Interlacing properties of system-poles, system-zeros and spectral-zeros in MIMO systems

Sandeep Kumar and Madhu N. Belur

Abstract—SISO systems with zeros interlacing poles (ZIP) have been extensively studied in the literature and have received interest widely. However, the ZIP property for MIMO systems has not been pursued sufficiently. In this paper, we pursue the ZIP property for MIMO systems and link this to strict passivity. For the class of systems that admit a symmetric state-space realization, which for the SISO case is equivalent to ZIP, we obtain sufficient conditions under which MIMO systems too have ZIP.

We also present new results in the context of 'spectral-zero' of a system, a notion that plays a key role in many optimal control and estimation problems. We formulate conditions under which the spectral-zeros of a MIMO system are real, and further, conditions that guarantee that the system-zeros, spectral-zeros, and the system-poles are all interlaced for MIMO systems.

Keywords—RC/RL realizability, MIMO impedance/admittance transfer matrices, real spectral-zeros, zeros interlacing poles (ZIP), spectral-zeros interlacing, symmetric state-space realizable systems

1. INTRODUCTION

It is well-known that SISO RLC electrical systems containing resistors and only one type of memory/storage element, namely capacitative or inductive, have only real poles/zeros, and further, that these are interlaced. In a related context, 'spectral-zeros' of a system is a well-studied notion: they play a key role in optimal control, minimum energy charging/discharging, model order reduction of large order passive circuits, in dissipativity studies, spectral factorization: more about this in Section 1-A.

In our opinion, a difficulty in extending SISO pole/zero interlacing properties to MIMO systems is pursuing with the appropriate notion of a system-zero, given the variety of (nonequivalent) definitions of MIMO system zeros in the literature. In this paper, with respect to an appropriate (and existing, welldefined) notion of system-zero, we formulate conditions that guarantee the poles and zeros are interlaced.

This paper also deals with spectral-zeros of a system. Spectral-zeros of a system play a key role in many filtering and control problems ever since this notion was formulated almost a century ago in the classic Wiener/Hopf filters. The role of spectral-zeros and the ensuing spectral factorization is central for almost all control problems involving a quadratic performance index, ranging from LQ control, asymptotic Kalman filter, robust estimation and control to minimum energy charging/discharging problems. Spectral-zeros being real signify that the optimal trajectories are exponential trajectories without oscillations. We formulate and prove conditions under which MIMO systems have real spectral-zeros, and further conditions for interlacing of system-zeros, spectral-zeros and system-poles. While many

Sandeep Kumar is in ACEM, DRDO, Nashik and also in the Department of Electrical Engineering, Indian Institute of Technology Bombay. Madhu Belur is in the Department of Electrical Engineering, Indian Institute Technology Bombay. of Emails: sandeepkumar.iitb@gmail.com, belur@iitb.ac.in

of the interlacing results are known for the SISO case only (see Section 1-A), some of this paper's MIMO-case conclusions hold under milder conditions for the SISO case, and are new results for the SISO case too; see after Corollary 4.2.

A. Background and related work

SISO systems with zeros-interlacing-poles (ZIP) have been well-studied, see, for example, [22], [14], [20], and references therein. It has been shown that such systems admit a symmetric state-space realization; see Definition 2.2. Passive systems which admit symmetric state-space realization correspond to physical systems which have only one "type" of energy storage possibility, e.g. only potential energy or only kinetic energy, but not both [22]. It has been noted there that Resistor-Inductor (RL) and Resistor-Capacitor (RC) have this property and, conversely, under mild assumptions, ZIP systems can be realized as impedance or admittance of RC/RL systems.

Beyond the classical areas of RC/RL realization, SISO passive systems, especially those having the ZIP property, have received much attention in the literature recently too: see [8], [20], for example. In the context of model order reduction. ZIP systems also find applications in the modelling of non-laminated axial magnetic bearings [10], and in biological systems [18]. In the context of the ability to compose a system as parallel interconnection of 'simple compartments', [2] brings out the close link with ZIP systems. In the context of Hankel singular values, [16] studies a class of linear dynamical systems, known as modally balanced systems, in which the system-poles are proportional to its Hankel singular values: these systems too are shown to exhibit the ZIP property. In the context of fractionalorder systems, [15] utilizes the pole-zero interlacing architecture for various applications like synthesis of fractional order PID controllers [3] and discrete time fractional operators.

However, all papers listed above, both classic and recent, focus only on SISO systems. Despite our best efforts in searching for interlacing related results in the literature on MIMO systems, just a mention that 'ZIP systems can also be defined for MIMO systems [25]' was found in [14], notwithstanding that [25] deals with a slightly different notion of interlacing called 'even interlacing' (also termed 'parity interlacing property'), in the context of stabilizing a MIMO system using a stable controller. Lack of progress in this direction is fairly expected since there are examples of multi-port RC circuits having driving point impedances with nonreal poles/zeros (and, together with mutual inductances, even nonminimum-phase zeros); see [19, Sec. 8.6]. Another reason explaining the difficulty in extending ZIP results to the MIMO case is the variety of (non-equivalent) definitions of MIMO system-zeros.

In order to obtain ZIP results for MIMO systems, and in the context of spectral-zeros of a system being real, we use *symmetric state-space* realizable systems (see Definition 2.2 below). SISO systems with such a realization, also called symmetric systems, have been well-studied: firstly, they exhibit ZIP [22],[20],[14]. Secondly, models of networks of systems often naturally give rise to a symmetric state-space realization: symmetry often coming because of a reciprocity in the interaction between neighbours. Such realizations have also found applications in multi-agent networks [6],[26].

We consider below a circuit to relate realizability of G(s) as RC or as RL when G satisfies the ZIP property.

B. RC/RL-networks, interlacing and spectral-zeros: example

Consider a strictly passive SISO system Σ with transfer function,

$$G(s) = \frac{(s+2)(s+5)}{(s+1)(s+3)} = 1 + \frac{2}{s+1} + \frac{1}{s+3}.$$

The system-zeros $\{-2, -5\}$ interlace the system-poles $\{-1, -3\}$. Obviously, the inverse system Σ^{-1} defined by the transfer function $G(s)^{-1}$ also has the ZIP property. A network realization of this system needs only a single type of energy storage element. The system can be realized as either RC or RL network depending on assigning the transfer function of the system as impedance Z(s) := G(s) or admittance Y(s) := G(s) of the network respectively. Though this is well-known, we motivate questions addressed in this paper using this example.

