Optimal Network Topology Design in Composite Systems for Structural Controllability

Shana Moothedath*, Prasanna Chaporkar†, and Madhu N. Belur†

***Abstract*—This paper deals with structural controllability of a Linear Time Invariant (LTI) *composite* system consisting of several circuits/subsystems. We consider subsystems with structurally similar state matrices, i.e., the zero/non-zero pattern of the state matrices of the subsystems are the same, but dynamics can be different due to different numerical values. Structurally similar subsystems arise in large circuits implemented using many similar smaller circuits and agent-based networks consisting of homogeneous agents. The subsystems may not be structurally controllable individually; however, structural controllability of the composite system can be achieved by sharing state information. Sharing of information among subsystems incurs cost and our aim is to design a structurally controllable composite system with minimum information sharing. Minimizing information sharing is the same as minimizing the number of interaction links between subsystems, referred to as *interconnections*. An *optimal network topology* is one with minimum number of interconnections. This paper presents a closed-form expression for the minimum number of interconnections for structural controllability and derives a polynomial time algorithm to find an optimal network topology. The algorithm is based on a minimum weight perfect matching algorithm and a so-called edge reconstruction process. The minimum number of interconnections required is formulated in terms of two indices we define in the paper: *maximum commonality index* ($\alpha_{\mathcal{N}}$) and *dilation index* ($\beta_{\mathcal{I}}$). Loosely speaking, more connectedness of subsystems leads to lower total value of $\alpha_{\mathcal{N}}$ and $\beta_{\mathcal{I}}$. Further, $\alpha_{\mathcal{N}}$ decreases if the subsystems have fewer number of connected components. We apply and verify our general result to special cases that arise in control, where the minimum number can be more directly obtained. The special cases considered are structurally cyclic subsystems, irreducible subsystems, and subsystems in controller canonical form.**

***Index Terms*—Structural controllability, Composite systems, Communication cost, Minimizing interconnections, Large-scale systems, Agent-based systems.**

1. Introduction

Recently there has been immense research advance in the area of large-scale dynamical systems collectively using concepts from control theory, network science and statistical physics [1]. Very often these networks consist of smaller entities called *subsystems* interacting with each other to form a collective system, referred to as the *composite* system. Composite systems are present in numerous applications, that include electric circuits [2], multi-agents [3], and power networks [4], where large networks are built by interconnecting smaller subsystems/circuits. Composite systems are also used in the parallelization of a larger big task into smaller subtasks which is to be implemented by smaller subsystems/circuits.

Many large systems and circuits can only be analyzed at a subsystem level. These subsystems can have exactly same intra-subsystem *structure* but are not necessarily identical due to different *paramters*. There exists a practically important class of composite systems, including robot swarms, power grids and biological systems, consisting of similar subsystems (circuits) interacting with each other for performing a desired task. Robots manufactured by the same manufacturer usually have the same structural pattern even if the numerical entries vary slightly due to the manufacturing variations of the components used. Design of optimal composite systems is relevant in a swarm of robots with homogeneous structure where the communication topology can change over time, or where robots may join or leave the swarm over time [5].

The interaction links between subsystems are referred to as *interconnections*. An interconnection is a directed edge between two states of distinct subsystems through which one state influences the other state by information transfer. In most of the applications, it is desired to achieve the intended performance, say controllability, by keeping the amount of information transfer the least because of security reasons, capacity constraints in communication, and to minimize the communication cost and delay. For instance, in multi-agent networked system, the robot swarm consists of many *agents*. In a formation control or consensus application with selected leaders, only a few agents receive external input and hence each agent is not necessarily controllable. Input addition to achieve controllability is not permissible as the input structure is predefined; however, interconnection links between agents are often allowed. Thus in a robot swarm, the agents communicate their state information with other agents [6]. In electric circuit design using interconnection network, the circuit modules are connected to a network to form an interconnected system over which the modules communicate to one another by sending packets instead of connecting them using dedicated physical wires [2]. An optimal network design is crucial here to obtain a modular design.

This paper aims to design a controllable *optimal network topology* of the composite system, optimal in the sense of amount of information transfer between subsystems. Since information transfer among subsystems are carried out using the interconnections and since each interconnection link corresponds to a transfer of one state information, an optimal network topology with minimum information transfer is a network topology with minimum number of interconnections. To this end, we identify *what* state information to be communicated and to *which* agents it should be communicated so as to obtain a controllable optimal network topology design. The minimum number of interconnections required is formulated in terms of two indices we define in the paper; *maximum commonality index* ($\alpha_{\mathcal{N}}$) and *dilation index* ($\beta_{\mathcal{I}}$).

The optimal network topology design problem with similar objectives has been addressed for numerical systems in [3], [7].

*Department of Electrical and Computer Engineering, University of Washington, Seattle, USA. Email: sm15@uw.edu.

†Department of Electrical Engineering, Indian Institute of Technology Bombay, India. Email: {chaporkar, belur}@ee.iitb.ac.in.

The approaches given in [3], [7] require the system matrices to be known precisely. Typically, complex networks are characterized by large system dimension and in most cases the system parameters are not known precisely because of various reasons that include uncertainties in the system model and time varying link weights of the graph [1]. Hence, for addressing system theoretic questions related to these networks, for instance controllability and feedback selection, many papers use the topological characteristics of the system. Control theoretic analysis of complex networks, when only the graph of the network is known, is done using 'structural analysis' [8]. We perform our analysis from a structural framework, where instead of the numerical matrices of the system, the structures or sparsity patterns of the system matrices are used. Given the sparsity pattern of a set of subsystems, we identify a minimum cardinality set of interconnections that has to be established between the subsystems so that the networked system achieves structural controllability.

We summarize this paper's contributions below.

• We introduce the indices, maximum commonality index ($\alpha_{\mathscr{N}}$) and dilation index ($\beta_{\mathcal{I}}$), to characterize the minimum number of interconnections required for structural controllability of composite system (Theorem 4.1). These indices quantify the connectedness of the system which we emphasize in Remark 4.9.

• Given a set of structured subsystems with identical structured state matrices, we find a closed-form expression for the minimum number of interconnections the subsystems should establish amongst one another such that the composite system is structurally controllable (Theorem 4.1).

• We propose a polynomial time complexity algorithm to find a set of minimum cardinality interconnection edges to solve the optimal network topology design problem given in Problem 2.3 (Algorithm 4.1 and Theorem 4.7).

• For each subsystem, the solution obtained using Algorithm 4.1 provides a set of subsystems that it should communicate with and the state information that must be communicated so that the composite system is structurally controllable using the minimum number of interconnections.

• The results and algorithm presented in this paper apply to the multi-input case. This is discussed in Section 5-C.

Related Work: Structural controllability of LTI systems is widely studied (see [1], [9], [10], [11] and references therein). For necessary and sufficient conditions related to various structural properties of the systems, see the survey paper [12]. For various applications in structural control see [13]. We briefly describe the most relevant literature here.

Composite systems consisting of subsystems with similar dynamics is studied in [14], [15]. Paper [14] formulated conditions for checking various system theoretic properties of composite systems when all subsystems are identical. Analysis of various system theoretic properties of composite systems with symmetrically interconnected subsystems is given in [15]. The controllability and observability of composite systems are addressed in papers [16], [17], [18] and [19]. Decentralized controller synthesis of composite systems is studied in [15] when the subsystems have identical dynamics and the interconnections are symmetric. Optimization problems in LTI composite systems are also studied in many papers for various problem settings. For instance, optimal network design for efficient average consensus of multi-agent systems is considered in [3]. Paper [20] addressed consensus in multi-agent systems when the communication topology of the agents have a spanning tree. On the other hand, design of optimal trajectory for establishing connectivity of spatially distributed dynamic agents is addressed in [21]. Optimal topology design is formulated as a mixed-integer semi-definite programming problem in [7] when there is a trade-off between cost of communication links and the closed-loop performance. While all these papers use numerical system matrices in their analysis, our approach is to use structural analysis for an optimal topology design in large complex networks. Moreover, our focus is on the structural controllability of the network.

Structural analysis of composite systems is studied in literature where various *conditions for checking* structural controllability of composite systems in terms of subsystems are given (see [5], [22], [23], [24], [25], [26] and references therein). The algorithm given in [5] accomplishes this using a distributed algorithm. In [26], a graphic notion referred to as 'g-cactus' is defined using which a sufficient condition is given for structural controllability of composite systems. This paper solves *optimal topology design* for structural controllability of composite systems by minimizing the number of communication links among subsystems. Specifically, we focus on composite systems composed of homogeneous (structurally identical) subsystems. We had addressed this problem for the case of irreducible[1]. heterogeneous subsystems in [27]. Recently, the constrained version of the problem, i.e., when the interconnection edges that can be included is constrained to a specified set, is shown to be NP-hard and a 2-approximation algorithm is given in [28]. Addition of the minimum number of edges in the system digraph to render a structurally controllable system are addressed in many papers including [29] and [30]. In contrast to [29], [30] in which insertion of an edge between any two states is allowed, we only allow interconnections between subsystems and the structure of the individual subsystems are unaltered.

Robustness of interdependent networks under node attacks is studied in [31] and a redundant design approach is proposed using node and edge backup (addition). Structural perturbations of structured systems in the context of strong structural controllability is studied in [32] and an upper bound on the number of edges that can be added/deleted to achieve strong structural controllability is derived. Paper [33] addressed structural perturbations in undirected networks with special topologies and proposed a method to synthesize structural and strong structural networks. In this paper, we analyze (weak) structural controllability in general (no structural constraint on the topology of the subsystem other than homogeneous) composite systems and derive the exact value of the minimum number of edges which when added into the composite system guarantee structural controllability. Further, we only consider edge perturbations and no addition of states or inputs are allowed as in [31]. Papers [34], [35] proved that selecting a minimum cost edge set that when added into the input pattern make the structured system structurally controllable is NP-hard. However, we focus on addition of minimum number of edges in the state matrix pattern of the composite system

[1]A subsystem is said to be irreducible if the associated state digraph is strongly connected

in contrast to addition into the input pattern. We prove that this problem is polynomial-time solvable and also provide a polynomial-time algorithm. The key difference between the problem considered in this paper and that in [34], [35] is that we only allow addition of interconnection edges and the structure of the individual subsystems are unaltered.

Organization: The organization of the rest of the paper is as follows: in Section 2, we formulate the optimal network topology design problem for structured composite systems. In Section 3, we give few graph theoretic preliminaries used in the sequel and some existing results. In Section 4, we first find the minimum number of interconnections required to make a composite system structurally controllable when state matrices of all the subsystems are structurally identical. Then we give a polynomial time algorithm for solving the optimal network topology design problem. In Section 5, we demonstrate our algorithm using an illustrative example and also discuss a few special cases and the extension to the multi-input case. In Section 6, we give concluding remarks and future directions.

2. Problem Statement

Structural representation of an LTI system with dynamics $\dot{x}(t) = Ax(t) + Bu(t)$, where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, is given by $\bar{A} \in \{0, \star\}^{n \times n}$ and $\bar{B} \in \{0, \star\}^{n \times m}$. Here $\mathbb{R}$ denotes the set of real numbers and $\star$ denotes a free independent parameter. The pair $(\bar{A}, \bar{B})$ structurally represents a system (A, B) if it satisfies the following:

$$\begin{aligned} \bar{A}_{pq} &= 0 \text{ whenever } A_{pq} = 0, \text{ and} \\ \bar{B}_{pq} &= 0 \text{ whenever } B_{pq} = 0. \end{aligned} \tag{1}$$

We refer to (A, B) that satisfies (1) as a *numerical realization* of the *structured system* $(\bar{A}, \bar{B})$. Note that $(\bar{A}, \bar{B})$ does not have numerical values but only indicates locations where non-zero entries are possible. Thus for a given (A, B), $(\bar{A}, \bar{B})$ structurally represents a class of control systems corresponding to all possible numerical realizations. The key idea in structural controllability is to determine controllability of the class of systems represented by $(\bar{A}, \bar{B})$. Specifically, we have the following definition.

