal{H} = \{\mathcal{S}_1, \ldots, \mathcal{S}_r\}$
Output: Structured system $(\bar{A}, \bar{B}, \bar{C})$ and forbidden set F

1: Define $x_1, \ldots, x_{3N+r+2}$ and $y_1, \ldots, y_{N+1}$ and $u_1, \ldots, u_{2N}$ to be interconnected by the following definition of $\bar{A}, \bar{B}, \bar{C}$
2: Define a structured system $(\bar{A}, \bar{B}, \bar{C})$ as follows:
3: $$\bar{A}_{ij} \leftarrow \begin{cases} \star, \text{ for } i = j, \text{ for } i \in \{1, \ldots, 3N+r+2\}, \\ \star, \text{ for } i \in \{N+1, \ldots, N+r\}, j \in \{1, \ldots, N\} \\ \quad \text{and } j \in \mathcal{S}_{i-N}, \\ \star, \text{ for } i = N+r+1, j \in \{N+1, \ldots, N+r\}, \\ \star, \text{ for } i = N+r+2, j = N+r+1, \\ \star, \text{ for } i \in \{1, \ldots, N\}, j = N+r+2+i, \\ \star, \text{ for } i \in \{2N+r+3, \ldots, 3N+r+2\}, \\ \quad j = i-(2N+r+2), \\ 0, \text{ otherwise.} \end{cases}$$
4: $$\bar{B}_{ii} \leftarrow \begin{cases} \star, \text{ for } i \in \{1, \ldots, N, N+r+3, \ldots, 2N+r+2\}, \\ 0, \text{ otherwise.} \end{cases}$$
5: $$\bar{C}_{jj} \leftarrow \begin{cases} \star, \text{ for } j \in \{N+r+2, 2N+r+3, \ldots, 3N+r+2\}, \\ 0, \text{ otherwise.} \end{cases}$$
6: Define forbidden set F as: $F \leftarrow \{(i,j) : i \in \{N+r+3, \ldots, 2N+r+2\}, j = N+r+2\} \cup \{(p,q) : p \in \{1, \ldots, N\}, q \in \{2N+r+3, \ldots, 3N+r+2\}\} \cup \{(g,h) : g \in \{N+r+3, \ldots, 2N+r+2\}, h \in \{2N+r+3, \ldots, 3N+r+2\}$ and $h \neq g+N\}$.
7: Define $\bar{K}^G := \{\bar{K}^G_{ij} = \star : (i,j) \notin F\}$
8: Given a solution $\bar{K}$ to Problem 1 on $(\bar{A}, \bar{B}, \bar{C})$ and $\bar{K}^G$ define: hitting set selected under $\bar{K}$, $\mathcal{U}'(\bar{K}) \leftarrow \{i : \bar{K}_{ij} = \star$ and $j = N+r+2\}$

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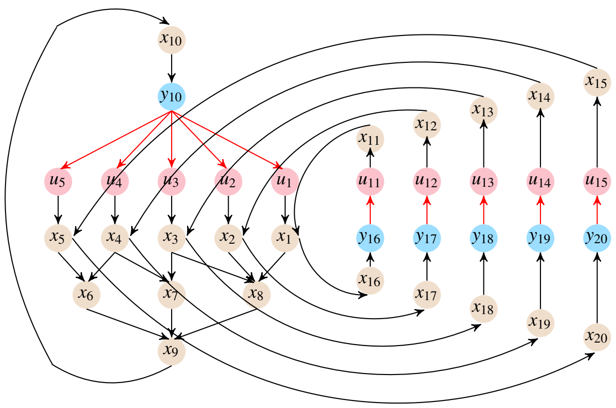

Figure 3: Illustrative example demonstrating construction of $\mathcal{D}(\bar{A},\bar{B},\bar{C})$ along with all non-forbidden feedback edges (shown in red colour) given in Algorithm 2 for a hitting set problem with $\mathcal{U}=\{1,\ldots,5\}$, $\mathcal{H}=\{\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3\}$, where $\mathcal{S}_1=\{1,2,3\}$, $\mathcal{S}_2=\{3,4\}$ and $\mathcal{S}_3=\{4,5\}$. For simplicity, self-loop of each state vertex x_k is not shown in the figure.

$\mathcal{U}'(\bar{K})$ is not a minimum hitting set. Then there exists a set $\mathcal{U}''\subset\mathcal{U}$ such that $\mathcal{U}''\cap\mathcal{S}_i\neq\emptyset$ for all $\mathcal{S}_i$'s in $\mathcal{H}$ and $|\mathcal{U}''|<|\mathcal{U}'(\bar{K})|$. Then $\bar{K}'':=\{\{\bar{K}''_{ij}=\star : i\in\{N+r+3,\ldots,2N+r+2\}$ and $j=i+N\}$ and $\{\bar{K}''_{pq}=\star : p\in\mathcal{U}''$ and $q=N+r+2\}\}$ is a feasible solution to Problem 1. Also, $\|\bar{K}''\|_0<\|\bar{K}\|_0$. This contradicts the assumption that $\bar{K}$ is an optimal solution to Problem 1. Therefore, $\mathcal{U}'(\bar{K})$ is an optimal solution to Problem 1. This completes the proof. $\square$

Theorem 4 concludes that Problem 1 is NP-hard with dedicated input-output if the set of feedback connections are constrained. Combining this with results in [9] and [8], we conclude that the optimal feedback selection problem is NP-hard for all other cases, except when both the input and the output sets are dedicated, feedback links are non-weighted and unconstrained feedback pattern. This paper thus identifies the only possible instance of optimal feedback selection problem that is polynomially solvable without imposing any additional assumption on the state digraph other than being structurally cyclic.

4. Conclusion

This paper dealt with sparsest feedback selection for structured systems with dedicated inputs and outputs. The objective here is to obtain a sparsest feedback matrix such that the resulting closed-loop system has no structurally fixed modes. This problem was considered earlier in [11] and later in [17]. We have shown recently in [9] that this problem is NP-hard when considering structurally cyclic systems with *non-dedicated* input-output. In this paper, we proved that solving this problem is not NP-hard on structurally cyclic systems with *dedicated* inputs and outputs, but in fact of polynomial complexity (Theorem 1). For the case $\bar{B}=\bar{C}=\mathbb{I}_n$, we provided an algorithm that has $O(n^2)$ complexity. When $\bar{B}$ and $\bar{C}$ are Z_m and Z_p, respectively, we provided an $O(n^3)$ algorithm (Theorem 3). We also show that for structurally cyclic systems with dedicated input-output and a *constrained* feedback pattern, Problem 1 is NP-hard (Theorem 4). This concludes that for structurally cyclic systems, optimal feedback selection problem has polynomial time complexity only when both the input and the output sets are dedicated and the feedback edges are unconstrained and without weights (Table I).

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