If we choose the transfer function as the impedance Z(s) := G(s) of the realized network, then the system is realized as a RC-network (Foster-I form), whereas if we choose G(s) =: Y(s) as the admittance of the network then the system is realized as a RL-network (Foster-II form): see Figure 1.



Fig. 1: RC/RL-network realization of Z(s)/Y(s)

In many control problems, the notion of spectral-zeros plays a key role. One important application is that of optimal charging and discharging i.e. charging the circuit to a specified state with the minimum supply of energy from the (multi-)port and that of discharging the circuit from a specified state with maximum energy extraction from the (multi-)port. The energy required for charging and the energy extractable by discharging are given by the solutions of an appropriate Algebraic Riccati equation (ARE). The current/voltage trajectories corresponding optimal charging and discharging are governed by, respectively, the antistable and stable spectral-zeros of the system. If the spectralzeros are real then the trajectories are purely exponential, but if two or more of the spectral-zeros are nonreal, then the optimal trajectories would contain oscillations. Hence an important question arises naturally for passive systems: when does a system have only exponential (and non-oscillatory) optimal charging/discharging trajectories? This is same as the question: when does a passive system have only real spectral-zeros?

Further, continuing with the property of zeros-interlacingpoles (ZIP) property, whose study has primarily been restricted

to SISO systems, this paper relates MIMO systems with symmetric state-space realizations and the ZIP property, using the appropriate notion of system-zero, and also relates their interlacing with that of spectral-zeros.

C. Contributions of the paper

In this section, we summarize the contributions in this paper. As mentioned in Section 1-A, though SISO systems with zeros-interlacing-poles (ZIP) property have been well-studied, extensions have seldom been pursued for MIMO systems; even recent papers dealing with ZIP property are limited to SISO systems only. In Section 3 we formulate and present new results (Theorem 3.1 and 3.2) which extend ZIP property to MIMO systems. In addition to proving the SISO results for the MIMO case (under mild conditions), we also show that

- for MIMO symmetric state-space systems, the interlacing holds: not just between the system-poles and system-zeros, but also together with the spectral-zeros: Theorem 3.6,
- a biproper MIMO system with a symmetric state-space realization that allows its feedthrough matrix *D* to be scaled exhibits ZIP for sufficiently large *D*: Corollary 3.5.

Section 4 contains new results specific to the SISO case.

- Some new MIMO results (of Section 3) about realness of spectral-zeros and its interlacing with system poles/zeros are novel for the SISO case too, more after Corollary 4.2.
- In Lemma 4.1, we formulate relations between the product and sum of squares of the spectral-zeros with the system-poles and system-zeros.
- We also show as a special case that for single-order SISO systems, the spectral-zero is the geometric mean of the system-pole and system-zero.

D. Organization of the paper

The rest of the paper is organized as follows. Section 2 contains some preliminaries required for the paper. Section 3 contains the main results for MIMO systems: interlacing properties of system-zeros, system-poles and results about interlacing of spectral-zeros. The interlacing w.r.t. spectral-zeros are new for the SISO case too, and together with some more SISO specific results, we focus on a simpler proof outline for the SISO case in Section 4. Section 5 contains some examples that illustrate the main results of the paper: one where we consider multiagent systems which naturally yield symmetric state space realizations, and another where we do get ZIP property for MIMO systems after appropriate scaling of the feedthrough matrix D. Finally, Section 6 contains concluding remarks and future directions.

2. PRELIMINARIES

In this paper we consider linear time-invariant dynamical system Σ with minimal i/s/o representation (A, B, C, D) and transfer function G(s).

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}, \ G(s) = C(sI - A)^{-1}B + D \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times p}$. In this paper, due to the notion of system-zero we will define later below in Section 2-D, we consider systems with m = p, and thus the

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feedthrough matrix D is square. Further, we assume controllable and observable systems with B full column rank and C full row rank: this rules out redundancy in inputs/outputs. We also assume that the number of states, n is greater than the number of inputs m.

A. Passivity and positive realness

Passive systems are a class of systems which contain no source of energy within, but only absorb externally supplied energy; they however can store energy supplied externally in the past. For LTI systems, positive realness of the transfer matrix is linked to passivity.

Definition 2.1. [1] A real rational transfer function matrix G(s) is said to be positive real if G(s) satisfies:

1)
$$G(s)$$
 is analytic for Re (s) > 0,

2) $G(s) + G(s)^* \ge 0$ for all Re (s) > 0.

It is well-known that an LTI system is passive if and only if its transfer function matrix is positive real and, further, for such systems with a state-space realization (A, B, C, D), we have $(D + D^T) \ge 0$. In addition, if $(D + D^T) > 0$ and if none of the system-poles/zeros lie on the imaginary axis, then we call such systems *strictly passive*.

B. Spectral-zeros

The spectral-zeros of a positive real system with transfer function G(s) are defined as $\mu \in \mathbb{C}$ such that:

$$\det[G(\mu) + G(-\mu)^T] = 0.$$

Considering controllable and observable *n*-th order systems for which $(D + D^T)$ is invertible, the spectral-zeros counted with their multiplicities are exactly the eigenvalues of the Hamiltonian matrix $H \in \mathbb{R}^{2n \times 2n}$ defined as:

$$H := \begin{bmatrix} A - B(D + D^{T})^{-1}C & B(D + D^{T})^{-1}B^{T} \\ -C^{T}(D + D^{T})^{-1}C & -(A - B(D + D^{T})^{-1}C)^{T} \end{bmatrix}.$$
 (2)

The spectral-zeros are symmetric about the imaginary axis $j\mathbb{R}$. For a strictly passive system, H does not have any eigenvalues on the imaginary axis $j\mathbb{R}$, and there are 2n spectral-zeros of the system of which *n*-spectral-zeros are in the \mathbb{C}^- (the open left half complex plane), and their n mirror images in \mathbb{C}^+ plane.