Definition 2.1 ([8])**.** *The structured system $(\bar{A}, \bar{B})$ is said to be structurally controllable if there exists at least one controllable numerical realization (A, B).*

Remark 2.2. *Even though the definition of structural controllability requires only one controllable realization, it is known that if a system is structurally controllable, then 'almost all' numerical realizations of the same structure is controllable [36]. In other words, structural controllability is a generic property.*

Now we describe structural representation of a composite system consisting of k subsystems with *structurally identical* state matrices. Two matrices are referred to as structurally identical if their zero/non-zero pattern are the same. Consider k subsystems with structured state matrix $\bar{A}_s \in \{0, \star\}^{n_s \times n_s}$. Thus $\bar{A}_s$ denotes the structured state matrix for every subsystem and each subsystem has dimension n_s. Although structured state matrix of every subsystem is $\bar{A}_s$, each subsystem can possibly have a different numerical realization of $\bar{A}_s$. This is important because building subsystems with exactly identical dynamics is not practically feasible. Let $\bar{B}_i \in \{0, \star\}^{n_s \times m_i}$ denote the structured input matrix for the i^{th} subsystem, for $i = 1, \ldots, k$. Different agents in a networked system receive control signals differently due to which the input matrices are not structurally identical. The pair $(\bar{A}_s, \bar{B}_i)$ is referred to as the i^{th} *subsystem* and is denoted by S_i. With this notation, the dynamics of S_i is

$$\dot{x}_i(t) = \bar{A}_s x_i(t) + \bar{B}_i u_i(t), \text{ for } i = 1, \ldots, k. \tag{2}$$

We do not assume that each subsystem is individually structurally controllable. To achieve structural controllability, one needs to interconnect subsystems. Let $\bar{E}_{ij} \in \{0, \star\}^{n_s \times n_s}$ denote the *structured connection matrix* from S_j to S_i. With the structured connection matrices, the composite structured system of k subsystems has the following dynamics:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} \bar{A}_s & \bar{E}_{12} & \cdots & \bar{E}_{1k} \\ \bar{E}_{21} & \bar{A}_s & \cdots & \bar{E}_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ \bar{E}_{k1} & \bar{E}_{k2} & \cdots & \bar{A}_s \end{bmatrix}}_{\bar{A}_T} x(t) + \underbrace{\begin{bmatrix} \bar{B}_1 & 0 & \cdots & 0 \\ 0 & \bar{B}_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{B}_k \end{bmatrix}}_{\bar{B}_T} u(t), \tag{3}$$

where $\bar{A}_T \in \{0, \star\}^{n_T \times n_T}$ with $n_T = k \times n_s$ and $\bar{B}_T \in \{0, \star\}^{n_T \times m_T}$ with $m_T = \sum_{i=1}^{k} m_i$. Here, $x = [x_1^T, \ldots, x_k^T]^T$ with $x_i = [x_1^i, \ldots, x_{n_s}^i]^T$ and $u = [u_1^T, \ldots, u_k^T]^T$ with $u_i = [u_1^i, \ldots, u_{m_i}^i]^T$. The system $(\bar{A}_T, \bar{B}_T)$ is said to be a *structured composite system* formed by subsystems $(\bar{A}_s, \bar{B}_1), \ldots, (\bar{A}_s, \bar{B}_k)$ interconnected through $\bar{E}_{ij}$'s, for $i, j \in \{1, \ldots, k\}$.

Our aim in this paper is to design a structurally controllable optimal network topology of $(\bar{A}_T, \bar{B}_T)$. Since we cannot change the dynamics of the individual subsystem, optimality is with respect to designing interconnection matrices. Formally, the optimization problem we consider is as follows:

Problem 2.3. *Given k subsystems with structurally identical state matrices $\bar{A}_s \in \{0, \star\}^{n_s \times n_s}$ and a structured input matrix $\bar{B}_T \in \{0, \star\}^{n_T \times m_T}$ consisting of input matrices $\bar{B}_i \in \{0, \star\}^{n_s \times m_i}$ as given in* (3)*, for $i \in \{1, \ldots, k\}$, where $n_T = k n_s$ and $m_T = \sum_{i=1}^{k} m_i$, find*

$$\bar{A}_T^{\star} \in \arg\min_{\bar{A}'_T \in \mathcal{K}} \left\| \bar{A}'_T \right\|_0,$$

where $\mathcal{K} := \{\bar{A}'_T \in \{0, \star\}^{n_T \times n_T} :$ all $(n_s \times n_s)$ diagonal submatrices of $\bar{A}'_T$ are $\bar{A}_s$ and $(\bar{A}'_T, \bar{B}_T)$ is structurally controllable$\}$.

Here, $\|\cdot\|_0$ denotes the zero matrix norm[2]. The set $\mathcal{K}$ denotes the set of all feasible solutions of Problem 2.3. Two matrices $\bar{A}'_T$ and $\bar{A}''_T$ in $\mathcal{K}$ differs only in their off-diagonal blocks. Note that, for $\bar{E}_{ij} = \{\star\}^{n_s \times n_s}$ for all i, j, the composite structured system is structurally controllable if $\bar{B}_T \neq 0$ and hence the set $\mathcal{K}$ is non-empty. Solving the minimum interconnection problem is same as minimizing the non-zero entries in matrices in $\mathcal{K}$, since for all matrices in $\mathcal{K}$ the diagonal blocks are fixed and optimization is possible only corresponding to the off-diagonal blocks. This in turn is same as minimizing the interconnections.

Remark 2.4. *Note that an interconnection link corresponds to a $\star$ entry in $\bar{E}_{ij}$ and it can be also considered as an output of subsystem S_j fed as input to subsystem S_i.*

[2]Although $\|\cdot\|_0$ does not satisfy some of the norm axioms, the number of non-zero entries in a matrix is conventionally referred to as the *zero norm*.

As a first step to solving Problem 2.3 and for better insight, we first consider a case where only the first subsystem receives an input. That is, $\bar{B}_1 \in \{0,\star\}^{n_s \times 1}$ and $\bar{B}_i = 0$ for all $i \in \{2,\ldots,k\}$. Then $\bar{B}_T \in \{0,\star\}^{n_T \times 1}$ and recall from the structure of $\bar{B}_T$ that an input of a particular subsystem connects to arbitrary number of states of that subsystem only. Interconnections are essential for the composite system to be structurally controllable. The single input case, with only one of the agent receiving an input is relevant in many applications, including single leader multi-agent systems [6] and single source drug target identification [13]. Later in Section 5-C we show that the results obtained for the single input case extend to the multi-input case also. The constructions described in Section 3 is given for a multi-input as these descriptions are later needed while explaining the multi-input case in Section 5-C.

3. Review of Essential Graph Theoretic Results

The key motivation behind considering graphs for analyzing structured systems is because we can represent the influences of states and inputs on each state through a directed graph. In order to capture the interactions of states and inputs efficiently, we construct digraphs corresponding to a structured system $(\bar{A},\bar{B})$ as described below.

Consider a structured system[3] $(\bar{A},\bar{B})$, where $\bar{A} \in \{0,\star\}^{n\times n}$ and $\bar{B} \in \{0,\star\}^{n\times m}$. Then the state digraph is $\mathcal{D}(\bar{A}) := (V_X, E_X)$, where $V_X = \{x_1,\ldots,x_n\}$ and $(x_p,x_q) \in E_X \Leftrightarrow \bar{A}_{qp} = \star$. Now we define the system digraph as $\mathcal{D}(\bar{A},\bar{B}) := (V_X \cup V_U, E_X \cup E_U)$, where $V_U = \{u_1,\ldots,u_m\}$ and $(u_p,x_q) \in E_U \Leftrightarrow \bar{B}_{qp} = \star$. A state x_q is said to be *accessible* if there exists a path from some input node u_p to x_q. A digraph is said to be strongly connected if for each ordered pair of vertices (v_p,v_q), there exists a path from v_p to v_q. Using the *strong connectedness* of the digraph $\mathcal{D}(\bar{A})$ one can check the accessibility of states $\{x_1,\ldots,x_n\}$. A *strongly connected component* (SCC) is a maximal strongly connected subgraph of a digraph [37]. All states of a structured system are accessible if and only if every SCC consists of at least one state that is accessible. We characterize the SCCs as per the following definition.

Definition 3.1. *In a digraph, an SCC $\hat{\mathcal{N}}$ is said to be* non-top linked *if there are no directed edges from the nodes of other SCCs into any node in $\hat{\mathcal{N}}$.*

All states in a subsystem are accessible if each non-top linked SCC contains an accessible state [11]. While accessibility of all states is necessary for structural controllability, it is not sufficient. In addition to accessibility the system digraph should satisfy a *no-dilation* condition. Presence of dilations in $\mathcal{D}(\bar{A},\bar{B})$ can be easily checked using a matching condition on the system bipartite graph $\mathcal{B}(\bar{A},\bar{B})$ defined below.

Given a bipartite graph $G_b = ((V_b,\widetilde{V}_b),\mathcal{E}_b)$, where $V_b \cup \widetilde{V}_b$ denotes the set of nodes satisfying $V_b \cap \widetilde{V}_b = \emptyset$ and $\mathcal{E}_b \subseteq V_b \times \widetilde{V}_b$ denotes the set of undirected edges, a matching M_b is a collection of edges $M_b \subseteq \mathcal{E}_b$ such that no two edges in the collection share the same endpoint. That is, for any (p,q) and $(w,v) \in M_b$, we have $p \neq w$ and $q \neq v$, where $p,w \in V_b$ and $q,v \in \widetilde{V}_b$. A vertex $q \in \widetilde{V}_b$ is said to be a *right unmatched vertex* with respect to a matching M_b if there does not exist a vertex $p \in V_b$ such that $(p,q) \in M_b$. A matching M_b is said to be a perfect matching of the bipartite graph G_b if $|M_b| = \min(|V_b|,|\widetilde{V}_b|)$. Further, given G_b and a cost function c from the set $\mathcal{E}_b$ to the set of non-negative real numbers $\mathbb{R}_+$, a minimum cost perfect matching, referred to as an *optimum matching*, is a perfect matching M_b such that $\sum_{e\in M_b} c(e) \leqslant \sum_{e\in M'_b} c(e)$, for every perfect matching M'_b in G_b [37]. There exists an equivalent matching condition on a bipartite graph denoted by $\mathcal{B}(\bar{A},\bar{B})$, for the no-dilation condition. The construction of $\mathcal{B}(\bar{A},\bar{B})$ is explained here in two stages. In the first stage, the state bipartite graph is $\mathcal{B}(\bar{A}) := ((V_{X'},V_X),\mathcal{E}_X)$ is constructed, where $V_X = \{x_1,x_2,\ldots,x_n\}$, $V_{X'} = \{x'_1,x'_2,\ldots,x'_n\}$ and $(x'_q,x_p) \in \mathcal{E}_X \Leftrightarrow (x_p,x_q) \in E_X$. Subsequently, the system bipartite graph is $\mathcal{B}(\bar{A},\bar{B}) := ((V_{X'},V_X \cup V_U),\mathcal{E}_X \cup \mathcal{E}_U)$ is constructed, where $V_U = \{u_1,u_2,\ldots,u_m\}$ and $(x'_q,u_p) \in \mathcal{E}_U \Leftrightarrow (u_p,x_q) \in E_U$. In $\mathcal{B}(\bar{A},\bar{B})$, the left vertex set indicates the equations, while the right vertex set indicated the variables. The following results relates $\mathcal{B}(\bar{A},\bar{B})$ and the no-dilation condition.

Proposition 3.2. *[38, Theorem 2] A digraph $\mathcal{D}(\bar{A},\bar{B})$ has no dilation if and only if the bipartite graph $\mathcal{B}(\bar{A},\bar{B})$ has a perfect matching.*

Using the state accessibility condition and the no-dilation condition, Lin proved the following result for structural controllability. Proof of the multi-input case is given in [39], [40].

Proposition 3.3. *[8, pp.207] The structured system $(\bar{A},\bar{B})$ is structurally controllable if and only if the associated digraph $\mathcal{D}(\bar{A},\bar{B})$ has no inaccessible states and has no dilations.*

Alternatively, a structured system is said to be structurally controllable if and only if all non-top linked SCCs are accessible and there exists a perfect matching in $\mathcal{B}(\bar{A},\bar{B})$.

4. Main Results

Consider a set of k subsystems $S_1,\ldots,S_k$ with structurally identical state matrices $\bar{A}_s$. Since the subsystems are structurally identical, they have the same number of nodes, i.e., n_s. We first describe construction of few graphs associated with these subsystems. The state digraph of a subsystem S_i has vertex set V_{X_i} and edge set E_{X_i}. Here, $V_{X_i} = \{x^i_1,\ldots,x^i_{n_s}\}$ and $(x^i_g,x^i_h) \in E_{X_i}$ if $\bar{A}_{s_{hg}} = \star$. Also the system digraph of S_i is denoted by $\mathcal{D}(\bar{A}_s,\bar{B}_i)$, with vertex set $V_{X_i} \cup V_{U_i}$ and edge set $E_{X_i} \cup E_{U_i}$. Here, $V_{U_i} = \{u^i_1,\ldots,u^i_{m_i}\}$ and $(u^i_g,x^i_h) \in E_{U_i}$ if $\bar{B}_{i_{hg}} = \star$. We assume that all states of the j^{th} subsystem can be connected to all states of the i^{th} subsystem. In other words, for any pair of subsystems S_i,S_j, there is no restriction on the structure of $\bar{E}_{ij}$. Thus the set of all possible interconnections, denoted by $E_{\mathcal{I}}$, consists of $(x^i_g,x^j_h) \in E_{\mathcal{I}}$ for all $i,j \in \{1,\ldots,k\}$, $i \neq j$, $x^i_g \in V_{X_i}$ and $x^j_h \in V_{X_j}$. Now we define the digraph $\mathcal{D}(\bar{A}_T) := (\cup^k_{i=1} V_{X_i}, \cup^k_{i=1} E_{X_i} \cup E_{\mathcal{I}})$ with vertex set $\cup^k_{i=1} V_{X_i}$ and edge set $\cup^k_{i=1} E_{X_i} \cup E_{\mathcal{I}}$. The digraph $\mathcal{D}(\bar{A}_T)$ consists of all subsystems along with all possible interconnections. Then we define the composite system digraph as $\mathcal{D}(\bar{A}_T,\bar{B}_T) := (\cup^k_{i=1} V_{X_i} \cup \cup^k_{i=1} V_{U_i}, \cup^k_{i=1} E_{X_i} \cup E_{\mathcal{I}} \cup \cup^k_{i=1} E_{U_i})$.

The state bipartite graph of subsystem S_i has vertex set $(V_{X'_i},V_{X_i})$ and edge set $\mathcal{E}_{X_i}$. Here, $V_{X'_i} = \{x'^i_1,\ldots x'^i_{n_s}\}$ and $(x'^i_h,x^i_g) \in \mathcal{E}_{X_i} \Leftrightarrow (x^i_g,x^i_h) \in E_{X_i}$. Then the *subsystem bipartite graph* $\mathcal{B}(\bar{A}_s,\bar{B}_i)$ is defined with vertex set $(V_{X'_i}, V_{X_i} \cup V_{U_i})$

[3]Typical structured system is denoted by $(\bar{A},\bar{B})$ and the related concepts can be extended to specific system under consideration.

and edge set $\mathcal{E}_{X_i} \cup \mathcal{E}_{U_i}$, where $({x'}^i_g, u^i_h) \in \mathcal{E}_{U_i} \Leftrightarrow (u^i_h, x^i_g) \in E_{U_i}$. Now we will discuss the construction of the bipartite graphs associated with the composite system. The state bipartite graph is $\mathcal{B}(\bar{A}_T) := ((\cup_{i=1}^k V_{X'_i}, \cup_{i=1}^k V_{X_i}), \cup_{i=1}^k \mathcal{E}_{X_i} \cup \mathcal{E}_{\mathcal{I}})$, where $({x'}^i_g, x^j_h) \in \mathcal{E}_{\mathcal{I}} \Leftrightarrow (x^j_h, x^i_g) \in E_{\mathcal{I}}$. Further, the system bipartite graph of the composite system is $\mathcal{B}(\bar{A}_T, \bar{B}_T) := ((\cup_{i=1}^k V_{X'_i}, \cup_{i=1}^k V_{X_i} \cup \cup_{i=1}^k V_{U_i}), \cup_{i=1}^k \mathcal{E}_{X_i} \cup \mathcal{E}_{\mathcal{I}} \cup \cup_{i=1}^k \mathcal{E}_{U_i})$, where $({x'}^i_g, u^i_h) \in \mathcal{E}_{U_i} \Leftrightarrow (u^i_h, x^i_g) \in E_{U_i}$. This completes the construction of the digraphs and the bipartite graphs associated with composite system composed of subsystems. Now we derive a closed-form expression for the minimum number of interconnections required for structural controllability and then present an algorithm to solve Problem 2.3.

A. Finding Minimum Number of Interconnections Required to Solve Problem 2.3

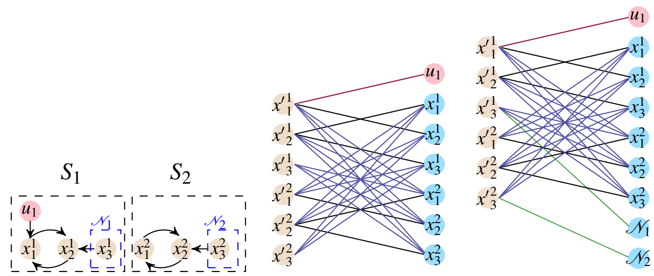
Using the constructions and definitions given above, now we analyze and solve Problem 2.3. Given a set of subsystems with structurally identical state matrices, we give an expression to find the minimum number of interconnections required to make the composite system structurally controllable. Subsequently, in subsection 4-B, we propose a polynomial time algorithm to identify an optimum set of interconnections and then prove its optimality. This algorithm solves Problem 2.3.

Let $\mathscr{N}_{\mathcal{H}} = \{\mathscr{N}_1, \ldots, \mathscr{N}_Q\}$ denotes the set of non-top linked SCCs of the subsystems that are inaccessible in the digraph with vertex set $\cup_{i=1}^k V_{X_i} \cup \cup_{i=1}^k V_{U_i}$ and edge set $\cup_{i=1}^k E_{X_i} \cup \cup_{i=1}^k E_{U_i}$. In other words, these are the non-top linked SCCs of the subsystems that are inaccessible[4] without using interconnections. Thus interconnections are essential to achieve accessibility of these non-top linked SCCs in the composite system. With some abuse of notation we denote a set of Q nodes that correspond to SCCs $\mathscr{N}_1, \ldots, \mathscr{N}_Q$ using the same notation. For the subsystems $S_1, \ldots, S_k$, we now define a bipartite graph $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathscr{N}_{\mathcal{H}}) := ((\cup_{i=1}^k V_{X'_i}, \cup_{i=1}^k V_{X_i} \cup \cup_{i=1}^k V_{U_i} \cup \mathscr{N}_{\mathcal{H}}), \cup_{i=1}^k \mathcal{E}_{X_i} \cup \mathcal{E}_{\mathcal{I}} \cup \cup_{i=1}^k \mathcal{E}_{U_i} \cup \mathcal{E}_{\mathscr{N}})$, where $({x'}^i_g, \mathscr{N}_h) \in \mathcal{E}_{\mathscr{N}} \Leftrightarrow x^i_g$ is a vertex in SCC $\mathscr{N}_h$. Figure 1 shows an example demonstrating construction of $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ and $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathscr{N}_{\mathcal{H}})$. Bipartite graphs with non-top linked SCCs of the structured system as nodes is used in the literature for structural analysis of LTI systems in different contexts [10], [41].

Our approach is to find minimum cardinality interconnections for structural controllability using minimum cost perfect matching. The edges in sets $\cup_{i=1}^k \mathcal{E}_{U_i}$ and $\cup_{i=1}^k \mathcal{E}_{X_i}$ are edges within the subsystems and hence are not to be optimized. However, a matching with maximum number of edges from the set $\cup_{i=1}^k \mathcal{E}_{U_i}$ will have more unmatched state nodes in the right vertex set, i.e., equations, of the bipartite graph which can then be used for interconnections. This motivates us to choose a weight 0 for edges in $\cup_{i=1}^k \mathcal{E}_{U_i}$ and 1 for $\cup_{i=1}^k \mathcal{E}_{X_i}$. Note that, it is essential that all non-top linked SCCs must have at least one incoming interconnection edge, in order to guarantee the accessibility, and further, an interconnection edge starting from a state node in a non-top linked SCC will contribute towards both accessibility and no-dilation conditions. This motivates the selection of choosing weight 2 for edges in $\mathcal{E}_{\mathscr{N}}$ and 3 for $\mathcal{E}_{\mathcal{I}}$. Note that, the numerical values in the cost function are not crucial, however, the order needs to be maintained. Now we define the cost function $c_{\mathcal{H}}$ on $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathscr{N}_{\mathcal{H}})$ as:

$$c_{\mathcal{H}}(e) := \begin{cases} 0, \text{ for } e \in \cup_{i=1}^k \mathcal{E}_{U_i}, \\ 1, \text{ for } e \in \cup_{i=1}^k \mathcal{E}_{X_i}, \\ 2, \text{ for } e \in \mathcal{E}_{\mathscr{N}}, \\ 3, \text{ for } e \in \mathcal{E}_{\mathcal{I}}. \end{cases} \tag{4}$$

With slight abuse of notation, we use Eq. (4) to denote the cost function on $\mathcal{B}(\bar{A}_T, \bar{B}_T)$, where $\mathcal{E}_{\mathscr{N}} = \emptyset$. There exists a perfect matching in both the bipartite graphs $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ and $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathscr{N}_{\mathcal{H}})$ as the set $\mathcal{K}$ is non-empty.



(a) Subsystems S_1, S_2 (b) $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ (c) $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathscr{N}_{\mathcal{H}})$

Figure 1: Illustrative example demonstrating construction of $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ and $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathscr{N}_{\mathcal{H}})$. The green coloured edges in Figure 1c correspond to $\mathcal{E}_{\mathscr{N}}$. The blue and red coloured edges in Figures 1b and 1c correspond to $\cup_{i=1}^k \mathcal{E}_{X_i}$ and $\cup_{i=1}^k \mathcal{E}_{U_i}$, respectively.

Let $M^\star_{\mathcal{H}}$ be an optimum matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathscr{N}_{\mathcal{H}})$ under cost function $c_{\mathcal{H}}$. Recall, that an optimum matching is a perfect matching with the minimum cost. Define $|M^\star_{\mathcal{H}} \cap \mathcal{E}_{\mathscr{N}}| := \alpha_{\mathscr{N}}$ and $|M^\star_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}| := \beta_{\mathcal{I}}$. As the number of edges in a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathscr{N}_{\mathcal{H}})$ is fixed and the cost of any minimum cost perfect matching is unique and the cost function $c_{\mathcal{H}}$ has different values associated with edges in $\mathcal{E}_{\mathscr{N}}$ and $\mathcal{E}_{\mathcal{I}}$, the indices $\alpha_{\mathscr{N}}$ and $\beta_{\mathcal{I}}$ are constant for a given subsystem $\bar{A}_s$ and number of subsystems, k. In a graph that is well connected fewer number of edges from sets $\mathcal{E}_{\mathscr{N}}$ and $\mathcal{E}_{\mathcal{I}}$ are present in an optimum matching $M^\star_{\mathcal{H}}$ and hence lower are values of $\alpha_{\mathscr{N}}$ and $\beta_{\mathcal{I}}$. Next, we give one of the main result of this paper.

Theorem 4.1. *Consider a structured composite system consisting of k subsystems with structurally identical state matrices. Let $\bar{A}^\star_T$ be an optimum solution to Problem 2.3 and let $E^\star_{\mathcal{I}}$ be the set of interconnection edges in $\bar{A}^\star_T$. Further, let $|\mathscr{N}_{\mathcal{H}}| = Q$ and let $M^\star_{\mathcal{H}}$ be an optimum matching in the bipartite graph $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathscr{N}_{\mathcal{H}})$ such that $|M^\star_{\mathcal{H}} \cap \mathcal{E}_{\mathscr{N}}| = \alpha_{\mathscr{N}}$ and $|M^\star_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}| = \beta_{\mathcal{I}}$. Then, $|E^\star_{\mathcal{I}}| = \beta_{\mathcal{I}} + Q$.*

To prove Theorem 4.1, we state and prove the following lemmas with respect to the bipartite graph $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. Since we assume single input, $\cup_{i=1}^k V_{U_i} = u^1_1$. For the sake of notational brevity, henceforth we denote u^1_1 as u_1. The optimality proof is a construction based proof which relies on the fact that an optimum matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ contains input node u_1 matched (Lemma 4.2) and also contains a right unmatched node which is input accessible (Lemma 4.3).

Lemma 4.2. *Let $(\bar{A}_T, \bar{B}_T)$ be a structured composite system consisting of k subsystems with structurally identical state*

[4]An SCC is said to inaccessible if it consists of at least one state node that is inaccessible.

matrices interconnected using all possible interconnections, $E_{\mathcal{I}}$, and let $\bar{B}_T$ be a single input matrix. Let $\overline{M}$ be an optimum matching obtained by solving the minimum cost perfect matching on the bipartite graph $\mathcal{B}(\bar{A}_T,\bar{B}_T)$ under cost function $c_{\mathcal{H}}$ given in (4)*. Then, $(x'^{1}_{a},u_1) \in \overline{M}$ for some $a \in \{1,\ldots,n_s\}$.*

Proof of Lemma 4.2 is given in the appendix.

Lemma 4.3. *Let $(\bar{A}_T,\bar{B}_T)$ be a structured composite system consisting of k subsystems with structurally identical state matrices interconnected using all possible interconnections, $E_{\mathcal{I}}$, and let $\bar{B}_T$ be a single input matrix. Consider an optimum perfect matching $\overline{M}$ in $\mathcal{B}(\bar{A}_T,\bar{B}_T)$ and let $\mathcal{E}_{\mathcal{I}'} = \overline{M} \cap \mathcal{E}_{\mathcal{I}}$. Then, there exists a right unmatched node in $\overline{M}$ such that it is accessible in the digraph constructed with vertex set $\cup_{i=1}^{k} V_{X_i} \cup \cup_{i=1}^{k} V_{U_i}$ and edge set $\cup_{i=1}^{k} E_{X_i} \cup \cup_{i=1}^{k} E_{U_i} \cup E_{\mathcal{I}'}$, where $(x_g^i,x_h^j) \in E_{\mathcal{I}'} \Leftrightarrow (x'^{j}_{h},x_g^i) \in \mathcal{E}_{\mathcal{I}'}$.*

See appendix for the proof of Lemma 4.3.