A noteworthy point here is that, for a strictly passive system, even if the system poles and zeros are real, the spectral-zeros need not be real: we consider a simple example to see this. Let the system Σ have the transfer function $G(s) = \frac{n(s)}{d(s)} = \frac{(s+1)(s+2)}{(s+3)(s+4)}$. Then, the spectral-zeros $\mu \in \mathbb{C}$ of the system Σ are the roots of $\xi(s) = n(s)d(-s) + n(-s)d(s) = 2s^4 - 14s^2 + 48$, i.e $\mu = \{\pm 2.05 \pm 0.84j\}$.

For a strictly passive system Σ , of order-*n*, we denote the complex spectral-zeros as $\mu(\Sigma) = (\pm \mu_1, \pm \mu_2, \dots, \pm \mu_n)$ with Re $(\mu_i) < 0$. We denote the set of stable spectral-zeros by $\mu(\Sigma)^-$ with individual elements being $\mu_i(\Sigma)^- = \mu_i$ and the set of antistable spectral zeros as $\mu(\Sigma)^+$ with elements $\mu_i(\Sigma)^+ = -\mu_i$. This paper focuses on formulating conditions such that systems have real spectral-zeros.

C. Symmetric state-space realization

We define a symmetric state-space realization [1], [14] as:

Definition 2.2. A state-space realization (A, B, C, D) is said to be state-space symmetric if $A = A^T$, $D = D^T$ and,

either
$$B = C^T$$
 or $B = -C^T$.

If a system with a given state-space realization can be transformed into the above form, then we call that system symmetric state-space realizable. State-space symmetric systems have been called internally symmetric [22] and are distinct from so-called externally symmetric systems defined as $G(s) = G(s)^T$. Passive systems which admit symmetric state-space realization correspond to physical systems which have only one "type" of energy storage possibility, e.g. only potential energy or only kinetic energy, but not both. Another family of examples which have only one type of storage is RC or RL electrical networks [22]. It is easily verified that a SISO symmetric state-space realization ensures realness of system-poles and system-zeros. Further, it is also known that SISO systems with zeros interlacing poles admit a symmetric state-space realization [20]: we review this next. Recall that this paper assumes minimality of state space realizations, system-order at least two and all poles distinct.

D. Zero-Interlacing-Poles (ZIP) systems

The notion of interlacing between poles and zeros of a system (i.e. ZIP) is well-known and has been widely studied in SISO systems. However, the ZIP property has not been extended for MIMO systems. A first point to note is that for MIMO systems, unlike the notion of system-pole, there are various notions of a system-zero. While there are some inter-relations (like set-inclusions) between these various nonequivalent definitions of zeros of a system [24], we use a notion of system-zero which is natural for systems where the inputs and outputs play quite a symmetric role: for example, passivity studies where the power delivered to a system is $u^T y$.

As stated above, we assume that the MIMO transfer matrix G(s) is square and invertible. For such a G(s), we define the system-zeros as the poles of the transfer matrix $G(s)^{-1}$. Further, we restrict ourselves to systems in which G(s) is biproper, i.e. the feedthrough matrix D in any state-space realization of G(s) is invertible. Under this assumption, the state-space equations: $\dot{x} = Ax + Bu$ and y = Cx + Du can be rewritten as:

$$\dot{x} = (A - BD^{-1}C)x + BD^{-1}y, u = -D^{-1}Cx + D^{-1}y.$$
(3)

Hence, the state space realization of $G(s)^{-1}$ is $(A - BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1})$. We define the system-poles and system-zeros of a given system as:

Definition 2.3. Consider a system Σ with a biproper transfer function matrix G(s). The system-zeros of Σ are defined as the poles of $G(s)^{-1}$.

Therefore for a system Σ , with state-space realization (A, B, C, D), the system-poles and system-zeros are given by the eigenvalues of A and $(A - BD^{-1}C)$ respectively. It is interesting to note that G(s) and the inverse system $G(s)^{-1}$ have the same Hamiltonian matrix and consequently they share the same set of spectral-zeros i.e. the spectral-zeros are invariant to i/o partition. Further, we define ZIP systems as:

Definition 2.4. A system Σ with biproper transfer function matrix G(s), having real system-poles $p_i < 0$ and real system-zeros $z_i < 0$, is said to have the zeros-interlacing-poles (ZIP)

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of the zeros, either (a) or (b) below holds:

$$\begin{array}{ll} (a) & z_1 < p_1 < z_2 < \dots < p_{n-1} < z_n < p_n < 0, \\ (b) & p_1 < z_1 < p_2 < \dots < z_{n-1} < p_n < z_n < 0. \end{array}$$

$$(5)$$

In the context of strict interlacing as above, we assume throughout the paper that the poles are distinct and $n \ge 2$ (except Corollary 4.2, where n = 1). The following result is classic (see [22] and, more recently, [20], for example) and has been included for completeness.

Proposition 2.5. Consider a SISO system Σ with biproper transfer function G(s) having all poles and zeros distinct, real and negative. Assume the feedthrough term $D = G(\infty)$ is positive. Then, the following are equivalent.

- 1) The system admits a symmetric state space realization.
- 2) The system Σ exhibits the ZIP property.

Symmetric state-space systems have wide applications. A class of well-studied systems with collocated actuators and sensors [21], [7], [9] result in $B = C^T$. The assumption of collocated sensors/actuators is also standard in the study of portcontrolled Hamiltonian systems; see [5]. Collocated sensors and actuators in decentralized control systems reduce the complexity and hence are economically advantageous (for example, w.r.t. number of sensor/actuator installations/locations, and signal interfacing). Symmetry within A arises due to, for example, a certain type of reciprocity in the interaction between subsystems in a network of such simpler systems: multi-agent networks with single integrator have been modelled to obtain a symmetric statespace realization [6], [26].

3. INTERLACING PROPERTIES IN SPECTRAL-ZEROS OF MIMO SYSTEMS

In this section we pursue MIMO systems and present the main results of the paper: interlacing properties of system zeros/poles for square, invertible transfer matrices, and later we formulate strict-passivity and spectral-zeros results for MIMO systems.

Theorem 3.1. A MIMO system Σ that admits a symmetric statespace realization with D > 0 and system-poles/zeros in the open left half complex plane satisfies the following:

- 1) The system Σ is strictly passive.
- 2) All spectral-zeros of the system Σ are real.

Proof. Consider a MIMO system Σ with symmetric state-space realization $(A = A^T, B = C^T, D = D^T > 0)$ and the systempoles/zeros lie in the open LHP. (The proof for $B = -C^T$ is identical and is not reproduced here.) The Hamiltonian matrix H is as follows:

$$H = \begin{bmatrix} A - B(D + D^{T})^{-1}C & B(D + D^{T})^{-1}B^{T} \\ -C^{T}(D + D^{T})^{-1}C & -(A - B(D + D^{T})^{-1}C)^{T} \end{bmatrix}.$$

Let $P := B(D + D^T)^{-1}B^T = C^T(D + D^T)^{-1}C$. Since $A = A^T$ and $P = P^T$, the Hamiltonian matrix H can be represented as

$$H = \begin{bmatrix} A - P & P \\ -P & -A + P \end{bmatrix}.$$

¹Ordering and indexing convention: Given a set of real eigenvalues, we order and index the elements $\lambda_1, \ldots, \lambda_n$ in a non-decreasing order to satisfy: $\lambda_{\min} = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n = \lambda_{\max}.$ (4)

property if, after appropriate ordering/indexing¹ of the poles and Using a similarity transformation of the Hamiltonian matrix $(T^{-1}HT)$ where $T = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$ and $T^{-1} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$ we get H =

$$\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} A - P & P \\ -P & -A + P \end{bmatrix} \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} = \begin{bmatrix} A & P \\ -2A & -A \end{bmatrix}$$

Computing the square of the Hamiltonian matrix, we get

$$H^{2} = \begin{bmatrix} A & P \\ -2A & -A \end{bmatrix} \begin{bmatrix} A & P \\ -2A & -A \end{bmatrix} = \begin{bmatrix} A^{2} - 2PA & AP - PA \\ 0 & A^{2} - 2AP \end{bmatrix}.$$
 (6)

Now since the block-diagonal entries satisfy: $(A^2 - 2AP)^T =$ $(A^2 - 2PA)$, eigenvalues of H^2 are same as the eigenvalues of $(A^2 - 2AP)$, but with multiplicities doubled.

Applying, similarity transform of $(A^2 - 2AP)$ using T := $\sqrt{-A}$, the square-root² we get:

$$A^{2} - 2AP = \sqrt{-A}^{-1} (A^{2} - 2AP)\sqrt{-A}$$
$$= A^{2} - 2\sqrt{-A} P\sqrt{-A} .$$

Now, $\sqrt{-A} P \sqrt{-A}$ is symmetric and hence so is $(A^2 - A) = A P \sqrt{-A}$ $2\sqrt{-A} P\sqrt{-A}$). Therefore, $(A^2 - 2AP)$ has real eigenvalues i.e. H^2 has real eigenvalues.

2P). Therefore, using Eqn. (13) Lemma 3.4 we get that $\lambda_i^2(H) \ge \lambda_i^2(-A) = \lambda_i^2(A)$. Now, $\lambda_i(A) < 0$ as the system poles/zeros are in the open left half complex plane. This implies that all the spectral-zeros of Σ are real. Since none of the spectral-zeros are on the imaginary axis, due to Hurwitz poles/zeros of the system and since D > 0, using [23, Theorem 6.4], it follows that the system is strictly passive.

It is important to note that, in order for a system to exhibit the ZIP property, it is essential for the system to have distinct poles. In the SISO case, a symmetric state-space realizable system which is controllable and observable automatically dictates that the system-poles are distinct and hence no additional assumptions are required. However, in the MIMO case a controllable/observable symmetric state-space realizable system does not guarantee distinct system-poles.

Our next main result formulates a sufficient condition for the interlacing of poles and zeros for the MIMO case. In this context, we define the difference between two consecutive system-poles of the ordered set $\{p_i(\Sigma)\}_{i=1}^n$ by $\nu(\Sigma)$:

 $\nu_i(\Sigma) := p_{i+1}(\Sigma) - p_i(\Sigma)$ for $i = 1, 2, \dots, (n-1)$

and $\nu_{\min}(\Sigma)$ is the minimum difference between the poles.

Theorem 3.2. Consider a biproper MIMO system Σ that admits controllable symmetric state-space realization with D > 0 and distinct system-poles in the open left half complex plane. If the minimum difference between the system-poles is greater than the largest eigenvalue of $(BD^{-1}B^T)$, i.e.

$$\nu_{\min}(\Sigma) > \lambda_{\max}(BD^{-1}B^T).$$
⁽⁷⁾

Then, the system-poles and system-zeros interlace strictly:

$$\begin{array}{l} for \ B = C^T &: z_1 < p_1 < z_2 < p_2 < z_3 < \cdots < p_{n-1} < z_n < p_n, \\ for \ B = -C^T &: p_1 < z_1 < p_2 < z_2 < p_3 < \cdots < z_{n-1} < p_n < z_n. \end{array}$$

For the case that $B = C^T$, it easily follows from Theorem 3.1 that the system Σ is strictly passive. However, for the case B =

²For a symmetric and positive definite matrix P, we define \sqrt{P} as the unique symmetric and positive definite matrix that satisfies $(\sqrt{P})^2 = P$ and denote its inverse as $\sqrt{P}^{-1} = P^{-\frac{1}{2}}$.

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 $-C^T$, under the additional condition that the system-zeros are also in the open left half complex plane, it follows that the system Σ is strictly passive.

Proof. We prove Theorem 3.2 for just the case of $B = C^T$ since proof for the other case $B = -C^T$ is analogous. Consider a biproper MIMO system Σ with controllable symmetric statespace realization $A = A^T, B = C^T, D = D^T > 0$ and distinct system-poles in the open left half complex plane. Next, the poles of the system Σ are $p(\Sigma) = \lambda(A)$ and system-zeros are $z(\Sigma) = \lambda(A - BD^{-1}B^T)$. Define $P := BD^{-1}B^T$ and express the system-zeros as: $z(\Sigma) = \lambda(A - P)$. The minimum difference between the system-poles is greater than the largest eigenvalue of $BD^{-1}B^T$: this means

$$\nu_{\min} > \lambda_{\max}(BD^{-1}B^T) = \lambda_{\max}(P)$$

As the system is (A, B) controllable, from the PBH test for controllability, we get that for every left-eigenvector w_i of Asuch that $w_i^T A = \lambda_i w_i^T$, we have $w_i^T B \neq 0$. Further, since $A = A^T$, the left and right eigenvectors are the same, we get that for every eigenvector x_i of A, the vector $B^T x_i \neq 0$. Also, since the system is strictly passive, D > 0, and hence we get:

$$Px_i = BD^{-1}B^T x_i \neq 0$$
 for all $i = 1, 2, ..., n$. (8)

Therefore, utilizing Lemma 3.3 and Eqn (12), we get $z_1 < p_1 < z_2 < \ldots < p_{n-1} < z_n < p_n$, thus completing the proof.