Lemma 4.3 concludes that with respect to any optimum perfect matching in $\mathcal{B}(\bar{A}_T,\bar{B}_T)$, there exists a *unique right unmatched accessible node*. Let $M^{\star}_{\mathcal{H}}$ be an optimum matching in $\mathcal{B}(\bar{A}_T,\bar{B}_T,\mathcal{N}_{\mathcal{H}})$ under cost function $c_{\mathcal{H}}$. Now we give the following result to show that the input node u_1 is matched in $M^{\star}_{\mathcal{H}}$. Even though the proof of Lemma 4.4 uses the same argument as that of the proof of Lemma 4.2, it is given in the appendix for the sake of completeness.

Lemma 4.4. *Consider a structured composite system consisting of k subsystems with structurally identical state matrices. Let $M^{\star}_{\mathcal{H}}$ be an optimum perfect matching obtained by solving the minimum cost perfect matching on the bipartite graph $\mathcal{B}(\bar{A}_T,\bar{B}_T,\mathcal{N}_{\mathcal{H}})$ using cost function $c_{\mathcal{H}}$ given in* (4)*. Then, $(x'^{1}_{a},u_1) \in M^{\star}_{\mathcal{H}}$ for some $a \in \{1,\ldots,n_s\}$.*

We present the Proof of Lemma 4.4 in the appendix.

By Lemma 4.4, $|M^{\star}_{\mathcal{H}} \cap \cup_{i=1}^{k} \mathcal{E}_{U_i}| = 1$, when $|\cup_{i=1}^{k} \mathcal{E}_{U_i}| = 1$. With respect to $M^{\star}_{\mathcal{H}}$, we now prove the following lemmas. Lemmas 4.3 and 4.4 are used in the proofs of Lemmas 4.5 and 4.6 which are later used in the proof of Theorem 4.1.

Lemma 4.5. *Consider a structured composite system consisting of k subsystems with structurally identical state matrices. Let $M^{\star}_{\mathcal{H}}$ be an optimum perfect matching in the bipartite graph $\mathcal{B}(\bar{A}_T,\bar{B}_T,\mathcal{N}_{\mathcal{H}})$ under cost function $c_{\mathcal{H}}$ such that $|M^{\star}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{N}}| = \alpha_{\mathcal{N}}$ and $|M^{\star}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}| = \beta_{\mathcal{I}}$. Then, any minimum cost perfect matching $\widetilde{M}_{\mathcal{H}}$ in $\mathcal{B}(\bar{A}_T,\bar{B}_T)$ satisfies $|\widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}| = \alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$.*

Proof of Lemma 4.5 is given in the appendix. From Lemma 4.5, we conclude that an optimum matching in $\mathcal{B}(\bar{A}_T,\bar{B}_T)$ under $c_{\mathcal{H}}$ has $\alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$ interconnections. An intuitive explanation of indices $\alpha_{\mathcal{N}}$ and $\beta_{\mathcal{I}}$ is given in Remark 4.9.

In the result below we prove the existence of an optimum perfect matching $\hat{M}_{\mathcal{H}}$ in $\mathcal{B}(\bar{A}_T,\bar{B}_T)$ that ensures accessibility of $\alpha_{\mathcal{N}}$ SCCs in $\mathcal{N}_{\mathcal{H}}$ using only the interconnections in $\hat{M}_{\mathcal{H}}$.

Lemma 4.6. *Let $(\bar{A}_T,\bar{B}_T)$ be the structured composite system consisting of k subsystems with structurally identical state matrices and interconnected using all possible interconnections, $E_{\mathcal{I}}$. Then, there exists an optimum matching $\hat{M}_{\mathcal{H}}$ in $\mathcal{B}(\bar{A}_T,\bar{B}_T)$ such that $|\hat{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}| = \alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$. Further, SCCs $\{\mathcal{N}_1,\ldots,\mathcal{N}_{\alpha_{\mathcal{N}}}\} \in \mathcal{N}_{\mathcal{H}}$ are accessible in the digraph consisting of vertex set $\cup_{i=1}^{k} V_{X_i} \cup \cup_{i=1}^{k} V_{U_i}$ and edge set $\cup_{i=1}^{k} E_{X_i} \cup \cup_{i=1}^{k} E_{U_i} \cup E_{\mathcal{I}'}$, where $(x_h^j,x_g^i) \in E_{\mathcal{I}'} \Leftrightarrow (x'^{i}_{g},x_h^j) \in \hat{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}$.*

We present the proof of Lemma 4.6 in the appendix.

Using Lemmas 4.5 and 4.6, now we prove Theorem 4.1.

Proof of Theorem 4.1: We prove this result in two steps. In Step (i) we show that $|E^{\star}_{\mathcal{I}}| \leqslant \beta_{\mathcal{I}} + \varrho$ and in Step (ii) we show that $|E^{\star}_{\mathcal{I}}| \geqslant \beta_{\mathcal{I}} + \varrho$. The result follows from Steps (i) and (ii).

Step (i): Here we will prove that $|E^{\star}_{\mathcal{I}}| \leqslant \beta_{\mathcal{I}} + \varrho$. From Lemmas 4.5 and 4.6, we know that there exists a perfect matching $\hat{M}_{\mathcal{H}}$ in $\mathcal{B}(\bar{A}_T,\bar{B}_T)$ that uses exactly $\alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$ interconnections and out of ϱ SCCs in $\mathcal{N}_{\mathcal{H}}$, $\alpha_{\mathcal{N}}$ SCCs are accessible using these interconnections. After using the interconnections in $\hat{M}_{\mathcal{H}}$, there are $\varrho - \alpha_{\mathcal{N}}$ number of SCCs in $\mathcal{N}_{\mathcal{H}}$ that are inaccessible. Accessibility of these SCCs can be achieved by adding $\varrho - \alpha_{\mathcal{N}}$ interconnections more. Using $(\alpha_{\mathcal{N}} + \beta_{\mathcal{I}}) + (\varrho - \alpha_{\mathcal{N}}) = \beta_{\mathcal{I}} + \varrho$ interconnections, all SCCs are accessible and there exists a perfect matching in $\mathcal{B}(\bar{A}_T,\bar{B}_T)$. Consequently, one can interconnect the subsystems using $\beta_{\mathcal{I}} + \varrho$ interconnections such that the composite system is structurally controllable. Hence $|E^{\star}_{\mathcal{I}}| \leqslant \beta_{\mathcal{I}} + \varrho$.

Step (ii): Here we will prove that $|E^{\star}_{\mathcal{I}}| \geqslant \varrho + \beta_{\mathcal{I}}$. We prove this using a contradiction argument. Suppose not. Then $|E^{\star}_{\mathcal{I}}| < \varrho + \beta_{\mathcal{I}}$. This implies $|E^{\star}_{\mathcal{I}}| \leqslant \varrho + \beta_{\mathcal{I}} - 1$. Without loss of generality, assume that $|E^{\star}_{\mathcal{I}}| = \varrho + \beta_{\mathcal{I}} - 1$. Then one can interconnect the subsystems using $\varrho + \beta_{\mathcal{I}} - 1$ interconnections such that the composite system is structurally controllable. Consider an optimum matching $\overline{M}_{\mathcal{H}}$ in $\mathcal{B}(\bar{A}_T,\bar{B}_T)$ under cost function $c_{\mathcal{H}}$. We know from Lemma 4.5 that $\overline{M}_{\mathcal{H}}$ consists of $\alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$ interconnections. Thus $[(\varrho + \beta_{\mathcal{I}} - 1) - (\alpha_{\mathcal{N}} + \beta_{\mathcal{I}})] = \varrho - \alpha_{\mathcal{N}} - 1$ interconnections are solely for achieving accessibility condition. This implies that out of ϱ inaccessible non-top linked SCCs, $(\alpha_{\mathcal{N}} + 1)$ SCCs are accessible using the interconnections in $\overline{M}_{\mathcal{H}}$. Note that SCCs, $\mathcal{N}_1,\ldots,\mathcal{N}_Q$, are those SCCs whose states do not have a directed path from the input node when interconnections are not used. Hence at least one node in each of the $(\alpha_{\mathcal{N}} + 1)$ SCCs are connected using interconnection edges in $\overline{M}_{\mathcal{H}}$. Now we will construct a matching in $\mathcal{B}(\bar{A}_T,\bar{B}_T,\mathcal{N}_{\mathcal{H}})$ from $\overline{M}_{\mathcal{H}}$. Note that $\overline{M}_{\mathcal{H}}$ is a perfect matching in $\mathcal{B}(\bar{A}_T,\bar{B}_T,\mathcal{N}_{\mathcal{H}})$ also. Let $\mathcal{E}_Q$ denotes the set of interconnections connecting one node each of $(\alpha_{\mathcal{N}} + 1)$ SCCs in $\overline{M}_{\mathcal{H}}$. Then, $|\mathcal{E}_Q| = \alpha_{\mathcal{N}} + 1$. Remove edge set $\mathcal{E}_Q$ from $\overline{M}_{\mathcal{H}}$ and connect them to $\alpha_{\mathcal{N}} + 1$ SCC nodes, say $\{\mathcal{N}_1,\ldots,\mathcal{N}_{\alpha_{\mathcal{N}}+1}\}$, in the right side of $\mathcal{B}(\bar{A}_T,\bar{B}_T,\mathcal{N}_{\mathcal{H}})$. Let this new set of edges is denoted by $\mathcal{E}_{\alpha_{\mathcal{N}}+1}$. Define $M''_{\mathcal{H}} := \{\overline{M}_{\mathcal{H}} \setminus \mathcal{E}_Q\} \cup \{\mathcal{E}_{\alpha_{\mathcal{N}}+1}\}$. Recall, $|M^{\star}_{\mathcal{H}} \cap \cup_{i=1}^{k} \mathcal{E}_{X_i}| = n_T - (1 + \alpha_{\mathcal{N}} + \beta_{\mathcal{I}})$. The cost of this new matching under $c_{\mathcal{H}}$ is $3[(\alpha_{\mathcal{N}} + \beta_{\mathcal{I}}) - (\alpha_{\mathcal{N}} + 1)] + 2(\alpha_{\mathcal{N}} + 1) + (n_T - (\alpha_{\mathcal{N}} + \beta_{\mathcal{I}} + 1)) = 2\beta_{\mathcal{I}} + \alpha_{\mathcal{N}} + n_T - 2$. Note that cost of optimum matching $M^{\star}_{\mathcal{H}}$ in $\mathcal{B}(\bar{A}_T,\bar{B}_T,\mathcal{N}_{\mathcal{H}})$ is $3\beta_{\mathcal{I}} + 2\alpha_{\mathcal{N}} + (n_T - (\alpha_{\mathcal{N}} + \beta_{\mathcal{I}} + 1)) = 2\beta_{\mathcal{I}} + \alpha_{\mathcal{N}} + n_T - 1$. Thus $c_{\mathcal{H}}(M''_{\mathcal{H}}) < c_{\mathcal{H}}(M^{\star}_{\mathcal{H}})$. This contradicts that $M^{\star}_{\mathcal{H}}$ is an optimum matching in $\mathcal{B}(\bar{A}_T,\bar{B}_T,\mathcal{N}_{\mathcal{H}})$ and the assumption $|E^{\star}_{\mathcal{I}}| < \varrho + \beta_{\mathcal{I}}$ is not true. Therefore, $|E^{\star}_{\mathcal{I}}| \geqslant \varrho + \beta_{\mathcal{I}}$. From Steps (i) and (ii), we get $|E^{\star}_{\mathcal{I}}| = \varrho + \beta_{\mathcal{I}}$. $\square$

The minimum number of interconnection required to solve Problem 2.3 is $\varrho + \beta_{\mathcal{I}}$. We present below an algorithm that identifies a set of interconnections, of cardinality $\varrho + \beta_{\mathcal{I}}$, that results in a structurally controllable composite system.

B. Algorithm to Solve Problem 2.3

In this subsection, we give an optimal algorithm to solve Problem 2.3 in polynomial time. The pseudocode of the proposed algorithm is given in Algorithm 4.1. Algorithm 4.1

incorporates a minimum weight perfect matching algorithm along with an edge reconstruction process. The key idea of the algorithm is to select interconnection edges in such a way that majority of them serve both the accessibility and the no-dilation condition. A brief explanation of the individual steps in the algorithm is given below.