The following lemmas help in the proof of the main results.

Lemma 3.3. Consider symmetric matrices $P, M \in \mathbb{R}^{n \times n}$ with P having distinct eigenvalues and M positive semidefinite symmetric matrix of rank r with (r < n) and with the eigenvalues of each matrix ordered as in Eqn. (4). Suppose the largest eigenvalue of M is at most the minimum difference between any two eigenvalues of P i.e.

$$\lambda_n(M) \leq \min_{i=1,\ldots,n-1} (\lambda_{i+1}(P) - \lambda_i(P)).$$
(9)

Then, the following statements hold.

1) The eigenvalues of P and (P + M) interlace, i.e.³

$$\lambda_i(P) \leq \lambda_i(P+M) \leq \lambda_{i+1}(P)$$
 for each $i = 1, 2, \dots, n.$ (10)

- 2) Further, if $Mx \neq 0$ for every eigenvector x of P and the inequality is strict in Eqn. (9), then
- a) the eigenvalues of (P + M) are distinct
- b) the eigenvalues of P and P+M interlace strictly:

$$\lambda_i(P) < \lambda_i(P+M) < \lambda_{i+1}(P) \text{ for each } i = 1, 2, \cdots, n.$$
(11)

The proof, available in [12], involves a meticulous use of Weyl's inequality theorem (see [11, Thm 4.3.1], for example).

Following the above Lemma 3.3, the relation between eigenvalues of P and (P - M) is given as:

$$\lambda_i(P-M) < \lambda_i(P) < \lambda_{i+1}(P-M) \quad \text{for } i = 1, 2, \dots, n \quad (12)$$

if, $\lambda_n(M) < \min_{\substack{i=1,\ldots,n-1 \\ eigenvector x \text{ of } P, Mx \neq 0.}} (\lambda_{i+1}(P) - \lambda_i(P))$ and for every

³Note that amongst the two inequalities within Eqn. (10), index *i* varies from 1 to *n* in the first, while varies from 1 to n-1 in the second. This slight abuse of indexing notation helps convey the interlacing property and avoids repetition. Same indexing is used within other such interlacing inequalities also.



Fig. 2: Scaling in feedforward path (Corollary 3.5)

Lemma 3.4. Suppose P and $M \in \mathbb{R}^{n \times n}$ are both symmetric, let P be positive definite and M be singular and positive semidefinite. Then, the following hold.

- 1) The set of eigenvalues of the products of P(P+M) and (P+M)P coincide, i.e. $\lambda(P(P+M)) = \lambda((P+M)P)$.
- 2) Eigenvalues of the product P(P + M) are real.
- 3) Eigenvalues of the product P(P + M) lie between the eigenvalues of P^2 and $(P + M)^2$.

$$\lambda_i^2(P) \le \lambda_i(P(P+M)) \le \lambda_i^2(P+M) \text{ for } i=1,2,...,n.$$
 (13)

4) Suppose for every eigenvector x of P, we have $Mx \neq 0$. Then each of the inequalities in Eqn. (13) are strict, i.e.

$$\lambda_i^2(P) < \lambda_i(P(P+M)) < \lambda_i^2((P+M)) \text{ for } i = 1, 2, \dots, n$$

The proof, available in [12], involves the use of the square-root of the matrices and a careful note of the non-commutativity of these matrices while proving the inequalities. The next result, a corollary to the main MIMO results, brings out the significance of the feedthrough term (vis-à-vis the contribution of the effect of the input on the output *through the state*): modelling system dynamics with a rescaled time-axis would amount to changing the relative contribution of the feedthrough term.

Corollary 3.5. Suppose a biproper MIMO system Σ admits a controllable symmetric state-space realization (A, B, C, D)with D > 0 and distinct system-poles in the open left half complex plane. Scaling⁴ D to define $\tilde{D} := \eta D$ with $\eta \in \mathbb{R}_+$, denote (A, B, C, \tilde{D}) by $\tilde{\Sigma}$. Then, for sufficiently large η , the system-poles/zeros of $\tilde{\Sigma}$ are strictly interlaced.

Proof. Consider the assumptions given in Corollary 3.5. The system-poles and system-zeros of $\tilde{\Sigma}$ are given as:

$$p(\Sigma) \coloneqq \lambda(A), \text{ and } z(\Sigma) \coloneqq \lambda(A - \frac{1}{\eta} B \tilde{D}^{-1} B^T).$$

From Theorem 3.2 we get that for the system $\tilde{\Sigma}$, the systemzeros and poles are interlaced if $\nu_{\min}(\Sigma) > \lambda_{\max}(B\tilde{D}^{-1}B^T)$, which is satisfied if $\nu_{\min}(\Sigma) > \frac{1}{\eta}\lambda_{\max}(BD^{-1}B^T)$. Thus, for η sufficiently large, the above inequalities are satisfied, and hence the system-poles and zeros are interlaced strictly. \Box Now we move to another interesting result with respect to the interlacing of spectral-zeros along with system-poles/zeros.

Theorem 3.6. Consider a biproper MIMO system Σ that admits controllable symmetric state-space realization with D > 0 and distinct system-poles/zeros in the open left half complex plane. If the system exhibits ZIP property, i.e. after ordering and indexing the poles/zeros as in Eqn. (4), we have:

$$\begin{array}{l} for \quad B = C^T \quad : z_1 < p_1 < z_2 < p_2 < z_3 < \cdots < p_{n-1} < z_n < p_n, \\ for \quad B = -C^T \quad : p_1 < z_1 < p_2 < z_2 < p_3 < \cdots < z_{n-1} < p_n < z_n. \end{array}$$

⁴The scaling of *D* by a positive scalar η is just one way to make the feedthrough term *D* (assumed symmetric and positive definite) 'sufficiently large relative to' the strictly proper part, namely, the input's contribution to the output *through* the state; figure 2 illustrates this.

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Then, the stable spectral-zeros of the system $\mu(\Sigma)^-$ are also of Lemma 3.4 and Eqn. (16) we get: interlaced strictly between the pair of system poles $p(\Sigma)$ and zeros $z(\Sigma)$:

for
$$B = C^T$$
 : $z_1 < \mu_1 < p_1 < z_2 < \mu_2 < p_2 < \dots < \mu_n < p_n$,
for $B = -C^T$: $p_1 < \mu_1 < z_1 < p_2 < \mu_2 < z_2 < \dots < \mu_n < z_n$.

The proof of Theorem 3.6 is presented within the proof of Theorem 3.7 as the former is a special case of the latter. The next result states that, even if the poles and system zeros of MIMO systems (that admit a symmetric state-space realization, etc) are not interlaced, the spectral-zeros still occur between the appropriate system-pole/zero pair.

Theorem 3.7. If a biproper MIMO system Σ admits symmetric state-space realization with D > 0 and distinct systempoles/zeros in the open left half complex plane, then each stable spectral-zero occurs between a system pole-zero pair. More precisely, suppose the system-poles $p_i(\Sigma)$, system-zeros $z_i(\Sigma)$ and the stable spectral-zeros $\mu_i(\Sigma)^-$ are ordered and indexed as in Eqn. (4). Then:

$$z_i(\Sigma) \leq \mu_i(\Sigma)^- \leq p_i(\Sigma) \quad \text{for } B = +C^T , p_i(\Sigma) \leq \mu_i(\Sigma)^- \leq z_i(\Sigma) \quad \text{for } B = -C^T .$$
(14)

Proof. Consider a biproper MIMO system Σ which admits symmetric state-space realization $(A = A^T, B = \pm C^T, D = D^T > 0)$ with distinct system-poles/zeros in the open left half complex plane. We prove below only for the case $B = C^T$, since the proof of the case $B = -C^T$ follows closely.

We know that the spectral-zeros $\mu(\Sigma)$ are the eigenvalues of the Hamiltonian matrix $H(\Sigma)$ with respect to the passivity supply rate. From the Eqn. (6) the eigenvalues of square of Hamiltonian matrix are expressed as:

$$\lambda(H^{2}(\Sigma)) = \lambda(A^{2} - 2B(D + D^{T})^{-1}B^{T}A) = \lambda(A^{2} - BD^{-1}B^{T}A)$$
$$= \lambda((-A)^{2} + BD^{-1}B^{T}(-A)) .$$

Define P := -A and $M := BD^{-1}B^T$ and the set of stable spectral-zeros as $\mu(\Sigma)^+$. The above equation is nothing but

$$\mu^2(\Sigma)^+ = \lambda((P+M)P).$$

We order and index the set of stable spectral-zeros $\mu(\Sigma)^+$ and eigenvalue sets $\lambda(P)$ and $\lambda(P+M)$ as per Eqn. (4). As P is symmetric and positive definite and M is symmetric and positive semi-definite, utilizing Lemma 3.4 we get that for i = 1, 2, ..., n:

and hence
$$\lambda_i^2(P) \le \mu_i^2(\Sigma)^+ \le \lambda_i^2(P+M)$$
$$\lambda_i(P) \le \mu_i(\Sigma)^+ \le \lambda_i(P+M) .$$
(15)

Next, note that the system-poles and system-zeros are given by the eigenvalues sets $-\lambda(P)$ and $-\lambda(P+M)$ respectively. However, while indeed $p(\Sigma) = -\lambda(P)$, the indexing convention followed in Footnote 1, would have reversal of element-wise inequalities, and thus from Eqn. (15) we get:

$$z_i(\Sigma) \leq \mu_i(\Sigma)^- \leq p_i(\Sigma)$$
 for each $i = 1, 2, \dots, n.$ (16)

This completes the proof of Theorem 3.7.

In order to prove Theorem 3.6, we use the PBH test of controllability, and using symmetry of A, we get that for every eigenvector x of A, $B^T x \neq 0$. Now P has the same set of orthogonal eigenvectors of A, therefore for every eigenvector xof P we get that $Mx \neq 0$. Therefore utilizing the Statement 4

$$z_i(\Sigma) < \mu_i(\Sigma)^- < p_i(\Sigma) \text{ for each } i = 1, 2, \dots, n.$$
(17)

Further, the system Σ exhibits ZIP property then:

$$z_1 < p_1 < z_2 < p_2 < z_3 < \dots < p_{n-1} < z_n < p_n .$$
 (18)

Therefore, combining Eqns. (18) and (17) we get:

$$z_1 < \mu_1 < p_1 < z_2 < \mu_2 < p_2 < z_3 < \ldots < p_{n-1} < z_n < \mu_n < p_n$$
.

This completes the proof of Theorem 3.6 also. Since none of the spectral-zeros lie on $j\mathbb{R}$, the system Σ is strictly passive.

We saw earlier in Eqn. (16) about how the poles and zeros need not be interlaced for MIMO systems: an extreme case being when two SISO systems are decoupled subsystems of a MIMO system. It is interesting to note that for any MIMO system in symmetric state-space realization, irrespective of ZIP property, each spectral-zero lies between a pole-zero pair, i.e., after appropriate ordering, $z_k(\Sigma_Z) \leq \mu_k(\Sigma_Z)^- \leq p_k(\Sigma_Z)$. Linear algebraic methods have also been pursued for passivity/dissipativity in [4].

4. FURTHER RESULTS FOR SISO SYSTEMS

In this section, we first show some of the new results for SISO systems (Lemma 4.1 and Corollary 4.2). Later in this section, we present an alternate and simpler proof for the SISO-specific case of the main result. Further, Theorem 3.6 is a new result for, not just the MIMO case, but for the SISO case too. The next result relates the spectral-zeros with the system poles/zeros.