Algorithm 4.1 Pseudocode for solving Problem 2.3 on structured subsystems consisting of structurally identical state matrices

---

Input: k structured subsystems with state matrices $\bar{A}_s$ and a structured composite input matrix $\bar{B}_T$

Output: Interconnections $\mathcal{I}_{\mathcal{H}}$

1: Construct the bipartite graph $\mathcal{B}(\bar{A}_T,\bar{B}_T) \leftarrow ((\cup_{i=1}^k V_{X_i'}, \cup_{i=1}^k V_{X_i} \cup \cup_{i=1}^k V_{U_i}), \cup_{i=1}^k \mathcal{E}_{X_i} \cup \mathcal{E}_{\mathcal{I}} \cup \cup_{i=1}^k \mathcal{E}_{U_i})$

2: Construct the bipartite graph $\mathcal{B}(\bar{A}_T,\bar{B}_T,\mathscr{N}_{\mathcal{H}}) \leftarrow ((\cup_{i=1}^k V_{X_i'}, \cup_{i=1}^k V_{X_i} \cup \cup_{i=1}^k V_{U_i} \cup \mathscr{N}_{\mathcal{H}}), \cup_{i=1}^k \mathcal{E}_{X_i} \cup \mathcal{E}_{\mathcal{I}} \cup \cup_{i=1}^k \mathcal{E}_{U_i} \cup \mathcal{E}_{\mathscr{N}})$

3: Define cost vector $c_{\mathcal{H}}(e) \leftarrow \begin{cases} 0, & \text{for } e \in \cup_{i=1}^k \mathcal{E}_{U_i}, \\ 1, & \text{for } e \in \cup_{i=1}^k \mathcal{E}_{X_i}, \\ 2, & \text{for } e \in \mathcal{E}_{\mathscr{N}}, \\ 3, & \text{for } e \in \mathcal{E}_{\mathcal{I}}. \end{cases}$

4: Find minimum cost maximum matching in $\mathcal{B}(\bar{A}_T,\bar{B}_T,\mathscr{N}_{\mathcal{H}})$ using cost function $c_{\mathcal{H}}$, say $M_{\mathcal{H}}^{\star}$

5: $M_{\mathcal{H}}' \leftarrow M_{\mathcal{H}}^{\star} \cap \{\cup_{i=1}^k \mathcal{E}_{X_i'} \cup \cup_{i=1}^k \mathcal{E}_{U_i}\}$

6: Find a perfect matching in $\mathcal{B}(\bar{A}_T,\bar{B}_T) \odot M_{\mathcal{H}}'$, say $M_{\mathcal{H}}''$

7: $\widetilde{M}_{\mathcal{H}} \leftarrow M_{\mathcal{H}}' \cup M_{\mathcal{H}}''$

8: $V_{\mathscr{N}'} \leftarrow \{x_g'^i : (x_g'^i, x_z^j) \in \widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}, x_g^i \in \mathscr{N}_o, \mathscr{N}_o$ not accessible using edges $\widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}\}$

9: Let x_a^d be the unique unmatched accessible node in $\widetilde{M}_{\mathcal{H}}$

10: **while** $|V_{\mathscr{N}'}| \neq 0$ **do**

11:     **if** $x_g'^i \in V_{\mathscr{N}'}$ and $i \neq d$ **then**

12:         $\widetilde{M}_{\mathcal{H}} \leftarrow \{\widetilde{M}_{\mathcal{H}} \setminus \{(x_g'^i, x_z^j)\}\} \cup \{(x_g'^i, x_a^d)\}$

13:     **else if** $x_g'^i \in V_{\mathscr{N}'}$ and $i = d$ **then**

14:         Find x_v^h such that for $\ell \in \{1,\ldots,n_s\}$, $(x_\ell'^j, x_v^h) \in \widetilde{M}_{\mathcal{H}}$

15:         $\widetilde{M}_{\mathcal{H}} \leftarrow \{\widetilde{M}_{\mathcal{H}} \setminus \{(x_\ell'^j, x_v^h)\}\} \cup \{(x_\ell'^j, x_a^d)\}$

16:     **end if**

17:     Update $V_{\mathscr{N}'}$ and the unique accessible unmatched node x_a^d

18: **end while**

19: $\mathcal{E}_{\mathcal{I}}^{Q} \leftarrow \{(x_v'^i, x_a^d)$: SCC $\hat{\mathscr{N}}$ is inaccessible in the digraph with vertex set $(\cup_{i=1}^k V_{X_i} \cup \cup_{i=1}^k V_{U_i})$ and edge set $\cup_{i=1}^k \mathcal{E}_{X_i} \cup \cup_{i=1}^k \mathcal{E}_{U_i} \cup (\widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}})$, $x_v^i \in \hat{\mathscr{N}}, x_a^d$ is accessible and $i \neq d\}$

20: $\mathcal{I}_{\mathcal{H}} \leftarrow \{(x_g^i, x_z^j) : (x_z'^j, x_g^i) \in \widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}\} \cup \{(x_a^d, x_v^i) : (x_v'^i, x_a^d) \in \mathcal{E}_{\mathcal{I}}^{Q}\}$

Steps 1-4: Initially we run a minimum cost perfect matching algorithm on the bipartite graph $\mathcal{B}(\bar{A}_T,\bar{B}_T,\mathscr{N}_{\mathcal{H}})$ using cost function $c_{\mathcal{H}}$. Let $M_{\mathcal{H}}^{\star}$ be the optimum matching. Then $|M_{\mathcal{H}}^{\star} \cap \mathcal{E}_{\mathcal{I}}| = \beta_{\mathcal{I}}$ and $|M_{\mathcal{H}}^{\star} \cap \mathcal{E}_{\mathscr{N}}| = \alpha_{\mathscr{N}}$ and from Lemma 4.4, we know that $|M_{\mathcal{H}}^{\star} \cap \cup_{i=1}^k \mathcal{E}_{U_i}| = 1$.

Steps 5-7: Now we define a matching $M_{\mathcal{H}}' := M_{\mathcal{H}}^{\star} \cap \{\cup_{i=1}^k \mathcal{E}_{X_i'} \cup \cup_{i=1}^k \mathcal{E}_{U_i}\}$. Note that $|M_{\mathcal{H}}'| = n_T - (\alpha_{\mathscr{N}} + \beta_{\mathcal{I}})$. Subsequently, we find the difference of $\mathcal{B}(\bar{A}_T,\bar{B}_T)$ and $M_{\mathcal{H}}'$, denoted as $\mathcal{B}(\bar{A}_T,\bar{B}_T) \odot M_{\mathcal{H}}'$. $\mathcal{B}(\bar{A}_T,\bar{B}_T) \odot M_{\mathcal{H}}'$ consists of only those nodes in $\mathcal{B}(\bar{A}_T,\bar{B}_T)$ that are not matched in $M_{\mathcal{H}}'$ and the edges between them. Moreover, there exists a perfect matching $M_{\mathcal{H}}''$ in $\mathcal{B}(\bar{A}_T,\bar{B}_T) \odot M_{\mathcal{H}}'$ such that $M_{\mathcal{H}}'' \subset \mathcal{E}_{\mathcal{I}}$ (see proof of Lemma 4.5). We define $\widetilde{M}_{\mathcal{H}}$ as the union of $M_{\mathcal{H}}'$ and $M_{\mathcal{H}}''$. Note that, $\widetilde{M}_{\mathcal{H}}$ is a perfect matching in $\mathcal{B}(\bar{A}_T,\bar{B}_T)$. Further, $|\widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}| = \alpha_{\mathscr{N}} + \beta_{\mathcal{I}}$ and $\alpha_{\mathscr{N}}$ interconnections connect to states in $\alpha_{\mathscr{N}}$ distinct SCCs. However, these $\alpha_{\mathscr{N}}$ SCCs need not be accessible using these $\alpha_{\mathscr{N}} + \beta_{\mathcal{I}}$ interconnections in $\widetilde{M}_{\mathcal{H}}$. Our aim is to update $\widetilde{M}_{\mathcal{H}}$ in such a way that in the new $\widetilde{M}_{\mathcal{H}}$, $|\widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}| = \alpha_{\mathscr{N}} + \beta_{\mathcal{I}}$ and $\alpha_{\mathscr{N}}$ SCCs are accessible using interconnections in $\widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}$.

Steps 8-18: For achieving the accessibility of $\alpha_{\mathscr{N}}$ SCCs, we first identify the $\alpha_{\mathscr{N}}$ interconnection edges in $\widetilde{M}_{\mathcal{H}}$ that connects to one state each in SCCs, say $\mathscr{N}_1,\ldots,\mathscr{N}_{\alpha_{\mathscr{N}}}$. Let $V_{\mathscr{N}'}$ is the set of $\alpha_{\mathscr{N}}$ left side nodes in $\mathcal{B}(\bar{A}_T,\bar{B}_T)$ belonging to SCCs that are matched through edges in $\mathcal{E}_{\mathcal{I}}$ in $\widetilde{M}_{\mathcal{H}}$. Further, these SCCs are inaccessible even after using interconnections in $\widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}$. By Lemma 4.3, we know that in $\widetilde{M}_{\mathcal{H}}$ there exists an unmatched accessible node. Let x_a^d be this node. Our approach is to remove the edges corresponding to nodes in $V_{\mathscr{N}'}$ from $\widetilde{M}_{\mathcal{H}}$ and make new interconnections using the node x_a^d such that SCCs become accessible. Consider an arbitrary vertex $x_g'^i \in V_{\mathscr{N}'}$. Let $x_g^i \in \mathscr{N}_o$. Then $x_g'^i$ satisfies one of the following cases: (a) $i \neq d$ or (b) $i = d$. In case (a), we update $\widetilde{M}_{\mathcal{H}}$ by removing the edge $(x_g'^i, x_z^j)$ and including the edge $(x_g'^i, x_a^d)$. Note that, in the updated $\widetilde{M}_{\mathcal{H}}$, SCC $\mathscr{N}_o$ is accessible. In case (b), the unique unmatched accessible node belongs to the same subsystem as x_g^i and hence the edge $(x_g'^i, x_a^d)$ cannot be formed. However, notice that since the unique unmatched node is in the i^{th} subsystem and $(x_g'^i, x_z^j) \in \widetilde{M}_{\mathcal{H}}$, there exists an edge $(x_\ell'^j, x_v^h)$ for some $\ell, v \in \{1,\ldots,n_s\}$ and $j \neq h$ in $\widetilde{M}_{\mathcal{H}}$. In case (b), we update $\widetilde{M}_{\mathcal{H}}$ by removing the edge $(x_\ell'^j, x_v^h)$ and including the edge $(x_\ell'^j, x_a^d)$. This results in node x_v^h being the unique unmatched accessible node. Sets $V_{\mathscr{N}'}$ and unique accessible unmatched node in $\widetilde{M}_{\mathcal{H}}$ are updated. Notice that if $i \neq h$, then in the next iteration of the **While** loop, $\mathscr{N}_o$ becomes accessible. On the other hand, if $i = h$, then by atmost n_s iterations of the **While** loop, node x_g^i of the i^{th} subsystem becomes accessible. This will make SCC $\mathscr{N}_o$ accessible. Notice that accessibility of $\mathscr{N}_o$ is achieved by keeping the number of interconnections the same as before, i.e., $\alpha_{\mathscr{N}} + \beta_{\mathcal{I}}$. By the end of the Step 18, $\alpha_{\mathscr{N}}$ SCCs are accessible using interconnections in $\widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}$. Consequently, the number of inaccessible SCCs are $\varrho - \alpha_{\mathscr{N}}$.

Steps 19-20: Now we add $\varrho - \alpha_{\mathscr{N}}$ interconnections one each to some state in these $\varrho - \alpha_{\mathscr{N}}$ SCCs from the accessible nodes in other subsystems. These set of edges that are added to attain accessibility of $\varrho - \alpha_{\mathscr{N}}$ SCCs is denoted by $\mathcal{E}_{\mathcal{I}}^{Q}$. Thus using $[(\alpha_{\mathscr{N}} + \beta_{\mathcal{I}}) + (\varrho - \alpha_{\mathscr{N}})] = \varrho + \beta_{\mathcal{I}}$ interconnections, we achieve accessibility of all SCCs $\{\mathscr{N}_1,\ldots,\mathscr{N}_Q\}$ and a perfect matching in $\mathcal{B}(\bar{A}_T,\bar{B}_T)$. The final interconnection edge set is given by $\mathcal{I}_{\mathcal{H}}$. This completes the description of Algorithm 4.1. Now we prove optimality of Algorithm 4.1 and its computational complexity below.

Theorem 4.7. *Algorithm 4.1 which takes as input k structured subsystems with state matrices $\bar{A}_s$ of dimension $(n_s \times n_s)$ and input matrix $\bar{B}_T$ gives as output the interconnection edges $\mathcal{I}_{\mathcal{H}}$ which is an optimal solution to Problem 2.3, i.e., $|\mathcal{I}_{\mathcal{H}}| = |E_{\mathcal{I}}^{\star}|$.*

Proof. By Theorem 4.1 we know that the minimum number

(a) For the matching $\widetilde{M}_{\mathcal{H}}$ (edges in $M''_{\mathcal{H}}$ are shown in red and the edges in $M'_{\mathcal{H}}$ are shown in blue) non-top linked SCCs $\mathcal{N}_1, \mathcal{N}_2$ are accessible. The unique accessible unmatched node with respect to this matching is x_5^2.

(b) For the matching $\widetilde{M}_{\mathcal{H}}$ (edges in $M''_{\mathcal{H}}$ are shown in red and the edges in $M'_{\mathcal{H}}$ are shown in blue) non-top linked SCCs $\mathcal{N}_1, \ldots, \mathcal{N}_4$ are accessible. The unique accessible unmatched node with respect to this matching is x_4^4.