Lemma 4.1. Consider a SISO ZIP system Σ with real and stable poles p_1, \ldots, p_n and zeros z_1, \ldots, z_n . Then, the following are true.

1) The product of the n-stable/antistable spectral-zeros equals the square root of the product of system-zeros and systempoles, i.e. ignoring the signs,

$$|\mu_1\mu_2\cdots\mu_n| = \sqrt{p_1p_2\cdots p_n \cdot z_1z_2\cdots z_n}.$$

2) The sum of the squares of the n-stable/antistable spectralzeros $\mu_1^2 + \mu_2^2 + \dots + \mu_n^2$ is

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} \sum_{k=i+1}^{n} p_i p_k - \sum_{i=1}^{n} \sum_{k=i+1}^{n} z_i z_k.$$

The proof is skipped and can be found in [12], but the same same proof techniques and claims hold for the more general case when poles and zeros need not be interlaced, and, in fact, need not even be real, and nor do the spectral-zeros have to be real; we do not digress into this since we focus on interlacing aspects for SISO/MIMO poles/zeros/spectral-zeros, which is relevant for the real case. A special case of the above lemma is when a SISO system has just one spectral-zero: namely passive SISO systems with only one pole and one zero, the spectral-zero is the geometric mean of the pole and zero values.

Corollary 4.2. Consider a biproper single-order SISO system with transfer function $G(s) = \frac{s-z}{s-p}$, with p, z < 0. Then the stable and anti-stable spectral-zeros satisfy $\pm \mu = \pm \sqrt{pz}$.

While many of the MIMO results of Section 3 are extension of known SISO results, some of the MIMO results are new for the SISO case also: this is elaborated below. Statement 2 of the Theorem 3.1 is novel for SISO case too. Further, Theorem 3.2's inferences hold for SISO systems even without imposing Inequality (7). For SISO systems, Theorem 3.6 reduces

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to the statement that for a minimum-phase, biproper stable system admitting a symmetric state space realization, not just are the system-zeros and poles interlaced, but in fact, the spectral-zeros too are interlaced, a novel result for the SISO case too. SISO-specific simpler proofs of Theorems 3.6 and 3.7 involve a meticulous use of Bolzano's theorem⁵ (about change of signs of a polynomial function over an interval when the interval contains a root of the polynomial) and the following lemma. Both the SISO-proofs, skipped here for paucity of space, can be found in [12].

Lemma 4.3. Consider the function $f(x) : \mathbb{C} \to \mathbb{C}$ defined by

$$f(x) \coloneqq \sum_{k=1}^{n} \frac{q_k}{x^2 - p_k^2}$$
(19)

with p_k, q_k real and $q_k > 0$ for k = 1, ..., n. Then, f(x) has only real zeros.

5. EXAMPLES

We illustrate the above presented theorems with two examples.

Example 5.1. (Decoupled subsystems:) We consider a MIMO transfer function matrix in which we have two subsystems that are 'decoupled', and we see how a sufficiently high value of the scaling parameter η causes the interlacing of, not just the poles and system-zeros, but also the spectral-zeros: like the SISO case.

$$G = \begin{bmatrix} 1 + \frac{1}{s+3} + \frac{1}{s+7} & 0\\ 0 & 1 + \frac{1}{s+4} + \frac{1}{s+8} \end{bmatrix} \ .$$

Using Gilbert's state-space realization:

$$A = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -8 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = C^T; \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore the poles, zeros and spectral-zeros of the system are:

The system-poles and system-zeros are not interlaced $(z_1 \text{ and } z_2 \text{ are smaller than } p_1)$ due to the choice of the two decoupled SISO subsystems. In order to see the effect of scaling of feedthrough matrix D, we choose a scalar scaling factor $\eta \in \mathbb{R}_+$ and define $D = \eta \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We increase η and tabulate its effect on the system-poles and system-zeros interlacing.

From Table I, it is evident that as the scaling factor (η) is increased system-zeros and system-poles move closer to interlace condition. When the feedthrough matrix D becomes *sufficiently large* i.e. $\eta \ge 2$ then the system-poles and system-zeros along with spectral-zeros get interlaced.

⁵Bolzano's theorem: Suppose a function $f : \mathbb{R} \to \mathbb{R}$ is continuous in the interval [a, b]. Suppose $f(a) \cdot f(b) < 0$ then there exists an x_0 in the interval [a, b] such that $f(x_0) = 0$. Conversely, if $f(x_1) \cdot f(x_2) > 0$ for each x_1, x_2 in the interval [a, b], then f(x) has no roots in the interval [a, b].

Table I: Effect of scaling parameter η on system-zeros/poles and spectral-zeros interlacing: of Example 5.1

Scaling parameter	System pole/zero	(1)	(2)	(3)	(4)
η , remark	properties				
$\eta = 1$	system-zeros (z_i)	-9.24	-8.24	-4.76	-3.76
system-poles/zeros: not interlaced	spectral-zeros (μ_i)	-8.52	-7.51	-4.40	-3.40
	system-poles (p_i)	-8.00	-7.00	-4.00	-3.00
$\eta = 1.2$	system-zeros (z_i)	-9.00	-8.00	-4.67	-3.67
system-poles/zeros: verge-of interlaced	spectral-zeros (μ_i)	-8.43	-7.43	-4.35	-3.34
	system-poles (p_i)	-8.00	-7.00	-4.00	-3.00
$\eta = 2$	system-zeros (z_i)	-8.56	-7.56	-4.44	-3.44
system-poles/zeros: after interlaced	spectral-zeros (μ_i)	-8.26	-7.25	-4.22	-3.22
	system-poles (p_i)	-8.00	-7.00	-4.00	-3.00
$\eta = 100$ system-poles/zeros: continue to be interlaced					
	system-zeros (z_i)	-8.01	-7.01	-4.01	-3.01
	spectral-zeros (μ_i)	-8.005	-7.005	-4.005	-3.005
	system-poles (p_i)	-8.00	-7.00	-4.00	-3.00

Example 5.2. (Multi-agent example:) Consider a multi-agent network arranged in a path graph as shown in the figure below: If we consider that the node-1 and node-4 are chosen as the



Fig. 3: Multi-agent network in path graph arrangement

controlled nodes and inputs to the nodes are current i_1 and i_4 injected in the node.