Figure 2: Illustrative example demonstrating Algorithm 4.1 on subsystems S_1, S_2, S_3 and S_4. The blue and the red edges corresponds to a matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. The blue edges are those edges which connects two nodes in the same subsystem and the red edges are the interconnections in the matching. The set $\mathcal{N}_{\mathcal{H}}$ consists of 4 non-top linked SCCs.

of interconnections that solve Problem 2.3 is $|E_{\mathcal{I}}^\star| = \beta_{\mathcal{I}} + \varrho$. Using the set of interconnections formed in Algorithm 4.1, there exists a perfect matching and the no-dilation condition of the composed structured system is satisfied. By the edge reconstruction process, it is ensured that all non-top linked SCCs $\{\mathcal{N}_1, \ldots, \mathcal{N}_Q\}$ are made accessible. This guarantess the accessibility of all the state nodes of the composite system. The number interconnections made in Algorithm 4.1 is $\beta_{\mathcal{I}} + \varrho$ (see description of Steps 19-20). By Theorem 4.1, output of Algorithm 4.1 is an optimal solution to Problem 2.3. $\square$

Theorem 4.8. *Algorithm 4.1 which takes as input k structured subsystems with state matrices $\bar{A}_s$ of dimension $(n_s \times n_s)$ and input matrix $\bar{B}_T$ and gives as output the interconnection edges $\mathcal{I}_{\mathcal{H}}$ has running time complexity $O(n_T^{2.5})$, where $n_T = k \times n_s$.*

Proof. The number of inputs of each subsystem is of the order of the number of states, i.e., $m_i = O(n_s)$. This gives $m_T = O(n_T)$. The number of edges in the bipartite graphs $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ and $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathcal{N}_{\mathcal{H}})$ are of $O(n_T^2)$ as the number of vertices are of $O(n_T)$. Finding all SCCs in the subsystems is of complexity $O({n_s}^{2.5})$ [42], where n_s denotes the number of nodes in each subsystem. Thus constructing the bipartite graphs $\mathcal{B}(\bar{A}_T, \bar{B}_T), \mathcal{B}(\bar{A}_T, \bar{B}_T, \mathcal{N}_{\mathcal{H}})$ is of complexity $O(n_T^{2.5})$, where $n_T = k \times n_s$. Solving minimum cost perfect matching problem on these bipartite graphs has complexity $O(n_T^{2.5})$ [42] as the number of vertices in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ and $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathcal{N}_{\mathcal{H}})$ are of $O(n_T)$. The rest of the constructions, which include finding an unmatched node in the obtained matching and addition and deletion of one edge each, are of linear complexity with maximum n_T iterations. Therefore, complexity of Algorithm 4.1 is $O(n_T^{2.5})$. $\square$

Remark 4.9. *Let Γ_A be the minimum cardinality subset of all interconnections which can be used to achieve accessibility. Then, $|\Gamma_A| = \varrho$ (since there are ϱ inaccessible non-top linked SCCs). Also, let Γ_D be the minimum set of interconnections which can be used to achieve the no-dilation condition. Then, $|\Gamma_D| = \alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$ (since optimum matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ has $\alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$ interconnections). The maximum cardinality of $\Gamma_A \cap \Gamma_D$ is the set of interconnections that can serve both the conditions, i.e., accessibility and the no-dilation. Thus, $\alpha_{\mathcal{N}}$ is the maximum cardinality of $\Gamma_A \cap \Gamma_D$. In other words, $\alpha_{\mathcal{N}}$ is the maximum number of interconnections present in sets Γ_A and Γ_D that can serve both the purposes. Hence $\beta_{\mathcal{I}} = |\Gamma_D| - \alpha_{\mathcal{N}}$, is the minimum number of interconnections in Γ_D that are needed to meet the no-dilation condition solely.*

We refer to $\alpha_{\mathcal{N}}$ as the *maximum commonality index* and $\beta_{\mathcal{I}}$ as the *dilation index*. Both these indices quantifies the connectedness of the system, specifically, $\alpha_{\mathcal{N}}$ depends on the strong connectivity. As the subsystems are more interconnected within themselves, the value of indices, $\alpha_{\mathcal{N}}$ and $\beta_{\mathcal{I}}$, decreases.

5. Illustrative Example, Multi-input Case and Special Cases

In this section, we first give an illustrative example to demonstrate Algorithm 4.1. Then, we discuss few special cases and extension to the multi-input case.

A. Illustrative Example

We demonstrate Algorithm 4.1 through an illustrative example in Figure 2. The subsystems are S_1, S_2, S_3, S_4. The set $\mathcal{N}_{\mathcal{H}} = \{\mathcal{N}_1, \ldots, \mathcal{N}_4\}$, where $\mathcal{N}_1 = x_4^1$, $\mathcal{N}_2 = x_4^2$, $\mathcal{N}_3 = x_4^3$, and $\mathcal{N}_4 = x_4^4$. We first obtain an optimum matching $M_{\mathcal{H}}^\star$ in $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathcal{N})$. Let the optimum matching $M_{\mathcal{H}}^\star = \{(x_1'^1, u_1), (x_2'^1, x_4^1), (x_3'^1, x_2^1), (x_4'^1, \mathcal{N}_1), (x_5'^1, x_4^2), (x_1'^2, x_2^2), (x_2'^2, x_1^2), (x_3'^2, x_1^1), (x_4'^2, \mathcal{N}_2), (x_5'^2, x_5^1), (x_1'^3, x_3^4), (x_2'^3, x_1^3), (x_3'^3, x_2^3), (x_4'^3, \mathcal{N}_3), (x_5'^3, x_5^4), (x_1'^4, x_2^4), (x_2'^4, x_1^4), (x_3'^4, x_3^3), (x_4'^4, \mathcal{N}_4), (x_5'^4, x_5^3)\}$. Here, $\alpha_{\mathcal{N}} = 4$ and $\beta_{\mathcal{I}} = 7$. Corresponding to $M_{\mathcal{H}}^\star$, we obtain matching $\widetilde{M}_{\mathcal{H}} = M'_{\mathcal{H}} \cup M''_{\mathcal{H}}$ as shown in Figure 2a. Here, $M'_{\mathcal{H}} = \{(x_1'^1, u_1), (x_2'^1, x_4^1), (x_3'^1, x_2^1), (x_2'^2, x_1^2), (x_1'^2, x_2^2), (x_2'^3, x_1^3), (x_3'^3, x_2^3), (x_2'^4, x_1^4), (x_1'^4, x_2^4)\}$ and $M''_{\mathcal{H}} = \{(x_3'^2, x_1^1), (x_5'^2, x_5^1), (x_5'^1, x_4^2), (x_4'^2, x_3^1), (x_4'^1, x_3^2), (x_4'^4, x_4^3), (x_4'^3, x_4^4), (x_5'^3, x_5^4), (x_5'^4, x_5^3), (x_3'^4, x_3^3), (x_1'^3, x_3^4)\}$ as given in Steps 5 and 6 of Algorithm 4.1. With respect to $\widetilde{M}_{\mathcal{H}}$ the inaccessible SCCs are $\{\mathcal{N}_3, \mathcal{N}_4\}$. The node set $V_{\mathcal{N}'} = \{x_4'^3, x_4'^4\}$ (condition in Step 10 satisfied) and the unique unmatched accessible node corresponding to $\widetilde{M}_{\mathcal{H}}$ is the blue coloured node, x_5^2.

Now we update $\widetilde{M}_{\mathcal{H}}$ as per Steps 11-15. In order to update $\widetilde{M}_{\mathcal{H}}$, we first remove the edge $(x_4'^3, x_4^4)$ from $\widetilde{M}_{\mathcal{H}}$ and include edge $(x_4'^3, x_5^2)$ as shown in Figure 2b. While $M'_{\mathcal{H}}$ remains the same as before the matching $M''_{\mathcal{H}}$ gets modified as $M''_{\mathcal{H}} = \{(x_3'^2, x_1^1), (x_5'^2, x_5^1), (x_5'^1, x_4^2), (x_4'^2, x_3^1), (x_4'^1, x_3^2), (x_4'^4, x_4^3), (x_4'^3, x_5^2), (x_5'^3, x_5^4), (x_5'^4, x_5^3), (x_3'^4, x_3^3), (x_1'^3, x_3^4)\}$ as given in Steps 5 and 6 of Algorithm 4.1. After this, SCCs $\mathcal{N}_3, \mathcal{N}_4$ also become accessible and $V_{\mathcal{N}'} = \emptyset$. The unique unmatched node in this stage is x_4^4.

In this example $\alpha_{\mathscr{N}} = \varrho = 4$. This implies $\mathcal{E}_{\mathcal{I}}^{Q} = \emptyset$. Here the minimum number of interconnections to make the composite system structurally controllable is equal to $\beta_{\mathcal{I}} + \varrho = 7 + 4 = 11$ as shown by the red edges in Figure 2b.

B. Special Cases

Now, we will focus on few special cases, where the minimum number of interconnections can be more directly obtained, and see the value of $|E_{\mathcal{I}}^{\star}|$ for these cases.

Structurally Cyclic Systems: The first case is when $\bar{A}_s$ is structurally cyclic. A structured system is said to be structurally cyclic if the vertices of the state digraph is spanned by disjoint cycles. The class of structurally cyclic systems are wide: self-damped systems including multi-agent systems and epidemic dynamic systems are structurally cyclic [43]. Then, $\mathcal{B}(\bar{A}_s)$ has a perfect matching. So the composite system does not have dilation even without using any interconnection. Thus only the accessibility condition has to be catered. For optimum matching $M_{\mathcal{H}}^{\star}$ in $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathscr{N}_{\mathcal{H}})$, our algorithm gives $\alpha_{\mathscr{N}} + \beta_{\mathcal{I}} = 0$. Hence, $|E_{\mathcal{I}}^{\star}| = \varrho$. In other words, the set of interconnections needed to solve Problem 2.3 equals the number of non-top linked SCCs that are inaccessible.

Irreducible Systems: Here we consider subsystems that are individually irreducible. The digraph $\mathcal{D}(\bar{A}_s)$ is an SCC. Except S_1 all other subsystems belong to $\mathscr{N}_{\mathcal{H}}$ giving $|\mathscr{N}_{\mathcal{H}}| = k-1$. The subsystems satisfy one of the following: (i) $\mathcal{B}(\bar{A}_s)$ has a perfect matching, or (ii) $\mathcal{B}(\bar{A}_s)$ does not have a perfect matching. In case (i), since no-dilation condition is satisfied by the subsystems individually, the number of interconnections needed for the composite system to be structurally controllable is $k-1$. By our algorithm, $\alpha_{\mathscr{N}} + \beta_{\mathcal{I}} = 0$ and $\varrho = k-1$. Therefore, $|E_{\mathcal{I}}^{\star}| = \varrho = k-1$. In case (ii), as the first subsystem receives an input, in any perfect matching in $\mathcal{B}(\bar{A}_s, \bar{B}_1)$ there exists an accessible unmatched node in the first subsystem. In any perfect matching in the bipartite graph $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathscr{N}_{\mathcal{H}})$, the unique accessible node of S_1 connects to some subsystem and its unmatched accessible node connects to another subsystem and so on. These interconnects caters both accessibility of the subsystem and also removes one dilation. Thus the number of interconnections required for structural controllability of the composite system is the same as the minimum number of interconnections for satisfying the no-dilation condition. Now, analysing this case using our algorithm, $\alpha_{\mathscr{N}} = k-1$. Since each subsystem is irreducible and the first subsystem receives an input, $\varrho = k-1$. As result, $\alpha_{\mathscr{N}} = \varrho = k-1$ and $|E_{\mathcal{I}}^{\star}| = \beta_{\mathcal{I}} + \varrho = \alpha_{\mathscr{N}} + \beta_{\mathcal{I}}$ is the number of interconnections for no-dilation condition.

Controller Canonical Form: Now we consider another class of systems, where $\bar{A}_s$ is in the controller canonical form and $\bar{B}_T = \star e_{n_T}$. Here, e_{n_T} is the last column of the $(n_T \times n_T)$ identity matrix. For example, $\bar{A}_s = \begin{bmatrix} 0 & \star & 0 \\ 0 & 0 & \star \\ \star & \star & \star \end{bmatrix}$. Notice that, when $\bar{A}_s$ is in the controller canonical form, then $\mathcal{B}(\bar{A}_s)$ has a perfect matching. The composite system does not have dilation even without using any interconnection edge. For an optimum matching $M_{\mathcal{H}}^{\star}$ in $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathscr{N}_{\mathcal{H}})$, we get $\alpha_{\mathscr{N}} + \beta_{\mathcal{I}} = 0$. Further, $\mathcal{D}(\bar{A}_s)$ is irreducible. This gives $|E_{\mathcal{I}}^{\star}| = \varrho = k-1$, since each subsystem is a non-top linked SCC and exactly one subsystem is accessible without using any interconnections.

C. Multi-input Case

The algorithm and results given in this paper extend to the multi-input case. We briefly explain the outline of the extension in this subsection.