Fig. 4: Multi-agent network with sources at two nodes

If the node voltages (V_1, V_2, V_3, V_4) are considered as the state variables and the inputs are i_1 and i_4 with outputs as the voltages across the current sources then, the state-space realization can be expressed as:

$$A = \begin{bmatrix} \frac{-1}{R_c C_c} - \frac{1}{R_p C_c} & \frac{1}{R_c C_c} & 0 & 0\\ \frac{1}{R_c C_c} & \frac{-2}{R_c C_c} - \frac{1}{R_p C_c} & \frac{1}{R_c C_c} & 0\\ 0 & \frac{1}{R_c C_c} & \frac{-2}{R_c C_c} - \frac{1}{R_p C_c} & \frac{1}{R_c C_c} \\ 0 & 0 & \frac{1}{R_c C_c} & \frac{-1}{R_c C_c} - \frac{1}{R_c C_c} \end{bmatrix}$$
$$B = \begin{bmatrix} \frac{1}{C_c} & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & \frac{1}{C_c} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 \end{bmatrix}, and D = \begin{bmatrix} R_s & 0\\ 0 & R_s \end{bmatrix}.$$

Letting $R_s = 0.1\Omega$, $R_c = 1\Omega$, $C_c = 1F$, $R_p = 10\Omega$, we get:

$$A = \begin{bmatrix} -1.1 & 1.0 & 0.0 & 0.0 \\ 1.0 & -2.1 & 1.0 & 0.0 \\ 0.0 & 1.0 & -2.1 & 1.0 \\ 0.0 & 0.0 & 1.0 & -1.1 \end{bmatrix}, \ B = C^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \ D = 0.1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the poles and zeros of the systems are:

System-poles : $p_1 = -3.51$, $p_2 = -2.10$, $p_3 = -0.69$, $p_4 = -0.10$, System-zeros : $z_1 = -11.22$, $z_2 = -11.20$, $z_3 = -2.98$, $z_4 = -1.0$.

We can see that the poles and zeros are not interlaced $(z_1 \text{ and } z_2 \text{ both are less than } p_1)$. Now to see the effect of the scaling of feedthrough matrix D, we choose a scalar scaling factor $\eta \in \mathbb{R}_+$ and define $D \coloneqq \eta \times \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$.

We increase the scaling factor and tabulate its effect on the system-poles and system-zeros interlacing. From Table II, we

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infer that as the scaling factor (η) is increased system-zeros and system-poles move closer to interlace condition. When the feedthrough matrix D is *sufficiently large* i.e. $\eta \ge 10$, then the system-poles and system-zeros together with spectral-zeros get interlaced.

Table II: Effect of scaling parameter η on system-zeros/poles and spectral-zeros interlacing: Example 5.2

	-				
Scaling parameter	System pole/zero	(1)	(2)	(3)	(4)
η , remark	properties				
η = 1, system-poles/zeros: not interlaced	system-zeros (z_i)	-11.22	-11.20	-2.98	-1.00
	spectral-zeros (μ_i)	-4.44	-3.91	-2.02	-0.39
	system-poles (p_i)	-3.51	-2.10	-0.69	-0.10
η = 5, system-poles/zeros: verge-of interlaced	system-zeros (z_i)	-4.10	-3.51	-2.10	-0.69
	spectral-zeros (μ_i)	-3.67	-2.56	-1.24	-0.28
	system-poles (p_i)	-3.51	-2.10	-0.69	-0.10
$\eta = 10,$ system-poles/zeros: after interlaced	system-zeros (z_i)	-3.72	-2.72	-1.48	-0.48
	spectral-zeros (μ_i)	-3.59	-2.34	-1.01	-0.22
	system-poles (p_i)	-3.51	-2.10	-0.69	-0.10
η = 100, system-poles/zeros: continue to be interlaced					
	system-zeros (z_i)	-3.53	-2.15	-0.77	-0.15
	spectral-zeros (μ_i)	-3.52	-2.12	-0.73	-0.12
	system-poles (p_i)	-3.51	-2.10	-0.69	-0.10

6. CONCLUDING REMARKS

In the context of MIMO systems, we first used the definition of system-zeros of a system with biproper transfer function G(s), as the poles of its inverse system $G(s)^{-1}$, and showed that both systems have the same Hamiltonian matrix and the same spectral-zeros; a property specific to the i/o invariant supply rate: $u^T y$. Next, pursuing with systems that admit a symmetric state-space realization, we next proved realness of all the spectral-zeros (Theorem 3.1). Using existing/new properties of differences in eigenvalues of pairs of symmetric matrices, we proved in Theorem 3.2 that if the poles of a MIMO system are 'relatively well-separated' (w.r.t. the B scaled appropriately by D), then the poles and zeros are interlaced. We also showed that this separation is ensured by systems with a sufficiently large feedthrough matrix (Lemma 3.5). We proved that stable MIMO systems with symmetric state-space realizations have not just ZIP but also spectral-zero interlacing: Theorems 3.6 & 3.7.

Then in Section 4 we presented some new results for the SISO systems. We showed in Lemma 4.1 the relation between the product and sum of squares of the spectral-zeros with the system-poles and system-zeros, and obtained as a special case that for an order-1 system, the spectral-zero is the geometric mean of the system-pole and system-zero. In Section 5, we elaborated on a few examples (linked to the RC/RL network of Section 1-B, and a multi-agent network), for which the results in our paper were applicable.

A possible direction for further work is to formulate milder conditions or conditions that are both necessary and sufficient for realness of spectral-zeros and/or interlacing properties of system poles/zeros. Another future direction of work may be development of efficient algorithms for computation of eigenvalues of the Hamiltonian matrix of such systems when it is a priori known that all the eigenvalues are real. In this context, guaranteeing that the spectral-zeros are *real* helps because the corresponding optimal trajectories are not oscillatory but pure exponentials. Further, it is then possible to develop and use numerical algorithms that are tailored for such cases and thus improved numerical accuracy and computational efficiency in

the context of, for example, data analysis; see [5]. Another direction of further research is to explore the extent to which Model-Order-Reduction methods developed for ZIP SISO systems [20] are extendable to symmetric state-space MIMO systems having the ZIP property.

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