In the multi-input case of Lemma 4.2, all the input nodes are matched in an optimum matching $\overline{M}$ of $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. In other words, $|\overline{M} \cap \cup_{i=1}^{k} \mathcal{E}_{U_i}| = m_T$, where m_T is the total number of inputs. Consequently, there are m_T number of right unmatched nodes in $\overline{M}$ that are accessible (using the same argument in Lemma 4.3). The multi-input version of Lemma 4.3 concludes that corresponding to an optimum matching $\overline{M}$ of $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ there are m_T number of unmatched accessible nodes. Similarly, Lemma 4.4 extends to the multi-input case with $|M_{\mathcal{H}}^{\star} \cap \cup_{i=1}^{k} \mathcal{E}_{U_i}| = m_T$, where $M_{\mathcal{H}}^{\star}$ is an optimum matching in $\mathcal{B}(\bar{A}, \bar{B}, \mathscr{N}_{\mathcal{H}})$. Lemma 4.5 is based on the existence of an unmatched accessible node in an optimum matching $\overline{M}$ of $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. Note that, the uniqueness of the unmatched accessible node is not required in the proof. The only difference in the proof of Lemma 4.5 is that the bipartite graph $\mathcal{B}(\bar{A}_T, \bar{B}_T) \odot M'_{\mathcal{H}}$ consists of $\alpha_{\mathscr{N}} + \beta_{\mathcal{I}}$ nodes on the left side and $\alpha_{\mathscr{N}} + \beta_{\mathcal{I}} + m_T$ nodes on the right side. $\mathcal{B}(\bar{A}_T, \bar{B}_T) \odot M'_{\mathcal{H}}$ has a perfect matching in the multi-input case because of the same reason in Lemma 4.5 and the rest of the proof follows. Finally, Lemma 4.6 and Theorem 4.1 hold for the multi-input case as they are based on previous lemmas, which hold for the multi-input case.

Now we explain the key steps in the algorithm for multi-input case. For a multi-input case, consider any optimum matching $M_{\mathcal{H}}^{\star}$ obtained in Step 4 of Algorithm 4.1. Then, $|M_{\mathcal{H}}^{\star} \cap \cup_{i=1}^{k} \mathcal{E}_{U_i}| = m_T$. If not, then one can use the same argument in Lemma 4.2 and Lemma 4.4 to construct another matching $M'_{\mathcal{H}}$ such that $|M'_{\mathcal{H}} \cap \cup_{i=1}^{k} \mathcal{E}_{U_i}| = m_T$, as each input node is connected to at least one state node, and $c_{\mathcal{H}}(M'_{\mathcal{H}}) < c_{\mathcal{H}}(M_{\mathcal{H}}^{\star})$ contradicting the optimality. Let $|M_{\mathcal{H}}^{\star} \cap \mathcal{E}_{\mathscr{N}}| = \alpha_{\mathscr{N}}$ and $|M_{\mathcal{H}}^{\star} \cap \mathcal{E}_{\mathcal{I}}| = \beta_{\mathcal{I}}$. The matching $\widetilde{M}_{\mathcal{H}}$ constructed in Step 7, consists of $\alpha_{\mathscr{N}} + \beta_{\mathcal{I}}$ interconnections. Further, there exists at least one unmatched accessible node corresponding to $\widetilde{M}_{\mathcal{H}}$. Thus, one can obtain a matching $\hat{M}_{\mathcal{H}}$ in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ with $|\hat{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}| = \alpha_{\mathscr{N}} + \beta_{\mathcal{I}}$ such that $\alpha_{\mathscr{N}}$ SCCs are accessible using the interconnections in $\hat{M}_{\mathcal{H}}$. In other words, one can achieve accessibility of $\alpha_{\mathscr{N}}$ non-top linked SCCs using the same number of interconnections as before. The remaining SCCs can be made accessible using extra interconnections as in Step 19 of Algorithm 4.1. Note that the proofs in this paper uses two concepts: (a) in an optimum matching $\widetilde{M}_{\mathcal{H}}$ there exists a node matched to some input node, and (b) there exists an unmatched accessible node in $\widetilde{M}_{\mathcal{H}}$. Both (a) and (b) continue to be true for the multi-input case also. Thus the algorithm and results apply to the multi-input case.

6. Conclusion

In this paper, we studied structural controllability of an LTI composite system consisting of several subsystems. The objective was to obtain an optimal network topology design, i.e., find a minimum cardinality set of interconnections among these subsystems, such that the composite system is structurally controllable using a specified input matrix. The analysis is done in a structured framework by using the sparsity pattern of the system matrices. In this paper, we considered subsystems whose state matrices have identical sparsity pattern and proposed a polynomial time algorithm for solving the optimal network topology design problem (Problem 2.3). Given a set of structured subsystems, we first gave a closed-form expression for the minimum number of interconnections required to make the composite system structurally controllable (Theorem 4.1), using two indices $\alpha_{\mathcal{N}}$ and $\beta_{\mathcal{I}}$ defined in the paper that quantifies the connectedness of the composite system as noted in Remark 4.9. Then we proposed an algorithm to obtain an interconnection edge set of minimum cardinality (Algorithm 4.1). The algorithm given in this paper identifies a neighbouring set of each subsystem, i.e., the set of subsystems that it must communicate with, to make the composite system structurally controllable with least possible number of interconnections (Theorem 4.7). We also proved that the proposed algorithm has polynomial time complexity (Theorem 4.8). For notational convenience and brevity, we discussed single input case in this paper. However, all the analysis carried out here directly extends to the multi-input case as discussed in Section 5-C. Needless to elaborate, due to duality between controllability and observability in LTI systems all results of this paper directly follow to the observability problem. Complexity analysis and deriving efficient algorithm for the case where neighbours of each subsystem is constrained is a topic of future research.

REFERENCES

[1] Y.-Y. Liu and A.-L. Barabási, "Control principles of complex systems," *Reviews of Modern Physics*, vol. 88, no. 3, pp. 035 006:1–58, 2016.

[2] W. J. Dally and B. Towles, "Route packets, not wires: On-chip interconnection networks," in *Proceedings of the IEEE Conference on Design Automation*, 2001, pp. 684–689.

[3] M. Rafiee and A. M. Bayen, "Optimal network topology design in multi-agent systems for efficient average consensus," in *Proceedings of the IEEE Conference on Decision and Control (CDC)*, Atlanta, USA, 2010, pp. 3877–3883.

[4] V. Terzija, G. Valverde, D. Cai, P. Regulski, V. Madani, J. Fitch, S. Skok, M. M. Begovic, and A. Phadke, "Wide-area monitoring, protection, and control of future electric power networks," *Proceedings of the IEEE*, vol. 99, no. 1, pp. 80–93, 2011.

[5] J. F. Carvalho, S. Pequito, A. P. Aguiar, S. Kar, and K. H. Johansson, "Composability and controllability of structural linear time-invariant systems: Distributed verification," *Automatica*, vol. 78, pp. 123–134, 2017.

[6] S. Jafari, A. Ajorlou, and A. G. Aghdam, "Leader localization in multi-agent systems subject to failure: A graph-theoretic approach," *Automatica*, vol. 47, no. 8, pp. 1744–1750, 2011.

[7] D. Groß and O. Stursberg, "Optimized distributed control and network topology design for interconnected systems," in *Proceedings of the IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, Florida, USA, 2011, pp. 8112–8117.

[8] C.-T. Lin, "Structural controllability," *IEEE Transactions on Automatic Control*, vol. 19, no. 3, pp. 201–208, 1974.

[9] C. Commault and J.-M. Dion, "The single-input minimal controllability problem for structured systems," *Systems & Control Letters*, vol. 80, pp. 50–55, 2015.

[10] S. Pequito, S. Kar, and A. P. Aguiar, "A framework for structural input/output and control configuration selection in large-scale systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 2, pp. 303–318, 2016.

[11] S. Moothedath, P. Chaporkar, and N. M. Belur, "A flow-network-based polynomial-time approximation algorithm for the minimum constrained input structural controllability problem," *IEEE Transactions on Automatic Control*, vol. 63, no. 9, pp. 3151 – 3158, 2018.

[12] J.-M. Dion, C. Commault, and J. Van Der Woude, "Generic properties and control of linear structured systems: a survey," *Automatica*, vol. 39, no. 7, pp. 1125–1144, 2003.

[13] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, "Controllability of complex networks," *Nature*, vol. 473, no. 7346, pp. 167–173, 2011.

[14] J. Lunze, "Dynamics of strongly coupled symmetric composite systems," *International Journal of Control*, vol. 44, no. 6, pp. 1617–1640, 1986.

[15] M. K. Sundareshan and R. M. Elbanna, "Qualitative analysis and decentralized controller synthesis for a class of large-scale systems with symmetrically interconnected subsystems," *Automatica*, vol. 27, no. 2, pp. 383–388, 1991.

[16] C.-T. Chen and C. Desoer, "Controllability and observability of composite systems," *IEEE Transactions on Automatic Control*, vol. 12, no. 4, pp. 402–409, 1967.

[17] E. Davison and S. Wang, "New results on the controllability and observability of general composite systems," *IEEE Transactions on Automatic Control*, vol. 20, no. 1, pp. 123–128, 1975.

[18] W. Wolovich and H. Hwang, "Composite system controllability and observability," *Automatica*, vol. 10, no. 2, pp. 209–212, 1974.

[19] T. Zhou, "On the controllability and observability of networked dynamic systems," *Automatica*, vol. 52, pp. 63–75, 2015.

[20] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 57, no. 1, pp. 213–224, 2010.

[21] R. Dai, J. Maximoff, and M. Mesbahi, "Optimal trajectory generation for establishing connectivity in proximity networks," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 49, no. 3, pp. 1968–1981, 2013.

[22] E. J. Davison, "Connectability and structural controllability of composite systems," *Automatica*, vol. 13, no. 2, pp. 109–123, 1977.

[23] B. D. Anderson and H.-M. Hong, "Structural controllability and matrix nets," *International Journal of Control*, vol. 35, no. 3, pp. 397–416, 1982.

[24] C. Rech and R. Perret, "About structural controllability of interconnected dynamical systems," *Automatica*, vol. 27, no. 5, pp. 877–881, 1991.

[25] G.-H. Yang and S.-Y. Zhang, "Structural properties of large-scale systems possessing similar structures," *Automatica*, vol. 31, no. 7, pp. 1011–1017, 1995.

[26] K. Li, Y. Xi, and Z. Zhang, "G-cactus and new results on structural controllability of composite systems," *International journal of systems science*, vol. 27, no. 12, pp. 1313–1326, 1996.

[27] S. Moothedath, P. Chaporkar, and M. Belur N., "Optimal selection of essential interconnections for structural controllability in heterogeneous subsystems," *Automatica*, vol. 103, pp. 424–434, 2019.

[28] S. Moothedath, P. Chaporkar, and A. Joshi, "Optimal network topology design in composite systems with constrained neighbors for structural controllability," in *Proceedings of the IEEE American Control Conference (ACC)*, 2019.

[29] W.-X. Wang, X. Ni, Y.-C. Lai, and C. Grebogi, "Optimizing controllability of complex networks by minimum structural perturbations," *Physical Review E*, vol. 85, no. 2, pp. 026 115: 1–5, 2012.

[30] X. Chen, S. Pequito, G. J. Pappas, and V. M. Preciado, "Minimal edge addition for network controllability," *IEEE Transactions on Control of Network Systems*, 2018.

[31] Z. Zhang, Y. Yin, X. Zhang, and L. Liu, "Optimization of robustness of interdependent network controllability by redundant design," *PLoS one*, vol. 13, no. 2, p. e0192874, 2018.

[32] S. S. Mousavi, M. Haeri, and M. Mesbahi, "Robust strong structural controllability of networks with respect to edge additions and deletions," in *Proceedings of the IEEE American Control Conference (ACC)*, Seattle, USA, 2017, pp. 5007–5012.

[33] S. S. Mousavi, M. Haeri, and M. Mesbahi, "On the structural and strong structural controllability of undirected networks," *IEEE Transactions on Automatic Control*, vol. 63, no. 7, pp. 2234–2241, 2018.

[34] Y. Zhang and T. Zhou, "On the edge insertion/deletion and controllability distance of linear structural systems," in *Proceedings of the IEEE Conference on Decision and Control (CDC)*, Melbourne, Australia, 2017, pp. 2300–2305.

[35] Y. Zhang and T. Zhou, "Minimal structural perturbations for network controllability: Complexity analysis," *ArXiv e-prints*, 2018.

[36] K. J. Reinschke, *Multivariable Control: a Graph Theoretic Approach*. Springer-Verlag, 1988.

[37] R. Diestel, *Graph Theory*. Springer: New York, 2000.

[38] A. Olshevsky, "Minimum input selection for structural controllability," in *Proceedings of the IEEE American Control Conference (ACC)*, Chicago, USA, 2015, pp. 2218–2223.

[39] R. Shields and J. Pearson, "Structural controllability of multiinput linear systems," *IEEE Transactions on Automatic control*, vol. 21, no. 2, pp. 203–212, 1976.

[40] K. Glover and L. Silverman, "Characterization of structural controllability," *IEEE Transactions on Automatic control*, vol. 21, no. 4, pp. 534–537, 1976.

[41] S. Assadi, S. Khanna, Y. Li, and V. M. Preciado, "Complexity of the minimum input selection problem for structural controllability," *IFAC-PapersOnLine, NecSys*, vol. 48, no. 22, pp. 70–75, Philadelphia, USA, 2015.

[42] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*. MIT press: Cambridge, 2001.

[43] A. Chapman and M. Mesbahi, "On strong structural controllability of networked systems: A constrained matching approach," in *Proceedings of the IEEE American Control Conference (ACC)*, Washington DC, USA, 2013, pp. 6126–6131.

APPENDIX

Proof of Lemma 4.2: Given $\overline{M}$ is an optimum matching of $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. We prove the result using a contradiction argument. Suppose $(x'^1_a, u_1) \notin \overline{M}$, for all $a \in \{1, \ldots, n_s\}$. Note that, in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ the input node u_1 connects to state node x^1_a for some $a \in \{1, \ldots, n_s\}$. Then, since $\overline{M}$ is a perfect matching, $(x'^1_a, x^g_h) \in \overline{M}$ for some node x^g_h. Construct a new matching $\overline{M}'$ by removing the edge (x'^1_a, x^g_h) and including the edge (x'^1_a, u_1), i.e., $\overline{M}' = \{\overline{M} \setminus (x'^1_a, x^g_h)\} \cup \{(x'^1_a, u_1)\}$. Notice that $c_{\mathcal{H}}(\overline{M}') < c_{\mathcal{H}}(\overline{M})$. This contradicts the assumption that $\overline{M}$ is an optimum matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ and hence $(x'^1_a, u_1) \in \overline{M}$, for some $a \in \{1, \ldots, n_s\}$.

Proof of Lemma 4.3: The bipartite graph $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ consists of n_T left side nodes and $n_T + 1$ right side nodes, where one extra node in the right side is the input node u_1. Any perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ has size n_T and hence in $\overline{M}$ there is one right unmatched node. Now we need to show that this unmatched node is accessible in the digraph constructed using vertex set $\cup_{i=1}^k V_{X_i} \cup \cup_{i=1}^k V_{U_i}$ and edge set $\cup_{i=1}^k E_{X_i} \cup \cup_{i=1}^k E_{U_i} \cup E_{\mathcal{I}'}$, where $(x^i_g, x^j_h) \in E_{\mathcal{I}'} \Leftrightarrow (x'^j_h, x^i_g) \in \overline{M} \cap \mathcal{E}_{\mathcal{I}}$. By Lemma 4.2, all optimum perfect matchings in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ contains an edge (x'^i_g, u_1) for some node x'^i_g. Let $(x'^i_g, u_1) \in \overline{M}$. Then the node x^i_g is accessible in the specified digraph. Now in the matching $\overline{M}$, the node x^i_g satisfies one of the following: (a) x^i_g is unmatched, or (b) x^i_g is matched. In case (a), the proof follows. In case (b), let $(x'^j_h, x^i_g) \in \overline{M}$. Then the node x^j_h is accessible. Recursively using the same argument as before, we can say that the unmatched node in $\overline{M}$ is accessible in the digraph constructed using vertex set $\cup_{i=1}^k V_{X_i} \cup \cup_{i=1}^k V_{U_i}$ and edge set $\cup_{i=1}^k E_{X_i} \cup \cup_{i=1}^k E_{U_i} \cup E_{\mathcal{I}'}$. The recursive argument terminates as the number of edges in the matching is finite.

Proof of Lemma 4.4: We know that $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathcal{N}_{\mathcal{H}})$ is a bipartite graph with n_T vertices on the left side and $n_T + 1 + \varrho$ vertices on the right side. Given $M^\star_{\mathcal{H}}$ is an optimum matching of $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathcal{N}_{\mathcal{H}})$. We prove the result using a contradiction argument. Suppose $(x'^1_a, u_1) \notin M^\star_{\mathcal{H}}$, for all $a \in \{1, \ldots, n_s\}$. Note that, in $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathcal{N}_{\mathcal{H}})$ input u_1 connects to state x^1_a for some $a \in \{1, \ldots, n_s\}$. Then, since $M^\star_{\mathcal{H}}$ is a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathcal{N}_{\mathcal{H}})$, $(x'^1_a, v) \in M^\star_{\mathcal{H}}$ for some node v. Construct a new matching $M'_{\mathcal{H}}$ by removing the edge (x'^1_a, v) and including the edge (x'^1_a, u_1), i.e., $M'_{\mathcal{H}} = \{M^\star_{\mathcal{H}} \setminus (x'^1_a, v)\} \cup \{(x'^1_a, u_1)\}$. Notice that $c_{\mathcal{H}}(M'_{\mathcal{H}}) < c_{\mathcal{H}}(M^\star_{\mathcal{H}})$. This contradicts the assumption that $M^\star_{\mathcal{H}}$ is an optimum matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathcal{N}_{\mathcal{H}})$.

Proof of Lemma 4.5: Given $M^\star_{\mathcal{H}}$ is an optimum perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathcal{N}_{\mathcal{H}})$. We first prove the existence of a perfect matching $\widetilde{M}_{\mathcal{H}}$ in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ satisfying $|\widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}| = \alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$. For this we construct $\widetilde{M}_{\mathcal{H}}$, a matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$, from $M^\star_{\mathcal{H}}$ such that $|\widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}| = \alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$. Given $M^\star_{\mathcal{H}}$ satisfies $|M^\star_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{N}}| = \alpha_{\mathcal{N}}$ and $|M^\star_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}| = \beta_{\mathcal{I}}$. By Lemma 4.4, $|M^\star_{\mathcal{H}} \cap \cup_{i=1}^k \mathcal{E}_{U_i}| = 1$. Thus, $|M^\star_{\mathcal{H}} \cap \cup_{i=1}^k \mathcal{E}_{X_i}| = n_T - (\alpha_{\mathcal{N}} + \beta_{\mathcal{I}} + 1)$. Let $M'_{\mathcal{H}} \subset M^\star_{\mathcal{H}}$ is defined as $M'_{\mathcal{H}} := M^\star_{\mathcal{H}} \cap \{\cup_{i=1}^k \mathcal{E}_{X_i} \cup \cup_{i=1}^k \mathcal{E}_{U_i}\}$. Then $|M'_{\mathcal{H}}| = n_T - \alpha_{\mathcal{N}} - \beta_{\mathcal{I}}$. Note that $M'_{\mathcal{H}}$ is a matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. Consider the bipartite graph $\mathcal{B}(\bar{A}_T, \bar{B}_T) \odot M'_{\mathcal{H}}$, where $\odot$ denotes a difference operation in which all nodes with non-zero degree in $M'_{\mathcal{H}}$ and all the edges associated with these nodes are removed from $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. More precisely, $\mathcal{B}(\bar{A}_T, \bar{B}_T) \odot M'_{\mathcal{H}}$ consists of only those nodes in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ that are not matched in $M'_{\mathcal{H}}$ and the edges in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ between those nodes. The bipartite graph $\mathcal{B}(\bar{A}_T, \bar{B}_T) \odot M'_{\mathcal{H}}$ consists of $\alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$ nodes on the left side and $\alpha_{\mathcal{N}} + \beta_{\mathcal{I}} + 1$ nodes on the right side. Notice that $\mathcal{B}(\bar{A}_T, \bar{B}_T) \odot M'_{\mathcal{H}}$ has a perfect matching. This is because since state matrices of all subsystems are structurally identical and $M'_{\mathcal{H}} = M^\star_{\mathcal{H}} \cap \{\cup_{i=1}^k \mathcal{E}_{X'_i} \cup \cup_{i=1}^k \mathcal{E}_{U_i}\}$, where $M^\star_{\mathcal{H}}$ is an optimum matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T, \mathcal{N}_{\mathcal{H}})$ under cost function $c_{\mathcal{H}}$, the number of nodes in $\mathcal{B}(\bar{A}_T, \bar{B}_T) \odot M'_{\mathcal{H}}$ corresponding to each subsystem is the same in both left and right sides except for one subsystem. For one subsystem (the i^{th} subsystem if $(x'^i_g, u_1) \in M'_{\mathcal{H}}$ for some $g \in \{1, \ldots, n_s\}$) either the number of nodes in the left side of $\mathcal{B}(\bar{A}_T, \bar{B}_T) \odot M'_{\mathcal{H}}$ is one <u>less</u> than other subsystems or the number of nodes in the right side is one <u>more</u> than the other subsystems. In both cases there exists a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T) \odot M'_{\mathcal{H}}$. Let $M''_{\mathcal{H}}$ be an optimum perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T) \odot M'_{\mathcal{H}}$ using cost function $c_{\mathcal{H}}$. Then $M''_{\mathcal{H}} \subset \mathcal{E}_{\mathcal{I}}$. This is because if an edge in $\cup_{i=1}^k \mathcal{E}_{X_i} \cup \cup_{i=1}^k \mathcal{E}_{U_i}$ is present in $M''_{\mathcal{H}}$, then it contradicts the optimality of $M^\star_{\mathcal{H}}$. Now $\widetilde{M}_{\mathcal{H}} = M'_{\mathcal{H}} \cup M''_{\mathcal{H}}$ is a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$. Note that, $|\widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}| = \alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$. This proves that there exists a perfect matching in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ consisting of $\alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$ interconnections. Since $\widetilde{M}_{\mathcal{H}}$ is constructed from $M^\star_{\mathcal{H}}$ which is an optimum matching under cost function $c_{\mathcal{H}}$, the optimality of $\widetilde{M}_{\mathcal{H}}$ follows.

Proof of Lemma 4.6: We know from Lemma 4.5 that there exists an optimum matching $\widetilde{M}_{\mathcal{H}}$ in $\mathcal{B}(\bar{A}_T, \bar{B}_T)$ such that $|\widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}| = \alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$. Out of these $\alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$ interconnections, $\alpha_{\mathcal{N}}$ interconnections has left side nodes from distinct SCCs, say $\mathcal{N}_1, \ldots, \mathcal{N}_{\alpha_{\mathcal{N}}}$. Hence, at least $\alpha_{\mathcal{N}}$ left side nodes in $\widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}$ are from $\alpha_{\mathcal{N}}$ distinct SCCs. Let x'^i_g be an arbitrary node such that $(x'^i_g, x^j_h) \in \widetilde{M}_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}$, $x^i_g \in \hat{\mathcal{N}}$, $\hat{\mathcal{N}} \in \mathcal{N}_{\mathcal{H}}$ and $\hat{\mathcal{N}}$ is inaccessible in the digraph with vertex set $\cup_{i=1}^k V_{X_i} \cup \cup_{i=1}^k V_{U_i}$ and edge set $\cup_{i=1}^k E_{X_i} \cup \cup_{i=1}^k E_{U_i} \cup E_{\mathcal{I}'}$. Now we describe an edge reconstruction process to make $\hat{\mathcal{N}}$ accessible without increasing the number of interconnections. By Lemma 4.3 we know that there exists a unique unmatched node in $\widetilde{M}_{\mathcal{H}}$ that is accessible, say $\hat{x}$. Then $\hat{x}$ satisfies one of the following cases: (a) $\hat{x}$ is in the same subsystem as of $\hat{\mathcal{N}}$, or (b) $\hat{x}$ is not in the subsystem of $\hat{\mathcal{N}}$. We will resolve case (b) first. Construct a new matching $\hat{M}_{\mathcal{H}}$ such that $\hat{M}_{\mathcal{H}} = \{\widetilde{M}_{\mathcal{H}} \setminus (x'^i_g, x^j_h)\} \cup \{(x'^i_g, \hat{x})\}$. Note that in $\hat{M}_{\mathcal{H}}$ the number of interconnections is the same as in $\widetilde{M}_{\mathcal{H}}$ and further the SCC $\hat{\mathcal{N}}$ is accessible. Now we will resolve case (a). In case (a), note that $\hat{x}$ is in the same subsystem as $\hat{\mathcal{N}}$. Since the unique unmatched node $\hat{x}$ is in the i^{th} subsystem and $(x'^i_g, x^j_h) \in \widetilde{M}_{\mathcal{H}}$, there exists an interconnection edge in $\widetilde{M}_{\mathcal{H}}$ matching a left side node in j^{th} subsystem to some node in a different subsystem, say $(x'^j_a, x^v_d) \in \widetilde{M}_{\mathcal{H}}$, $j \neq v$. Construct a new matching $\hat{M}'_{\mathcal{H}} = \widetilde{M}_{\mathcal{H}} \setminus \{(x'^j_a, x^v_d)\} \cup \{(x'^j_a, \hat{x})\}$. This is possible since $i \neq j$. Notice that $|\hat{M}'_{\mathcal{H}} \cap \mathcal{E}_{\mathcal{I}}| = \alpha_{\mathcal{N}} + \beta_{\mathcal{I}}$ and with respect to $\hat{M}'_{\mathcal{H}}$ the unique accessible unmatched vertex is x^v_d. If $x^v_d \in \hat{\mathcal{N}}$, then $\hat{\mathcal{N}}$ is accessible and $\hat{M}_{\mathcal{H}} = \hat{M}'_{\mathcal{H}}$. Otherwise, by a sequence of removal and inclusion of edges (atmost n_s times) node x^i_g becomes accessible resulting in SCC $\hat{\mathcal{N}}$ being accessible. Since x'^i_g is arbitrary the proof follows.